Examples of noncommutative geometrical spaces

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Abstract

We develop some examples of spectral triples in noncommutative geometry, on the theme of the noncommutative torus but with several variations.

Contents

Introduction

The purpose of these lectures is to survey several examples of a central construction in Noncommutative Geometry. These examples, collectively called spectral triples, have as their common motivation the search for a suitable description of quantum spacetime. In quantum theory, observable quantities are represented by operators on Hilbert spaces, that in general do not commute. Thus we need a mathematical toolbox which incorporates spaces with noncommuting coordinates. Or rather, we replace familiar coordinate algebras of ordinary spaces by noncommutative algebras represented on Hilbert spaces, and somewhat improperly declare that these are "coordinate algebras of noncommutative spaces".

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Our first example is the noncommutative torus, which was historically the first genuinely noncommutative space carrying a family of differentiable functions [9]; later, it reemerged in the mathematical description of aperiodic solids and the quantum Hall effect [4], and more recently as an important model is string theory [16, 43]. We discuss briefly how the K-theory of the noncommutative torus yields a "gap labelling" of band spectra that arise in solid-state physics.

We then examine how spectral triples appear in the "commutative" context of ordinary (spin) manifolds [14, 23], and compute the Dirac spectra in detail for tori and spheres of dimension 3. Motivated by these examples, we lay out the general framework of spectral triples in the noncommutative case. Historically, this framework arose from an attempt to interpret the families of known elementary particles in terms of the metric on a mildly noncommutative spacetime. This yielded a considerable, if partial, success as a noncommutative interpretation of the Standard Model at the classical level [13, 28, 32].

Much recent attention has been given to the construction of new examples of noncommutative spaces, capable of supporting spectral triples. We consider two fairly elementary constructions of this type, each of which is a variant of the noncommutative torus. The first involves a noncommutative twisting of compact manifolds foliated by the action of a product of (at least two) circles [15, 18, 49]. The second concerns the Moyal plane, familiar from the phase-space approach to quantum mechanics, which may be regarded as a noncompact covering of the noncommutative torus [21], and is the starting point of many studies in noncommutative field theory: it provides a test case for developing the noncommutative geometry of noncompact spaces.

1 The noncommutative torus and its algebraic toolkit

In noncommutative geometry (NCG from now on), a (locally compact, Hausdorff) topological space X may be described indirectly by its algebra $C_0(X)$ of continuous functions vanishing at infinity. When X is compact, we may omit "vanishing at infinity" and the constant function 1 is a unit in $C(X)$. The Gelfand–Naĭmark theorem assures us that any commutative unital C^*-algebra is of the form $C(X)$ for a suitable compact Hausdorff space X, and there is a contravariant functor $X \mapsto C(X)$ making the passage from topological spaces to commutative C^*-algebras a two-way street. Thus, at the topological level, NCG simply drops the commutativity of the algebra and concentrates on the theory of C^*-algebras.

When X is a torus $\mathbb{T}^l$, we can expand elements of $C(\mathbb{T}^l)$ as Fourier series:

$$f(\phi_1, \dots, \phi_l) = \sum_{r\in\mathbb{Z}^l} c_r\, e^{2\pi i(r_1\phi_1+\cdots+r_l\phi_l)}, \tag{1}$$

with $0 \leqslant \phi_k < 1$ for $k = 1, \dots, l$. Setting $u_k(\phi_1, \dots, \phi_l) := e^{2\pi i\phi_k}$, we can rewrite this as

$$f = \sum_{r\in\mathbb{Z}^l} c_n\, u_1^{r_1} u_2^{r_2} \cdots u_l^{r_l},$$

and it is clear that the $u_1, u_2, \dots, u_l$ generates a dense subalgebra (the trigonometric polynomials) of $C(\mathbb{T}^l)$. Thus, the coordinate algebra of the torus $\mathbb{T}^l$ is the unital C^*-algebra generated by l elements $u_1, \dots, u_l$. These are *unitary* elements, namely

$$u_k u_k^* = u_k^* u_k = 1,$$

they commute:

$$u_k u_j = u_j u_k,$$

and there are no further algebraic relations among them.

The noncommutative torus $\mathbb{T}^l_\Theta$ may be described abstractly as the unital C^*-algebra densely generated by l unitary elements $u_1, \dots, u_l$, where the commutativity constraint is replaced by

$$u_k u_j = e^{2\pi i \theta_{jk}}\, u_j u_k, \tag{2}$$

given a fixed parameter matrix $\Theta = [\theta_{jk}]$, which is a real, skewsymmetric $l \times l$ matrix. When $l = 2$, we abbreviate $\theta := \theta_{12}$, $u := u_1$, $v := u_2$, so that the single commutation relation becomes

$$vu = e^{2\pi i \theta}\, uv. \tag{3}$$

This relation appears at the very beginning of quantum mechanics. Indeed, on the Hilbert space $L^2(\mathbb{R})$ of a one-dimensional configuration space $\mathbb{R}$, there are unitary operators of translation V_θ and phase modulation U_θ given by

$$V_\theta \psi(t) := \psi(t + \theta), \qquad U_\theta \psi(t) := e^{2\pi i t}\, \psi(t).$$

One checks immediately that $V_\theta U_\theta = e^{2\pi i \theta}\, U_\theta V_\theta$. In order to have a "compact" noncommutative space, we replace $L^2(\mathbb{R})$ by $L^2(\mathbb{T})$ – regarded as periodic functions with period 1 – so that V_θ is reinterpreted as the rigid rotation by an angle $2\pi\theta$. The unitary U_θ generates the C^*-algebra $C(\mathbb{T})$ realized as multiplication operators on $L^2(\mathbb{T})$, and $V_\theta^* f V_\theta = \alpha(f)$, where $\alpha(f)(\phi) = f(\phi + \theta)$ for $f \in C(\mathbb{T})$. In other words, the C^*-algebra generated by V_θ and U_θ is the *crossed product* C^*-algebra [34],

$$A^2_\theta := C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}.$$

It turns out that the obvious $*$-homomorphism from $C(\mathbb{T}^2_\theta)$ to A^2_θ determined by $u \mapsto U_\theta$, $v \mapsto V_\theta$ is an isomorphism, so that A^2_θ is the universal C^*-algebra generated by two unitaries u, v satisfying (3).

► The isomorphism type of A^2_θ is highly dependent on θ. Indeed, it is clear that for *integral* $\theta = m$, the commutation relation (3) is trivial, and $A^2_m \simeq C(\mathbb{T}^2)$ by the aforementioned universality result. Furthermore, there is an isomorphism $A^2_\theta \simeq A^2_{m+\theta}$ for any $m \in \mathbb{Z}$, since (3) is unchanged by $\theta \mapsto m+\theta$. Also, since $uv = e^{2\pi i (1-\theta)}\, vu$, the switch $u \leftrightarrow v$ yields an isomorphism $A^2_\theta \simeq A^2_{1-\theta}$. These are in fact the only isomorphisms between the 2-torus algebras: the interval $[0, \frac{1}{2}]$ parametrizes a family of nonisomorphic C^*-algebras.

When θ is rational, $\theta = p/q$ with p, q coprime and $q > 0$, we can represent the relation $RS = \lambda\, SR$, where $\lambda = e^{2\pi i p/q}$, by a pair of unitary "clock and shift" matrices:

$$R := \begin{pmatrix} 1 & & & & 0 \\ & \lambda & & & \\ & & \lambda^2 & & \\ & & & \ddots & \\ 0 & & & & \lambda^{q-1} \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & 1 & 0 \end{pmatrix}, \tag{4}$$

satisfying $R^q = S^q = 1$. Then $A^2_{p/q}$ can be realized as a subalgebra of $B = C(\mathbb{T}^2) \otimes M_q(\mathbb{C})$ under the correspondence $v \mapsto u_1 \otimes R$, $u \mapsto u_2 \otimes S$, namely, the subalgebra which is fixed under the action on B of the finite abelian group $H = (\mathbb{Z}/q\mathbb{Z})^2$ generated by the transformations $f(z_1, z_2) \otimes T \mapsto f(z_1, \lambda^{-1} z_2) \otimes RTR^{-1}$ and $f(z_1, z_2) \otimes T \mapsto f(\lambda z_1, z_2) \otimes STS^{-1}$. We can regard B as the space of continuous sections of a trivial matrix bundle $E_q \to \mathbb{T}^2$ with fibre $M_q(\mathbb{C})$, with an action of H by bundle automorphisms generated by $(z_1, z_2; T) \mapsto (z_1, \lambda^{-1} z_2; RTR^{-1})$ and $(z_1, z_2; T) \mapsto (\lambda z_1, z_2; STS^{-1})$. Thus, $A^2_{p/q}$ is isomorphic to the space of continuous sections of the matrix bundle $E_q/H \to \mathbb{T}^2/H$, and $\mathbb{T}^2/H$ may be identified with $\mathbb{T}^2$ through the wrapping map $(z, w) \mapsto (z^q, w^q)$. In summary, $A^2_{p/q}$ is isomorphic to the space of continuous sections of a bundle $E_q/H \mapsto \mathbb{T}^2$, whose fibres are copies of $M_q(\mathbb{C})$; its centre is the algebra of scalar-matrix sections, isomorphic to $C(\mathbb{T}^2)$.

When the parameter θ is irrational, A^2_θ is often called an *irrational rotation algebra*. In this case, the centre is trivial; indeed, any irrational A^2_θ is a *simple* C^*-algebra, as we shall see later.

► For higher-dimensional tori ($l > 2$), we can avoid much cumbersome notation by introducing a *Weyl system* of unitary elements $\{ u^r : r \in \mathbb{Z}^l \}$. For $l = 3$, we take [3, 23]

$$u^r := \exp\{\pi i (r_1\theta_{12}r_2 + r_1\theta_{13}r_3 + r_2\theta_{23}r_3)\}\, u_1^{r_1} u_2^{r_2} u_3^{r_3};$$

for $l > 3$, the exponent is $\pi i \sum_{j<k} r_j \theta_{jk} r_k$; these coefficients entail that $(u^r)^* = u^{-r}$ in all cases. The product rule is

$$u^r\, u^s = \sigma(r, s)\, u^{r+s}, \qquad \sigma(r, s) := \exp\bigl\{-\pi i \textstyle\sum_{j,k} r_j \theta_{jk} s_k\bigr\}. \tag{5}$$

Notice that $|\sigma(r, s)| = 1$ and $\sigma(r, \pm r) = 1$ by skewsymmetry of θ. This σ is in fact a 2-cocycle for the abelian group $\mathbb{Z}^l$: that is to say, it satisfies the *cocycle relation*:

$$\sigma(r, s + t)\sigma(s, t) = \sigma(r, s)\sigma(r + s, t) \quad \text{for all} \quad r, s, t \in \mathbb{Z}^l, \tag{6}$$

which incorporates the associativity of the products of three Weyl elements. There is a universal C^*-algebra $C^*(\mathbb{Z}^3, \sigma)$ generated by the u^r subject to (5), called a "twisted group C^*-algebra". We write A^l_Θ or $C(\mathbb{T}^l_\Theta)$ instead of $C^*(\mathbb{Z}^3, \sigma)$ in order to speak colloquially of this algebra as if it consisted of continuous functions on a "noncommutative space" called $\mathbb{T}^l_\Theta$.

► The C^*-algebras $C(\mathbb{T}^l_\Theta)$, for Θ skewsymmetric, come naturally equipped with an action of the abelian Lie group $\mathbb{T}^l$. In the commutative case $\Theta = 0$, this is just given by the action of the torus on itself by rotations. In the general case, the action is fully specified by

$$z \cdot u^r := z_1^{r_1} \cdots z_l^{r_l}\, u^r, \quad \text{for all} \quad z = (z_1, \dots, z_l) \in \mathbb{T}^l. \tag{7}$$

This is consistent with the relations (5). (The one-dimensional subspace generated by u^r, for each $r \in \mathbb{Z}^l$, is therefore a spectral subspace for this Lie group action.)

To move on to differential geometry, we need to introduce the subalgebra of *smooth* functions on these compact noncommutative spaces. The starting point is again the Fourier series expansions on ordinary tori (1). Indeed, for many continuous functions this expansion is formal, because convergence is not guaranteed. However, it is known, and easy to check, that the smooth functions are precisely those for which the coefficient l-sequence is rapidly decreasing: we say that "$c_r \to 0$ rapidly" if $(1 + r^2)^k\, |c_r|^2$ is bounded for all $k = 1, 2, 3, \dots$. (This condition is clearly needed in order to take partial derivatives term-by-term in the Fourier series.) We therefore introduce

$$C^\infty(\mathbb{T}^l_\Theta) := \{ a = \textstyle\sum_r a_r\, u^r : a_r \to 0 \text{ rapidly} \}, \tag{8}$$

using the Weyl system as "elementary powers" in the expansion. Within this algebra, we can manipulate the series term-by-term without hesitation. Under the seminorms p_k defined by $p_k(a)^2 := \sup_r (1 + r^2)^k \, |a_r|^2$, the algebra $C^\infty(\mathbb{T}^l_\Theta)$ is Fréchet, i.e., it has a complete, metrizable locally convex topology.

The subalgebra $C^\infty(\mathbb{T}^l_\Theta)$ of $C(\mathbb{T}^l_\Theta)$ is a *pre-C^*-algebra*: that is to say, it is stable under the holomorphic functional calculus defined for the whole C^*-algebra. In particular, if $a \in C^\infty(\mathbb{T}^l_\Theta)$ is invertible in $C(\mathbb{T}^l_\Theta)$, then a^{-1} lies already in $C^\infty(\mathbb{T}^l_\Theta)$; for Fréchet algebras, this closure-under-inversion condition is in fact equivalent to stability under holomorphic functional calculus. This property guarantees that the K-theories of $C^\infty(\mathbb{T}^l_\Theta)$ and of $C(\mathbb{T}^l_\Theta)$ coincide: see, for instance, [23, §3.8].

By averaging the effect of the torus action on the C^*-algebra $C(\mathbb{T}^l_\Theta)$, we obtain an operator

$$E(a) := \int_{\mathbb{T}^l} (z \cdot a) \, dz$$

(where dz denotes the *normalized* Haar measure on $\mathbb{T}^l$), mapping $C(\mathbb{T}^l_\Theta)$ continuously onto the fixed-point subalgebra of the action; indeed, $\|E(a)\| \leqslant \|a\|$. For smooth elements $a = \sum_r a_r u^r$, it is clear that

$$E(a) = \sum_{r \in \mathbb{Z}^l} a_r \biggl(\int_{\mathbb{T}^l} z^r \, dz \biggr) u^r = a_0 \, 1.$$

Since smooth elements are dense, we see that the torus action on $C(\mathbb{T}^l_\Theta)$ is ergodic, namely, it fixes only the one-dimensional subalgebra of scalars.

► The next order of business is to find a Hilbert space on which the noncommutative torus algebra may be represented. The critical property is that the C^*-algebra $C(\mathbb{T}^l_\Theta)$ comes naturally equipped with a special *tracial state*, given by $E(a) =: \tau(a) \, 1$. On the dense subalgebra $C^\infty(\mathbb{T}^l_\Theta)$, we may use the formula

$$\tau\bigl(\textstyle\sum_r a_r u^r\bigr) = a_0.$$

Clearly, $\tau(1) = 1$, and τ is a faithful positive functional since

$$\tau(a^* a) = \textstyle\sum_r |a_r|^2 > 0 \quad \text{unless} \quad a = 0.$$

The tracial property of τ follows at once from

$$\tau(ab) = \tau\bigl(\textstyle\sum_{r,s} a_r b_s \sigma(r,s) \, u^{r+s}\bigr) = \sum_r a_r b_{-r} \sigma(r,-r) = \sum_r a_r b_{-r}.$$

By definition, the functional τ is invariant under the torus action (7).

We can now get a Hilbert space $\mathcal{H}_\tau = L^2(\mathbb{T}^l_\Theta, \tau)$ by completing $C(\mathbb{T}^l_\Theta)$ in the Hilbert-space norm $\|a\|_2 := \sqrt{\tau(a^* a)}$. We write $\underline{c}$ for an element $c \in C(\mathbb{T}^l_\Theta)$ regarded as a vector in this Hilbert space. There is an obvious representation of $C(\mathbb{T}^l_\Theta)$ on $\mathcal{H}_\tau$ given simply by "left multiplication operators":

$$\pi_0(a) : \underline{c} \mapsto \underline{ac}. \tag{9}$$

This is the GNS representation [34] of $C(\mathbb{T}^l_\Theta)$ with respect to τ.

The involution $a \mapsto a^*$ induces an antilinear operator J_0 on $\mathcal{H}_\tau$, given by

$$J_0(\underline{a}) := \underline{a^*}.$$

Clearly, $J_0^2 = 1$; and J_0 is an isometry since $\tau(aa^*) = \tau(a^*a)$, so it is antiunitary. The operator

$$\pi_0'(b) := J_0 \pi_0(b^*) J_0 : \underline{c} \mapsto J_0 \underline{b^* c^*} = \underline{cb}$$

is a "right multiplication operator" on $\mathcal{H}_\tau$, so π_0' is an antirepresentation of $C(\mathbb{T}^l_\Theta)$, or equivalently, a representation of its opposite algebra. It is clear that $\pi_0(a)$ and $\pi_0'(b)$ commute for all $a, b \in C(\mathbb{T}^l_\Theta)$.

► The canonical trace τ is a cyclic 0-cocycle on the algebra $\mathcal{A}_\Theta = C^\infty(\mathbb{T}^l_\Theta)$. The vector space of cyclic 0-cocycles is one-dimensional if the canonical trace is the unique tracial state on $C(\mathbb{T}^l_\Theta)$. One can show that τ is unique, and indeed $C(\mathbb{T}^l_\Theta)$ is a simple C^*-algebra, if and only if the matrix Θ satisfies a certain irrationality condition; when $l = 2$, this condition is simply that θ be an irrational number. We shall come back to this point a little later on.

To get higher cyclic cocycles, we make use of the following canonical derivations $\delta_1, \dots, \delta_l$ on $\mathcal{A}_\Theta$, generalizing the ordinary partial derivatives of Fourier series:

$$\delta_j\big(\textstyle\sum_r a_r u^r\big) := 2\pi i \textstyle\sum_r r_j\, a_r u^r.$$

These are symmetric derivations which kill the trace:

$$\delta_j(ab) = (\delta_j a)b + a(\delta_j b), \qquad \delta_j(a^*) = (\delta_j a)^*, \qquad \tau(\delta_j a) = 0.$$

With τ and these derivations, we can build up certain rotation-invariant cyclic cocycles on $\mathcal{A}_\Theta$. For instance, there are cyclic 1-cocycles ψ_j for $j = 1, \dots, l$ and cyclic 2-cocycles ψ_{jk} for $j < k$, given by

$$\begin{aligned} \psi_j(a, b) &:= \tau(a\, \delta_j b), \\ \psi_{jk}(a, b, c) &:= \frac{1}{2\pi i} \tau(a\, \delta_j b\, \delta_k c - a\, \delta_k b\, \delta_j c), \end{aligned}$$

and more generally, to each r-element subset K of $\{1, \dots, l\}$ there corresponds a cyclic r-cocycle given by

$$\psi_K(a_0, a_1, \dots, a_r) := \frac{1}{(2\pi i)^{r-1}} \sum_\sigma (-1)^\sigma \tau(a_0\, \delta_{\sigma(k_1)} a_1 \dots \delta_{\sigma(k_r)} a_r),$$

where the sum ranges over all permutations of $K = \{k_1, \dots, k_r\}$. To verify that ψ_j is a cyclic 1-cocycle, we just apply the Hochschild coboundary operator b:

$$b\psi_j(a, b, c) = \tau(ab\, \delta_j c - a\, \delta_j(bc) + ca\, \delta_j b) = \tau(ab\, \delta_j c - a\, \delta_j(bc) + a(\delta_j b)c) = 0.$$

When $\Theta = 0$, these cocycles incorporate the homological structure of the ordinary l-torus: for instance, $\psi_j(a, b)$ matches the line integral of the 1-form $a\, db$ over the jth generating circle of the torus $\mathbb{T}^l$. This is an instance of the general theorem of Connes [10] that, for a compact boundaryless smooth manifold M, the periodic cyclic cohomology of $C^\infty(M)$ is given by the de Rham homology of M.

There is a dual theory, relating the cyclic homology of $C^\infty(M)$ with the de Rham cohomology of M. In a slightly simpler fashion, Hochschild homology classes of $C^\infty(M)$ can be matched with differential forms on M. A *volume form* on M, if it exists, is a nonvanishing differential form on M

of top degree. Its counterpart for the coordinate algebra $\mathcal{A} = C^\infty(M)$ is a Hochschild cycle of degree $\dim M$. For the ordinary torus $\mathbb{T}^l$, the standard normalized volume form is

$$d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_l = (2\pi i)^{-l} (u_1 u_2 \dots u_l)^{-1}\, du_1 \wedge du_2 \wedge \cdots \wedge du_l,$$

where $u_j(\phi) = e^{2\pi i \phi_j}$. We can write down the corresponding Hochschild l-chain by replacing the wedge products of forms by tensor products, with an explicit skewsymmetrization (and a convenient normalization):

$$\mathbf{c} := \frac{(-i)^{\lfloor l/2 \rfloor}}{l!(2\pi)^l} \sum_{\sigma \in S_l} (-1)^\sigma u_{\sigma(l)}^{-1} \dots u_{\sigma(2)}^{-1} u_{\sigma(1)}^{-1} \otimes u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(l)}. \tag{10}$$

Now it happens that this same formula defines a Hochschild n-cycle on any noncommutative torus algebra $C^\infty(\mathbb{T}^l_\Theta)$, irrespective of Θ. One must check that it is killed by the Hochschild boundary operator:

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_l) := a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_l + \sum_{j=1}^{l-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^l a_l a_0 \otimes a_1 \otimes \cdots \otimes a_{l-1}.$$

Terms in $b\,\mathbf{c}$ arising from the first and last summands of b cancel at once. Terms arising from intermediate summands of b also cancel in pairs, for a more subtle reason: the commutation relations (2) yield

$$u_j^{-1} u_k^{-1} \otimes u_k u_j = u_k^{-1} u_j^{-1} \otimes u_j u_k,$$

and this is enough to check that $b\,\mathbf{c} = 0$.

► To complete the geometrical framework, we introduce Dirac operators on our noncommutative tori. On the torus $\mathbb{T}^l$ with the flat metric and periodic boundary conditions, the Dirac operator is given simply by

$$D\!\!\!\!/ = -i\,\gamma^j \otimes \partial_j, \quad \text{where} \quad \partial_j = \frac{\partial}{\partial \phi_j},$$

where the (Euclidean) gamma matrices satisfy $\gamma^j \gamma^k + \gamma^k \gamma^j = 2\,\delta^{jk}$; they act on a vector space of dimension 2^m, where either $l = 2m$ or $l = 2m + 1$. The *spinor space* of $\mathbb{T}^l$ is $\mathbb{C}^{2^m} \otimes L^2(\mathbb{T}^l)$, and $D\!\!\!\!/$ is an essentially selfadjoint first-order differential operator on this Hilbert space.

For noncommutative tori, we simply replace $L^2(\mathbb{T}^l)$ by the GNS representation space $\mathcal{H}_\tau$, and the partial derivatives ∂_j by the derivations δ_j, and define

$$D = -i\,\gamma^j \otimes \delta_j, \quad \text{acting on} \quad \mathcal{H} = \mathbb{C}^{2^m} \otimes \mathcal{H}_\tau.$$

For the cases $l = 2$ and $l = 3$, the gamma matrices are just the 2×2 Pauli matrices:

$$D := -i(\sigma_1\,\delta_1 + \sigma_2\,\delta_2) = -i \begin{pmatrix} 0 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & 0 \end{pmatrix} \quad (l = 2),$$

$$D := -i(\sigma_1\,\delta_1 + \sigma_2\,\delta_2 + \sigma_3\,\delta_3) = -i \begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -\delta_3 \end{pmatrix} \quad (l = 3). \tag{11}$$

An immediate use to which we can put the Dirac operator is to represent the Hochschild l-cycle $\mathbf{c}$ of (10) by an operator on the spinor space. Elements of $\mathcal{A} = C^\infty(\mathbb{T}^l_\Theta)$ are represented reducibly on $\mathcal{H}$ by $\pi(a) := 1_{2^m} \otimes \pi_0(a)$, where $\pi_0(a)$ is the "left multiplication" operator (9) of the GNS representation of τ. We now write

$$\pi_D(a_0 \otimes a_1 \otimes \cdots \otimes a_l) := \pi(a_0)\,[D, \pi(a_1)] \dots [D, \pi(a_l)], \tag{12}$$

and extend π_D by linearity to the space $C_n(\mathcal{A}, \mathcal{A})$ of Hochschild n-chains with values in $\mathcal{A}$. Suppressing π_0 for convenience, we find $[D, \pi(u_j)] = -i\,\gamma^j \otimes \pi_0(\delta_j u_j) = 2\pi\,\gamma^j \otimes \pi_0(u_j)$ (with no summation on j), and thus

$$\pi_D(\mathbf{c}) = \frac{(-i)^m}{l!} \sum_\sigma (-1)^\sigma \gamma^{\sigma(1)} \dots \gamma^{\sigma(l)} \otimes 1 = (-i)^m \gamma^1 \dots \gamma^l \otimes 1 =: \gamma^{l+1} \otimes 1.$$

Here, if l is even, γ^{l+1} is the chirality element of the Clifford algebra generated by $\gamma^1, \dots, \gamma^l$: it anticommutes with every other γ^j and satisfies $(\gamma^{l+1})^2 = 1$. Thus, $\chi := \gamma^{l+1} \otimes 1$ is the chiral grading operator on spinor space. We find that $\gamma^{l+1} = 1$ if l is odd. The normalization of $\mathbf{c}$ was chosen so that the following statement holds: the l-cycle $\mathbf{c}$ satisfies $\pi_D(\mathbf{c}) = \chi$ if l is even, $\pi_D(\mathbf{c}) = 1$ if l is odd.

We can extend the antiunitary operator J_0 on $\mathcal{H}_\tau$ to an antiunitary operator J on $\mathcal{H}$ in such a way that

$$\pi'(b) := J\pi(b^*)J^{-1} = 1 \otimes \pi'_0(b)$$

commutes with any $\pi(a)$, for $a, b \in C(\mathbb{T}^l_\Theta)$. For instance, if $l = 2$ or 3, we can use

$$J := -i\gamma^2 \otimes J_0 = \begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}.$$

For higher l, the first tensor factor will be a suitable product of gamma matrices making J antiunitary, such that in all cases $J\gamma^\mu J^{-1} = \mp\gamma^\mu$, with $-$ sign except when $l \equiv 1 \bmod 4$. One finds that $J^2 = \pm 1$, where the sign depends on l; indeed, $J^2 = -1$ for $l = 2$ or 3. Notice that $[D, \pi(a)] = -i\,\gamma^j \otimes \pi_0(\delta_j a)$ commutes with $1 \otimes \pi'_0(b)$, so that for all a, b the relation

$$\big[\, [D, \pi(a)], J\pi(b^*)J^{-1} \,\big] = 0 \tag{13}$$

holds, since D depends linearly on the elementary derivations δ_j. The relation (13) gives the noncommutative formulation of the property of being a "first-order differential operator".

The condition (13) also allows one to extend the formula (12) to represent Hochschild cycles with values in $\mathcal{A} \otimes \mathcal{A}^\circ$, which is more suitable when $\mathcal{A}$ is not commutative [14]. Namely, one replaces (12) with the formula

$$\pi_D((a_0 \otimes b^\circ) \otimes a_1 \otimes \cdots \otimes a_l) := \pi(a_0)\, J\pi(b^*)J^{-1}\, [D, \pi(a_1)] \dots [D, \pi(a_l)]. \tag{14}$$

In the examples we discuss in these notes, the entry b will be 1 in each case, so we shall continue to write formulas for $\mathcal{A}$-valued cocycles only.

2 Topological properties of noncommutative tori

How can one describe topological properties of "spaces" which may have few points, or none at all? In line with the Gelfand–Naĭmark result, a point x of a commutative coordinate algebra $C_0(X)$ is given by a character of the algebra, i.e., the scalar-valued $*$-homomorphism $f \mapsto f(x)$. Given an arbitrary C^*-algebra A, commutative or not, any nonzero character $\mu\colon A \to \mathbb{C}$ is sometimes called a "classical point" of A. But for $A_\theta^2 = C(\mathbb{T}_\theta^2)$ with θ irrational, the C^*-algebra is simple, so there are no codimension-one ideals $\ker\mu$, and therefore no classical points.

To check the simplicity of A_θ^2, we first notice that, for θ irrational, *the tracial state τ is unique*. To see that, we compute that, for $r, s \in \mathbb{Z}^2$,

$$u^s u^r u^{-s} = \sigma(s,r)^2\, u^r = e^{2\pi i\theta(r_1 s_2 - r_2 s_1)}\, u^r =: z(s)^r\, u^r = z(s)\cdot u^r,$$

for $z(s) = (e^{2\pi i\theta s_2}, e^{-2\pi i\theta s_1}) \in \mathbb{T}^2$. The irrationality of θ is equivalent to the condition that the countable set $\{\, z(s) : s \in \mathbb{Z}^2 \,\}$ be dense in $\mathbb{T}^2$. Since the Weyl elements u^r densely generate A_θ^2, we conclude that $u^s a u^{-s} = z(s)\cdot a$ for $a \in A_\theta^2$ and $s \in \mathbb{Z}^2$. Now let $\tau'\colon A \to \mathbb{C}$ be any positive trace such that $\tau'(1)$ finite; it is automatically continuous as a linear functional on the C^*-algebra A [34]. For $s \in \mathbb{Z}^2$, the trace property implies that $\tau'(z(s)\cdot a) = \tau'(u^s a u^{-s}) = \tau'(a)$; continuity of τ' and irrationality of θ then show that $\tau'(z\cdot a) = \tau'(a)$ for all $z \in \mathbb{T}^2$. On integrating over $\mathbb{T}^l$, we arrive at

$$\tau'(a) = \int_{\mathbb{T}^2} \tau'(z\cdot a)\, dz = \tau'(E(a)) = \tau'(1)\,\tau(a). \tag{15}$$

In particular, if τ' is a tracial *state*, i.e., if $\tau'(1) = 1$, then $\tau' = \tau$.

The simplicity of A_θ^2 is established by a similar argument. Suppose that J is a nonzero closed ideal in A_θ^2, and let $a \in J$, $a \neq 0$. Then $a^*a \in J$, and also $z(s)\cdot a^*a = u^s a^*a u^{-s} \in J$ for each $s \in \mathbb{Z}^2$. Since J is closed and θ is irrational, we can conclude that $z\cdot a^*a \in J$ for each $z \in \mathbb{T}^2$. Averaging over $\mathbb{T}^2$ gives $\tau(a^*a)\,1 = \int_{\mathbb{T}^2} z\cdot a^*a \in J$, and since τ is faithful and $a \neq 0$ so that $\tau(a^*a) > 0$, we conclude that $1 \in J$ and thus $J = A_\theta^2$.

For higher-dimensional cases, the same arguments go through to show that $C(\mathbb{T}_\Theta^l)$ has a unique tracial state and is simple, provided only that the analogous set $\{\, z(s) : s \in \mathbb{Z}^l \,\}$ is dense in $\mathbb{T}^l$, where $z(s)_j = \exp(2\pi i \sum_k \theta_{jk} s_k)$. This happens if and only if the lattice in $\mathbb{R}^l$ generated by the columns of the matrix Θ is wrapped into a dense subset of $\mathbb{T}^l$ by the exponential map: in that case we may say that Θ is "quite irrational" [23]. For such parameter matrices Θ, it is clearly pointless to pursue a topological description of the C^*-algebra by means of its characters.

► A rougher handle on the topology of noncommutative spaces is given by (C^*-algebraic) K-theory. It is known that $K_0(C(X)) \simeq K^0(X)$, as a consequence of the Serre–Swan theorem, and a similar result holds for (topological) K_1-groups. Thus we would like to determine $K_0(A_\theta)$ and $K_1(A_\theta)$.

The group $K_1(A_\theta)$ is fairly easy to describe. There are two generating unitaries, u_1 and u_2, yielding classes $[u^r]$ in $K_1(A_\theta)$ for each $r \in \mathbb{Z}^2$. These can be paired with the cyclic 1-classes $[\psi_1]$ and $[\psi_2]$ by setting

$$\langle [\psi_j], [u^r] \rangle := (2\pi i)^{-1} \psi_j(u^{-r}, u^r) = (2\pi i)^{-1} \tau(u^{-r}\, \delta_j u^r) = r_j \in \mathbb{Z},$$

so that every u^r lies in a different connected component of the unitary group of $\mathcal{A}_\theta$. (In the commutative case, r_1 and r_2 are the winding numbers of u^r on the two generating circles of $\mathbb{T}^2$.)

It does not help to pass to $k \times k$ matrices over A_θ, because on replacing τ by $\tau \otimes \mathrm{tr}_k$ we can still distinguish the homotopy classes of unitaries in $M_k(A_\theta)$. Therefore $K_1(A_\theta)$ contains at least a copy of $\mathbb{Z}^2$. It can be shown that there are no further K_1-classes: $K_1(A_\theta) \simeq \mathbb{Z}^2$.

It turns out that $K_0(A_\theta) \simeq \mathbb{Z}^2$ also, but in a much less trivial manner. The unit 1 yields a generator [1], and $\tau \otimes \mathrm{tr}_k(1_k) = k$ where 1_k is the unit of $M_k(A_\theta)$. Powers and Rieffel [37] found a nontrivial projector $p = p^2 = p^*$ in A_θ such that $\tau(p) = \theta$, assuming that $\frac{1}{2} \leqslant \theta < 1$, as we may. We look for p of the form

$$p = gv + f + hv^{-1},$$

where f, g, h are smooth real functions of u (i.e., smooth real elements of the copy of $C(\mathbb{T})$ generated by u, which we shall write as periodic functions with period 1). The condition $p^* = p$ is fulfilled if $vh = gv$, i.e., if $h(t) \equiv g(t - \theta)$. We choose f to be a smooth increasing function on $[0, 1 - \theta]$, define $f(t) := 1$ if $1 - \theta \leqslant t \leqslant \theta$, and $f(t) := 1 - f(t - \theta)$ if $\theta \leqslant t \leqslant 1$, so that the graph of f on $[0, 1]$ is symmetric about $t = \frac{1}{2}$; next, we define $g(t) := 0$ for $0 \leqslant t \leqslant \theta$ and $g(t) := \sqrt{f(t) - f(t)^2}$ for $\theta \leqslant t \leqslant 1$. One checks that $p - p^2 = f(1 - f) - g^2 - h^2 = 0$ (terms with nonzero powers of v cancel). Moreover,

$$\tau(p) = \int_0^1 f(t)\,dt = \int_0^{1-\theta} f(t)\,dt + (\theta - (1 - \theta)) + \int_\theta^1 f(t)\,dt = (2\theta - 1) + \int_0^{1-\theta} dt = \theta.$$

Also, $\tau(1 - p) = 1 - \theta$, so that a similar argument holds if $0 < \theta \leqslant \frac{1}{2}$. (By passing to matrices over $\mathcal{A}_\theta$ if necessary, we may dispense with the condition that $0 < \theta < 1$.)

The group homomorphism $\tau_*\colon K_0(A_\theta) \to \mathbb{C}$ given by $\tau_*[q] := \tau \otimes \mathrm{tr}_k(q)$, for q a projector in $M_k(A_\theta)$, therefore has a range which includes the countable dense subgroup $\mathbb{Z} \oplus \theta\mathbb{Z}$ of $\mathbb{R}$. Results of Pimsner and Voiculescu [35] now show that this subgroup is the whole range, and indeed that $\tau_*\colon K_0(A_\theta) \to \mathbb{Z} \oplus \theta\mathbb{Z}$ is an isomorphism of ordered groups.

► In some applications, noncommutative tori have appeared in pairs, where each algebra lies within the commutator of the other. This depends, of course, on the concrete representations involved. Such was the case in the seminal paper of Connes, Douglas and Schwarz [16] on toroidal compactifications of Matrix theory. This phenomenon can be observed in a more elementary fashion using the canonical commutation relations of quantum mechanics in the Weyl form; namely, we consider the following 2-parameter family of operators [20, 50] on $L^2(\mathbb{R})$:

$$W_\theta(a, b)\psi : t \longmapsto e^{-\pi i\theta ab}\, e^{2\pi i\theta bt}\, \psi(t - a).$$

These linearly generate a Lie algebra, since

$$[W_\theta(a, b), W_\theta(c, d)] = -2i \sin\bigl(\pi\theta(ad - bc)\bigr)\, W_\theta(a + c, b + d),$$

so that $W_\theta(a, b)$ and $W_\theta(c, d)$ commute if and only if $\theta(ad - bc)$ is an integer. Taking $U_\theta := W_\theta(1, 0)$ and $V_\theta := W_\theta(0, 1)$ yields the commutation relation $V_\theta U_\theta = e^{2\pi i\theta}\, U_\theta V_\theta$, so that the C^*-algebra generated by these two operators is isomorphic to A_θ. If we take its weak closure (or double commutant) within $\mathcal{L}(L^2(\mathbb{R}))$, we get the von Neumann algebra

$$A_\theta'' := \{\, W_\theta(m, n) : m, n \in \mathbb{Z} \,\}''.$$

The commutant of this turns out to be [38]:

$$A_\theta' := \{\, W_\theta(r/\theta, s/\theta) : r, s \in \mathbb{Z} \,\}''.$$

Now take $U'_\theta := W_\theta(1/\theta, 0)$, $V'_\theta := W_\theta(0, 1/\theta)$; one checks immediately that

$$V'_\theta U'_\theta = e^{2\pi i/\theta}\, U'_\theta V'_\theta,$$

so that V'_θ and U'_θ generate a copy of $A_{1/\theta}$. This algebra is not isomorphic to A_θ, but they are Morita equivalent. If θ is irrational, then both von Neumann algebras are II_1-factors.

► One of the earliest physical applications of noncommutative geometry was an interpretation of the Brillouin zone of an aperiodic solid in terms as a generalization of a noncommutative torus. We cannot do justice to this matter here, and refer the reader to Bellissard's lectures [5] for the physical background. We shall instead give a much oversimplified discussion, which does however capture the essence of the noncommutativity argument.

Consider first a perfect crystal of dimension d, whose atoms occupy sites of a lattice Γ generating $\mathbb{R}^d$ as a vector space. One sets up a Schrödinger equation $i\hbar \frac{\partial}{\partial t}\psi = H\psi$, with a Γ-periodic potential; thus one takes as Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, and one-particle Hamiltonian $H = p^2/2m + V$ where V is a Γ-periodic real function. The translations $U(s)\psi(x) := \psi(x - s)$, for $s \in \Gamma$, commute with the Hamiltonian:

$$U(s)\, H\, U(s)^{-1} = H \quad \text{for } s \in \Gamma.$$

The characters $s \mapsto e^{2\pi i k\cdot s}$ are labelled by the torus $\mathbb{B} := \mathbb{R}^d/\Gamma^\perp$, where $\Gamma^\perp$ is the dual lattice $\{\, t \in \mathbb{R}^d : t \cdot s \in \mathbb{Z} \text{ for } s \in \Gamma \,\}$. Then H can be decomposed into operators H_k on spectral subspaces $\mathcal{H}_k = \{\, \psi \in \mathcal{H} : \psi(x + s) = e^{2\pi i k\cdot s}\psi(x) \,\}$, for each $k \in \mathbb{B}$. Each H_k has a discrete eigenvalue spectrum, and the spectrum of H, being the union of these, is a so-called band spectrum which typically has many *gaps* (intervals excluded from the spectrum).

Now we replace the perfect crystal by a thin lamina of metal (specializing to $d = 2$, as an idealization), with atoms at lattice sites Γ, subject to a uniform magnetic field perpendicular to the plane of the metal. The movement of electrons within the metal effectively modifies the Hamiltonian to one of the form

$$H = (p - eA)^2/2m + V, \quad \text{where } V \text{ is } \Gamma\text{-periodic.}$$

Now H is still invariant under the action of the group Γ, but now the $U(s)$ form a *projective* representation of Γ, with the former $U(s) = e^{ip\cdot s/\hbar}$ being replaced by $U(s) = e^{i(p+eA)\cdot s/\hbar}$, for $s \in \Gamma$. The upshot is that the several $U(s)$ commute with H, but not with each other! Taking $U := U(s_1)$ and $V := U(s_2)$ where s_1, s_2 are generators of Γ, we get a relation of the form

$$VU = e^{2\pi i\theta'}\, UV,$$

identical to (3), where θ' depends on the choice of generators up to a Möbius transformation from $SL(2, \mathbb{Z})$.

Thus bounded functions of H belong to the commutant of the algebra $A_{\theta'}$ generated by the operators U and V. By adducing other operators with the same symmetries, one can in fact embed these functions in another noncommutative torus A_θ included in $A''_{\theta'}$, in the context of the integer quantum Hall effect [4]: see also the remarks in [12, IV.6.γ].

The trace on the noncommutative torus enters the picture as follows. One may restrict the Hamiltonian H to a bounded rectangle $\Lambda \subset \mathbb{R}^2$, under suitable boundary conditions. This restriction H_Λ is then a selfadjoint operator which is bounded below and has discrete spectrum, and we may consider the (finite) number of eigenvalues less than a given E, which can be expected to be of order $O(|\Lambda|)$. Using a suitable sequence of rectangles, we may form the "integrated density of states":

$$N(E) := \lim_{\Lambda \uparrow \mathbb{R}^2} \frac{1}{|\Lambda|} \#\{ E' \in \operatorname{sp} H_\Lambda : E' \leqslant E \}.$$

A formula of Shubin [5] expresses this quantity in terms of a certain trace $\tilde{\tau}$ (depending on the precise way in which the large-area limit is formulated) of a spectral projector of H:

$$N(E) = \tilde{\tau}(P_E(H)), \qquad P_E(H) := \chi(H \leqslant E).$$

We again emphasize that we have restricted our algebra of observables to a noncommutative torus C^*-algebra A_θ which includes $\{ f(H) : F \in C_0(\mathbb{R}) \}$. For an irrational parameter θ, we know that the normalized trace τ is unique, so $\tilde{\tau} = \tilde{\tau}(1)\, \tau$.

However, in general the indicator function $\chi(\cdot \leqslant E)$ is not a continuous function on $\operatorname{sp} H$, so that the continuous functional calculus cannot be applied to assert that $P_E(H) \in A_\theta$. But there is no problem *if E belongs to a spectral gap* of H, because in that case the indicator function *is* continuous on the spectrum; if preferred, one can replace it by a continuous function falling from 1 to 0 within a small interval of the real line which lies outside the spectrum of H. Moreover, it is clear that $P_E(H) = P_{E'}(H)$ if E and E' are values in the same spectral gap. The numerical value $\tau(P_E(H))$ therefore depends only on the gap itself; and these values provide a labelling of the gaps.

Finally, we should observe that for each such gap, $P_E(H)$ is a *projector* lying in A_θ. Thus $\tau(P_E(H))$ lies in the countable set $(\mathbb{Z} + \theta\mathbb{Z}) \cap [0, 1]$ determined by evaluating τ on the classes in $K_0(A_\theta)$), and the further knowledge of $\tilde{\tau}(1)$ determines the countable set of possible values of the $N(E)$ on spectral gaps.

► We briefly mention in passing a related formula, which invokes the cyclic cohomology of the NC torus. For the full story, we refer the reader to the authoritative paper of Bellissard, van Elst and Schulz-Baldes [6]. In the integer quantum Hall effect, which involves the electron flow in a thin metal plate at very low temperatures where quantum effects predominate, a quantity called the Hall conductivity σ_H was experimentally observed to attain predominantly certain integer multiples of e^2/h. The relevant spectral variable is called the Fermi level μ; when this level lies in a gap of the Hamiltonian spectrum, the corresponding projector P_μ lies in the C^*-algebra A_θ.

A formula due to Kubo may be used to compute σ_H. The observation of Bellissard et al is that the right hand side of this formula boils down to evaluating the cyclic 2-cocycle

$$\tau_2(a, b, c) := \frac{1}{2\pi i} \tau(a\, \delta_1 b\, \delta_2 c - a\, \delta_2 b\, \delta_1 c)$$

on the projector P_μ (whenever μ lies in a spectral gap):

$$\sigma_H = \frac{e^2}{h}\, \tau_2(P_\mu, P_\mu, P_\mu) = \frac{e^2}{h} \langle [\tau_2], [P_\mu] \rangle.$$

Here the expression in angle brackets in the evaluation of the periodic cyclic class of τ_2 on the K-theory class of the projector μ; therefore it represents an integer! In other words, the index theory of the noncommutative torus gives a reason for the integrality of the Hall conductivity – under the appropriate physical conditions – and contributes to understanding the robustness of the Hall effect.

3 Dirac operators and spectral triples

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ has three main ingredients: an algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ on which $\mathcal{A}$ acts as bounded operators, and an (in general, unbounded) selfadjoint operator D on $\mathcal{H}$. A fundamental idea of noncommutative geometry is that the operator D encodes a *metric structure* which generalizes that of a Riemannian metric on a differential manifold. To justify such a claim, we examine briefly some algebraic properties of Dirac operators on spin manifolds.

Thus, we begin with a compact boundaryless smooth manifold M endowed with a Riemannian metric g. The metric $g = [g_{ij}]$ determines a unique g-compatible, torsion-free connection, the Levi-Civita connection ∇^g on M, which is locally expressed through the Christoffel symbols Γ^k_{ij}:

$$\nabla^g_{\partial_i} \partial_j = \Gamma^k_{ij}\, \partial_k, \qquad \Gamma^k_{ij} = \tfrac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

We assume that M carries a spin structure, which is expressed by the existence of a spinor bundle $S \to M$: the smooth spinors can be completed in a standard (g-dependent) inner product to yield the Hilbert space $\mathcal{H} = L^2(M, S)$ of square-integrable spinors. If we choose a local orthonormal basis of vector fields $E_1, \dots, E_n$, say by taking $E_a = e^r_a\, \partial_r$ in terms of a vielbein e^r_a for which $e^r_a e^s_b \delta^{ab} = g^{rs}$, we may recompute the Christoffel symbols in this basis:

$$\nabla^g_{E_a} E_b = \widehat{\Gamma}^c_{ab}\, E_c,$$

where $\widehat{\Gamma}^c_{ab} + \widehat{\Gamma}^b_{ac} = 0$ because $g(E_b, E_c) = 0$. Since the $\widehat{\Gamma}^c_{ab}$ are skewsymmetric in the indices b, c, we may apply the infinitesimal spin representation $\mathfrak{so}(n) \to \operatorname{End} S_x$ at each point to promote the Levi-Civita connection ∇^g to a *spin connection* ∇^S, acting on smooth spinors by

$$\nabla^S_{E_a} \psi = E_a \cdot \psi + \tfrac{1}{4} \widehat{\Gamma}^c_{ab}\, \gamma^b \gamma_c\, \psi.$$

Here $E_a \cdot \psi$ denotes the action of the vector field E_a on the spinor ψ; we may write $\hat{\omega}_a := \tfrac{1}{4} \widehat{\Gamma}^c_{ab}\, \gamma^b \gamma_c$ for the components of the spin connection. The Dirac operator $\not{D}$ may now be defined (locally, but consistently) by

$$\not{D} := -i\, \gamma^a \big(E_a + \tfrac{1}{4} \widehat{\Gamma}^c_{ab}\, \gamma^b \gamma_c\big), \tag{16}$$

in the local orthonormal basis (to get it in a coordinate basis, one can rewrite $\widehat{\Gamma}^c_{ab}$ in terms of the original Christoffel symbols by employing the vielbein).

For instance, if M is the 3-sphere $\mathbb{S}^3$, then both the tangent bundle and the spinor are parallelizable, because $\mathbb{S}^3$ is the underlying manifold of the Lie group $\mathrm{SU}(2)$. There is no ambiguity in the choice of spinor bundle, since the sphere with its usual orientation has a unique spin structure. Suppose that g is the round metric on $\mathbb{S}^3$: it is invariant under left and right translations of $\mathrm{SU}(2)$, and therefore is given by a bilinear form on the Lie algebra $\mathfrak{su}(2) = T_1 \mathbb{S}^3$ which is itself rotation-invariant (i.e., preserved by the adjoint action of $\mathrm{SU}(2)$ on $\mathfrak{su}(2)$); thus g is unique up to normalization. The Christoffel symbols may be determined as follows [25]: consider the general formula for the Levi-Civita connection:

$$\begin{aligned} 2\, g(\nabla^g_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g(Y, [Z,X]) + g(Z, [X,Y]) - g(X, [Y,Z]). \end{aligned} \tag{17}$$

The first three terms on the right vanish if X, Y, Z are left-invariant vector fields. A local orthonormal basis $\{E_1, E_2, E_3\}$ is determined by choosing an orthonormal basis for $T_1\mathbb{S}^3$ and extending by left invariance; and the Lie brackets are given by the Lie algebra relations of $\mathfrak{su}(2)$, namely

$$[E_a, E_b] = 2\varepsilon^c_{ab}\, E_c,$$

where ε^c_{ab} is the totally skewsymmetric tensor. Plugging this into (17) gives

$$\nabla^g_{E_a} E_b = \varepsilon^c_{ab}\, E_c, \tag{18}$$

so that $\widehat{\Gamma}^c_{ab} = \varepsilon^c_{ab}$.

Therefore, the Dirac operator on $\mathbb{S}^3$ is of the form

$$D\!\!\!/ := -i\,\gamma^a\Big(E_a + \frac{1}{4}\varepsilon^c_{ab}\,\gamma^b\gamma_c\Big) = -i\,\gamma^a\,E_a + \frac{3}{2}\chi, \tag{19}$$

and $\chi = -i\,\gamma^1\gamma^2\gamma^3$ acts as the identity on the spinor space, since the dimension is odd.

Remark 1. If one computes the spinor Laplacian $\Delta^S = -\delta^{ab}(\nabla^S_{E_a}\nabla^S_{E_a} - \widehat{\Gamma}^c_{ab}\nabla^S_{E_c})$ for $\mathbb{S}^3$, one finds that it differs from $D\!\!\!/^2$ only by a scalar; indeed,

$$D\!\!\!/^2 = \Delta^S - \frac{3}{4} + \frac{9}{4} = \Delta^S + \frac{3}{2},$$

whereas the Lichnerowicz formula asserts that $D\!\!\!/^2 = \Delta^S + \frac{1}{4}s$, where s is the scalar curvature of the sphere $\mathbb{S}^3$. In the units implicitly chosen here, one can indeed show by direct computation of the Riemann curvature tensor that $s = 6$. In other words, the constant curvature of the 3-sphere is responsible for the "extra" $3/2$ in the Dirac operator formula (19).

► Two properties of the Dirac operator deserve special attention. The first is a formula which expresses the Riemannian distance function on the manifold M in terms of $D\!\!\!/$, noted originally by Connes [11].

From the defining formula (16), it is clear that for $a \in C^\infty(M)$,

$$[D\!\!\!/, a]\,\psi = D\!\!\!/(a\psi) - a\,D\!\!\!/\psi = -i(\partial_j a)\,\gamma^j\psi = -i\,\mathrm{grad}\,a\cdot\psi,$$

where $\mathrm{grad}\,a$ is the gradient vector field of a. Thus

$$\|[D\!\!\!/, a]\| = \|\,\mathrm{grad}\,a\|_\infty,$$

where the norm on the left-hand side is the operator norm of $\mathcal{L}(\mathcal{H})$.

Now if p and q are two distinct points of M and if γ is a piecewise smooth curve from p to q whose arc length is $\ell(\gamma)$, then

$$a(q) - a(p) = \int_0^1 \frac{d}{dt}\,a(\gamma(t))\,dt = \int_0^1 da_{\gamma(t)}(\dot\gamma(t))\,dt = \int_0^1 g_{\gamma(t)}((\mathrm{grad}\,a)_{\gamma(t)}, \dot\gamma(t))\,dt,$$

and therefore

$$|a(q) - a(p)| \leqslant \|\,\mathrm{grad}\,a\|_\infty \int_0^1 |\dot\gamma(t)|\,dt = \|\,\mathrm{grad}\,a\|_\infty\,\ell(\gamma).$$

It follows that

$$\sup\{\, |a(p) - a(q)| : a \in C^\infty(M),\ \|\operatorname{grad} a\|_\infty \leqslant 1 \,\} \leqslant \inf_{\gamma: p \to q} \ell(\gamma) =: d(p,q),$$

and this inequality is attained as an equality, by considering the function $a_p(q) := d(p,q)$; although a_p is not smooth, it is continuous and Lipschitz. We end up with the formula

$$\begin{aligned} d(p,q) &= \sup\{\, |a(p) - a(q)| : a \in C(M),\ \|\operatorname{grad} a\|_\infty \leqslant 1 \,\} \\ &= \sup\{\, |a(p) - a(q)| : a \in C(M),\ \|[D\!\!\!/, a]\| \leqslant 1 \,\}. \end{aligned} \tag{20}$$

We may now turn this formula around, and rephrase it in a much more general context of a noncommutative algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$. If D is a selfadjoint operator on $\mathcal{H}$ and *if the operator* $[D, a]$ *is bounded* (more precisely, is bounded on $\operatorname{Dom} D$ and thus extends to a bounded operator on $\mathcal{H}$) *for all* $a \in \mathcal{A}$, then the formula

$$d(\mu, \nu) := \sup\{\, |\mu(a) - \nu(a)| : a \in C(M),\ \|[D, a]\| \leqslant 1 \,\}$$

defines a distance function between states of the algebra $\mathcal{A}$. Unfortunately, this distance is in general difficult to compute, but it is the starting point of the theory of noncommutative metric spaces developed by Rieffel and coworkers [41].

The condition that each $[D, a]$ be bounded is a mild restriction on the algebra $\mathcal{A}$: instead of a whole C^*-algebra, we must limit our focus to a dense subalgebra of elements which are "Lipschitz with respect to D". Provided that this subalgebra is stable under the holomorphic functional calculus, nothing essential is lost.

► The second property of Dirac operators which holds our interest is the observation that the spectral growth of the Dirac operator determines the dimension n of the manifold M. Since up to now we have been treating compact manifolds only, our Dirac operator will have compact resolvent, and so its spectrum is discrete with finite multiplicity. If $N_{|D\!\!\!/|}(\lambda) = \#\{\, \mu \in \operatorname{sp}|D\!\!\!/| : \mu \leqslant \lambda \,\}$, it is known that

$$N_{|D\!\!\!/|}(\lambda) \sim C_n \operatorname{vol} M\, \lambda^n \quad \text{as } \lambda \to +\infty, \tag{21}$$

where $C_n = 2^m \operatorname{vol}(\mathbb{S}^{n-1})/n(2\pi)^n$ is a constant depending only on the dimension. This is a consequence of Lichnerowicz' formula $D\!\!\!/^2 = \Delta^S + \frac{1}{4}s$ and Weyl's theorem for the (scalar) Laplacian Δ, which states that

$$N_\Delta(\lambda) \sim C'_n \operatorname{vol} M\, \lambda^{n/2} \quad \text{as } \lambda \to +\infty,$$

where $C'_n = \operatorname{vol}(\mathbb{S}^{n-1})/n(2\pi)^n$. The extra factor 2^m in (21) is the rank of the spinor bundle.

Before looking at examples in more detail, we pause to proclaim a formal definition of "spectral triple", together with a full set of requirements for "spin geometry on noncommutative spaces".

► A (compact) **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ consists of a unital pre-C^*-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ carrying a faithful representation of $\mathcal{A}$ by bounded operators – so that we can regard $\mathcal{A}$ as a subalgebra of $\mathcal{L}(\mathcal{H})$ – and a selfadjoint operator D on $\mathcal{H}$, with compact resolvent, such that the commutator $[D, a]$ is also a bounded operator on $\mathcal{H}$, for each $a \in \mathcal{A}$.

A spectral triple is called *even* if there is a $\mathbb{Z}_2$-grading operator χ on $\mathcal{H}$ (i.e., $\chi = \chi^* \in \mathcal{L}(\mathcal{H})$ with $\chi^2 = 1$), such that $\chi a = a\chi$ for all $a \in \mathcal{A}$, and $\chi D = -D\chi$. Otherwise, the spectral triple is called *odd*; for convenience, we write $\chi = 1$ in the odd case.

The "standard commutative example" is given by $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(M, S)$, $D = \not{D}$, where M is a compact oriented smooth manifold without boundaryless, having a spin structure with spinor bundle S and a Riemannian metric g; and $\not{D}$ is the corresponding Dirac operator.

The following extra conditions [14] are verified for the commutative case and also for noncommutative tori and some further examples of interest. Any spectral triple satisfying all these conditions deserves to be called a **noncommutative spin geometry** [23, §10.5].

1. *Metric dimension*: There is a unique nonnegative integer n, the "metric dimension" of the geometry, for which the eigenvalue sums

$$\sigma_N(|D|^{-n}) := \sum_{0 \leqslant k < N} \lambda_k(|D|^{-n}) \quad \text{satisfy} \quad \sigma_N(|D|^{-n}) \sim C \log N \quad \text{as } N \to \infty,$$

with $0 < C < \infty$; on the orthogonal complement of the finite-dimensional kernel of D, the positive operator $|D|^{-n}$ is compact, and the condition $0 < C < \infty$ determines the value of n uniquely. Moreover, n will be even if and only if the spectral triple is even. We write $C =: ⨍ |D|^{-n}$.

(In the standard commutative case, the eigenvalue growth rate of $\not{D}$ shows that $n = \dim M$.) The nontrivial coefficient C holds much interest, because it equals the "Dixmier trace" of the operator $|D|^{-n}$: see [23, §7.5] for the definition of the Dixmier trace.

2. *Regularity*: Not only are the operators a and $[D, a]$ bounded, but they lie in the smooth domain of the derivation $\delta(T) := [|D|, T]$.

In the standard commutative case, this smooth domain is precisely $C^\infty(M)$, as can be shown with pseudodifferential calculus. For noncommutative tori, it coincides with the smooth algebra of rapidly decreasing Fourier–Weyl series, already described.

3. *Finiteness*: The space of smooth vectors $\mathcal{H}^\infty := \bigcap_k \operatorname{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$-module. There is a natural $\mathcal{A}$-valued Hermitian pairing $(\cdot \mid \cdot)$ on $\mathcal{H}^\infty$, implicitly defined by the relation $⨍ (\psi \mid \phi)\, |D|^{-n} = \langle \phi \mid \psi \rangle$.

In the standard commutative case, $\mathcal{H}^\infty = \Gamma^\infty(M, S)$ consists of the smooth spinors, i.e., C^∞ sections of the spinor bundle. The spinors are paired by the Hermitian form $(\psi \mid \phi) := \phi^\dagger \psi$.

4. *Reality*: There is an antiunitary operator J on $\mathcal{H}$, such that $[a, Jb^*J^{-1}] = 0$ for all $a, b \in \mathcal{A}$ (thus $b \mapsto Jb^*J^{-1}$ is a commuting representation on $\mathcal{H}$ of the "opposite algebra" $\mathcal{A}^\circ$); moreover, $J^2 = \pm 1$, $JD = \pm DJ$, and $J\chi = \pm \chi J$ in the even case, where the signs depend only on n mod 8, according to the following table:

n mod 8	0	2	4	6
$J^2 = \pm 1$	+	−	−	+
$JD = \pm DJ$	+	+	+	+
$J\chi = \pm \chi J$	+	−	+	−

n mod 8	1	3	5	7
$J^2 = \pm 1$	+	−	−	+
$JD = \pm DJ$	−	+	−	+

In the standard commutative case, J is the charge conjugation operator on spinors.

5. *First order*: The bounded operators $[D, a]$ commute with the opposite algebra representation: $[[D, a], Jb^* J^{-1}] = 0$ for all $a, b \in \mathcal{A}$.

   In the standard commutative case, this follows because $\not{D}$ is a first-order differential operator on spinors.

6. *Orientation*: There is a *Hochschild n-cycle* $\mathbf{c}$ on $\mathcal{A}$ with values in $\mathcal{A} \otimes \mathcal{A}^\circ$, whose representative operator (14) on $\mathcal{H}$ is given by

$$\pi_D(\mathbf{c}) = \chi \text{ (even case)}; \quad \text{or} \quad \pi_D(\mathbf{c}) = 1 \text{ (odd case)}.$$

   In the standard commutative case, $\mathbf{c}$ is an algebraic transcription of the volume form ν on the oriented manifold M, obtained from its explicit expression in local coordinates.

7. *Poincaré duality*: The index map of D determines a nondegenerate pairing on the K-theory of the algebra $\mathcal{A}$.

   In the standard commutative case, the Chern homomorphism transfers this nondegeneracy into the Poincaré duality theorem for de Rham cohomology of compact manifolds.

The fundamental reconstruction theorem of Connes [14] states that, given a spectral triple with $\mathcal{A} = C^\infty(M)$, the spin structure, metric, and Dirac operator can be fully recovered from these seven conditions: a detailed proof is given in [23].

► For the noncommutative 3-torus $\mathcal{A} = C^\infty(\mathbb{T}^3_\Theta)$, we have already constructed a spectral triple, with $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_\tau$ and $D = -i(\sigma_1 \delta_1 + \sigma_2 \delta_2 + \sigma_3 \delta_3)$ given by (11). Most of the listed properties above are readily checked, except for Poincaré duality. The Hochschild 3-cycle representing the volume form is given by (10) with $l = 3$. The regularity condition is easily checked, and the J operator fulfils the reality condition. What remains to be verified is that the metric dimension is indeed 3; we do so by computing the spectrum of D in detail.

The Weyl system of unitary elements $\{ u^r \in \mathcal{A} : r \in \mathbb{Z}^3 \}$ given rise to pairs of spinors

$$\psi_r^+ := \begin{pmatrix} u^r \\ 0 \end{pmatrix}, \qquad \psi_r^- := \begin{pmatrix} 0 \\ \underline{u^r} \end{pmatrix}, \qquad r \in \mathbb{Z}^3,$$

which indeed form an orthonormal basis for $\mathcal{H}$. Since $D^2 = -(\vec{\sigma} \cdot \vec{\delta})^2 = -(\delta_1^2 + \delta_2^2 + \delta_3^2)$ and $\delta_j u^r = 2\pi i\, r_j\, u^r$, it follows immediately that $D^2 \psi_r^\pm = 4\pi^2 |r|^2\, \psi_r^\pm$, and so

$$|D| \psi_r^\pm = 2\pi |r|\, \psi_r^\pm, \qquad |r| = \sqrt{r_1^2 + r_2^2 + r_3^2}. \tag{22}$$

It is enough for our purposes to know the spectrum of the positive operator $|D| := (D^2)^{1/2}$, with its multiplicities, which is apparent from (22). However, it is easy to check that, for $r \neq 0$ in $\mathbb{Z}^3$,

$$D\phi_r^\pm = \pm 2\pi |r|\, \phi_r^\pm, \quad \text{where} \quad \phi_r^\pm := (r_3 \pm |r|)\psi_r^+ + (r_1 + ir_2)\psi_r^-.$$

There are also two zero modes, ψ_0^+ and ψ_0^-, which span $\ker D$: these we ignore when dealing with $|D|^{-1}$. We have just exhibited all eigenvalues and eigenspinors of D. Notice that the spectrum of D is symmetric about 0, although in this "odd case" there is no chirality operator available which would anticommute with D.

For $R > 0$, let N_R be the number of lattice points $r \in \mathbb{Z}^3$ in the sphere $|r| \leqslant R$; then $N_R = O(R^3)$ and $\log N_R \sim 3 \log R$ as $R \to \infty$. Therefore

$$\sigma_{N_R}(|D|^{-s}) \sim 2 \sum_{1 \leqslant |r| \leqslant R} (2\pi|r|)^{-s} \sim \int_1^R \frac{8\pi t^2\, dt}{(2\pi t)^s}.$$

For $s > 3$, the integral converges, and for $s < 3$ it diverges faster than $\log R$. But for $s = 3$, we get

$$\sigma_{N_R}(|D|^{-s}) \sim \frac{1}{\pi^2} \log R \sim \frac{1}{3\pi^2} \log N_R.$$

More generally, if $N_R \leqslant N \leqslant N_S$, then $(\log N_R)^{-1}\sigma_{N_S} \leqslant (\log N)^{-1}\sigma_N \leqslant (\log N_S)^{-1}\sigma_{N_R}$; by taking R, S so that $\log S/\log R \to 1$, we conclude that

$$\sigma_N(|D|^{-s}) \sim \frac{1}{3\pi^2} \log N \quad \text{as } N \to +\infty.$$

Thus we find that the metric dimension equals 3, as claimed, and the coefficient of logarithmic divergence is $⨍ |D|^{-3} = 1/3\pi^2$. Notice that $1/3\pi^2 = 2(4\pi)/3(2\pi)^3 = C_3$ of (21): in the commutative case $\Theta = 0$, we have parametrized the torus $\mathbb{T}^3$ by the unit cube: $\mathrm{vol}(\mathbb{T}^3) = 1$, so that $1/3\pi^2$ is precisely the coefficient to be expected for the 3-dimensional *commutative* torus. In the next section, we shall explain that this is not a mere coincidence, by exhibiting this noncommutative toral spin geometry as an "isospectral deformation" of the commutative one.

More generally, a similar calculation shows that the metric dimension of this "standard" spectral triple for any noncommutative l-torus $\mathcal{A} = C^\infty(\mathbb{T}^l_\Theta)$ is l, and that $⨍ |D|^{-l} = C_l$.

► Our second example is commutative but not flat: we reconsider the case of the Dirac operator $\not{D}$ for the round metric on the 3-sphere $\mathbb{S}^3$. While Weyl's theorem already gives us the desired asymptotics of the spectrum, and hence confirms that the metric dimension is 3, it is instructive to determine the spectrum exactly.

The spectrum of $\not{D}$ on $\mathbb{S}^3$ first appears, to our knowledge, in Hitchin [25], and in recent years there have been several treatments giving the eigenspinors more or less explicitly: we may mention [1, 8, 26, 31, 47]. Here we follow Homma's very concrete treatment [26].

The sphere $\mathbb{S}^3$ can be seen as the Lie group SU(2), or as the homogeneous space SO(4)/SO(3), or better yet as Spin(4)/Spin(3): recall that Spin(n) is the simply connected double cover of SO(n) for $n \geqslant 3$. Now $\mathrm{Spin}(4) \approx \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{Spin}(3) \approx \mathrm{SU}(2)$; to be fully explicit, we regard $\mathbb{S}^3$ as the symmetric space G/H, where $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and H is its diagonal SU(2) subgroup. The principal spin bundle over $\mathbb{S}^3$ is this $G \to G/H$; there is an obvious trivialization $\mathrm{Spin}(4) \approx \mathbb{S}^3 \times \mathrm{SU}(2)$ given by $(p, q) \mapsto (pq^{-1}, p)$ for $p, q \in \mathrm{SU}(2)$. If ρ denotes the fundamental representation of SU(2) on $\mathbb{C}^2$, the associated spinor bundle is likewise trivial:

$$S = \mathrm{Spin}(4) \times_\rho \mathbb{C}^2 \simeq \mathbb{S}^3 \times \mathbb{C}^2 \quad \text{by} \quad [(p, q), \xi] \leftrightarrow (pq^{-1}, \rho(p)\xi).$$

We compute the action of the invariant vector fields E_1, E_2, E_3 on spinors. On writing $p = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$, subject to the constraint $z\bar{z} + w\bar{w} = 1$, we obtain

$$E_3 \cdot \psi(p) = i \frac{d}{dt}\bigg|_{t=0} \psi(\exp(-it\sigma_3)p) = i \frac{d}{dt}\bigg|_{t=0} \psi \begin{pmatrix} e^{-it}z & e^{-it}w \\ -e^{it}\bar{w} & e^{it}\bar{z} \end{pmatrix},$$

and $E_1 \cdot \psi(p)$, $E_2 \cdot \psi(p)$ may be computed similarly. We get

$$\begin{aligned}
(E_1 \cdot\,) &= -\bar{w}\frac{\partial}{\partial z} + w\frac{\partial}{\partial \bar{z}} + \bar{z}\frac{\partial}{\partial w} - z\frac{\partial}{\partial \bar{w}}, \\
(E_2 \cdot\,) &= i\bar{w}\frac{\partial}{\partial z} + iw\frac{\partial}{\partial \bar{z}} - i\bar{z}\frac{\partial}{\partial w} - iz\frac{\partial}{\partial \bar{w}}, \\
(E_3 \cdot\,) &= z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{w}}.
\end{aligned}$$

Since $D\!\!\!/\,\psi = -i\sigma_j E_j \cdot \psi + \frac{3}{2}\psi$, it is now fairly straightforward (but tedious) to write down formulas for the eigenspinors in these coordinates.

The eigenvalues and their multiplicities may be found more directly from the representation theory of compact groups: see, for instance, [45]. Since $D\!\!\!/$ is a G-invariant operator on $\mathcal{H} = L^2(\mathbb{S}^3, S) \simeq \mathbb{C}^2 \otimes L^2(\mathbb{S}^3)$, we can decompose the spinor space into isotypical components for the irreducible unitary representations $\sigma \in \widehat{G}$ of the group G:

$$\mathcal{H} \simeq \bigoplus_{\sigma\in\widehat{G}} \mathcal{H}_\sigma \otimes \mathrm{Hom}_G(\mathcal{H}_\sigma, \mathcal{H}). \tag{23}$$

The factor $\mathrm{Hom}_G(\mathcal{H}_\sigma, \mathcal{H})$ is a finite-dimensional vector space labelling the various copies (if any) of σ as a subrepresentation of the G-space $\mathcal{H}$. Now, this representation of G on $\mathcal{H}$ is *induced* from a representation of the subgroup H on the spinor fibre $\mathbb{C}^2$, which is a copy of the fundamental representation $\rho = \pi_{\frac{1}{2}}$ of $\mathrm{SU}(2)$. The Frobenius reciprocity theorem for induced representations [45] allows us to simplify the right hand side of (23) to

$$\mathcal{H} \simeq \bigoplus_{\sigma\in\widehat{G}} \mathcal{H}_\sigma \otimes \mathrm{Hom}_H(\mathbb{C}^2, \mathcal{H}_\sigma).$$

Since $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, its irreducible unitary representations are pairs of $\mathrm{SU}(2)$ irreps, so that $\sigma = \pi_j \otimes \pi_k$ for $j, k \in \frac{1}{2}\mathbb{N}$. This breaks up according to the Clebsch–Gordan rule:

$$\pi_j \otimes \pi_k \simeq \pi_{j+k} \oplus \cdots \oplus \pi_{|j-k|}.$$

Schur's lemma then says that $\mathrm{Hom}_H(\mathbb{C}^2, \mathcal{H}_\sigma) = 0$ unless $|j - k| = \frac{1}{2}$, and is one-dimensional in this case. Therefore the only representations of H which occur in the decomposition of $\mathcal{H}$ are $\sigma = \pi_{l+\frac{1}{2}} \otimes \pi_l$ and $\sigma = \pi_l \otimes \pi_{l+\frac{1}{2}}$, which both occur once (i.e., without multiplicity). We denote the representation spaces by V_l and V_{-l}, respectively, for $l \in \frac{1}{2}\mathbb{N}$; notice that $\dim V_l = \dim V_{-l} = (2l+1)(2l+2)$, for each l. Thus,

$$\mathcal{H} = L^2(\mathbb{S}^3, S) \simeq \bigoplus_{l\in\frac{1}{2}\mathbb{N}} V_l \oplus V_{-l}.$$

The invariance of $D\!\!\!/$ and the one-dimensionality of each $\mathrm{Hom}_H(\mathbb{C}^2, V_{\pm l})$ shows that $D\!\!\!/$ restricts to each subspace $V_{\pm l}$ and is given by a scalar operator there. In other words, the $V_{\pm l}$ are eigenspaces of $D\!\!\!/$, with the announced multiplicities. Some further calculation with eigenspinors selected from these subspaces [26] reveals that

$$D\!\!\!/ = 2l + \tfrac{3}{2} \text{ on } V_l, \qquad D\!\!\!/ = -(2l + \tfrac{3}{2}) \text{ on } V_{-l}.$$

Notice that in this case there are no zero modes: the lowest eigenvalue of $|D|$ is $\frac{3}{2}$.

Finally, $|D|^{-s}$ has eigenvalues $(m+\frac{3}{2})^{-s}$ with multiplicity $2(m+1)(m+2)$, for each $m \in \mathbb{N}$. Let $N_R := \sum_{m=0}^{R} 2(m+1)(m+2) = \frac{2}{3}(R^3+6R^2+11R)$ so that $\log N_R \sim 3\log R$. Then

$$\sigma_{N_R}(|D|^{-s}) = \sum_{m=0}^{R} 2(m+1)(m+2)(m+\tfrac{3}{2})^{-s} \sim 2\sum_{m=0}^{R}(m+\tfrac{3}{2})^{2-s} \sim 2\int_{3/2}^{R+3/2} t^{2-s}\,dt,$$

and this diverges logarithmically if and only if $s = 3$. Moreover,

$$\sigma_{N_R}(|D|^{-s}) \sim 2\log R \sim \frac{2}{3}\log N_R,$$

so the coefficient of logarithmic divergence is $⨍ |D|^{-3} = 2/3$ in this case. On dividing this by the coefficient $1/3\pi^2$ already obtained for the unit-volume case, we recover the well-known formula: $\operatorname{vol}(\mathbb{S}^3) = 2\pi^2$.

4 Twisted noncommutative spaces and isospectral deformations

In the previous section, we have seen that for certain spin manifolds with a great deal of symmetry, it is possible to compute the spectrum of the Dirac operator exactly: and that much of the geometrical information about the manifold can be recovered from this spectrum. An old question, ¿can you hear the shape of a drum? asks whether $\operatorname{sp}(D\!\!\!/)$ in fact determines the manifold M, and in general the answer is no: there are manifolds which are not diffeomorphic but are *isospectral*. Actually, isospectrality usually refers to the Laplacian rather than the Dirac operator, but this distinction is of minor importance: examples are constructed in [2] of two “space forms” (quotients of spheres $\mathbb{S}^n$ by finite groups acting without fixed points) which have identical Dirac spectrum but are not isometric, in dimensions $n = 19+4k$, $k \in \mathbb{N}$. In the world of noncommutative spaces, there is greater scope for isospectrality: we shall show that the deformation procedure of quantum mechanics leads to a large class of noncommutative spaces capable of carrying spin geometries.

A still unresolved issue in noncommutative geometry is whether, given a noncommutative spin geometry over a commutative algebra, an underlying manifold M, for which $\mathcal{A} = C^\infty(M)$, can be extracted from the spectral data alone. The key ingredient for doing so, whenever it can be done, is the Hochschild cycle $\mathbf{c}$ for which $\pi_D(\mathbf{c}) = \chi$ or 1. In his millennium survey of noncommutative geometry [15], Connes showed how to recover the 2-sphere from purely homological data.

To begin, we recall the standard volume form on $\mathbb{S}^2$, in Cartesian coordinates:

$$\nu = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy. \qquad (24)$$

The corresponding Hochschild 2-cycle is

$$\mathbf{c} := \frac{i}{2}\sum_{\text{cyclic}}(x\otimes y\otimes z - x\otimes z\otimes y) \in \Omega^2(C^\infty(\mathbb{S}^2)), \qquad (25)$$

where $\sum_{\text{cyclic}}$ runs over cyclic permutations of x, y, z. The condition $\pi_D(\mathbf{c}) = \chi$ becomes

$$\frac{i}{2}\sum_{\text{cyclic}}\big(x\,[D,y]\,[D,z] - x\,[D,z]\,[D,y]\big) = \chi.$$

Abbreviating $\mathrm{d}x := [D, x]$, we may rewrite this as

$$\tfrac{i}{2}\big(x\,[\mathrm{d}y, \mathrm{d}z] + y\,[\mathrm{d}z, \mathrm{d}x] + z\,[\mathrm{d}x, \mathrm{d}y]\big) = \chi. \tag{26}$$

Now the C^*-algebra $A = C(\mathbb{S}^2)$ is generated by the three commuting coordinates x, y, z, subject to the constraint $x^2 + y^2 + z^2 = 1$ – and one may eventually obtain the smooth subalgebra $C^\infty(\mathbb{S}^2)$ by imposing regularity conditions. From this generators-and-relations point of view, the orientation condition (26) can be thought of as a (nonlinear) abstract differential equation for the coordinates x, y, z.

It is useful to collect the three coordinates x, y, z as entries of a single matrix in $M_2(A)$:

$$e = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \tag{27}$$

This is easily seen to be a *projector*, i.e., a selfadjoint idempotent in $M_2(A)$, that is, $e = e^2 = e^*$. It is indeed a version of the Bott projector, whose class $[e] \in K_0(A) = K^0(\mathbb{S}^2)$ is nontrivial. We note also the trivial identity $\mathrm{tr}(e - \frac{1}{2}) = 0$.

It is easy to reexpress the volume form (24) in terms of e:

$$\mathrm{tr}\big((e - \tfrac{1}{2})\, de \wedge de\big) = \tfrac{i}{2}\,\nu.$$

Comparison with (26) gives a corresponding operator equation:

$$\mathrm{tr}\big((e - \tfrac{1}{2})\, \mathrm{d}e\, \mathrm{d}e\big) = \chi, \tag{28}$$

and indeed, $\mathrm{tr}\big((e - \frac{1}{2}) \otimes e \otimes e\big)$ is the Hochschild 2-cycle corresponding to the volume form.

What happens in this 2-dimensional case is that the discussion can be reversed. Suppose that an even spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given, together with an element $e \in M_2(\mathcal{A})$, such that

$$e = e^2 = e^* \quad \text{and} \quad \mathrm{tr}(e - \tfrac{1}{2}) = 0.$$

Suppose also that the entries of e generate a dense subalgebra of $\mathcal{A}$ and that (28) holds. Let x, y, z be defined by (27); then $e^* = e$ says that x, y, z are selfadjoint elements of $\mathcal{A}$ and the positivity of the projector e implies $-1 \leqslant z \leqslant 1$. The idempotence $e^2 = e$ boils down to a set of equations

$$(1 \pm z)^2 + x^2 + y^2 \pm i[x, y] = 2(1 \pm z),$$
$$(1 \mp z)(x \pm iy) + (x \pm iy)(1 \pm z) = 2(x \pm iy),$$

which simplify to

$$[x, y] = [y, z] = [z, x] = 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

These relations tells us that the C^*-completion A of $\mathcal{A}$ is a commutative C^*-algebra whose generators satisfy the equation of a sphere. It follows that $A \simeq C(V)$, where V is a closed subset of $\mathbb{S}^2$. The condition (28) obliges V to be the support of the volume form ν on $\mathbb{S}^2$, namely the whole sphere: $V = \mathbb{S}^2$. Thus, $\mathbb{S}^2$ emerges from the homological data.

► The expression $\mathrm{tr}\big((e - \frac{1}{2}) \otimes e \otimes e\big)$ plays a leading role here because it appears as the second-degree term in the Chern character (in cyclic homology) of the class $[e] \in K_0(\mathcal{A})$: it is the 2-cycle corresponding to the volume form. One can formulate a similar homological problem for 3-spheres, but in terms of the Chern character of a unitary matrix of elements representing a K_1-class; this is the theme of recent work by Connes and Dubois-Violette [17]. However, it is perhaps simpler to set up a 4-sphere problem, as follows.

Suppose that A is a C^*-algebra and that we are given a projector $e = e^* = e^2$ in $M_4(\mathcal{A})$, of the form

$$e := \frac{1}{2}\begin{pmatrix}(1+z)1_2 & q \\ q^* & (1-z)1_2\end{pmatrix}, \quad \text{where} \quad 1_2 = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \quad q = \begin{pmatrix}a & b \\ -b^* & a^*\end{pmatrix}.$$

Then $z = z^*$ in $\mathcal{A}$ satisfies $-1 \leqslant z \leqslant 1$, and

$$e^2 = e \implies qq^* = (1 - z^2)\, 1_2 = q^*q, \quad [z1_2, q] = 0. \tag{29}$$

Since $qq^* = q^*q$ is diagonal, we find that z, a, a^*, b, b^* are commuting elements of A, subject to the constraint $aa^* + bb^* = 1 - z^2$: these are coordinate relations for a closed subset of $\mathbb{S}^4$. The degree-four analogue of the condition (28) produces the standard volume form supported on the full sphere, leading to $A = C(\mathbb{S}^4)$, so that $\mathbb{S}^4$ can also be described in purely homological language.

It was pointed out by Landi that one could modify the "quaternion" q to be of the form

$$q = \begin{pmatrix}a & b \\ -\lambda b^* & a^*\end{pmatrix}, \quad \text{with} \quad \lambda \in \mathbb{C},$$

motivated by the usual presentation of the fundamental unitary matrix for the quantum group $\mathrm{SU}_q(2)$ – where λ is usually taken to be real. But computation of the Chern character of $[e]$ yields an unwanted 2-chain with coefficient $(1 - \lambda\bar{\lambda})$, so to get a "noncommutative 4-sphere" one should instead impose the restriction $|\lambda| = 1$, that is, $\lambda = e^{-2\pi i\theta}$ for some $\theta \in \mathbb{R}$. In that case, the relations (29) become, after some simplification,

$$\begin{gathered} ab = e^{-2\pi i\theta}\, ba, \qquad a^*b = e^{2\pi i\theta}\, ba^*, \\ aa^* = a^*a, \qquad bb^* = b^*b, \qquad aa^* + bb^* = 1 - z^2. \end{gathered} \tag{30}$$

These are generators for the C^*-algebra $A = C(\mathbb{S}^4_\theta)$ of Connes and Landi [18].

► To get a firmer grip on this noncommutative space, we make the change of variables

$$a =: u \sin\psi \cos\phi, \qquad b =: v \sin\psi \sin\phi, \qquad z =: \cos\psi, \tag{31}$$

where ϕ, ψ are ordinary angular coordinates. It is clear that this is equivalent to (30), provided u, v are unitaries satisfying $vu = e^{2\pi i\theta}\, uv$. Thus the algebra A contains a copy (indeed, many copies) of the noncommutative torus algebra $C(\mathbb{T}^2_\theta)$.

Recall the canonical action of $\mathbb{T}^2$ on $C(\mathbb{T}^2_\theta)$, given by $z \cdot u := z_1\, u$, $z \cdot v := z_2\, v$. On plugging these into (31), we also obtain an action of $\mathbb{T}^2$ on $A = C(\mathbb{S}^4_\theta)$. This action decomposes A into spectral subspaces indexed by $r = (r_1, r_2) \in \mathbb{Z}^2$. The rth subspace consists of elements $f_r = u^{r_1} v^{r_2} h(\phi, \psi)$ satisfying $z \cdot f_r = z_1^{r_1} z_2^{r_2}\, f_r$. To describe the algebra A, it is enough to specify the product of "homogeneous" elements in a pair of spectral subspaces. One finds that this product is given by $f_r * g_s := e^{2\pi i\theta r_2 s_1}\, f_r g_s$. It helps to modify this phase factor to one with a skewsymmetric exponent: the transformation $f_r \mapsto e^{\pi i\theta r_1 r_2}\, f_r$ extends to a bijective linear isomorphism of A with itself, but modifies the product to

$$f_r \star g_s := \sigma(r, s)\, f_r g_s, \tag{32a}$$

where now $\sigma(r, s) = e^{\pi i\theta(r_2 s_1 - r_1 s_2)}$.

More generally, if M is a Riemannian manifold carrying a group action of $\mathbb{T}^l$ by isometries, and if $\Theta = [\theta_{jk}]$ is a skewsymmetric matrix, there is a decomposition of $C(M)$ into spectral subspaces indexed by $\mathbb{Z}^l$, and up to a $*$-isomorphism the algebra structure is given by (32a) where the cocycle σ is now

$$\sigma(r, s) := \exp\bigl\{-\pi i \textstyle\sum_{j,k=1}^{l} r_j \theta_{jk} s_k\bigr\} = \exp(-\pi i r \cdot \Theta s). \tag{32b}$$

These phase factors make up a 2-cocycle on the additive group $\mathbb{Z}^l$. The cocycle relation (6) ensures that the new product is associative.

For $M = \mathbb{T}^l$, this procedure gives an alternative construction of the noncommutative l-torus algebra $C(\mathbb{T}^l_\Theta)$. Indeed, for general manifolds M carrying an isometric action of $\mathbb{T}^l$, we are able to define the smooth subalgebra $C^\infty(M_\Theta)$: when $\Theta = 0$, any $f \in C^\infty(M)$ can be expressed as a rapidly convergent "Fourier series" $f = \sum_r f_r$ with ordinary multiplication. These same series, with the *twisted product* (32) instead, form the smooth subalgebra $C^\infty(M_\Theta)$ of $C(M_\Theta)$.

► The component-wise product (32) is closely related with the *Moyal product* of ordinary quantum mechanics. For an invertible skewsymmetric $l \times l$ matrix Θ (only possible if l is even), the Moyal product of two Schwartz functions f, h on $\mathbb{R}^l$ is defined as an oscillatory integral:

$$f \star_\Theta h(x) = (\pi\theta)^{-l} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} f(x+s)\, h(x+t)\, e^{-2is\cdot\Theta^{-1}t}\, ds\, dt, \qquad \theta^l = \det\Theta. \tag{33a}$$

(In the phase-space approach to quantum mechanics, x is a vector of phase-space coordinates and $\theta = \hbar$ is Planck's constant.) The invertibility of Θ and the restriction to even l may be dispensed with by a change of variables, which reduces this to an ordinary Fourier integral:

$$f \star_\Theta h(x) := (2\pi)^{-l} \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} f(x - \tfrac{1}{2}\Theta u)\, h(x+t)\, e^{-iu\cdot t}\, du\, dt. \tag{33b}$$

It was realized by Rieffel [39] that, since the integrand is the product of certain translates of the functions f and h, this formula can be generalized at once to define a twisted product on any C^*-algebra A which carries a strongly continuous action α of the abelian group $\mathbb{R}^l$:

$$a \star_\Theta b := \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \alpha_{\frac{1}{2}\Theta u}(a) \alpha_{-t}(b)\, e^{2\pi i u\cdot t}\, du\, dt. \tag{34}$$

For this to make sense, a and b should be smooth elements of A, that is, the A-valued functions $t \mapsto \alpha_t(a)$, $t \mapsto \alpha_t(b)$ should be smooth: thus the product is defined, a priori, only on the dense subalgebra $\mathcal{A}$ of such smooth elements (which, by the way, is a pre-C^*-algebra).

When the action α of $\mathbb{R}^l$ is *periodic*, so that $\alpha_t = \mathrm{id}_A$ for each t in the lattice $\mathbb{Z}^l$, then α is effectively an action of the torus $\mathbb{T}^l$. Then $\mathcal{A}$ decomposes into spectral subspaces indexed by $\mathbb{Z}^l$ where $\alpha_t(a_r) = e^{2\pi i r\cdot t} a_r$ for a homogeneous element a_r in the rth spectral subspace. It is then easy to check that

$$a_r \star_\Theta b_s = e^{-\pi i r\cdot\Theta s} a_r b_s$$

in all cases. Thus our previous recipe (32) is none other than a *periodic Moyal product* on the smooth commutative algebra $C^\infty(M)$.

At the level of C^*-algebras, there is an important nuance, because neither the component-wise product nor the Moyal double integral make sense beyond the smooth subalgebra. However, one can overcome this difficulty with a little representation theory. Given A, $\alpha\colon \mathbb{R}^l \to \operatorname{Aut}(A)$, and Θ, one constructs a Hilbert space of functions $f\colon \mathbb{R}^l \to A$ by setting

$$(f \mid g)_A := \int_{\mathbb{R}^l} f(x)^* g(x)\,dx \,\in A, \qquad |||f||| := \sqrt{\|(f \mid f))_A\|}.$$

(This an example of a C^*-module over the C^*-algebra A.) For $a \in \mathcal{A}$ and suitable f, the following integral makes sense:

$$\lambda(a) f(x) := \int_{\mathbb{R}^l} \int_{\mathbb{R}^l} \alpha_{x+\frac{1}{2}\Theta s}(a) f(x+t)\, e^{2\pi i s\cdot t}\, ds\, dt,$$

and defines a $*$-homomorphism on $(\mathcal{A}, \star_\Theta)$:

$$\begin{aligned}
\lambda(a \star_\Theta b) f(x) &= \iint \alpha_{x+\frac{1}{2}\Theta s}(a \star_\Theta b) f(x+t)\, e^{2\pi i s\cdot t}\, ds\, dt \\
&= \iiiint \alpha_{x+\frac{1}{2}\Theta s+\frac{1}{2}\Theta u}(a)\alpha_{x+\frac{1}{2}\Theta s+v}(b) f(x+t)\, e^{2\pi i(s\cdot t+u\cdot v)}\, du\, dv\, ds\, dt \\
&= \iiiint \alpha_{x+\frac{1}{2}\Theta u'}(a)\alpha_{x+v+\frac{1}{2}\Theta s}(b) f(x+v+t')\, e^{2\pi i(s\cdot t'+u'\cdot v)}\, ds\, dt'\, du'\, dv \\
&= \iint \alpha_{x+\frac{1}{2}\Theta u'}(a)\lambda(b) f(x+v)\, e^{2\pi i u'\cdot v}\, du'\, dv \\
&= \lambda(a)\lambda(b) f(x).
\end{aligned}$$

The calculation uses only the elementary changes of variable $t' := t - v$, $u' := s + u$, for which $s \cdot t + u \cdot v = s \cdot t' + u' \cdot v$. Thus $\mathcal{A}$ is represented (faithfully) as an algebra of bounded operators on this Hilbert space, and the operator norm provides a C^*-norm whereby $(\mathcal{A}, \star_\Theta)$ may be completed to a C^*-algebra, call it A_Θ.

Rieffel investigated this construction in detail, and showed that if Φ is another skewsymmetric $l\times l$ real matrix, then $(A_\Theta)_\Phi \simeq A_{\Theta+\Phi}$; by taking $\Phi = -\Theta$, one finds that these C^*-algebra deformations are reversible. Moreover, the action α on A yields a corresponding isometric action of $\mathbb{R}^l$ on A_Θ, and it turns out that the smooth subalgebra of A_Θ for this action is exactly the original smooth algebra $\mathcal{A}$. Thus, A and A_Θ are two C^*-completions of the same $\mathcal{A}$ with two different products. (This, in turn, justifies the assertion that all calculations need only be performed within the common smooth subalgebra.)

► So far, we have seen that if a compact smooth boundaryless manifold M carries an action of the compact abelian group $\mathbb{T}^l$ by isometries, then there is a Moyal product defined on $C^\infty(M)$ which in turn determines a family of noncommutative C^*-algebras $C(M_\Theta) := C(M)_\Theta$, parametrized (somewhat redundantly) by skewsymmetric matrices Θ. This twisting of the algebra product may be expressed on $C^\infty(M)$ either by the Moyal integral formula or by a cocycle formula on spectral subspaces.

To obtain a spectral triple on such a noncommutative space, we shall further assume that M *carries a spin structure*. An important class of examples is given by the homogeneous manifolds G/H where G is a compact connected Lie group, and $H \simeq \mathbb{T}^l$ is a toral subgroup in G. When G is semisimple and H is a maximal torus, the flag manifold G/H is simply connected and carries a unique spin structure (with a given orientation); and left multiplication by H on G induces an action of H on $M = G/H$, which is isometric with respect to the G-invariant metric on M. Other examples with unique spin structure are the spheres $\mathbb{S}^3$ and $\mathbb{S}^4$: the usual embedding $\mathbb{S}^3 \hookrightarrow \mathbb{C}^2$ is stable under the obvious action of $\mathbb{T}^2$ on $\mathbb{C}^2$ by multiplication, and this action on $\mathbb{S}^3$ extends to another on the suspension $\mathbb{S}^4$; to see that, just put $\psi = \frac{\pi}{2}$ in (31).

On M we choose a metric which is invariant under the action of $\mathbb{T}^l$ (to find one, start with an arbitrary Riemannian metric and average its translates by the $\mathbb{T}^l$ action). This lifts to an action of $\mathbb{T}^l$ by bundle automorphisms $\{ \sigma_s : s \in \mathbb{T}^l \}$ on the tangent bundle $TM \to M$, which has structure group $\mathrm{SO}(n)$; to get symmetries of the Dirac operator $D\!\!\!/$ on $\mathcal{H} = L^2(M, S)$, we need to lift this to the spinor bundle $S \to M$, which has structure group $\mathrm{Spin}(n)$. This can be achieved [17] by finding a double cover $\pi\colon \widetilde{\mathbb{T}}^l \to \mathbb{T}^l$ and a homomorphism $V\colon \widetilde{\mathbb{T}}^l \to \operatorname{Aut} S$ making the following diagram commute:

$$\begin{array}{ccc} \widetilde{\mathbb{T}}^l & \xrightarrow{V_{\tilde{s}}} & \operatorname{Aut} S \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \mathbb{T}^l & \xrightarrow{\sigma_s} & \operatorname{Aut} TM. \end{array}$$

To find such a homomorphism π, we write $\mathbb{T}^l = \mathbb{R}^l/(2\pi\mathbb{Z})^l$ and replace $\mathbb{Z}^l$ by the lattice $\mathbb{Z}^l + \widehat{\mathbb{Z}}^l$, where $\widehat{\mathbb{Z}}^l$ is the translate of $\mathbb{Z}^l$ by $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Then define $\widetilde{\mathbb{T}}^l := \mathbb{R}^l/((2\pi\mathbb{Z})^l + (2\pi\widehat{\mathbb{Z}})^l)$; the map π is determined by matching a set of generators for each lattice and passing to the quotient. The upshot of this construction is that one can find a group of unitary operators $\{ V_{\tilde{s}} : \tilde{s} \in \widetilde{\mathbb{T}}^l \}$ on spinors covering the isometry group $\{ \sigma_s : s \in \mathbb{T}^l \}$ acting on $C^\infty(M)$, where $s = \pi(\tilde{s})$, in such a way that

$$V_{\tilde{s}}(f\psi) = (\sigma_s f)\, V_{\tilde{s}}\psi \quad \text{and} \quad (V_{\tilde{s}}\phi \mid V_{\tilde{s}}\psi) = \sigma_s(\phi \mid \psi),$$

for $f \in C^\infty(M)$ and $\phi, \psi \in \mathcal{H}$. The abelian group of unitaries $V_{\tilde{s}}$ is generated by commuting selfadjoint operators $p_1, \dots, p_l$ on $\mathcal{H}$, and the periodicity of the action of $\mathbb{T}^l = \pi(\widetilde{\mathbb{T}}^l)$ implies that the spectrum of each p_j is integral or half-integral: $\operatorname{sp} p_j \subseteq \frac{1}{2}\mathbb{Z}$.

We now define a family of unitary operators on $\mathcal{H}$, indexed by $t \in \mathbb{R}^l$, by setting

$$\sigma(p, t) := \exp\bigl(-\pi i \textstyle\sum_{j,k} p_j \theta_{jk} t_k\bigr), \tag{35}$$

on formally replacing the parameters r_j in (32b) with the operators p_j; the inverse of $\sigma(p, t)$ is the similarly defined operator $\sigma(t, p)$. These operators commute with each other and also with $D\!\!\!/$ (this is the concrete expression of the $\mathbb{T}^l$-invariance of $D\!\!\!/$), but *they do not commute with the representation of* $C^\infty(M)$ on $\mathcal{H} = L^2(M, S)$ by multiplication operators.

To analyze the interaction of $D\!\!\!/$ with the representation of the Moyal product algebra $C(M_\Theta) \equiv C(M)_\Theta$ on spinors, we use the unitaries $\sigma(p, t)$ to define an action of $\mathbb{R}^l$ on *all* bounded operators on $\mathcal{H}$ by conjugation:

$$\alpha_t(T) := \sigma(p, t)\, T\, \sigma(p, t)^{-1} = \sigma(p, t)\, T\, \sigma(t, p).$$

On account of the integrality of $\operatorname{sp}(2p_j)$ for each j, this action is also periodic and partitions $\mathcal{L}(\mathcal{H})$ into spectral subspaces indexed by $\mathbb{Z}^l$. More concretely, any bounded operator T in the smooth domain of this action has a (norm-summable) decomposition $T = \sum_{r \in \mathbb{Z}^l} T_r$, whose components satisfy the commutation rules:

$$\sigma(p, r)\, T_s = T_s\, \sigma(p + s, r) \quad \text{for all} \quad r, s \in \mathbb{Z}^l.$$

For multiplication operators coming from $C^\infty(M)$, we recover the previous decomposition $f = \sum_{r\in\mathbb{Z}^l} f_r$.

One can now introduce a "left twist" of smooth operators [18,49] by defining

$$L(T) := \sum_{r\in\mathbb{Z}^l} T_r\, \sigma(p,r).$$

For $f,g \in C^\infty(M)$, the group cocycle property (6) of σ can now be invoked to show that

$$\begin{aligned} L(f)L(g) &= \sum_{r,s} f_r\, \sigma(p,r)\, g_s\, \sigma(p,s) = \sum_{r,s} f_r\, g_s\, \sigma(p+s,r)\, \sigma(p,s) \\ &= \sum_{r,s} f_r\, g_s\, \sigma(r,s)\, \sigma(p,r+s) = L(f \star_\Theta g). \end{aligned}$$

Therefore, L yields a representation of the deformed algebra $C^\infty(M_\Theta)$ on the spinor space $\mathcal{H}$. That is to say, the Moyal product gives not only an abstract deformation of the algebra $C^\infty(M)$, but it also yields a deformation of its representation on the original Hilbert space.

► It should now be clear that the spectral triple we seek is none other than

$$(\mathcal{A}, \mathcal{H}, D) = (C^\infty(M_\Theta), L^2(M,S), D\!\!\!/)$$

where the original commutative algebra is deformed with the Moyal product but the Hilbert space and Dirac operator remain the same. The other terms of the commutative spectral triple, namely the charge conjugation J on spinors [23] and the grading χ if $\dim M$ is even, are also unchanged. This deformation is *isospectral* [18] in the tautological sense that the spectrum in question is that of the operator $D\!\!\!/$, which remains the same.

We must still check that the new spectral triple satisfies the conditions governing a spin geometry. First of all, each $[D\!\!\!/, L(f)]$, for $f \in C^\infty(M)$, must be a bounded operator; this is ensured because $D\!\!\!/$ commutes with each $\sigma(p,r)$, so

$$[D\!\!\!/, L(f)] = \sum_r [D\!\!\!/, f_r]\, \sigma(p,r) = L([D\!\!\!/, f]),$$

since each $[D\!\!\!/, f]$ is bounded, and $[D\!\!\!/, f]_r = [D\!\!\!/, f_r]$ since $D\!\!\!/$ is invariant under conjugation by the $\sigma(p,t)$ unitaries. The grading operator χ is likewise invariant since the metric is taken to be $\mathbb{T}^l$-invariant: this implies $L(\chi) = \chi$. The previous equation then shows that the orientation condition $\pi_{D\!\!\!/}(\mathbf{c}) = \chi$ survives the application of L to both sides.

The charge conjugation operator J is fixed by the $\mathbb{T}^l$ action, because the metric is invariant: therefore, J commutes with all $\sigma(p,r)$. From the explicit formula (35) and the antilinearity of J we obtain $Jp_jJ^{-1} = -p_j$ for each j. We can now define a "right twist":

$$R(T) := JL(T)^*J^{-1} = \sum_{r\in\mathbb{Z}^l} \sigma(r,p)\, JT_r^*J^{-1} = \sum_{r\in\mathbb{Z}^l} JT_r^*J^{-1}\, \sigma(r,p).$$

Now, J intertwines multiplication operators from $C^\infty(M)$ with their complex conjugates: $Jf^*J^{-1} = f$. Therefore, $R(f) = \sum_{r\in\mathbb{Z}^l} f_r\, \sigma(r,p)$, from which one can check that

$$R(f)R(g) = R(g \star_\Theta f);$$

thus, R gives an antirepresentation of $C^\infty(M_\Theta)$ on $\mathcal{H}$. This commutes with the L representation:

$$[L(f), R(g)] = \sum_{r,s} \sigma(p,r)\, [f_r, g_s]\, \sigma(s,p) = 0.$$

The first-order property of the spin geometry follows immediately:

$$[[D\!\!\!/, L(f)], R(g)] = \sum_{r,s} \sigma(p,r)\, [[D\!\!\!/, f_r], g_s]\, \sigma(s,p) = 0,$$

since $[[D\!\!\!/, f_r], g_s] = 0$ in the commutative case.

Regularity and finiteness are assured because the smooth subalgebra $C^\infty(M)$ does not grow or shrink under deformations, and consequently its module $\mathcal{H}^\infty$ of smooth spinors is unchanged. Poincaré duality also holds, on account of another theorem of Rieffel [40], which says that the K-theory of the pre-C^*-algebras remains unaffected by Moyal deformations.

To sum up: whenever M is a spin manifold carrying an isometric action of a torus of rank $l \geqslant 2$, there exists a family of isospectral deformations $(C^\infty(M_\Theta), L^2(M,S), D\!\!\!/)$ of the standard commutative spectral triple. These are labelled by skewsymmetric $l \times l$ matrices Θ, up to isomorphisms among the various noncommutative tori $C(\mathbb{T}^l_\Theta)$. There remains the challenge of going beyond these "noncommutative spaces foliated by tori", which is raised by the existence of noncommutative 3-spheres (and beyond) which are inaccessible by Moyal deformation [17].

5 The Moyal plane: a noncompact noncommutative space

Historically, the "axiom scheme" for spectral triples emerged from the study of a specific case, namely, the structure of spacetime which underlies the Standard Model of particle physics [13, 14, 32]. We indicate briefly what is involved. The Minkowski space M^4, or rather its smooth coordinate algebra $C_0^\infty(M^4)$, is replaced by an "almost commutative" algebra [28] $\mathcal{A} = C_0^\infty(M^4) \otimes \mathcal{A}_F$, where the finite-dimensional algebra $\mathcal{A}_F$ takes account of "internal degrees of freedom" yielding the several fermionic particles of the theory, i.e., leptons and quarks. These particles appear as labels in a particular representation $\pi_F\colon \mathcal{A}_F \to \mathcal{L}(\mathcal{H}_F)$, and the details of this representation determine the nature of $\mathcal{A}_F$; in particular, the familiar gauge symmetries (electroweak and colour) are manifest in the representation π_F.

The candidate which emerged from these considerations is $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The Dirac operator on M^4 is amplified to a "Dirac–Yukawa" operator $D = D\!\!\!/ \otimes 1 + \chi \otimes D_F$ on $L^2(M^4, S) \otimes \mathcal{H}_F$, with D_F being a matrix of Yukawa mass parameters. An early success of this approach was the emergence of the Higgs boson as simply a "noncommutative gauge field" [19, 29, 32].

In fact, however, the "commutative part" of the model, namely $D\!\!\!/$ on M^4, was treated in an ad-hoc fashion, because noncommutative geometry did not, at that time, cover spaces of Minkowskian signature, where $D\!\!\!/$ is not essentially selfadjoint. Since then, a few proposals have been put forward [30, 46, 48], but it is fair to say that the matter is not yet settled. Even if one retreats to the Euclidean signature, as we shall shortly do, one must still formulate spectral triples over noncompact manifolds. The minimal problem, then, is to properly describe noncompact noncommutative spectral triples.

Once one moves beyond *unital* algebras – recall that compactness of M is equivalent to unitality of $C^\infty(M)$ – one needs to rework the general framework for spectral triples. An interesting generalization has been proposed by Rennie [36] based on the concept of a "local ideal" $\mathcal{A}_c \subset \mathcal{A}$, by analogy with the compact-support subspace $C_c^\infty(M) \subset C_0^\infty(M)$. However, the even-dimensional Euclidean space $\mathbb{R}^{2m}$, with a Moyal product, is not "local" in Rennie's sense. We shall now examine these Moyal planes in greater detail.

► We have already given two formulas for the Moyal product of a pair of Schwartz functions (33), which we recall:

$$\begin{aligned} f \star_\Theta h(x) &= (\pi\theta)^{-l} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} f(x+s)\, h(x+t)\, e^{-2is\cdot\Theta^{-1}t}\, ds\, dt, \\ &= (2\pi)^{-l} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} f(x - \tfrac{1}{2}\Theta u)\, h(x+t)\, e^{-iu\cdot t}\, du\, dt. \end{aligned}$$

The first formula only makes sense if $l = 2m$ *is even* and $\theta^{2m} = \det\Theta \neq 0$. Any symplectic matrix can be put in the canonical form as a direct sum of matrices $\begin{pmatrix} 0 & \theta_j \\ -\theta_j & 0 \end{pmatrix}$, and to simplify the discussion, we shall assume that $\theta_1 = \cdots = \theta_m = \theta$. (This is standard in the phase-space approach to quantum mechanics, where $\theta = \hbar$.) Thus we write $\Theta = \theta S$ where $S = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$, and from now on we shall write $\star_\theta$ instead of $\star_\Theta$ for the Moyal product. Therefore, we can rewrite it as

$$f \star_\theta h(x) := (\pi\theta)^{-2m} \int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} f(y)h(z)\, e^{\frac{2i}{\theta}(x-y)\cdot S(x-z)}\, dy\, dz. \tag{36}$$

We may first treat this expression formally: a rigorous interpretation as an oscillatory integral can be given in several ways [22,27,39]. Several algebraic properties are immediately evident. First of all, complex conjugation of functions is an involution: $(f \star_\theta h)^* = h^* \star_\theta f^*$. Associativity may also be checked directly. Differentiation under the integral sign gives the Leibniz rule for partial derivatives:

$$\partial_j(f \star_\theta h) = \partial_j f \star_\theta h + f \star_\theta \partial_j h.$$

Finally, the ordinary Lebesgue integral is a *trace* for this product:

$$\int f \star_\theta h(x)\, dx = \int h \star_\theta f(x)\, dx = \int f(x)\, h(x)\, dx \equiv \langle f, h\rangle.$$

Another consequence of the formula (36) is that the operation of left multiplication $\lambda_\theta(f) : h \mapsto f \star_\theta h$ is a pseudodifferential operator, with symbol

$$\sigma[\lambda_\theta(f)](x,\xi) = f(x - \tfrac{1}{2}\theta S\xi).$$

This is the starting point of Hörmander's theory of "Weyl pseudodifferential calculus" [27]. Of course, the precise symbol class of $\lambda_\theta(f)$ depends delicately on f; for $f \in \mathcal{S} := \mathcal{S}(\mathbb{R}^{2m})$, the Schwartz space of smooth rapidly decreasing functions, $\lambda_\theta(f)$ is a smoothing operator.

Using (36) and suitable norm estimates, it is not hard to show [22, 39] that for $f, h \in \mathcal{S}$, the product $f \star_\theta h$ lies in $\mathcal{S}$ also, and the product is a *continuous* bilinear map from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{S}$. We abbreviate $\mathcal{S}_\theta := (\mathcal{S}, \star_\theta)$; this is a nonunital Fréchet algebra.

The trace property of the integral, plus associativity, gives

$$\langle f \star_\theta g, h\rangle = \int (f \star_\theta g \star_\theta h)(x)\, dx = \langle f, g \star_\theta h\rangle,$$

and now by duality, we may replace one factor by $T \in \mathcal{S}'$ (a tempered distribution):

$$\langle T \star_\theta f, g\rangle := \langle T, f \star_\theta g\rangle, \qquad \langle f \star_\theta T, g\rangle := \langle T, g \star_\theta f\rangle.$$

Finally, one may multiply two distributions that belong to the Moyal multiplier algebra $\mathcal{M}_\theta$, defined as

$$\mathcal{M}_\theta := \{\, T \in \mathcal{S}' : T \star_\theta h,\ h \star_\theta T \in \mathcal{S} \text{ for all } h \in \mathcal{S}\,\}.$$

This multiplier algebra, however, is much too large (for example, it contains all polynomials). An alternative possibility is to use L^2-multipliers rather than Schwarz multipliers:

$$A_\theta := \{\, T \in \mathcal{S}' : T \star_\theta g,\ g \star_\theta T \in L^2(\mathbb{R}^{2m}) \text{ for all } g \in L^2(\mathbb{R}^{2m})\,\}.$$

This is a C^*-algebra – known to be isomorphic to $\mathcal{L}(L^2(\mathbb{R}^m))$ – of which $\mathcal{S}_\theta$ is a subalgebra (not a dense subalgebra, however).

The behaviour of L^2-type norms on $\mathcal{S}$ becomes transparent by exhibiting a special orthogonal basis $\{\, f_{pq} : p, q \in \mathbb{N}^m \,\}$ of $L^2(\mathbb{R}^{2m})$ whose members all lie in $\mathcal{S}$. These functions appear in phase-space quantum mechanics [24] as eigentransitions for the harmonic oscillator Hamiltonian $H = \frac{1}{2}(x_1^2 + \cdots + x_{2m}^2)$. If $H_j := \frac{1}{2}(x_j^2 + x_{j+m}^2)$ so that $H = H_1 + \cdots + H_m$, they satisfy

$$H_j \star_\theta f_{pq} = \theta(p_j + \tfrac{1}{2}) f_{pq}, \qquad f_{pq} \star_\theta H_j = \theta(q_j + \tfrac{1}{2}) f_{pq},$$

and may be expressed by associated Laguerre functions in polar coordinates on $\mathbb{R}^{2m}$. They fulfil the following identities, whereby $\mathcal{S}_\theta$ (for $\theta \neq 0$) may justly be regarded as "an algebra of rapidly decreasing infinite matrices":

$$f_{pq} \star_\theta f_{rs} = \delta_{qr} f_{ps}, \qquad f_{pq}^* = f_{qp}.$$

Using explicit formulas [21], one finds $\int_{\mathbb{R}^{2m}} |f_{pq}(x)|^2\, dx = (2\pi\theta)^m$, so that the $(2\pi\theta)^{-m/2} f_{pq}$ form an orthonormal basis for $L^2(\mathbb{R}^{2m})$. For $f = \sum_{p,q} c_{pq} f_{pq}$, the seminorms r_k defined by

$$r_k(f)^2 := (2\pi\theta)^{-m} \sum_{p,q \in \mathbb{N}^m} \theta^{2|k|} (p + \tfrac{1}{2})^k (q + \tfrac{1}{2})^k |c_{pq}|^2, \qquad k \in \mathbb{N}^m,$$

are all finite if and only if $f \in \mathcal{S}$, and they determine the topology of $\mathcal{S}_\theta$ as a nonunital Fréchet algebra.

Since $f = \sum_{p,q} c_{pq} f_{pq}$ and $h = \sum_{r,s} d_{rs} f_{rs}$ imply $f \star_\theta h = \sum_{p,s} \Big(\sum_q c_{pq}\, d_{qs}\Big) f_{ps}$, the Schwarz inequality for double sequences shows that

$$\|f \star_\theta h\|_2^2 \leqslant (2\pi\theta)^{-m} \|f\|_2^2\, \|h\|_2^2. \tag{37}$$

We note an immediate consequence: left Moyal multiplication by any $f \in \mathcal{S}$ extends to a bounded operator $\lambda_\theta(f)$ on $L^2(\mathbb{R}^{2m})$, with the norm estimate $\|\lambda_\theta(f)\| \leqslant (2\pi\theta)^{-m/2} \|f\|_2$.

► It turns out that none of the algebras with Moyal product exhibited so far is the optimal candidate for the algebra of a Moyal spectral triple: the "very regular" algebra $\mathcal{S}_\theta$ is too small, while the unital algebras A_θ and $\mathcal{M}_\theta$ are too large. There is, however, a *unitization* $\mathcal{B}_\theta$ of $\mathcal{S}_\theta$ (that is, a unital algebra in which $\mathcal{S}_\theta$ embeds as an essential ideal) which is no larger than necessary. It satisfies

$\mathcal{B}_\theta \subset A_\theta \cap \mathcal{M}_\theta$, so that it can be regarded as an algebra of multipliers in either sense. Also, $\mathcal{B}_\theta$ contains all *plane waves* $u_k(x) := e^{ik\cdot x}$ for $k \in \mathbb{R}^{2m}$. A formal calculation, using (36), shows that

$$u_k \star_\theta u_l = e^{-\frac{i}{2}\theta\, k\cdot Sl}\, u_{k+l}.$$

More rigorously, one may check that $u_k \star_\theta u_l \star_\theta h = e^{-\frac{i}{2}\theta\, k\cdot Sl}\, u_{k+l} \star_\theta h$ for $h \in \mathcal{S}$, arriving at the same product formula.

To describe this algebra $\mathcal{B}_\theta$, let us first consider the commutative case where $\theta = 0$. In that case, the multiplier algebra of $\mathcal{S}$ is $\mathcal{O}_M(\mathbb{R}^{2m})$, the space of smooth functions each of whose derivatives (of any order) is bounded by a polynomial. Included in $\mathcal{O}_M$ is the unital algebra [42]

$$\mathcal{B} := \{\, f \in C^\infty(\mathbb{R}^{2m}) : \partial^\alpha f \text{ is bounded, for all } \alpha \in \mathbb{N}^{2m} \,\}.$$

Clearly, $u_k \in \mathcal{B}$ for all k, but no quadratic exponential such as $\exp(ix \cdot Ax)$ lies in $\mathcal{B}$. The topology of $\mathcal{B}$ is given by the countable family of seminorms

$$q_k(f) := \sup\{\, |\partial^\alpha f(x)| : x \in \mathbb{R}^{2m},\ |\alpha| \leqslant k \,\},$$

and the Leibniz rule $\partial^\alpha(f \star_\theta h) = \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta}\, \partial^\beta f \star_\theta \partial^{\beta-\alpha} h$ together with an explicit estimate on the integral form of each $\partial^\beta f \star_\theta \partial^{\beta-\alpha} h(x)$ shows that $q_k(f \star_\theta h) \leqslant C_{k,\theta}\, q_k(f)\, q_k(h)$ for suitable constants $C_{k,\theta}$ [21, 39]. We conclude that $\mathcal{B}$ is closed under the Moyal product, and that $\mathcal{B}_\theta := (\mathcal{B}, \star_\theta)$ is a unital Fréchet algebra. Indeed, as the space of smooth elements for the action of $\mathbb{R}^l$ on A_θ, it is a pre-C^*-algebra.

If $f \in C^\infty(\mathbb{R}^{2m})$ is *periodic*, with a rectangular period parallelogram of side 1, say, then f can be identified with a smooth function on the torus $\mathbb{T}^{2m}$. Since we can identify $C^\infty(\mathbb{T}^{2m}_{\theta S})$ with the function space $C^\infty(\mathbb{T}^{2m})$ endowed with a Moyal product given by same formula (36), we see that there is an embedding of $C^\infty(\mathbb{T}^{2m}_{\theta S}) \hookrightarrow \mathcal{B}_\theta$ as its subalgebra of periodic smooth functions.

There is an important (nonunital) algebra intermediate between $\mathcal{S}_\theta$ and $\mathcal{B}_\theta$. Schwartz [42] denotes by $\mathcal{D}_{L^2}$ the space of smooth functions whose higher derivatives are all square-integrable, and proves that $\mathcal{S} \subset \mathcal{D}_{L^2} \subset \mathcal{B}$. (Indeed, one can replace the 2 by any p with $1 \leqslant p \leqslant \infty$, to get $\mathcal{S} \subset \mathcal{D}_{L^p} \subset \mathcal{B}$.) On combining the Leibniz rule with (37), we get the norm estimate:

$$\|\partial^\alpha(f \star_\theta h)\|_2 \leqslant (2\pi\theta)^{m/2} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \|\partial^\beta f\|_2\, \|\partial^{\alpha-\beta} h\|_2.$$

This entails that $\mathcal{D}_{L^2}$ is also closed under the Moyal product, and that $\mathcal{A}_\theta := (\mathcal{D}_{L^2}, \star_\theta)$ is a Fréchet algebra with a continuous product. Moreover, $\mathcal{A}_\theta$ is an ideal in $\mathcal{B}_\theta$ (but not a closed ideal), and can also be shown to be a pre-C^*-algebra [21].

In fine, we obtain that $\mathcal{S}_\theta \subset \mathcal{A}_\theta \subset \mathcal{B}_\theta$ with continuous inclusions. Each of these three algebras plays a role in the formulation of the Moyal spectral triple.

► We turn now to the other data for a spectral triple. The selfadjoint operator will be just the usual Dirac operator $D\!\!\!\!/ = -i\,\gamma^j \otimes \partial_j$ acting on $\mathcal{H} = L^2(\mathbb{R}^{2m}, S) \simeq \mathbb{C}^{2^m} \otimes L^2(\mathbb{R}^{2m})$. Indeed, the spinor bundle $S \to \mathbb{R}^{2m}$ is trivial, of rank 2^m, and all Christoffel symbols are zero. The $\mathbb{Z}_2$-grading operator on $\mathcal{H}$ is $\chi := \gamma^{2m+1} \otimes 1$, where $\gamma^{2m+1} := (-i)^m \gamma^1\gamma^2 \dots \gamma^{2m}$. The charge conjugation operation J is that of the commutative case: it is an antilinear operator on $\mathcal{H}$ satisfying $J(\gamma^j \otimes 1)J^{-1} = -\gamma^j \otimes 1$, and therefore, $J D\!\!\!\!/ J^{-1} = D\!\!\!\!/$.

We opt for the algebra $\mathcal{A}_\theta = (\mathcal{D}_{L^2}, \star_\theta)$, represented on $\mathcal{H}$ as left Moyal multipliers:

$$L_\theta(f)\,\psi := (1_{2^m} \otimes \lambda_\theta(f))\,\psi = (f \star_\theta \psi_\alpha)_{\alpha=1}^{2^m}.$$

Since J is antilinear and complex conjugation reverses the Moyal product, we get also the antirepresentation

$$R_\theta(h)\,\psi := J L_\theta(h^*) J^{-1} \psi = (\psi_\alpha \star_\theta h)_{\alpha=1}^{2^m},$$

acting on spinors as “right multiplication operators”. Clearly, $L_\theta(f)$ and $R_\theta(h)$ commute.

It is immediate from the definition of L_θ that $[D\!\!\!/, L_\theta(f)] = L_\theta(D\!\!\!/ f)$, with a slight abuse of notation: we write $L_\theta(D\!\!\!/ f)$ as an abbreviation for $-i\,\gamma^j \otimes \lambda_\theta(\partial_j f)$. It is then clear that

$$[[D\!\!\!/, L_\theta(f)], R_\theta(h)] = [L_\theta(D\!\!\!/ f), R_\theta(h)] = 0,$$

so the first-order condition holds. The estimate (37) now shows that

$$\|L_\theta(D\!\!\!/ f)\| \leqslant (2\pi\theta)^{-m/2} \max_{j=1,\dots,2m} \|\partial_j f\|_2,$$

where the right hand side is finite since $f \in \mathcal{A}_\theta$. Not so obvious, but still true [21], is that both $L_\theta(f)$ and $[D\!\!\!/, L_\theta(f)]$ lie in the smooth domain of the derivation $\delta = \operatorname{ad}|D\!\!\!/|$ whenever $f \in \mathcal{A}_\theta$. In fact, we only need (sup-norm) boundedness of the derivatives $\partial^\alpha f$ to assure regularity: thus both the algebras $\mathcal{A}_\theta$ and $\mathcal{B}_\theta$ fulfil the regularity condition for noncommutative manifolds.

We now declare the data set of the Moyal spectral triple to be

$$(\mathcal{A}_\theta, \mathcal{B}_\theta; L^2(\mathbb{R}^{2m}, S), D\!\!\!/; J, \chi),$$

where $\mathcal{B}_\theta$ is explicitly mentioned as a preferred unitization of $\mathcal{A}_\theta$.

► The remaining issue is the determination of the metric dimension from spectral theory. The matter is too involved for a complete treatment here, but we indicate the highlights. First of all, the Dirac operator on $\mathbb{R}^{2m}$ does not have compact resolvent: indeed, $D^2 = 1_{2^m} \otimes \Delta$ has continuous spectrum. However, from the standard treatment of K-theory for nonunital algebras, we realize that the appropriate compactness condition in the nonunital case should instead be that

$$L_\theta(f)\,(D\!\!\!/ - \lambda)^{-1} \text{ is compact, for } \lambda \notin \operatorname{sp} D\!\!\!/,\ f \in \mathcal{A}_\theta.$$

For $\lambda = \pm i\varepsilon$ and $T = L_\theta(f)\,(D\!\!\!/ \mp i\varepsilon)^{-1}$, we get $TT^* = L_\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-1}\,L_\theta(f^*)$, so we can equivalently show that $L_\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-1/2}$ is compact for any $f \in \mathcal{A}_\theta$ and any $\varepsilon > 0$.

For that, an old trick of scattering theory is useful. Apart from the nuance of $\mathcal{H} = L^2(\mathbb{R}^{2m}, S)$ versus a single copy of $L^2(\mathbb{R}^{2m})$, the issue is nicely discussed in Simon's book [44]. Write $g(k) := (k^2 + \varepsilon^2)^{-1/2}$ for $k \in \mathbb{R}^{2m}$, so that $(D\!\!\!/^2 + \varepsilon^2)^{-1/2} = g(D\!\!\!/)$. We then find

$$[L_\theta(f)\,g(D\!\!\!/)\psi](x) = (2\pi)^{-2m} \iint f(x - \tfrac{1}{2}\theta S\xi)\,g(\xi)\psi(y)\,e^{i\xi\cdot(x-y)}\,dy,$$

so that $L_\theta(f)\,g(D\!\!\!/)$ is a Hilbert–Schmidt operator with square-integrable kernel

$$K(x,y) = (2\pi)^{-2m} \int f(x - \tfrac{1}{2}\theta S\xi)\,g(\xi)\,e^{i\xi\cdot(x-y)}\,d\xi,$$

whose L^2-norm is $(2\pi)^{-m}\|f\|_2\,\|g\|_2$. Therefore, independently of θ as it happens, we get an equality

$$\|L_\theta(f)\,g(D\!\!\!/)\|_{\mathcal{L}^2} = (2\pi)^{-m}\|f\|_2\,\|g\|_2. \tag{38a}$$

On the other hand, the bounded-operator norm of $L_\theta(f)\,g(D\!\!\!/)$ satisfies

$$\|L_\theta(f)\,g(D\!\!\!/)\| \leqslant \|L_\theta(f)\|\,\|g(D\!\!\!/)\| = \|L_\theta(f)\|\,\|g\|_\infty \leqslant (2\pi\theta)^{-m/2}\|f\|_2\,\|g\|_\infty. \tag{38b}$$

Faced with both relations (38), one can apply the so-called complex method of interpolation theory of Banach spaces to get intermediate estimates for each p with $2 < p < \infty$; one finds that

$$\|L_\theta(f)\,g(D\!\!\!/)\|_{\mathcal{L}^p} \leqslant C_{m,p}\,\theta^{-m(\frac{1}{2}-\frac{1}{p})}\,\|f\|_2\,\|g\|_p.$$

On the right hand side, the L^p-norm $\|g\|_p$ of the fixed function g is finite for $p > 2m$. On the left hand side, the norm is $\|T\|_{\mathcal{L}^p} := \big(\operatorname{Tr}(T^*T)^{p/2}\big)^{-1/p}$ whose finiteness determines the Schatten space $\mathcal{L}^p$ of compact operators. Thus, for $p > 2m$ and $f \in \mathcal{A}_\theta$ – notice that only the L^2 norms of f and some of its derivatives are involved – both $L_\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-1/2}$ and $[D\!\!\!/, L_\theta(f)]\,(D\!\!\!/^2 + \varepsilon^2)^{-1/2}$ lie in $\mathcal{L}^p$.

One can show that, in consequence, both $L_\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-m}$ and $[D\!\!\!/, L_\theta(f)]\,(D\!\!\!/^2 + \varepsilon^2)^{-m}$ lie in $\mathcal{L}^p$ for all $p > 1$, provided $f \in \mathcal{A}_\theta$. This is almost, but not quite, enough to show that the partial sums of singular values of these operators diverge logarithmically. To close the gap takes a lot of effort, and it turns out that it cannot be done for all $f \in \mathcal{A}_\theta$; however, in [21] it is shown that logarithmic divergence can be achieved provided f lies in the ideal $\mathcal{S}_\theta$. The coefficient of logarithmic divergence turns out to be

$$⨍ L_\theta(f)\,(D\!\!\!/^2 + \varepsilon^2)^{-m} = ⨍ L_\theta(f)\,(|D\!\!\!/| + \varepsilon)^{-2m} = \frac{1}{m!\,(2\pi)^m}\int_{\mathbb{R}^{2m}} f(x)\,dx.$$

This result is independent of ε (trivially) and of θ (interestingly). Since $2m$ is indeed the critical exponent for the operator $(|D\!\!\!/| + \varepsilon)^{-1}$, the metric dimension of the Moyal-deformed $\mathbb{R}^{2m}$ equals $2m$, as expected.

► At the price of adding one extra datum, the preferred unitization $\mathcal{B}_\theta$, we have achieved the construction of a particular noncompact noncommutative spin geometry. We finish by pointing out the three places where the unitization plays a role:

1. For regularity, we require that the operators $L_\theta(f)$ and $[D\!\!\!/, L_\theta(f)]$ not only be bounded, but also lie in the smooth domain of the derivation $\delta = \operatorname{ad}|D\!\!\!/|$. Thus f should be smooth and have bounded derivatives of all orders, that is, $f \in \mathcal{B}_\theta$ at any rate. This indicates that, as a unitization, $\mathcal{B}_\theta$ is not too large.

2. For orientation, the Hochschild $2m$-cycle $\mathbf{c}$ must correspond to the volume form in local coordinates. Taking into account the noncommutativity of the Moyal product, it is no surprise that $\mathbf{c}$ *is given by exactly the same formula as for the noncommutative torus* (10):

$$\mathbf{c} := \frac{(-i)^m}{(2m)!(2\pi)^{2m}} \sum_{\sigma\in S_{2m}} (-1)^\sigma \big(u_{\sigma(1)}u_{\sigma(2)}\dots u_{\sigma(2m)}\big)^{-1} \otimes u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(2m)}.$$

With this difference: these unitaries u_j are not the torus-algebra generators, but rather *plane waves* in orthogonal directions, which happen to satisfy exactly the same commutation relations as the toral generators – indeed, this is how the embedding of the torus algebra in $\mathcal{B}_\theta$ is achieved. Now, the plane waves are not square-integrable, so this Hochschild cycle must be taken over the unitization: $\mathbf{c} \in H_{2m}(\mathcal{B}_\theta, \mathcal{B}_\theta)$. The moral is that the unitization $\mathcal{B}_\theta$ is large enough to encompass the volume form.

3. A third role for the unital algebra $\mathcal{B}_\theta$ comes from the finiteness condition, namely that the smooth spinors should form a finitely generated projective (left) module over the algebra. Such a condition would make little sense for nonunital algebras. What one must do is to build a finitely generated projective module over the unitization: $\mathcal{E} := p\,\mathcal{B}_\theta^N$, where $p = p^2 = p^*$ in $M_N(\mathcal{B}_\theta)$. Now we can "pull back" to the corresponding $\mathcal{A}_\theta$-module by

$$\mathcal{E}\big|_{\mathcal{A}_\theta} := p\,\mathcal{A}_\theta^N, \quad \text{with } p \text{ still in } M_N(\mathcal{B}_\theta),$$

which makes sense because $\mathcal{A}_\theta$ is an ideal in $\mathcal{B}_\theta$. In the case at hand, $\mathcal{H}^\infty$ is actually a free module of rank 2^m, so that $N = 2^m$ and $p = 1_N$. Then $\mathcal{E}\big|_{\mathcal{A}_\theta}$ is just $\mathbb{C}^N \otimes \mathcal{A}_\theta = \mathbb{C}^N \otimes \mathcal{D}_{L^2}$: the spinor components are drawn from $\mathcal{D}_{L^2}$. The lesson here is that the "correct" algebra $\mathcal{A}_\theta$ can be pinned down by the chosen unitization and the finiteness condition: the original candidate $\mathcal{S}_\theta$ will not do the job. (The algebra $\mathcal{S}_\theta$ is relegated to the auxiliary but still vital role of determining the metric dimension.)

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