The sparse T1 Theorem

Presented by Darío Mena Arias

(joint work with Michael T. Lacey)

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Background

- Control over the norm: Lerner, 2013.
- Point-wise approach: Conde-Alonso, Rey, 2015.
- Point-wise stopping time: Lacey, 2015. Different points of view: Bernicot, Frey, Petermichl; Domelevo, Petermichl; Lerner; Volberg, Zorin-Kranich (2016).
- Bilinear form approach: Culiuc, Di Plinio, Ou; Benea, Bernicot, Luque; Lacey, Spencer (2016).

Introduction

A collection of cubes $\mathcal{S}$ is c-sparse if for each $S \in \mathcal{S}$ there is $E_S \subseteq S$ such that

1. $|E_S| > c|S|$,
2. $\| \sum_{S \in \mathcal{S}} \mathbb{1}_{E_S} \|_\infty \leq c^{-1}$.

If $\langle f \rangle_S = |S|^{-1} \int_S f(x)\, dx$, a bilinear sparse form is defined by

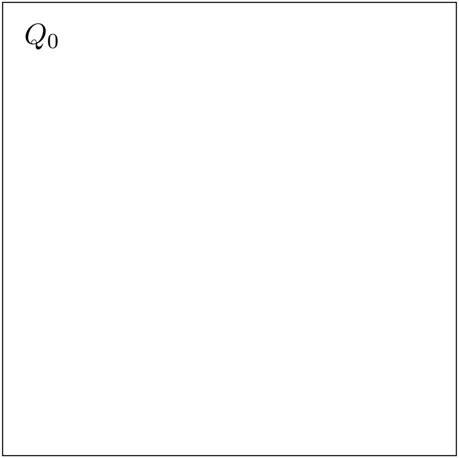
$$\Lambda_{\mathcal{S}}(f, g) = \sum_{S \in \mathcal{S}} \langle f \rangle_S \langle g \rangle_S |S|.$$

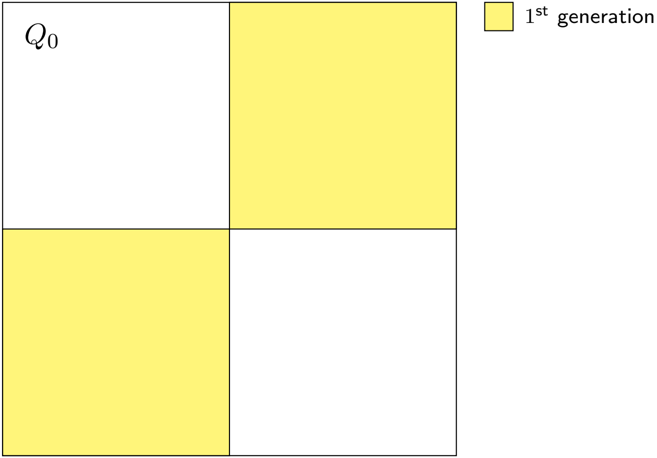
Usually we take $\mathcal{S}$ a subcollection of a dyadic grid such that

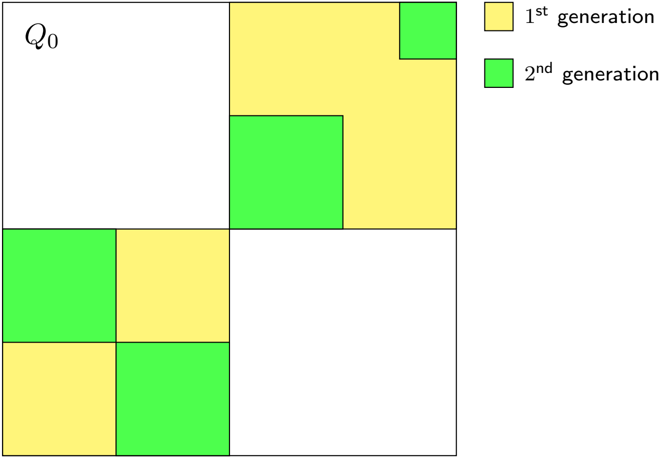
$$\sum_{S' \in \mathrm{Ch}_{\mathcal{S}}(S)} |S'| \leq \frac{1}{2}|S|.$$

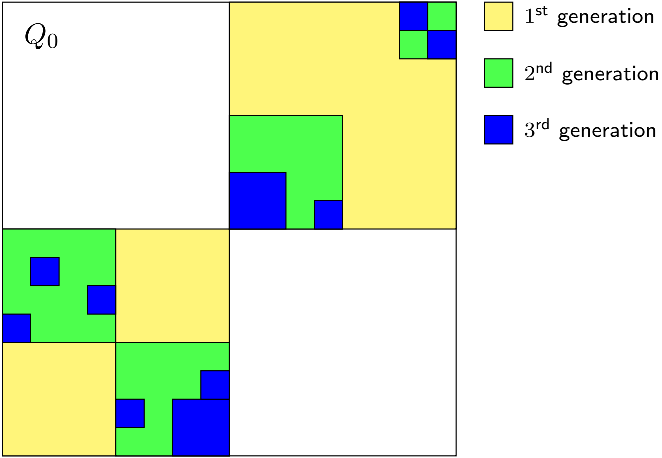
Here, $\mathrm{Ch}_{\mathcal{S}}(S) = \{ S' \in \mathcal{S}$ maximal : $S' \subsetneq S \}$.

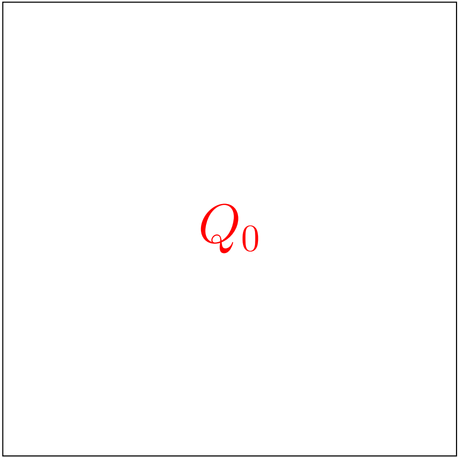
Then take $E_S = S \setminus \bigcup_{S' \in \mathrm{Ch}_{\mathcal{S}} S}$ S'

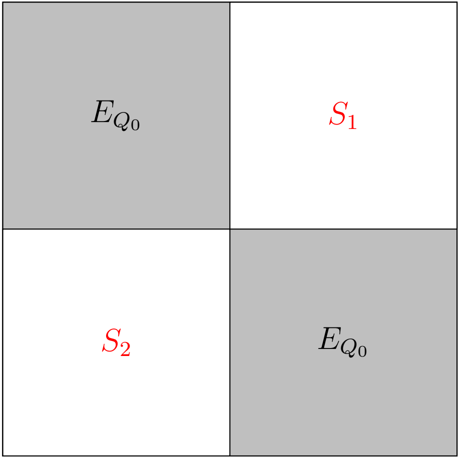

Q_0

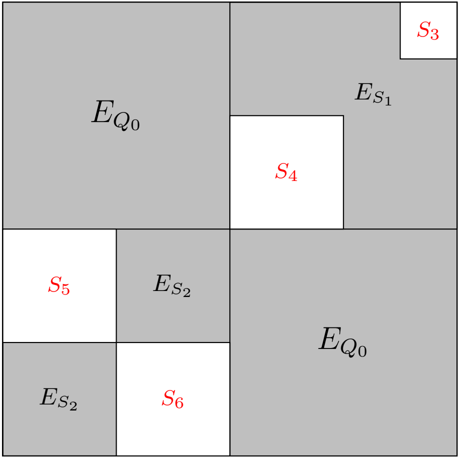

Q_0
1^{st} generation


Q_0
1st generation
2nd generation


Q_0
1st generation
2nd generation
3rd generation


Q_0


E_{Q_0}
S_1
S_2
E_{Q_0}


E_{Q_0}
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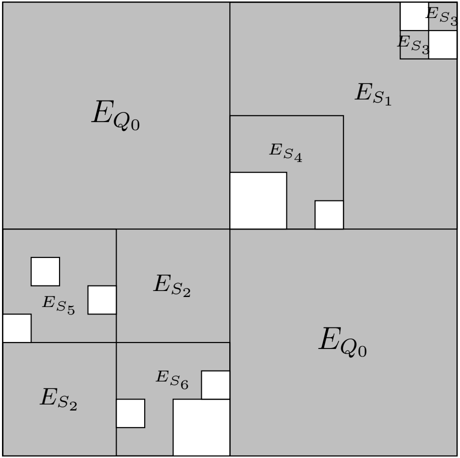
E_{Q_0}
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T1 Theorem

A function K on $\Omega = \mathbb{R}^d \times \mathbb{R}^d \backslash \{(x,x) : x \in \mathbb{R}^d\}$ is called a standard kernel if there are $C_K, \eta > 0$ such that

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2. $\forall x, x', y \in \Omega$, s.t. $2|x-x'| < |x-y|$, we have

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Let $T : \mathcal{S} \to \mathcal{S}'$ s.t. for $f, g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with disjoint supports $\langle Tf, g \rangle = \iint K(x,y) f(y) g(x)\, dy\, dx$. If T extends to a bounded operator on L^2, then it's called a Calderón-Zygmund operator.

Original T1 Theorem

Theorem (David, Journé)

Let T be a continuous operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ associated with a standard kernel. Then T can be extended to a bounded operator from $L^2(\mathbb{R}^n)$ to itself if and only if the three following conditions are satisfied:

1. $T1 \in \mathrm{BMO}$
2. $T^*1 \in \mathrm{BMO}$
3. T *has the weak boundedness property: for every ball* B, $|\langle T\mathbb{1}_B, \mathbb{1}_B\rangle| = O(|B|)$.

Our formulation

Theorem (Lacey, M.)

Suppose that T is a Calderón-Zygmund operator on $\mathbb{R}^d$, and moreover there is a constant $\mathfrak{T}$ so that for all cubes Q and functions $|\phi| \leq \mathbf{1}_Q$, there holds

$$|\langle T\mathbb{1}_Q, \phi\rangle| + |\langle T\phi, \mathbb{1}_Q\rangle| \leq \mathfrak{T}|Q|.$$

Then there is a constant $C = C(C_K, \mathfrak{T}, d, \eta)$ so that for all bounded compactly supported functions f, g, there is a sparse operator Λ so that

$$|\langle Tf, g\rangle| \leq C\Lambda(|f|, |g|).$$

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The proof doesn't appeal to any structural theory of Calderon-Zygmund, for example, boundedness of maximal truncations or Hytönen's representation (approach followed by Culiuc-Di Plinio-Ou).

Consequences of the sparse bound

1. Weak type $(1,1)$ inequality, L^p inequalities for $1 < p < \infty$:

$$\begin{aligned}\Lambda(f,g) &= \sum_{S\in\mathcal{S}} \langle f\rangle_S \langle g\rangle_S |S| \lesssim \sum_{S\in\mathcal{S}} \langle f\rangle_S \langle g\rangle_S |E_S| \\ &= \int_{\mathbb{R}^d} \sum_{S\in\mathcal{S}} \langle f\rangle_S \langle g\rangle_S \mathbb{1}_{E_S}(x)\,dx \lesssim \int_{\mathbb{R}^d} \mathcal{M}f(x)\mathcal{M}g(x)\,dx \\ &\leq \|\mathcal{M}f\|_p \|\mathcal{M}g\|_{p'} \lesssim \|f\|_p \|g\|_{p'}\end{aligned}$$

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3. The exponential integrability results of Karagulyan.

Tools

Random dyadic grids:

- $D^\omega = \{ Q + \sum_{j:2^{-j}<\ell Q} 2^{-j}\omega_j : Q \in \mathcal{D} \}$, $\mathcal{D}$ standard dyadic grid.
- Orthogonal decomposition: $f(x) = \sum_{Q\in\mathcal{D}^\omega} \Delta_Q f(x)$.
- Notion of good and bad intervals (associated to a positive integer r and a real number $\gamma > r^{-1}$).
- It is enough to prove for good projections.

$$\sum_{\substack{P\in\mathcal{D} \\ P \text{ is good}}} \sum_{\substack{Q\in\mathcal{D} \\ Q \text{ is good}}} \langle T(\Delta_P f), \Delta_Q g\rangle \lesssim \Lambda(|f|, |g|).$$

Special bilinear forms:

Let $i_P = \log_2(\ell P)$. Let $D_k f = \sum_{P\,:\,\ell P = 2^k} \Delta_P f$, and define

$$B^{u,v}(f,g) = \sum_P \langle |D_{i_P - u} f| \rangle_{3P} \langle |D_{i_P - v} g| \rangle_{3P} |P|$$

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We have

$$|B^{u,v}(f,g)| \leq \int S_u f(x) S_v g(x)\, dx,$$

with $S_u f(x) = \sum_P \langle |D_{i_P - u} f| \rangle_{3P}^2 \, \mathbb{1}_P$.

Lemma

We have the inequality below, valid for all integers $u \geq 0$

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Lemma

For all $u, v \geq 0$*, all bounded compactly supported functions* f, g*, there is a sparse collection* $\mathcal{S}$ *so that*

$$B^{u,v}(f,g) \lesssim (1+u)(1+v)\Lambda(f,g).$$

Universal domination:

There is one sparse form *"to rule them all"*...

Lemma

Given finitely supported functions f, g*, there is a sparse form* Λ^* *and a constant* $C > 0$ *such that for any other sparse operator* Λ *we have*

$$\Lambda(f,g) \leq C\Lambda^*(f,g).$$

Additional estimates :

1. Off-diagonal estimate:

   If $Q \Subset P$ good, there is $\eta' > 0$ s.t

   $$\langle T\mathbb{1}_{\mathbb{R}^d \setminus P}, g \rangle \lesssim \left[\frac{\ell Q}{\ell P}\right]^{\eta'} \|g\|_1.$$

   $Q \Subset P$ means $Q \subseteq P$ and $2^r \ell Q \leq \ell P$.

2. Hardy's inequality.

Sketch of the proof:

It is enough to prove for f, g compactly supported on a good large cube P_0 (almost all dyadic grids satisfy it).

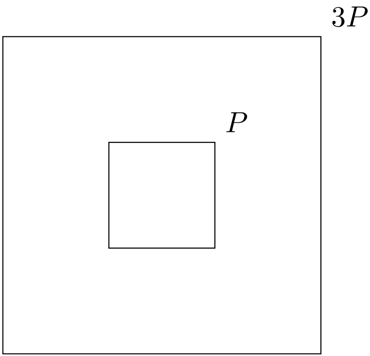
We consider only $\ell Q \leq \ell P$, the rest is addressed by duality.

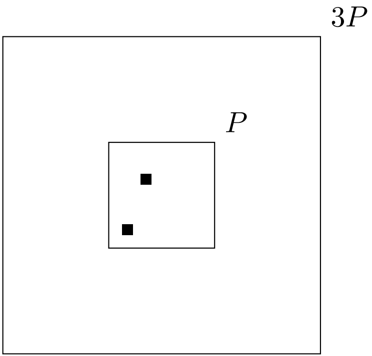
With this, we only consider

$$\sum_{P\,:\,P \subset P_0} \sum_{\substack{Q\,:\,Q \subset P_0 \\ \ell Q \leq \ell P}} \langle T \Delta_P f, \Delta_Q g \rangle$$

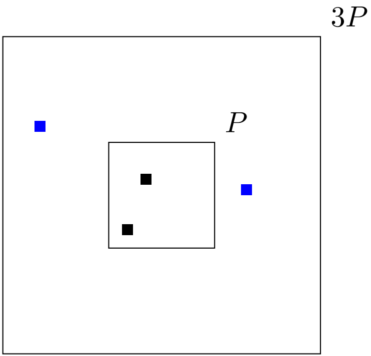
And we decompose it as follows:

$$
\begin{aligned}
\sum_{P\,:\,P\subset P_0} \sum_{\substack{Q\,:\,Q\subset P_0\\ \ell P\geq \ell Q}} \langle T\Delta_P f, \Delta_Q g\rangle
&= \sum_{P\,:\,P\subset P_0} \sum_{Q\,:\,Q\Subset P} \langle T\Delta_P f, \Delta_Q g\rangle && \text{(inside)}\\
&+ \sum_{P\,:\,P\subset P_0} \sum_{\substack{Q\,:\,2^r\ell Q\leq \ell P\\ Q\subset 3P\setminus P}} \langle T\Delta_P f, \Delta_Q g\rangle && \text{(near)}\\
&+ \sum_{P\,:\,P\subset P_0} \sum_{\substack{Q\,:\,\ell Q\leq \ell P\\ Q\cap 3P=\emptyset}} \langle T\Delta_P f, \Delta_Q g\rangle && \text{(far)}\\
&+ \sum_{P\,:\,P\subset P_0} \sum_{\substack{Q\,:\,\ell Q\leq \ell P\leq 2^r\ell Q\\ Q\cap 3P\neq\emptyset}} \langle T\Delta_P f, \Delta_Q g\rangle. && \text{(neighbors)}
\end{aligned}
$$


3P
P

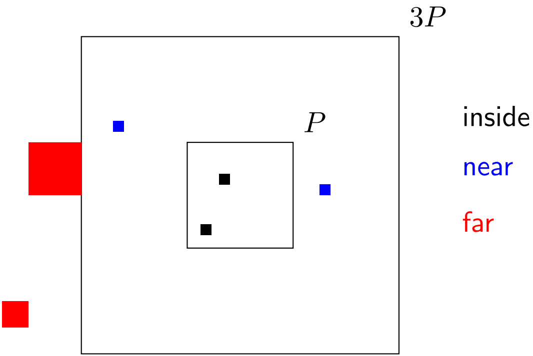


inside



inside

near


3P
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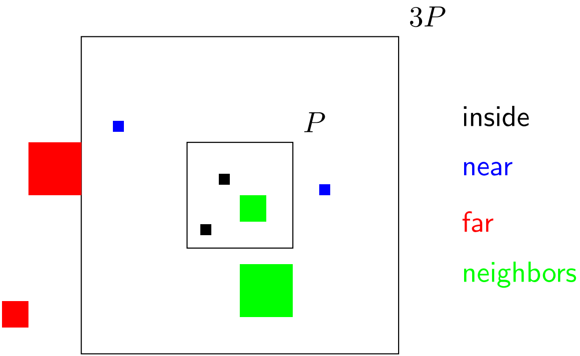
3P
P
inside
near
far
neighbors

How to define the sparse collection $\mathcal{S}$?

Stopping time argument: Add P_0 to $\mathcal{S}$. Recursively, for $S \in \mathcal{S}$, define the sets

- $F_S^1 = \bigcup \{ S' \in \mathcal{D}(S) : \langle |f| \rangle_{S'} > C_0 \langle |f| \rangle_S , \; S' \text{ maximal} \}$.
- $F_S^2 = \bigcup \{ S' \in \mathcal{D}(S) : \langle |g| \rangle_{S'} > C_0 \langle |g| \rangle_S , \; S' \text{ maximal} \}$.
- $F_S^3 = \bigcup \{ S' \in \mathcal{D}(S) : \langle |T\mathbb{1}_S| \rangle_{S'} > C_0 \mathcal{T}, \; S' \text{ maximal} \}$.

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Let $F_S = F_S^1 \cup F_S^2 \cup F_S^3$, and $\mathcal{F}_S$ be the family of dyadic components of F_S. Add $\mathcal{F}_S$ to $\mathcal{S}$. For C_0 big enough, the collection is sparse.

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Q^σ: smallest stopping cube containing Q
Q^τ: the smallest stopping cube strongly containing Q.

We illustrate the proof with one case and two sub-cases.

The inside terms: $$\sum_{P\,:\,P\subset P_0}\ \sum_{Q\,:\,Q\Subset P} \langle T\Delta_P f, \Delta_Q g\rangle$$

If P_Q is the child of P containing Q, write

$$\begin{aligned}\Delta_P f &= \Delta_P f \mathbb{1}_{P\setminus P_Q} + \langle \Delta_P f\rangle_{P_Q} \mathbb{1}_{P_Q} \\ &= \Delta_P f \mathbb{1}_{P\setminus P_Q} + \langle \Delta_P f\rangle_{P_Q} \cdot \begin{cases} \mathbb{1}_S - \mathbb{1}_{S\setminus P_Q} S = Q^\tau \supset P_Q \\ \mathbb{1}_S + \mathbb{1}_{P_Q\setminus S} S = Q^\tau \subsetneq P_Q \end{cases}\end{aligned}$$

We first look at $\left\langle T(\Delta_P f \mathbb{1}_{P\setminus P_Q}), \Delta_Q g\right\rangle$

We fix the relative sizes of P and Q, by considering $\ell P = 2^v \ell Q$. By the off-diagonal estimates

$$\begin{aligned} |\langle T(\Delta_P f \mathbb{1}_{P \setminus P_Q}), \Delta_Q g\rangle| &\lesssim [\ell Q/\ell P]^{\eta'} \langle |\Delta_P f|\rangle_P \|\Delta_Q g\|_1. \\ &= 2^{-\eta' v} \langle |\Delta_P f|\rangle_P \|\Delta_Q g\|_1. \end{aligned}$$

And further simplifications lead to

$$\sum_P \sum_{\substack{Q\,:\,Q \subseteq P \\ 2^v \ell Q = \ell P}} |\langle T(\Delta_P f \mathbb{1}_{P \setminus P_Q}), \Delta_Q g\rangle| \lesssim 2^{-\eta' v} B^{0,v}(f,g)$$

Use “universal domination” and sum over v.

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Use "universal domination" and sum over v.

Second sub-case: $\mathbf{Q}^\tau \subseteq \mathbf{P_Q}$**.** Fix $S \in \mathcal{S}$, we look at

$$\sum_{Q\,:\,Q^\tau = S} \; \sum_{P\,:\,Q \Subset P} \langle T(\Delta_P f \mathbb{1}_S), \Delta_Q g \rangle \,.$$

For $S = Q^\tau$, define $\{\varepsilon_Q\}$ by

$$\epsilon_Q \langle |f| \rangle_S := \sum_{P \in \mathcal{D},\; Q \Subset P_Q} \langle \Delta_P f \rangle_{P_Q} \,.$$

By the first stopping condition, $\{\varepsilon_Q\}$ is uniformly bounded. Then, the following is a martingale transform:

$$\Pi_S^\epsilon g = \sum_{Q\,:\,Q^\tau = S} \varepsilon_Q \Delta_Q g.$$

We can write the sum as

$$\sum_{Q\,:\,Q^\tau=S}\ \sum_{P\,:\,Q\Subset P} \langle T(\Delta_P f \mathbb{1}_S), \Delta_Q g\rangle = \langle |f|\rangle_S \langle T\mathbb{1}_S, \Pi^\epsilon_S g\rangle\,.$$

We apply the second and third stopping times (control over average of g and testing condition) to get that the previous sum is controlled by $\langle |f|\rangle_S\langle |g|\rangle_S|S|$. Summing over S we get a sparse bound.

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Rest of the terms?

Follow similar arguments to the first sub-case. By fixing relative sizes of Q and P, we can find bounds of the form

$$2^{-\eta'(u+v)} B^{u,v}(f,g).$$

Universal domination does the job.