Reconstruction of manifolds
in noncommutative geometry

Adam Rennie\textsuperscript{1,2} and Joseph C. Várrily\textsuperscript{3}

\textsuperscript{1} Institute for Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark
\textsuperscript{2} Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia
\textsuperscript{3} Escuela de Matemática, Universidad de Costa Rica, 11501 San José, Costa Rica


Abstract
We show that the algebra $\mathcal{A}$ of a commutative unital spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfying several additional conditions, slightly stronger than those proposed by Connes, is the algebra of smooth functions on a compact spin manifold.

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1 Introduction

Noncommutative Geometry, as developed over the past several years by Connes and coworkers, has produced a profusion of examples of “noncommutative spaces” [22], many of which partake of the characteristics of smooth Riemannian manifolds, whose metric and differential structure is determined by a generalized Dirac operator. To find a common framework for those examples, Connes proposed in [21] an axiomatic framework for “noncommutative spin manifolds”.

The geometry is carried by the notion of spectral triple $(A, \mathcal{H}, D)$; the familiar Riemannian spin geometry is recovered when $A$ is a coordinate algebra of smooth functions on a manifold, $\mathcal{H}$ a Hilbert space of spinors, and $D$ the Dirac operator determined by the spin structure and Riemannian metric. The question of reconstruction is whether the operator-theoretic framework proposed by Connes, or some variation of it, suffices to determine this spin manifold structure whenever the algebra $A$ is commutative.

In [21], Connes held out the hope that it could be so; but the extraction of a manifold from these postulates has proved elusive. In [50], a first attempt at doing so was presented, but was subsequently shown to fall short of the goal [28]. A detailed description of the reconstruction of many geometric features of a spin manifold was presented in [30], but there the starting algebra $A$ was assumed $a priori$ to be the smooth functions on a compact manifold.

In this paper, using a slightly stronger set of conditions on a spectral triple, we show that from the further assumption of a commutative coordinate algebra $A$ one can indeed recover a compact boundaryless manifold whose smooth functions coincide with $A$.

One of the key themes of the axioms proposed by Connes was Poincaré duality in $K$-theory. Earlier results of Sullivan [58] indicated that in high dimensions, in the absence of 2-torsion and in the simply connected setting, Poincaré duality in $KO$-theory characterizes the homotopy type of a compact manifold. While as a guiding principal such an idea is very attractive, we have not found a way to implement this approach to reconstruct a manifold.

Instead, we utilize an earlier formulation of Poincaré duality in noncommutative geometry which is phrased at the level of Hochschild chains, and thus is more useful for elaborating a proof. This concrete version of Poincaré duality is described by a “closedness condition” [18, VI.4.γ], which historically arose from attempts to fine-tune the Lagrangian of the Standard Model of elementary particles, and conceptually is an analogue of Stokes’ theorem.

Poincaré duality in $K$-theory plays no role in our reconstruction of a manifold as a compact space $X$ with charts and smooth transition functions. However, once that has been achieved, it is needed to show that $X$ carries a spin$^c$ structure and to identify the class of $(A, \mathcal{H}, D)$ as the fundamental class of the spin$^c$ manifold. The key to this is Plymen’s characterization of spin$^c$ structures [48] as Morita equivalence bimodules for the Clifford action induced by the metric. Indeed, it would be economical to replace Poincaré duality by postulating instead the existence of such bimodules; we
touch on this in our final section.

Compactness of the manifold, or equivalently the condition that the coordinate algebra have a unit, is an essential technical feature of our proof. However, the reconstruction of noncompact manifolds should also be possible, under some alternative conditions along the lines suggested in [27, 51]. Indeed, many of the crucial arguments used in reconstructing the coordinate charts are completely local.

The proof that the Gelfand spectrum $X = \text{sp}(A)$ is a differential manifold is quite long, but may be conceptually broken into two steps. The first is to construct a vector bundle over $X$ which plays the role of the cotangent bundle. Already at this stage we need to deploy all the conditions on our spectral triple (except Poincaré duality in $K$-theory and a metric condition). In particular, we identify local trivializations and bases of this bundle in terms of the ‘1-forms’ given by the orientability condition. These 1-forms $[\mathcal{D}, a^j_\alpha]$, for $j = 1, \ldots, p$, $\alpha = 1, \ldots, n$, generate the sections of this bundle, and the aim now is to show that the maps $a^j_\alpha = (a^1_\alpha, \ldots, a^p_\alpha) : X \to \mathbb{R}^p$ provide coordinates on suitable open subsets of $X$.

This is accomplished by proving that $a^j_\alpha$ is locally one-to-one and open. The tools used here are a Lipschitz functional calculus, some measure theoretic results of Voiculescu [61], some basic point set topology and properties of the map $a^j_\alpha$, and finally the unique continuation properties for Dirac-type operators [5, 36].

The main tools in the proof are a multivariate $C^\infty$ functional calculus for regular spectral triples [51], which we present here; as well as a Lipschitz functional calculus. The first of these enables us to construct partitions of unity and local inverses within the algebra $A$.

▶ The plan of the paper is as follows. In Section 2 we give some standard definitions and background results, including the $C^\infty$ functional calculus and its immediate consequences. In Section 3, we introduce the conditions on a spectral triple needed to establish our main result.

Section 4 details the construction of the cotangent bundle, while Section 5 develops a Lipschitz functional calculus needed to deal with the topology of our coordinate charts. Sections 6 and 7 contain the detailed proof that we do indeed recover a manifold. We develop the necessary point set topology to establish that the spectrum of our algebra is a manifold: the main issue is the absence of branch points in the chart domains. We show that the algebra generated by $A$ and $[\mathcal{D}, A]$ is locally a direct sum of Clifford actions arising from one or several Riemannian metrics, for which $\mathcal{D}$ is (again, locally) a direct sum of Dirac-type operators. Then we use the unique continuation properties of Dirac-type operators and the local description of $\mathcal{D}$ to banish any branch points and thereby get a manifold. That done, we assemble the Clifford actions globally, and so produce the Clifford action of a single Riemannian metric.

In Section 8 we explain in some detail how the (unique) spin$^c$ structure arises from Poincaré duality in $K$-theory. The Dirac operator is shown to differ from $\mathcal{D}$ by at most an endomorphism of the corresponding spinor bundle.

Section 9 collects some further remarks on our postulates and their possible variants.

Appendix A establishes some basic results about Hermitian pairings on finite projective modules. Appendix B examines additional results about our conditions, in particular the redundancy of the metric condition.
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2 Spectral triples and smooth functional calculus

The central notion of this paper is that of a spectral triple [19] over a commutative algebra. We begin by recalling several basic definitions, in order to establish a suitable functional calculus for them.

Definition 2.1. A spectral triple $(A, \mathcal{H}, D)$ is given by:

1. A faithful representation $\pi: A \to \mathcal{B}(\mathcal{H})$ of a unital $*$-algebra $A$ by bounded operators on a Hilbert space $\mathcal{H}$; and

2. A selfadjoint operator $D$ on $\mathcal{H}$, with dense domain $\text{Dom} \, D$, such that for each $a \in A$, $[D, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$ and $\pi(a)(1 + D^2)^{-1/2}$ is a compact operator.

The spectral triple is said to be even if there is an operator $\Gamma = \Gamma^* \in \mathcal{B}(\mathcal{H})$ such that $\Gamma^2 = 1$ (this determines a $\mathbb{Z}_2$-grading on $\mathcal{H}$), for which $[\Gamma, \pi(a)] = 0$ for all $a \in A$ and $\Gamma D + D \Gamma = 0$ (i.e., $\pi(A)$ is even and $D$ is odd with respect to the grading). If no such grading is available, the spectral triple is called odd.

Remark 2.1. Since $A$ is faithfully represented on $\mathcal{H}$, we may and shall omit $\pi$, regarding $A$ as a subalgebra of $\mathcal{B}(\mathcal{H})$. As such, its norm closure $\overline{A} = A$ is a $C^*$-algebra.

Remark 2.2. In this paper, we shall always assume that $A$ is unital. Nonunital spectral triples have been studied in [51, 52] under the assumption that $A$ has a dense ideal with local units. Another class of nonunital spectral triples are those arising from Moyal products, analyzed in detail in [27] (and anticipated in [29]). The Moyal example shows that it is important to treat a certain unitization of $A$ as part of the data of a (nonunital) spectral triple, so that it is proper to focus first on the unital case.

Definition 2.2. The operator $D$ gives rise to two (commuting) derivations of operators on $\mathcal{H}$; we shall denote them by

$$d_x := [D, x], \quad \delta x := [|D|, x], \quad \text{for} \quad x \in \mathcal{B}(\mathcal{H}).$$

According to Definition 2.1, $A$ lies within $\text{Dom} \, \delta := \{ x \in \mathcal{B}(\mathcal{H}) : [D, x] \in \mathcal{B}(\mathcal{H}) \}$.

A spectral triple $(A, \mathcal{H}, D)$ is called $QC^\infty$ if

$$A \cup \delta A \subseteq \bigcap_{m=1}^{\infty} \text{Dom} \, \delta^m.$$
Remark 2.3. The terminology $QC^\infty$ was introduced in [10], to distinguish “quantum” differentiability of operators from “classical” differentiability of smooth functions. One can also define $QC^k$, for $k \in \mathbb{N}$, by requiring only that $\mathcal{A} \cup \mathcal{D} \subseteq \text{Dom} \delta^m$ for $m = 1, \ldots, k$. Such spectral triples are more often referred to as regular [21, 30], and have been called smooth in [51].

Definition 2.3. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a $QC^\infty$ spectral triple, the family of seminorms

$$q_m(a) := \|\delta^m a\| \quad \text{and} \quad q'_m(a) := \|\delta^m([\mathcal{D}, a])\|, \quad m = 0, 1, 2, \ldots$$

(2.1)
determine a locally convex topology on $\mathcal{A}$ which is finer than the norm topology of $\mathcal{A}$ (that is given by $q_0$ alone) and in which the involution $a \mapsto a^*$ is continuous. Let $\mathcal{A_\delta}$ denote the completion of $\mathcal{A}$ in the topology of (2.1).

We quote Lemma 16 of [51].

Lemma 2.4. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $QC^\infty$ spectral triple. The Fréchét algebra $\mathcal{A_\delta}$ is a pre-$C^*$-algebra, and $(\mathcal{A_\delta}, \mathcal{H}, \mathcal{D})$ is also a $QC^\infty$ spectral triple. \hfill \Box

Recall that a pre-$C^*$-algebra is a dense subalgebra of a $C^*$-algebra which is stable under the holomorphic functional calculus of that $C^*$-algebra. There is little loss of generality in assuming that $\mathcal{A}$ is complete in the topology given by (2.1), thus is a Fréchét pre-$C^*$-algebra, and we shall do so. This condition guarantees that the spectrum of an element $a \in \mathcal{A}$ coincides with its spectrum in the $C^*$-completion $\mathcal{A}$, and that any character of the pre-$C^*$-algebra $\mathcal{A}$ extends to a character of $\mathcal{A}$ as well. We shall denote the character space by $X := \text{sp}(\mathcal{A}) = \text{sp}(\mathcal{A})$.

Moreover, when $\mathcal{A}$ is a Fréchét pre-$C^*$-algebra, so also is the algebra $M_n(\mathcal{A})$ of $n \times n$ matrices with entries in $\mathcal{A}$, whose $C^*$-completion is $M_n(A)$; for a proof, see [57]. By a theorem of Bost [6, 30], the (topological) $K$-theories of $\mathcal{A}$ and $\mathcal{A}$ coincide: $K_i(\mathcal{A}) = K_i(\mathcal{A})$ for $i = 0, 1$.

By replacing any seminorm $q$ by $a \mapsto q(a) + q(a^*)$ if necessary, we may suppose that $q(a) = q(a^*)$ for all $a \in \mathcal{A}$. We note in passing that the multiplication in the Fréchét algebra $\mathcal{A}$ is jointly continuous [43].

Lemma 2.5. Let $\mathcal{A}$ be a Fréchét pre-$C^*$-algebra and let $\mathcal{A}$ be its $C^*$-completion. If $\tilde{q} \in \mathcal{A}$ is a projector (i.e., a selfadjoint idempotent), and if $0 < \varepsilon < 1$, then there is a projector $q \in \mathcal{A}$ such that $\|q - \tilde{q}\| < \varepsilon$.

Proof. Choose $\delta$ with $0 < \delta < \varepsilon/32$, and let $b = b^* \in \mathcal{A}$ be such that $\|b - \tilde{q}\| < \delta$. Observe that

$$\|b^2 - b\| = \|b^2 - \tilde{q}^2 + \tilde{q} - b\| \leq (\|b + \tilde{q}\| + 1)\|b - \tilde{q}\| \leq (3 + \delta)\|b - \tilde{q}\| < \delta(3 + \delta) < 4\delta.$$

Since $\mathcal{A}$ is a Fréchét pre-$C^*$-algebra, one may, provided $\delta$ is sufficiently small, use holomorphic functional calculus to construct a homotopy within $\mathcal{A}$ from $b$ to $e \in \mathcal{A}$ such that $e^2 = e$ and $\|e - b\| < 2\|b^2 - b\|$; see [30, Lemma 3.43], for instance.

Let $q := e e^* (ee^* + (1 - e^*)(1 - e))^{-1}$. Then $q$ is a projector in $\mathcal{A}$, and it lies in $\mathcal{A}$ since $ee^* + (1 - e^*)(1 - e)$ is invertible in the pre-$C^*$-algebra $\mathcal{A}$. By taking $\mathcal{A}$ to be faithfully represented on a Hilbert space $\mathcal{H}$, we can write $e$, $q$, and $b$ as operators on $\mathcal{H} = e \mathcal{H} \oplus (1 - e) \mathcal{H}$, as follows:

$$e = \begin{pmatrix} 1 & T \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} R & V \\ V^* & S \end{pmatrix},$$
with \( R, S \) selfadjoint and \( V, T : (1 - e)\mathcal{H} \to e\mathcal{H} \) bounded. Then \( \|e - b\| < 8\delta < \varepsilon/4 \) means 
\[
\| (e - b)^*(e - b) \| < \frac{\varepsilon^2}{16},
\]
so that \( \| V \| < \varepsilon/4, \| V - T \| < \varepsilon/4 \), and therefore \( \| q - e \| = \| T \| < \varepsilon/2 \). Hence
\[
\| q - \tilde{q} \| \leq \| q - e \| + \| e - b \| + \| b - \tilde{q} \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \delta < \varepsilon.
\]

A \( QC^\infty \) spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) for which \( \mathcal{A} \) is complete has not only a holomorphic functional calculus for \( \mathcal{A} \), but also a \( C^\infty \) functional calculus for selfadjoint elements: we quote [51, Prop. 22].

**Proposition 2.6** (\( C^\infty \) Functional Calculus). Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be a \( QC^\infty \) spectral triple, and suppose \( \mathcal{A} \) is complete. Let \( f : \mathbb{R} \to \mathbb{C} \) be a \( C^\infty \) function in a neighbourhood of the spectrum of \( a = a^* \in \mathcal{A} \). If we define \( f(a) \in \mathcal{A} \) using the continuous functional calculus, then in fact \( f(a) \) lies in \( \mathcal{A} \). \( \square \)

**Remark 2.7.** For each \( a = a^* \in \mathcal{A} \), the \( C^\infty \)-functional calculus defines a continuous homomorphism \( \Psi : C^\infty(U) \to \mathcal{A} \), where \( U \subset \mathbb{R} \) is any open set containing the spectrum of \( a \), and the topology on \( C^\infty(U) \) is that of uniform convergence of all derivatives on compact subsets.

The following proposition extends this result to the case of smooth functions of several variables, yielding a *multivariate* \( C^\infty \) functional calculus. Before stating it, we recall the continuous functional calculus for a finite set \( a_1, \ldots, a_n \) of commuting selfadjoint elements of a unital \( C^* \)-algebra \( \mathcal{A} \). These generate a unital \( * \)-algebra whose closure in \( \mathcal{A} \) is a \( C^* \)-subalgebra \( C^* \langle 1, a_1, \ldots, a_n \rangle \); let \( \Delta \) be its (compact) space of characters. Evaluation of polynomials \( p \mapsto p(a_1, \ldots, a_n) \) yields a surjective morphism from \( C(\prod_{j=1}^n \text{sp} a_j) \) onto \( C^* \langle 1, a_1, \ldots, a_n \rangle \cong C(\Delta) \) which corresponds, via the Gelfand functor, to a continuous injection \( \Delta \hookrightarrow \prod_{j=1}^n \text{sp} a_j \); this joint spectrum \( \Delta \) may thus be regarded as a compact subset of \( \mathbb{R}^n \). If \( h \in C(\Delta) \), we may define \( h(a_1, \ldots, a_n) \) as the image of \( h|_\Delta \) in \( C^* \langle 1, a_1, \ldots, a_n \rangle \) under the Gelfand isomorphism.

**Proposition 2.8.** Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be a \( QC^\infty \) spectral triple. Let \( a_1, \ldots, a_n \) be mutually commuting selfadjoint elements of \( \mathcal{A} \), and let \( \Delta \subset \mathbb{R}^n \) be their joint spectrum. Let \( f : \mathbb{R}^n \to \mathbb{C} \) be a \( C^\infty \) function supported in a bounded open neighbourhood \( U \) of \( \Delta \). Then \( f(a_1, \ldots, a_n) \) lies in \( \mathcal{A}_\delta \).

**Proof.** We first define the operator \( f(a_1, \ldots, a_n) \) lying in \( \mathcal{A} \), the \( C^* \)-completion of \( \mathcal{A} \), using the continuous functional calculus.

Since \( f \) is a compactly supported smooth function on \( \mathbb{R}^n \), we may define \( f(a_1, \ldots, a_n) \in \mathcal{A} \) alternatively by a Fourier integral:
\[
f(a_1, \ldots, a_n) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(s_1, \ldots, s_n) \exp(i s \cdot a) \, d^n s,
\]
where \( s \cdot a = s_1 a_1 + \cdots + s_n a_n \). Since \( \delta \) (and likewise \( \text{d} = \text{ad} \mathcal{D} \)) is a norm-closed derivation from \( \mathcal{A} \) to \( \mathcal{B}(\mathcal{H}) \), we may conclude that \( f(a_1, \ldots, a_n) \in \text{Dom} \delta \) with
\[
\delta(f(a_1, \ldots, a_n)) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(s_1, \ldots, s_n) \delta(\exp(i s \cdot a)) \, d^n s.
\]
provided we can establish dominated convergence for the integral on the right hand side [7]. Just as in the one-variable case [51], since each \(a_j \in \text{Dom } \delta\), we find that \(\exp(is \cdot a) = \prod_j \exp(is_j a_j)\) lies in \(\text{Dom } \delta\) also: its factors are given by the expansion

\[
\delta(\exp(is \cdot a)) = is_j \int_0^1 \exp(its_j a_j) \delta(a_j) \exp(i(1-t)s_j a_j) \, dt,
\]

and in particular,

\[
\|\delta(\exp(is \cdot a))\| \leq C \sum_j |s_j|, \quad C = \max_j \left(\|\delta(a_j)\| \prod_{i \neq j} \|a_i\|\right).
\]

A norm bound which dominates the right hand side of (2.3) is thus given by

\[
\int_{\mathbb{R}^n} |\hat{f}(s_1, \ldots, s_n)| \|\delta(\exp(is \cdot a))\| \, d^n s \leq C \sum_{j=1}^n (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\hat{f}(s_1, \ldots, s_n)| |s_j| \, d^n s.
\]

Let \(A_0\) be the completion of \(\mathcal{A}\) for the norm \(\|a\|_D := \|a\| + \|da\|\); notice that \(A_0 \subseteq A\). Replacing \(\delta\) by \(\mathfrak{d}\) in the previous argument, we find that

\[
\|f(a_1, \ldots, a_n)\|_D \leq \|\hat{f}\|_1 + \|da\| \sum_{j=1}^n (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\hat{f}(s_1, \ldots, s_n)| |s_j| \, d^n s.
\]

Therefore, \(f(a_1, \ldots, a_n)\) can be approximated, in the \(\| \cdot \|_D\) norm, by Riemann sums for (2.2) belonging to \(\mathcal{A}\), and thus \(f(a_1, \ldots, a_n) \in A_0\).

Since \(\delta\) and \(\mathfrak{d}\) are commuting derivations, we obtain that \(\delta(f(a_1, \ldots, a_n)) \in \text{Dom } \mathfrak{d}\) and \(\mathfrak{d}(f(a_1, \ldots, a_n)) \in \text{Dom } \delta\) for \(a \in \mathcal{A}\), and \(\|\delta(\mathfrak{d}(f(a_1, \ldots, a_n)))\|\) is bounded by a linear combination of expressions

\[
\|\delta(\mathfrak{d}a_j)\| \int |\hat{f}(s_1, \ldots, s_n)| |s_j| \, d^n s \quad \text{and} \quad \|\delta a_j\| \|\mathfrak{d}a_k\| \int |\hat{f}(s_1, \ldots, s_n)| |s_j s_k| \, d^n s.
\]

In particular, \(\|\delta(f(a_1, \ldots, a_n))\|_D\) also has a bound of this type.

For each \(m = 1, 2, 3, \ldots\), let \(A_m\) be the completion of \(\mathcal{A}\) for the norm \(\sum_{k \leq m} \|\delta^k(a)\|_D\). Then \(\delta\) extends to a norm-closed derivation from \(A_m\) to \(\mathfrak{B}(\mathfrak{H})\), and an ugly but straightforward induction on \(m\) shows that each \(\delta^k(f(a_1, \ldots, a_n))\) and \(\delta^k(\mathfrak{d}(f(a_1, \ldots, a_n)))\) lies in its domain, using the convergence of \(\int |\hat{f}(s_1, \ldots, s_n)| |p(s_1, \ldots, s_n)| \, d^n s\) for \(p\) a polynomial of degree \(\leq m + 1\). Thus, \(f(a_1, \ldots, a_n) \in A_m\). Since \(A_\delta = \bigcap_{m \geq 0} A_m\), we conclude that \(f(a_1, \ldots, a_n) \in A_\delta\). \(\square\)

Remark 2.9. For most of this paper, the spectral triples we consider will be commutative and will satisfy the first order property [18, VI.4.7], meaning that \([D, a], b = 0\) for all \(a, b \in \mathcal{A}\). In such cases, to define \([D, f(a_1, \ldots, a_n)]\), we may note that for any polynomial \(p\), the first-order property allows us to write

\[
[D, p(a_1, \ldots, a_n)] = \sum_{j=1}^n \partial_j p(a_1, \ldots, a_n) [D, a_j],
\]

with \(\partial_j p\) being the \(j\)-th partial derivative of \(p\). By a \(C^1\)-approximation argument – see Proposition 5.1 below – we obtain \([D, f(a_1, \ldots, a_n)] = \sum_j \partial_j f(a_1, \ldots, a_n) [D, a_j]\) for any \(f\) satisfying the hypotheses of Proposition 2.8. Since \((\mathcal{A}, \mathfrak{H}, D)\) is \(QC^\infty\), one sees immediately that the right hand side of (2.5) belongs to the smooth domain of \(\delta\).
We now use the $C^\infty$ functional calculus to prove the existence of certain elements of $\mathcal{A}$, where $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a (unital) commutative $QC^\infty$ spectral triple. The algebra elements we are looking for are smooth partitions of unity and local inverses.

**Lemma 2.10.** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $QC^\infty$ spectral triple where $\mathcal{A}$ is commutative and complete. Let \( \{ U_\alpha : \alpha = 1, \ldots, n \} \) be any finite open cover of the compact Hausdorff space $X = \text{sp}(\mathcal{A})$. Then there exist $\phi_\alpha \in \mathcal{A}$, for $\alpha = 1, \ldots, n$, such that

\[
\text{supp} \phi_\alpha \subseteq U_\alpha, \quad 0 \leq \phi_\alpha \leq 1, \quad \text{and} \quad \sum_{\alpha=1}^n \phi_\alpha = 1.
\]  

**Proof.** Since $\mathcal{A} = \mathcal{A}_\delta$ is a pre-$C^*$-algebra, its character space is the same as that of its $C^*$-completion, $\mathcal{A}$; thus $X = \text{sp}(\mathcal{A})$ is a compact Hausdorff space. Now, $X$ always admits a continuous partition of unity [24] subordinate to the cover $\{ U_\alpha \}$, i.e., we can find $\tilde{\phi}_1, \ldots, \tilde{\phi}_n \in \mathcal{A}$ satisfying (2.6). Let $p \in M_n(\mathcal{A})$ be the matrix whose $(\alpha, \beta)$-entry is $(\tilde{\phi}_\alpha \tilde{\phi}_\beta)^{1/2}$; then $p$ is a projector, that is, a selfadjoint idempotent: $p^2 = p = p^*$. Since $M_n(\mathcal{A})$ is a Fréchet $C^*$-algebra, it is known [6, 30] that the inclusion $M_n(\mathcal{A}) \hookrightarrow M_n(\mathcal{A})$ induces a homotopy equivalence between the respective sets of idempotents in these algebras. Thus, there is a norm-continuous path of idempotents $t \mapsto e_t = e_t^2 \in M_n(\mathcal{A})$ linking $p = e_0$ to an idempotent $e_1 \in M_n(\mathcal{A})$. Moreover, such a path may be chosen so that each $\| p - e_t \| < \varepsilon$ for a preassigned $\varepsilon > 0$ [30, Lemma 3.43]. We choose $\varepsilon < 1/3n$. Replacing $e_t$ by $q_t := e_t e_t^*(e_t e_t^* + (1 - e_t^2)(1 - e_t))^{-1}$, we may link $p$ to $q = q_1$ by a path of projectors in $M_n(\mathcal{A})$; by the proof of Lemma 2.5, we obtain $\| p - q_1 \| < \varepsilon < 1/n$ for $0 \leq t \leq 1$. Since the positive element $e_1 e_1^* + (1 - e_1^2)(1 - e_1)$ is invertible in the pre-$C^*$-algebra $M_n(\mathcal{A})$, $q$ lies in $M_n(\mathcal{A})$.

By the Serre–Swan theorem [59], the projectors $p$ and $q$ define vector bundles over $X$ of the same rank: the rank is given by the matrix trace $\text{tr} p = \text{tr} q$, a locally constant integer-valued function in $A = C(X)$. Write $\psi_\alpha := q_{\alpha \alpha} \in \mathcal{A}$, and notice that $\psi_\alpha \geq 0$ in $\mathcal{A}$; then

\[
\sum_{\alpha=1}^n \psi_\alpha = \text{tr} q = \text{tr} p = \sum_{\alpha=1}^n \tilde{\phi}_\alpha = 1.
\]  

We now modify the elements $\psi_\alpha$ to obtain a partition of unity subordinate to the cover $\{ U_\alpha \}$. By construction, $\| \psi_\alpha - \tilde{\phi}_\alpha \| \leq \| q - p \| < 1/n$ for each $\alpha$. Choose a smooth function $g : \mathbb{R} \to [0, 1]$ with support in $[\varepsilon, 1 + \varepsilon]$ such that $0 < g(t) \leq 1$ for $\varepsilon < t < 1$. Then define $V_\alpha := \{ x \in X : \psi_\alpha(x) > \varepsilon \} \subset U_\alpha$. Setting $\chi_\alpha(x) := g(\psi_\alpha(x))$ gives $\chi_\alpha > 0$ on $V_\alpha$ and $\text{supp} \chi_\alpha \subset U_\alpha$. For all $x \in X$, there is some $\chi_\beta$ with $\chi_\beta(x) > 0$: for if not, then $\psi_\beta(x) \leq \varepsilon$ for each $\beta$, and $\sum_\beta \psi_\beta(x) \leq n \varepsilon < 1$, contradicting (2.7). We now define $\phi_\alpha := \chi_\alpha / \sum_\beta \chi_\beta$, which clearly satisfies (2.6). Since $\chi_\alpha = g(\psi_\alpha)$, Proposition 2.6 shows that $\chi_\alpha \in \mathcal{A}$. Moreover, $\sum_\beta \chi_\beta$ is invertible in $\mathcal{A}$, and hence $\phi_\alpha \in \mathcal{A}$ for each $\alpha$, as required.

**Corollary 2.11.** Given a $QC^\infty$ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where $\mathcal{A}$ is commutative and complete, let $K \subset U \subset \text{sp}(\mathcal{A})$ where $K$ is compact and $U$ is open. Then there is some $\psi \in \mathcal{A}$ such $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $K$, and $\psi \equiv 0$ outside $U$.

**Proof.** There is a partition of unity $\{ \psi, 1 - \psi \}$ subordinate to the open cover $\{ U, \text{sp}(\mathcal{A}) \setminus K \}$, with $\psi \in \mathcal{A}$. □
Lemma 2.12. Let \((A, \mathcal{H}, \mathcal{D})\) again be a QC\(^\infty\) spectral triple with \(A\) commutative and complete. Let \(a \in A\) have compact support contained in an open subset \(U \subset \text{sp}(A)\). Then there exists \(\phi \in A\) such that \(\phi a = a\) and \(\text{supp} \phi \subset U\).

Proof. Choose \(b \in A = C(X)\), by Urysohn’s lemma, such that \(0 \leq b \leq 1\), \(b(x) = 1\) for \(x \in \text{supp} a\) and \(b(x) = 0\) for \(x \notin U\). Pick \(\delta \in (0, \frac{1}{4})\) and choose \(\psi \in A\) satisfying \(\|b - \psi\| < \frac{1}{2}\delta\). Let \(f : \mathbb{R} \to [0, 1]\) be a compactly supported smooth function such that \(f(t) = 0\) for \(t \leq \delta\) and \(f(t) = 1\) for \(1 - \delta \leq t \leq 2\). Then \(\phi := f(\psi)\) lies in \(A\) by the C\(^\infty\)-functional calculus. Also, for all \(x \in \text{supp}(a)\), the estimates \(1 - \frac{1}{2}\delta \leq \psi(x) \leq 1 + \frac{1}{2}\delta\) hold, and so \(\phi(x) = f(\psi(x)) = 1\); this shows that \(\phi a = a\).

The continuity of \(b\) shows that
\[
\text{supp}(f(\psi)) \subseteq \{x : \psi(x) > \delta\} \subseteq \{x : b(x) > \frac{1}{2}\delta\} \subseteq \{x : b(x) \geq \frac{1}{2}\delta\} \subset U.
\]
Thus, \(\text{supp} \phi \subset U\), as required. \(\square\)

Proposition 2.13. Let \((A, \mathcal{H}, \mathcal{D})\) be a QC\(^\infty\) spectral triple with \(A\) commutative and complete. Let \(U \subset \text{sp}(A)\) be an open subset, and let \(h \in A\) satisfy \(h(x) \neq 0\) for all \(x \in U\). Then whenever \(a \in A\) with \(\text{supp} a \subset U\), \(A\) contains the element \(ah^{-1} \in C(\text{sp}(A))\) defined by
\[
(ah^{-1})(x) := \begin{cases} 
  a(x)/h(x) & \text{if } h(x) \neq 0, \\
  0 & \text{otherwise}. 
\end{cases} \quad (2.8)
\]

Proof. The formula (2.8) clearly defines a continuous function on \(\text{sp}(A)\), so that \(ah^{-1} \in A = C(\text{sp}(A))\). To check that it lies in \(A\), it is enough to replace \(h\) by an invertible element \(\tilde{h} \in A\) for which \(\tilde{h}(x) = h(x)\) whenever \(a(x) \neq 0\): since \(A = A_\delta\) is a pre-C\(^*\)-algebra, \(\tilde{h}^{-1}\) will lie in \(A\), and thus \(ah^{-1} = a\tilde{h}^{-1} \in A\).

Let \(\varepsilon := \inf\{ |h(x)| : x \in \text{supp} a \}\); since \(A\) is unital, \(\text{supp} a\) is compact and therefore \(\varepsilon > 0\). Let \(V := U \cap \{ x : |h(x)| > \varepsilon/2 \}\). By Corollary 2.11, we can find \(\phi \in A\) so that \(0 \leq \phi \leq 1\), \(\phi \equiv 0\) on \(\text{supp} a\) and \(\phi \equiv 1\) outside \(V\). The element \(h + \frac{1}{2}\varepsilon \phi \in A\) coincides with \(h\) on \(\text{supp} a\) and vanishes only on the compact set \(\{ x \notin V : h(x) = -\frac{1}{2}\varepsilon \}\). If this set is nonvoid, we can likewise find \(\psi \in A\), which is nonzero on this set and vanishes on \(V\), so that \(\tilde{h} := h + \frac{1}{2}\varepsilon \phi + \psi\) vanishes nowhere on \(\text{sp}(A)\). This gives the required \(\tilde{h} \in A\) such that \(\tilde{h} \equiv h\) on \(\text{supp} a\). \(\square\)

Corollary 2.14. Let \((A, \mathcal{H}, \mathcal{D})\) be a QC\(^\infty\) spectral triple with \(A\) commutative and complete. Let \(U \subset \text{sp}(A)\) be an open subset, and let \(h \in M_k(A)\) be such that \(h(x) \in M_k(C)\) is invertible for all \(x \in U\). Let \(a \in A\) with \(\text{supp} a \subset U\); then the element \(ah^{-1} \in M_k(A)\) defined by
\[
(ah^{-1})(x) := \begin{cases} 
  (a(x) \otimes 1_k) h(x)^{-1} & \text{if } h(x) \neq 0, \\
  0 & \text{otherwise}, 
\end{cases}
\]
actually lies in the subalgebra \(M_k(A)\).

Proof. The proof of Proposition 2.13 goes through with minor modifications; for instance, one may take \(V := U \cap \{ x : |\det(h(x))| > \varepsilon/2 \}\). By adding to \(h\) suitable scalar matrices which vanish on \(\text{supp} a\), one constructs an invertible element \(\tilde{h} \in M_k(A)\) such that \(ah^{-1} = (a \otimes 1_k)\tilde{h}^{-1}\), where \(\tilde{h}^{-1} \in M_k(A)\) since \(M_k(A)\) is a pre-C\(^*\)-algebra. \(\square\)
3 Geometric properties of noncommutative manifolds

The conditions on a spectral triple that we introduce below will control several interdependent features. Before introducing these conditions, we first discuss several such features: summability, metrics and differential structures.

We recall the symmetric operator ideals \( \mathcal{L}^p, \infty (\mathcal{H}) \), for \( 1 \leq p < \infty \); these are discussed in detail in [18, IV.2.α] and [30, 7.C]; in [16] and [30] they are called \( \mathcal{L}^{p+}(\mathcal{H}) \). The Dixmier ideal \( \mathcal{L}^{1,\infty}(\mathcal{H}) \) is the common domain of the Dixmier traces

\[
\text{Tr}_\omega : \mathcal{L}^{1,\infty}(\mathcal{H}) \to \mathbb{C}.
\]

These are labelled by an uncountable index set of generalized limits (\( \omega \)-limits [9]), but they are effectively computable only on the subspace of “measurable” operators \( T \) for which all values \( \text{Tr}_\omega T \) coincide; for instance, trace-class operators satisfy \( \text{Tr}_\omega T = 0 \). Once we have established that a certain operator \( T \) is indeed measurable, we shall write \( \overline{\text{Tr}} \) instead of \( \text{Tr}_\omega T \) to denote the common value of its Dixmier traces. If the limit

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=0}^n \mu_k(T)
\]

exists, where the \( \mu_k(T) \) are the singular values of \( T \) in nonincreasing order, then \( T \) is measurable and this limit equals \( \overline{\text{Tr}} T \). A partial converse has been established by Lord, Sedaev and Sukochev [42]: for positive \( T \), measurability is equivalent to the existence of this limit.

**Definition 3.1.** A spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is \( p^+ \)-summable, with \( 1 \leq p < \infty \), if \((1 + D^2)^{-1/2} \in \mathcal{L}^p, \infty (\mathcal{H}) \).

For convenience, we abbreviate

\[
\langle \mathcal{D} \rangle := (1 + D^2)^{1/2},
\]

recalling that \( \langle \mathcal{D} \rangle - |\mathcal{D}| \) is bounded, by functional calculus.

**Remark 3.1.** If \( \langle \mathcal{D} \rangle^{-1} \in \mathcal{L}^{p,\infty}(\mathcal{H}) \), it follows that \( A := \langle \mathcal{D} \rangle^{-p} \in \mathcal{L}^{1,\infty}(\mathcal{H}) \), and hence that \( \text{Tr}_\omega \langle \mathcal{D} \rangle^{-p} \) is finite for any Dixmier trace \( \text{Tr}_\omega \). Now if \( q > p \), then \( \langle \mathcal{D} \rangle^{-q} = A^{q/p} \) lies in the ideal \( \mathcal{L}^1(\mathcal{H}) \) of trace-class operators, so \( \text{Tr}_\omega \langle \mathcal{D} \rangle^{-q} = 0. \) It follows that there is at most one value of \( p \) (independent of \( \omega \)) for which \( \text{Tr}_\omega \langle \mathcal{D} \rangle^{-p} \) can be both finite and positive. Since

\[
\text{Tr}_\omega \langle \mathcal{D} \rangle^{-p} = \omega\text{-lim}_{n \to \infty} \frac{1}{\log n} \sum_{k=0}^n \mu_k(\langle \mathcal{D} \rangle^{-p})
\]

and [9] for any bounded positive sequence \( \{t_n\} \), one can estimate:

\[
\liminf_{n \to \infty} t_n \leq \omega\text{-lim}_{n \to \infty} t_n \leq \limsup_{n \to \infty} t_n,
\]

then if \( \lim \inf t_n > 0 \), every \( \omega\text{-lim} t_n \) is also positive. If \( 0 < \text{Tr}_\omega \langle \mathcal{D} \rangle^{-p} < \infty \) for all \( \omega \), we shall call \( p \) the metric dimension of the spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\).
If $\eta: A \to A/\mathbb{C}1$ is the linear quotient map, then $\|[D, a]\|$ depends only on the image $\eta(a)$ of $a \in A$. Suppose that the set
\[ \{ \eta(a) \in A/\mathbb{C}1 : \|[D, a]\| \leq 1 \} \] (3.1)
is norm bounded in the Banach space $A/\mathbb{C}1$. Then the following formula defines a bounded metric distance on the state space of $A$, as follows from [17]:
\[ d(\phi, \psi) := \sup \{ |\phi(a) - \psi(a)| : \|[D, a]\| \leq 1 \}. \] (3.2)
(In Appendix B, we show that any irreducible unital spectral triple $(A, \mathcal{H}, D)$ determines a possibly unbounded distance function, which actually suffices for the purposes of our proof.)

When $A$ is commutative, the distance function (3.2) is determined by its restriction to the subspace of pure states, which may be identified with $X = \text{sp}(A)$. In the case of $A = C^\infty(M)$ where $M$ is a compact spin$^c$ manifold, and $D$ is a Dirac operator arising from a Riemannian metric $g$ on $M$, this $d$ coincides [17] with the Riemannian distance function determined by $g$.

**Definition 3.2.** If $(A, \mathcal{H}, D)$ is a commutative spectral triple for which the set (3.1) is bounded, we define the **metric topology** on the pure state space of $A$ to be the topology defined by the distance function (3.2).

**Remark 3.2.** The equation (3.2) entails the inequality
\[ |\phi(a) - \psi(a)| \leq \|[D, a]\| d(\phi, \psi), \] (3.3)
so that all $a \in A$ are Lipschitz for the metric topology on $X$. When $[D, a] \neq 0$ for $a \notin \mathbb{C}1$, the two topologies coincide [47, 53] if and only if the set (3.1) is precompact in $A/\mathbb{C}1$.

**Remark 3.3.** A priori, the metric topology may be finer than the original weak$^*$ topology on $X$. In particular, the metric topology need not be compact unless the two topologies coincide. We shall henceforth adopt the convention, when discussing continuous functions on $X$ and so forth, that the topology of $X$ is by default its weak$^*$ topology, unless the metric topology is explicitly invoked.

To exhibit the differential structure of a spectral triple, we first recall the universal graded differential algebra $\Omega^*A$ of any associative algebra $A$ [15]. It is generated as an algebra by symbols $a, da$ for $a \in A$ subject to the preexisting algebra relations of $A$, the derivation rule $d(ab) = a db + da b$, and the relations
\[ d(a_0 da_1 \cdots da_k) = da_0 da_1 \cdots da_k, \quad d(da_1 da_2 \cdots da_k) = 0. \]
We may then identify $\Omega^*A$ with the (normalized) Hochschild complex of $A$, that is,
\[ \Omega^kA \simeq C^k(A) := A \otimes (A/\mathbb{C}1)^{\otimes k}, \quad a_0 da_1 \cdots da_k \leftrightarrow a_0 \otimes \eta(a_1) \otimes \cdots \otimes \eta(a_k). \]
When $A$ is a Fréchet algebra, one generally uses the projective topological tensor product to topologize $\Omega^*A$.

**Definition 3.3.** If $(A, \mathcal{H}, D)$ is a spectral triple, we shall use the notation $\mathcal{C}_D(A)$ for the subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $A$ and $\mathbb{D}A = [D, A]$. We can define an (algebra) representation $\pi_D: \Omega^*A \to \mathcal{C}_D(A)$ by setting
\[ \pi_D(a_0 da_1 \cdots da_k) := a_0 [D, a_1] \cdots [D, a_k]. \]
We may regard $\Omega^*A$ as an involutive algebra by setting $(da)^* := -d(a^*)$; then $\pi_D$ is a $*$-representation of the Hochschild chains of $A$ as operators on $\mathcal{H}$. 

11
However, this $\mathcal{C}_D(\mathcal{A})$ is not a graded algebra (although the count of $[\mathcal{D}, a]$ factors does give a filtration), and $\pi_D$ is not a representation of graded differential algebras. As is well known from physical examples [18, 44, 56], there may be nontrivial “junk forms” $\omega \in \Omega^e \mathcal{A}$ such that $\pi_D(\omega) = 0$ but $\pi_D(d \omega) \neq 0$. On quotienting out the junk, we obtain a graded differential algebra [18, VI.1]:

$$\Lambda^e \mathcal{A} := \mathcal{C}_D(\mathcal{A})/J_\pi, \quad \text{where} \quad J_\pi = \pi_D(d(\ker \pi_D)).$$

In particular, $J_\pi$ is a differential ideal. The subspaces $\Lambda^k_{\mathcal{D}} \mathcal{A} = \pi_D(\Omega^k \mathcal{A})/\pi_D(d(\Omega^{k-1} \mathcal{A} \cap \ker \pi_D))$ give the grading; the differential $d$ on $\Lambda^e \mathcal{A}$ is defined on equivalence classes of operators, modulo junk terms, by

$$d(a_0 [\mathcal{D}, a_1] \ldots [\mathcal{D}, a_k] + \text{junk}) := [\mathcal{D}, a_0] [\mathcal{D}, a_1] \ldots [\mathcal{D}, a_k] + \text{junk}.$$ 

Here $\Lambda^0_{\mathcal{D}} \mathcal{A} = \mathcal{A}$, and $\Lambda^1_{\mathcal{D}} \mathcal{A}$ may be identified with the $\mathcal{A}$-bimodule of operators of the form $\sum_j a_j [\mathcal{D}, b_j]$ (finite sum), with $a_j, b_j \in \mathcal{A}$. Nontrivial junk terms appear in higher degrees.

### 3.1 Axiomatic conditions on commutative spectral triples

From now on, let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple whose algebra $\mathcal{A}$ is commutative (and unital). We shall also assume that the $\text{C}^*$-algebra $A = \overline{\mathcal{A}}$ is separable.

In [20], Connes introduces several conditions on such an $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ in order to specify axiomatically what a noncommutative spin geometry should be. We now list these conditions (for the commutative case), as well as a few supplementary requirements which we need to establish our main results.

**Condition 1** (Dimension). The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $p^+$-summable for a fixed positive integer $p$, for which $\text{Tr}_\omega (\mathcal{D})^{-p} > 0$ for all $\omega$.

By the remark after Definition 3.1, this condition determines $p$ uniquely; we then say that the critical summability parameter $p$ is the “metric dimension” of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

**Remark 3.4.** A priori, there is no reason why the growth of the eigenvalues of $\mathcal{D}$ should be such that $p$ is an integer. However, the orientability condition below introduces another dimensionality parameter $p$ as the degree of a certain Hochschild cycle, which is necessarily an integer, and we require that these two quantities coincide.

This formulation excludes certain interesting “0-dimensional” cases, such as occur when $\mathcal{A}$ has finite (linear) dimension. For noncommutative 0-dimensional spectral triples built over matrix algebras, we refer to [32, 37, 46].

**Condition 2** (Metric). The set $\{ \eta(a) \in \mathcal{A}/\mathcal{C} 1 : \| [\mathcal{D}, a] \| \leq 1 \}$ is norm-bounded in the Banach space $\mathcal{A}/\mathcal{C} 1$. This ensures that the character space $X = \text{sp}(\mathcal{A})$ is a metric space [17] with the metric distance (3.2). (See Appendix B).

**Remark 3.5.** Since we have assumed that $\mathcal{A}$ is separable, the space $X$ with its weak* topology is metrizable. However, the metric on $X$ defined by the equation (3.2) does not necessarily give the weak* topology.

**Condition 3** (Regularity). The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $QC^\infty$, as set forth in Definition 2.2. Without loss of generality, we assume that $\mathcal{A}$ is complete in the topology given by (2.1) and so is a Fréchet pre-$\text{C}^*$-algebra.
Condition 4 (Finiteness). The dense subspace of $\mathcal{H}$ which is the smooth domain of $\mathcal{D}$,
\[
\mathcal{H}_\infty := \bigcap_{m \geq 1} \text{Dom} \mathcal{D}^m
\]
is a finitely generated projective $A$-module. Moreover, there exists a Hermitian pairing
\[
(\cdot | \cdot) : \mathcal{H}_\infty \times \mathcal{H}_\infty \to A,
\]
making $(\mathcal{H}_\infty, (\cdot | \cdot))$ a full pre-$C^*$ right $A$-module; and such that, for some particular Dixmier trace $\text{Tr}_\Omega$, the following relation holds:
\[
\text{Tr}_\Omega((\xi | \eta) \langle \mathcal{D} \rangle^{-p}) = \langle \xi | \eta \rangle, \quad \text{for all } \xi, \eta \in \mathcal{H}_\infty.
\]
(3.4)
Here $\langle \cdot | \cdot \rangle$ denotes the scalar product on $\mathcal{H}$.

Remark 3.6. Since $A$ is commutative, we are free to regard $\mathcal{H}_\infty$ as either a right or left $A$-module. As a dense subspace of $\mathcal{H}$, it is naturally a left module via the representation $\pi$, but it is algebraically more convenient to treat it as a right module; thus $\mathcal{H}_\infty \cong qA^m$ where $q \in M_m(A)$ is (a selfadjoint) idempotent.

Remark 3.7. It is proved in Appendix A that (up to positive scalar multiples) the only Hermitian form which can satisfy the conditions listed here is the standard one, expressible as $(\xi | \eta) = \sum_{j,k} \xi_j^* q_{jk} \eta_k$ on identifying $\mathcal{H}_\infty$ with $qA^m$ with $q$ selfadjoint.

Condition 5 (Absolute Continuity). For all nonzero $a \in A$ with $a \geq 0$, and for any $\omega$-limit, the following Dixmier trace is positive:
\[
\text{Tr}_\omega(a\langle \mathcal{D} \rangle^{-p}) > 0.
\]
Remark 3.8. The absolute continuity condition actually subsumes the dimension condition, and they should properly be regarded as one and the same. This formulation eases the adaptation of the conditions for nonunital algebras [27, 52].

Condition 6 (First Order). The bounded operators in $[\mathcal{D}, A]$ commute with $A$; in other words,
\[
[[\mathcal{D}, a], b] = 0 \quad \text{for all } a, b \in A.
\]
(3.5)
This condition says that operators in $[\mathcal{D}, A]$ can be regarded as endomorphisms of the $A$-module $\mathcal{H}_\infty$; and more generally, that $C_\mathcal{D}(A) \subseteq \text{End}_A(\mathcal{H}_\infty)$.

Remark 3.9. For spectral triples over noncommutative algebras, the first-order condition is more elaborate: as well as the representation $\pi : A \to \mathcal{B}(\mathcal{H})$ we require a commuting representation $\pi^\circ : A^\circ \to \mathcal{B}(\mathcal{H})$ of the opposite algebra $A^\circ$ (or equivalently, an antirepresentation of $A$ that commutes with $\pi$): writing $a$ for $\pi(a)$ as usual, and $b^\circ$ for $\pi^\circ(b)$, we ask that $[a, b^\circ] = 0$. Now $\mathcal{H}_\infty$ can be regarded as a right $A$-module under the action $\xi \cdot b := \pi^\circ(b) \xi$. The first-order condition is then expressed as: $[[\mathcal{D}, a], b^\circ] = 0$ for $a, b \in A$; and once again it entails that $C_\mathcal{D}(A) \subseteq \text{End}_A(\mathcal{H}_\infty)$.

Coming back to the commutative case, we take $\pi^\circ = \pi$ from now on. (But see [37], for instance, for examples of commutative algebras with different left and right actions on $\mathcal{H}$.)
**Condition 7** *(Orientability).* Let \( p \) be the metric dimension of \((\mathcal{A}, \mathcal{H}, \mathcal{D})\). We require that the spectral triple be even, with \( \mathbb{Z}_2 \)-grading \( \Gamma \), if and only if \( p \) is even. For convenience, we take \( \Gamma = 1 \) when \( p \) is odd. We say the spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is orientable if there exists a Hochschild \( p \)-cycle

\[
e = \sum_{a=1}^{n} a_0^a \otimes a_1^a \otimes \cdots \otimes a_p^a \in Z_p(\mathcal{A}, \mathcal{A})
\]

(3.6a)

whose Hochschild class \([e] \in HH_p(\mathcal{A})\) may be called the “orientation” of \((\mathcal{A}, \mathcal{H}, \mathcal{D})\), such that

\[
\pi_D(e) \equiv \sum_a a_0^a [\mathcal{D}, a_1^a] \cdots [\mathcal{D}, a_p^a] = \Gamma.
\]

(3.6b)

**Condition 8** *(Poincaré duality).* The spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) determines a \( K \)-homology class for \( \mathcal{A} \otimes \mathcal{A} \). Let \( \mu = [(\mathcal{A} \otimes \mathcal{A}, \mathcal{H}, \mathcal{D})] \in K^\bullet(\mathcal{A} \otimes \mathcal{A}) = K^\bullet(\mathcal{A} \otimes \mathcal{A})\) denote this \( K \)-homology class. We require that \( \mu \) be a fundamental class, i.e., that the Kasparov product [31]

\[
- \otimes_{\mathcal{A}} \mu : K_*(\mathcal{A}) \rightarrow K^*(\mathcal{A})
\]

be an isomorphism. (More on this in Section 8).

**Condition 9** *(Reality).* There is an antilinear operator \( J : \mathcal{H} \rightarrow \mathcal{H} \) such that \( Ja^*J^{-1} = a^* \) for all \( a \in \mathcal{A} \); and moreover, \( J^2 = \pm 1 \), \( J^*J^{-1} = \pm \mathcal{D} \) and also \( J^*J^{-1} = \pm \Gamma \) in the even case, according to the following table of signs depending only on \( p \) mod 8:

<table>
<thead>
<tr>
<th>( p ) mod 8</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>( p ) mod 8</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J^2 = \pm 1 )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>( J^2 = \pm 1 )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( J^*J^{-1} = \pm \mathcal{D} )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>( J^*J^{-1} = \pm \mathcal{D} )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( J^*J^{-1} = \pm \Gamma )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the origin of this sign table in \( KR \)-homology, we refer to [30, Sec. 9.5].

**Remark 3.10.** For a noncommutative algebra \( \mathcal{A} \), we would require \( Ja^*J^{-1} = a^* \) or, more precisely, \( J\pi(a)^*J^{-1} = \pi^\circ(a) \). Thus, \( J \) implements on \( \mathcal{H} \) the involution \( \tau : a \otimes b^\circ \mapsto b^* \otimes a^\circ \) of \( \mathcal{A} \otimes \mathcal{A}^\circ \).

**Condition 10** *(Irreducibility).* The spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is irreducible: that is, the only operators in \( \mathcal{B}(\mathcal{H}) \) (strongly) commuting with \( \mathcal{D} \) and with all \( a \in \mathcal{A} \) are the scalars in \( \mathbb{C}1 \).

The foregoing conditions are an elaboration of those set forth in [21] for the reconstruction of a spin manifold. We add a final condition, which provides a cohomological version of Poincaré duality: see the discussion in [18, VI.4.γ].

**Condition 11** *(Closedness).* The \( p^* \)-summable spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) satisfies the following closedness condition: for any \( a_1, \ldots, a_p \in \mathcal{A} \), the operator \( \Gamma [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p] \langle \mathcal{D} \rangle^{-p} \) has vanishing Dixmier trace; thus, for any \( \text{Tr}_\omega \),

\[
\text{Tr}_\omega(\Gamma [\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p] \langle \mathcal{D} \rangle^{-p}) = 0.
\]

(3.7)
Proof. It is enough to show that there are no nontrivial projectors in the space $\mathcal{D}$ since $\mathcal{D}$ is commutative: the first order condition gives the last equality. Hence $q$ be a projector; by Lemma 2.5, we can find a projector $q_\mathcal{A}$ such that $q_\mathcal{A}$ is invertible. Thus, $q$ commutes with $\mathcal{D}$, and with all $a \in \mathcal{A}$ since $\mathcal{A}$ is commutative: by irreducibility, $q$ must be a scalar, either $q = 0$ or $q = 1$.

We quote Lemma 3 of [18, VI.4], adapted to the present case where $\mathcal{A}$ is commutative and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $p^+$-summable.

Lemma 3.12 (Connes). Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be $p^+$-summable and satisfy Condition 6 (first order). Then for each $k = 0, 1, \ldots, p$ and $\eta \in \Omega^k \mathcal{A}$, a Hochschild cocycle $C_\eta \in Z^{p-k}(\mathcal{A}, \mathcal{A}^*)$ is defined by

$$C_\eta(a_0, \ldots, a^{p-k}) := Tr_\omega(\Gamma \pi_\mathcal{D}(\eta) a_0 [\mathcal{D}, a_1] \cdots [\mathcal{D}, a^{p-k}] \langle \mathcal{D} \rangle^{-p}).$$

Moreover, if Condition 11 (closedness) also holds, then $C_\eta$ depends only on the class of $\pi_\mathcal{D}(\eta)$ in $\Lambda^k \mathcal{A}$, and $B_0C_\eta = (-1)^k C_{d\eta}$. □

3.2 First consequences of the geometric conditions

We now describe some immediate consequences of these conditions, which already give a reasonable picture of the spaces and bundles we shall employ. In this subsection, $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ will always denote a $QC^\infty$ spectral triple whose algebra $\mathcal{A}$ is commutative and complete (and unital, too). In other words, Condition 3 (regularity) is taken for granted. We shall write, as before, $X = \text{sp}(\mathcal{A}) = \text{sp}(A)$ where $A$ is the separable $C^*$-completion of $\mathcal{A}$; it is a metrizable compact Hausdorff space under its weak* topology.

Lemma 3.13. Under Conditions 6 and 10 (first order, irreducibility), the algebra $\mathcal{A}$ contains no nontrivial projector.

Proof. Let $q \in \mathcal{A}$ be a projector. Then

$$[\mathcal{D}, q] = [\mathcal{D}, q^2] = q [\mathcal{D}, q] + [\mathcal{D}, q] q = 2q [\mathcal{D}, q],$$

where the first order condition gives the last equality. Hence $(2q - 1)[\mathcal{D}, q] = 0$, implying $[\mathcal{D}, q] = 0$ since $2q - 1$ is invertible. Thus, $q$ commutes with $\mathcal{D}$, and with all $a \in \mathcal{A}$ since $\mathcal{A}$ is commutative: by irreducibility, $q$ must be a scalar, either $q = 0$ or $q = 1$. □

Corollary 3.14. Under the same Conditions 6 and 10, the space $X$ is connected.

Proof. It is enough to show that there are no nontrivial projectors in the $C^*$-algebra $A$. Let $\tilde{q} \in A$ be a projector; by Lemma 2.5, we can find a projector $q \in \mathcal{A}$ such that $\|q - \tilde{q}\| < \frac{1}{2}$. Since $q$ is either 0 or 1 by Lemma 3.13, the same must be true of $\tilde{q}$. Therefore, $C(X)$ contains no nontrivial projectors, and so $X$ is connected. □

Lemma 3.15. Under Condition 4 (finiteness), the dense subspace $\mathcal{H}_\infty \subset \mathcal{H}$ consists of continuous sections of a complex vector bundle $S \to X$. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is also irreducible, then $S$ has constant rank.

Proof. Condition 4 says that there is an integer $m > 0$ and a projector $q \in M_m(\mathcal{A})$ such that $\mathcal{H}_\infty \cong q \mathcal{A}^m$ as a (right) $\mathcal{A}$-module. We may regard $q$ as an element of $M_m(\mathcal{A})$; the $\mathcal{A}$-module $q \mathcal{A}^m$ is well-defined as a finitely generated projective module over $A = C(X)$. By the Serre–Swan
theorem [59], \( qA^m \simeq \Gamma(X, S) \), the space of continuous sections of a complex vector bundle \( S \to X \). From the finiteness axiom, the Hermitian pairing on \( qA^m \) gives \( S \to X \) the structure of a Hermitian vector bundle.

If \((A, \mathcal{H}, \mathcal{D})\) is irreducible, then \( X \) is connected by Corollary 3.14, and the rank of \( S \) must be constant. We shall denote this rank by \( N \).

\[\Box\]

**Lemma 3.16.** Under Conditions 4 to 7 (finiteness, absolute continuity, first order, orientability), the algebra \( \mathcal{C}_D(A) \) is a unital selfadjoint subalgebra of \( \Gamma(X, \text{End} \, S) \). The operator norm of each \( T \in \mathcal{C}_D(A) \) coincides with its norm as an endomorphism of \( S \).

**Proof.** Since \((A, \mathcal{H}, \mathcal{D})\) is \( QC^{\infty} \), each operator \( T \in \mathcal{C}_D(A) \) maps \( \mathcal{H}_\infty \) into itself; and the first order condition ensures that \( T \) is an \( A \)-linear map on \( \mathcal{H}_\infty \). If \( T = \sum_j a_j [\mathcal{D}, b_j] \in \Lambda^1 A = \pi_D(\Omega^1 A) \), the adjoint operator

\[ T^* = - \sum_j [\mathcal{D}, b_j^*] a_j^* = \sum_j b_j^* [\mathcal{D}, a_j^*] - [\mathcal{D}, b_j a_j^*] \]

lies in \( \Lambda^1 A \) also, so \( \Lambda^1 A \) is a selfadjoint linear subspace of \( \mathcal{B}(\mathcal{H}) \). Thus, the algebra \( \mathcal{C}_D(A) \) generated by \( A \) and \( \Lambda^1 A \) is a *-subalgebra of \( \mathcal{B}(\mathcal{H}) \). Moreover, since the pairing on \( \mathcal{H}_\infty \) is determined by the scalar product on \( \mathcal{H} \) via (3.4), we conclude that \( (\xi \mid T\eta) = (T^* \xi \mid \eta) \) for each \( T \in \mathcal{C}_D(A) \). Consequently, \( T \) yields an adjointable \( A \)-module map of the \( C^*\)-module \( \Gamma(X, S) \); that is, \( \mathcal{C}_D(A) \subset \text{End}_A(\Gamma(X, S)) = \Gamma(X, \text{End} \, S) \). The algebra \( \mathcal{C}_D(A) \) contains \( \Gamma^2 = 1 \).

We use the inequality [49, Cor. 2.22] between positive elements of the \( C^*\)-algebra \( A \):

\[ (T\xi \mid T\xi) \leq \|T\|^2_{\text{End} \, S} (\xi \mid \xi), \]

where \( \|T\|_{\text{End} \, S} \) denotes the norm of \( T \) in the \( C^*\)-algebra \( \Gamma(X, \text{End} \, S) \). Therefore, when \( \xi \in \mathcal{H}_\infty \),

\[ (T\xi \mid T\xi) = \text{Tr}_\Omega((T\xi \mid T\xi) (\Omega)^{-p}) \leq \|T\|^2_{\text{End} \, S} \text{Tr}_\Omega((\xi \mid \xi) (\Omega)^{-p}) = \|T\|^2_{\text{End} \, S} (\xi \mid \xi). \]

Majorizing this inequality over \( \{ \xi \in \mathcal{H}_\infty : (\xi \mid \xi) \leq 1 \} \), we obtain \( \|T\| \leq \|T\|_{\text{End} \, S} \).

To see that these norms are indeed equal, suppose that \( 0 \leq M < \|T\|^2_{\text{End} \, S} \), so that \( M - T^* T \) is selfadjoint and not positive in \( \Gamma(X, \text{End} \, S) \). Then we can find a nonzero \( \xi \in \Gamma(X, S) \) such that \( (T\xi \mid T\xi) - M (\xi \mid \xi) \) is positive and nonzero. In view of Condition 5, this implies that

\[ M (\xi \mid \xi) = \text{Tr}_\Omega(M (\xi \mid \xi) (\Omega)^{-p}) < \text{Tr}_\Omega((T\xi \mid T\xi) (\Omega)^{-p}) = (T\xi \mid T\xi) = \|T\|^2 (\xi \mid \xi), \]

so that \( M < \|T\|^2 \) since \( \xi \neq 0 \). This is true for all \( M < \|T\|^2_{\text{End} \, S} \), thus \( \|T\| = \|T\|_{\text{End} \, S}. \)

\[\Box\]

**Corollary 3.17.** Under the same Conditions 4 to 7, the algebra of sections \( \mathcal{C}_D(A) \) is pointwise a direct sum of matrix algebras:

\[ (\mathcal{C}_D(A))_x \simeq M_{k_1}(C) \oplus \cdots \oplus M_{k_r}(C) \quad \text{for} \quad x \in X, \]

where \( k_1 + \cdots + k_r = N \).
Proof. Lemma 3.16 shows that \((C_D(A))_x\) is a selfadjoint subalgebra of the finite-dimensional algebra \(\text{End} S_x\); hence it is a direct sum of full matrix algebras. This subalgebra has full rank, since \(\Gamma = \pi_D(e)\) lies in \(C_D(A)\), so that \(\Gamma^2_x\) is the identity element in \(\text{End} S_x\). □

The following result allows us to sidestep several questions of domains.

**Proposition 3.18.** Let \(T : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty\) be \(A\)-linear. Then \(T\) extends to a bounded operator on \(\mathcal{H}\).

Proof. Let \(\xi_1, \ldots, \xi_m \in \mathcal{H}_\infty \cong qA^m\) be defined by \(\xi_j := qe_j = \sum_k q_k e_k\) where \(e_j \in A^m\) is the column-vector with 1 in the \(j\)th slot and zeroes elsewhere. Every \(\xi \in \mathcal{H}_\infty\) can be written in the form \(\xi = \sum_j \xi_j a_j\), for some \(a_j \in A\). By the properties of our chosen Hermitian pairing, we get

\[
(\xi_j | \xi_k)_{\mathcal{H}_\infty} = (\sum_r q_{rj} e_r | \sum_s q_{sk} e_s)_{A^m} = \sum_{r,s} q_{rj} q_{sk} \delta_{rs} = q_{jk}.
\]

Thus, as already noted, \((\xi | \xi) = \sum_{j,k} a_j^* q_{jk} a_k\).

The \(A\)-linearity of \(T\) gives \(T\xi = \sum_{j=1}^m (T\xi_j) a_j\), and by hypothesis, \(T\xi_j \in \mathcal{H}_\infty\). Therefore,

\[
(T\xi_j | T\xi_k) = \left(\sum_n q_{nj} e_n | \sum_l q_{lk} e_l\right) = \left(\sum_{n,r} q_{nr} q_{rj} e_n | T \sum_{l,s} q_{ls} q_{sk} e_l\right) = \sum_{r,s} q_{jr} (T\xi_r | T\xi_s) q_{sk}.
\]

Denote by \(\Theta_{\xi, \eta}\), for \(\xi, \eta \in \mathcal{H}_\infty\), the “ketbra” operator \(\rho \mapsto \xi(\eta | \rho)\). The pairing \((T\xi | T\xi)\) may be expanded as follows:

\[
(T\xi | T\xi) = \sum_{j,k} a_j^* (T\xi_j | T\xi_k) a_k = \sum_{j,k,r,s} a_j^* (\xi_j | \xi_r) (T\xi_r | T\xi_s) (\xi_s | \xi_k) a_k
\]

\[
= \sum_{r,s} (\xi | \xi_r) (T\xi_r | T\xi_s) (\xi_s | \xi) = \sum_{r,s} (T\xi_r (\xi_r | \xi) | T\xi_s (\xi_s | \xi))
\]

\[
= \sum_{r,s} (\Theta_{T\xi_r, \xi_s} \xi | \Theta_{T\xi_s, \xi_r} \xi) = \sum_{r,s} \left(\Theta_{\xi_s (T\xi_r | T\xi_r), \xi_r} \xi | \xi\right)
\]

\[
\leq \sum_{r,s} \left\|\Theta_{\xi_s (T\xi_r | T\xi_r), \xi_r}\right\| (\xi | \xi).
\]

In the last line here the norm is both the operator norm and the endomorphism norm, which coincide by Lemma 3.16. The norm of each ketbra is finite, since each \(T\xi_r \in \mathcal{H}_\infty\) by hypothesis. Now we can estimate the operator norm of \(T\); for \(\xi \in \mathcal{H}_\infty\) we get the bound

\[
(T\xi | T\xi) = \text{Tr}_\Omega((T\xi | T\xi) (\Omega)^{-p})
\]

\[
\leq \sum_{r,s} \left\|\Theta_{\xi_s (T\xi_r | T\xi_r), \xi_r}\right\| \text{Tr}_\Omega((\xi | \xi) (\Omega)^{-p})
\]

\[
= \sum_{r,s} \left\|\Theta_{\xi_s (T\xi_r | T\xi_r), \xi_r}\right\| (\xi | \xi).
\]

Since the \(\xi_r\) are a fixed finite set of vectors, the operator norm of \(T\) is finite, with

\[
\|T\|^2 \leq \sum_{r,s} \|(T\xi_s | T\xi_r)\| \|(\xi_r | \xi_r)\|^{1/2} \|(\xi_s | \xi_s)\|^{1/2}.
\]

This last expression for the norm follows from [49, Lemma 2.30] or [30, Lemma 4.21]. □
Lemma 3.19. Under Conditions 1, 4, 5 and 7 \((p^+\text{-summability, finiteness, absolute continuity, orientability})\), the \(\mathcal{A}\text{-valued Hermitian pairing on } \mathcal{H}_\infty\) given by (3.4) is independent of the choice of Dixmier trace.

**Proof.** Connes’ character theorem [18, Thm. IV.2.8] – we refer to [30] and [10] for its detailed proof – shows that any operator of the form

\[
T = \Gamma \sum a a_0^0 [D, a_a^1] \cdots [D, a_{ap}^p] \langle D \rangle^{-p},
\]

(3.8)

where \(c = \sum a a_0^0 \otimes a_a^1 \otimes \cdots \otimes a_{ap}^p\) is a Hochschild cycle, is a measurable operator. Condition 7 provides us with such a Hochschild \(p\)-cycle \(c\) for which \(\pi(c) = \Gamma\). Using \(\Gamma^2 = 1\), we can rewrite (3.4) as

\[
\langle \xi | \eta \rangle = Tr_\Omega((\langle \xi | \eta \rangle \langle D \rangle)^{-p}) = Tr_\Omega(\Gamma(\langle \xi | \eta \rangle \Gamma \langle D \rangle^{-p}).
\]

(3.9)

If \(a = (\langle \xi | \eta \rangle\), then \(ac\) is also a Hochschild cycle for \(\mathcal{A}\) – as an easy consequence of the cycle property of \(c\) and the commutativity of \(\mathcal{A}\) – so the right hand side of (3.9) is \(Tr_\Omega(T)\), where \(T = \Gamma \pi_D(ac) \langle D \rangle^{-p}\) is indeed of the form (3.8). Thus \(Tr_\Omega\) may be replaced, in the formula (3.4), by any other Dixmier trace \(Tr_\omega\).

It was noted in [19] that the orientability condition yields the following expression for \(D\) in terms of commutators \(\Delta a = [D, a]\) and \([D^2, a]\).

**Lemma 3.20.** Under Condition 7 (orientability), the operator \(D\) verifies the following formula (as an operator on \(\mathcal{H}_\infty\)):

\[
D = \frac{1}{2}(-1)^{p-1}\Gamma \sum_{a=1}^n \sum_{j=1}^p (-1)^{j-1} a_0^a \Delta a_a^1 \cdots \Delta a_a^{j-1} [D^2, a_a^j] \Delta a_a^{j+1} \cdots \Delta a_a^p + \frac{1}{2}(-1)^{p-1}\Gamma \Delta \Gamma,
\]

(3.10)

where \(\Gamma = \sum_{a=1}^n a_0^a \Delta a_a^1 \cdots \Delta a_a^p\) and we write \(\Delta \Gamma := \sum_{a=1}^n \Delta a_a^0 \Delta a_a^1 \cdots \Delta a_a^p\).

**Proof.** First note that on the domain \(\mathcal{H}_\infty\), the derivation \(\text{ad } D^2\) may be written as

\[
[D^2, a] = D \Delta a + \Delta a \text{ } D \quad \text{for all } a \in \mathcal{A}.
\]

(3.11)

Thus, the summation over \(j\) in (3.10) telescopes, to give

\[
\sum_{a=1}^n \sum_{j=1}^p (-1)^{j-1} a_0^a \Delta a_a^1 \cdots \Delta a_a^{j-1} [D^2, a_a^j] \Delta a_a^{j+1} \cdots \Delta a_a^p
\]

\[
= \sum_{a=1}^n (a_0^a \Delta a_a^1 \cdots \Delta a_a^p + (-1)^{p-1} a_0^a \Delta a_a^1 \cdots \Delta a_a^p \Delta D)
\]

\[
= -\sum_{a=1}^n \Delta a_a^0 \Delta a_a^1 \cdots \Delta a_a^p + \Delta \Gamma + (-1)^{p-1} \Gamma \Delta D
\]

\[
= -\Delta \Gamma + 2(-1)^{p-1} \Gamma \Delta D,
\]

and (3.10) follows on multiplying both sides by \(\frac{1}{2}(-1)^{p-1} \Gamma\). □
Corollary 3.21. Under Conditions 6 and 7 (first order, orientability), the commutator \([\mathcal{D}, a]\), for \(a \in \mathcal{A}\), has the expansion

\[
[\mathcal{D}, a] = \frac{1}{2} (-1)^{p-1} \Gamma \sum_{a=1}^{n} \sum_{j=1}^{p} (-1)^{j-1} a^0_a \, da^1_a \cdots (da^j_a \, da + da \, da^j_a) \cdots da^p_a.
\]

Proof. The first order condition entails that \(a\) commutes with all operator factors in the expansion (3.10), except the \([\mathcal{D}^2, a^j_a]\) factors. For those, (3.11) and \([[\mathcal{D}, a^j_a], a] = 0\) imply

\[
[[\mathcal{D}^2, a^j_a], a] = [\mathcal{D}[\mathcal{D}, a^j_a], a] + [[\mathcal{D}, a^j_a] \mathcal{D}, a] = [\mathcal{D}, a] [\mathcal{D}, a^j_a] + [\mathcal{D}, a^j_a] [\mathcal{D}, a].
\]

\(\square\)

4 The cotangent bundle

Throughout this section, \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) will be a spectral triple whose algebra \(\mathcal{A}\) is (unital and) commutative and complete; and \(X = \text{sp}(\mathcal{A})\) will be its metrizable compact Hausdorff character space. Moreover, we shall assume that Conditions 1, 3–7 and 11 hold, namely that \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is \(p^+\)-summable, \(QC^\infty\) and has the properties of finiteness, absolute continuity, first order, orientability and closedness.

Lemma 4.1. The operator \([\mathcal{D}, a][\mathcal{D}, b] + [\mathcal{D}, b][\mathcal{D}, a]\) is a junk term, for any \(a, b \in \mathcal{A}\).

Proof. We must show that \(da \, db + db \, da\) belongs to \(d(\ker \pi_{\mathcal{D}})\) in the universal graded differential algebra \(\Omega^* \mathcal{A}\). Since \(da \, db + db \, da = d(a \, db - d(ba) + b \, da) = d(a \, db - db \, a)\), it is enough to notice that the first-order condition gives

\[
\pi_{\mathcal{D}}(a \, db - db \, a) = a [\mathcal{D}, b] - [\mathcal{D}, b] a = 0.
\]

\(\square\)

Lemma 4.2. The image of \(\Gamma = \pi_{\mathcal{D}}(c)\) in \(\Lambda^p_{\mathcal{D}} \mathcal{A}\) is nonzero.

Proof. The Hochschild cycle \(c \in Z_p(\mathcal{A}, \mathcal{A})\) defines a Hochschild 0-cocycle (a trace) \(C_c\) on \(\mathcal{A}\) by Lemma 3.12. Taking into account Lemma 3.19, it is given by

\[
C_c(a) = \oint \Gamma \pi_{\mathcal{D}}(c) \, a(\mathcal{D})^{-p} = \oint a(\mathcal{D})^{-p}.
\]

Since

\[
C_c(1) = \oint (\mathcal{D})^{-p} > 0, \quad (4.1)
\]

this cocycle does not vanish. Moreover, Condition 11 entails that \(C_c\) depends only on the class of \(\pi_{\mathcal{D}}(c)\) in \(\Lambda^p_{\mathcal{D}} \mathcal{A}\). If this class were zero, so that \(\pi_{\mathcal{D}}(c) \in \pi_{\mathcal{D}}(d(\ker \pi_{\mathcal{D}}))\), then we could write it as a finite sum of the form \(\pi_{\mathcal{D}}(c) = \sum_\beta \Delta b^1_\beta \cdots \Delta b^p_\beta\). But the closedness condition would then apply to show that

\[
C_c(1) = \oint \Gamma \pi_{\mathcal{D}}(c)(\mathcal{D})^{-p} = \sum_\beta \oint \Gamma \Delta b^1_\beta \Delta b^2_\beta \cdots \Delta b^p_\beta (\mathcal{D})^{-p},
\]

contradicting (4.1). Hence, the class of \(\Gamma\) has a nonzero image in \(\Lambda^p_{\mathcal{D}} \mathcal{A}\). \(\square\)
Corollary 4.3. Let $\Gamma' \in \mathfrak{C}_D(\mathcal{A})$ be defined by
\begin{equation}
\Gamma' : = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^{\sigma} \sum_{\alpha} a^0_\alpha \, d a^{\sigma(1)}_\alpha \, d a^{\sigma(2)}_\alpha \ldots d a^{\sigma(p)}_\alpha,
\end{equation}
on skewsymmetrizing the expression for $\Gamma$ obtained from (3.6). If $a \in \mathcal{A}$ is positive and nonzero, then $a \Gamma' \neq 0$.

Proof. Let $a \in \mathcal{A}$ be positive, $a \neq 0$. Since $\mathcal{A}$ is commutative and $c \in Z_p(\mathcal{A}, \mathcal{A})$, the product $ac$ is also a Hochschild $p$-cycle. Now the absolute continuity condition implies that
\begin{equation}
C_{ac}(1) = C_c(a) = \int a(\mathcal{D})^{-p} > 0.
\end{equation}
Since $\pi_D(ac) = a\Gamma$, the proof of Lemma 4.2 shows that the class $[a\Gamma]$ in $\Lambda^p_D:\mathcal{A}$ is nonzero. Now, Lemma 4.1 shows that $[a\Gamma'] = [a\Gamma]$. In particular, $a\Gamma' \neq 0$ as an element of $\mathcal{B}(\mathcal{H})$.

Thus, the skewsymmetrization $\Gamma'$ of $\Gamma$ given by (4.2) is nonzero as an operator on $\mathcal{H}$, and \textit{a fortiori} as a section in $\Gamma(X, \text{End} S)$. In fact, this section vanishes nowhere on $X$, as the proof of the following Proposition shows.

\begin{itemize}
\item In what follows, whenever $T$ is a continuous (local) section of $\text{End} S \to X$, we write either $T(x)$ or $T_x$ to denote its value in the fibre $\text{End} S_x$. The \textit{support} of $T$ will mean its support as a section, namely, the complement of the largest open subset $V \subseteq X$ such that $T(x) = 0$ in $\text{End} S_x$ for all $x \in V$.
\item Proposition 4.4. There is an open cover $\{U_1, \ldots, U_n\}$ of $X$ such that, for each $\alpha = 1, \ldots, n$, the operators $[\mathcal{D}, a^1_\alpha], \ldots, [\mathcal{D}, a^n_\alpha]$ are pointwise linearly independent sections of $\Gamma(U_\alpha, \text{End} S)$.
\end{itemize}

Proof. Let $Z := \{ x \in X : \Gamma'(x) = 0 \}$ be the zero set of $\Gamma'$. If $V$ were a nonvoid open subset of $Z$, then, using Lemma 2.10, we could find a nonzero positive $b \in \mathcal{A}$ such that $\text{supp} b \subset V$; but this would imply $b\Gamma' = 0$, contradicting Corollary 4.3. Therefore, $Z$ has empty interior.

The pairing on $H_{\infty}$, or rather, on the completed $A$-module $\Gamma(X, S)$, induces a $C^*$-norm on $\Gamma(X, \text{End} S) = \text{End}_A(\Gamma(X, S))$ which in turn determines a norm on each fibre $\text{End} S_x$, so that $\|T\|_{\text{End} S} = \text{sup}_{x \in X} \|T(x)\|_{\text{End} S_x}$ for $T \in \Gamma(X, \text{End} S)$. Choose $\epsilon > 0$; unless $Z = \emptyset$, there is an open set $W \supseteq Z$ such that $\text{supp}_{y \in W} \|\Gamma'(y)\|_{\text{End} S_y} < \epsilon$. Next choose $a \in \mathcal{A}$, positive and nonzero, with $\text{supp} a \subset W$. By Lemma 2.12, there exists $\psi \in \mathcal{A}$ such that $0 < \psi \leq 1$, $\psi a = a$ and $\text{supp} \psi \subset W$. Hence, by Lemma 3.16, $\|\psi\Gamma'\| = \|\psi\Gamma'\|_{\text{End} S} < \epsilon$.

By Corollary 4.3, the Hochschild 0-cocycles $C_{a\Gamma}$ and $C_{a\Gamma'}$ are equal, and they define positive functionals on $\mathcal{A}$. For any Dixmier trace $\text{Tr}_\omega$, we know that
\begin{align*}
C_{a\Gamma}(1) &= C_{\Gamma}(a) = \text{Tr}_\omega(a(\mathcal{D})^{-p}), \\
C_{a\Gamma'}(1) &= \text{Tr}_\omega(\Gamma\Gamma' a(\mathcal{D})^{-p}) = \text{Tr}_\omega(\Gamma\Gamma' \psi a(\mathcal{D})^{-p}).
\end{align*}
This yields the following estimate:
\begin{align*}
|C_{a\Gamma}(1)| = C_{a\Gamma'}(1) &= \text{Tr}_\omega(\Gamma\Gamma' \psi a(\mathcal{D})^{-p}) \\
&\leq \|\Gamma\Gamma' \psi\| \text{Tr}_\omega(a(\mathcal{D})^{-p}) < \epsilon \text{Tr}_\omega(a(\mathcal{D})^{-p}).
\end{align*}
Since $\text{Tr}_\omega(a(\mathcal{D})^{-p}) > 0$, this is inconsistent with (4.3) when $0 < \epsilon < 1$. We conclude that the set $Z$ must necessarily be empty, so that $\Gamma'(x)$ is nonvanishing on $X$. 

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Proposition 4.8. The operators $a^0_\alpha(x) \neq 0$ and $\partial a^1_\alpha(x), \ldots, \partial a^p_\alpha(x)$ in End $S_x$ have a nonzero skewsymmetrized product, and therefore are linearly independent. We may now define $U_\alpha$ to be the open set of all $x$ for which this linear independence holds; and $U_1 \cup \cdots \cup U_n = X$ from the nonvanishing of $\Gamma'$.

□

Lemma 4.5. Fix $\alpha \in \{1, \ldots, n\}$ and let $a \in A$, writing $a =: a^{p+1}_\alpha$ for notational convenience. Then

$$\left(\begin{array}{c}
\frac{(-1)^{p-1}}{2(p+1)!} \sum_{\sigma \in S_{p+1}} (-1)^\sigma a^0_\alpha [\mathcal{D}, a^{(1)}_{\alpha}] \cdots [\mathcal{D}, a^{(p)}_{\alpha}] [\mathcal{D}, a^{(p+1)}_{\alpha}] = 0.
\end{array}\right) \quad (4.4)$$

Proof. By Corollary 3.21, we may write

$$\Gamma [\mathcal{D}, a] = \frac{1}{2} (\mathcal{D} a + a \mathcal{D})$$

for each $\alpha$, every term in the sum over $j$ contains a symmetric product of one-forms, so its skewsymmetrization vanishes.

Remark 4.6. For brevity, we shall denote by $\Gamma'_\alpha$ the $\alpha$th summand of $\Gamma'$ in (4.2), and by $\Gamma'_\alpha \wedge \partial a$ the operator on the left hand side of (4.4). Now, $\partial a$ and each $\partial a^j_\alpha$, and therefore each $\Gamma'_\alpha$, is an endomorphism of $\mathcal{H}_\alpha$; Lemma 4.5 shows that $\Gamma'_\alpha(x) \wedge \partial a(x) = 0$ in End $S_x$, for each $x \in U_\alpha$, where the notation $\wedge$ now denotes skewsymmetrization with respect to the several $\partial a^j_\alpha(x)$.

Remark 4.7. From now on we shall assume, without any loss of generality, that each $a^j_\alpha$, for $j = 1, \ldots, p$, $\alpha = 1, \ldots, n$, is selfadjoint. (Otherwise, we just take selfadjoint and skewadjoint parts, allowing some repetition of the sets $U_\alpha$.) Consequently, each $[\mathcal{D}, a^j_\alpha]$ is skewadjoint.

Proposition 4.8. The operators $[\mathcal{D}, a^j_\alpha]$, for $\alpha = 1, \ldots, n$ and $j = 1, \ldots, p$, generate $\Lambda^j_\alpha A$ as a finitely generated projective $A$-module.

Proof. Let $a \in A$; choose (and fix) $\alpha$ such that $a \Gamma'_\alpha \neq 0$. Then by Lemma 4.5, $\Gamma'_\alpha \wedge \partial a = 0$ in $\mathcal{C}_\mathcal{D}(A)$, and thus $\Gamma'_\alpha(x) \wedge \partial a(x) = 0$ in End $S_x$, for each $x \in U_\alpha$. Let $E_x$ be the (complex) vector subspace of End $S_x$ spanned by the endomorphisms $\partial a^1_\alpha(x), \ldots, \partial a^p_\alpha(x)$. The exterior algebra $\Lambda^* E_x$ is represented on $S_x$ by

$$(\vee^1 \wedge \cdots \wedge \vee^k) \cdot \xi := \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \vee^{(1)} \cdots \vee^{(k)} \cdot \xi,$$

and this representation is faithful, on account of Proposition 4.4. Similarly, we can represent on $S_x$ the exterior algebra of the vector subspace

$$E'_x := \text{span}\{\partial a^1_\alpha(x), \ldots, \partial a^p_\alpha(x), \partial a(x)\} \supseteq E_x.$$

Now Lemma 4.5 implies that $\Lambda^* E'_x = \Lambda^* E_x$, and thus $E'_x = E_x$; therefore, $\partial a(x)$ lies in $E_x$, for all $x \in U_\alpha$. 

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Choose a partition of unity $\{\phi_a\}_{a=1}^n \subset \mathcal{A}$ subordinate to the open cover $\{U_a\}$, as in Lemma 2.10. Then for any $a \in \mathcal{A}$ we may write $[\mathcal{D}, a] = \sum_{a} \phi_a [\mathcal{D}, a]$. Then for each $x \in U_a$, the linear independence of the $\mathfrak{d}a^j_a(x)$ yields unique constants $c_{ja}(x)$ such that

$$
\phi_a(x) \mathfrak{d}a(x) = \sum_{j=1}^p c_{ja}(x) \mathfrak{d}a^j(x). \quad (4.5)
$$

Since supp $\phi_a \subset U_a$, (4.5) defines a continuous local section in $\Gamma(U_a, \text{End } S)$. By uniqueness, $c_{ja}(x) = 0$ outside supp $\phi_a$, so we may extend $c_{ja}$ by zero to a function on all of $X$, and thus we may regard these local sections as elements of $\Gamma(X, \text{End } S)$. We claim that each $c_{ja}$ lies in $\mathcal{A}$ (and in particular is continuous).

Define an $\mathcal{A}$-valued Hermitian pairing on $\Lambda^1 \mathcal{A}$ by setting

$$
(\mathfrak{d}a | \mathfrak{d}b) := C_p \text{tr}((\mathfrak{d}a)^* \mathfrak{d}b), \quad (4.6)
$$

where $C_p$ is a suitable positive normalization constant, and $\text{tr}$ denotes the matrix trace in $\text{End}_A \mathcal{H}_\infty = qM_m(\mathcal{A})q$, where $\Gamma(X, S) = qA^m$. To see that this pairing takes values in $\mathcal{A}$, we use the following localization argument.

Choose a finite open cover $\{V_\rho\}$ of $X$ such that $S$ is trivial over each $V_\rho$. For each $\rho$, choose $N = \text{rank } S$ elements $\xi_1^\rho, \ldots, \xi_N^\rho \in \mathcal{H}_\infty$ which, when regarded as sections of $S$, are linearly independent over $V_\rho$. Moreover, these sections can be chosen so that $\{\xi_1^\rho(x), \ldots, \xi_N^\rho(x)\}$ is an orthonormal basis in each fibre $S_x$, for $x \in V_\rho$; this means that for all $b \in \mathcal{A}$ with supp $b \subset V_\rho$ the orthogonality relations $b(\xi_i^\rho | \xi_j^\rho) = b\delta_{ij}$ hold. That may be achieved by Gram–Schmidt orthogonalization, on invoking Proposition 2.13 to see that local inverses of elements in $\mathcal{A}$ also lie in $\mathcal{A}$.

Next, choose a partition of unity $\{\psi_\rho\}$ subordinate to the cover $\{V_\rho\}$, with $\psi_\rho \in \mathcal{A}$, as in Lemma 2.10. Then

$$
\text{tr}((\mathfrak{d}a)^* \mathfrak{d}b) = \sum_\rho \psi_\rho \text{tr}((\mathfrak{d}a)^* \mathfrak{d}b) = \sum_\rho \sum_{j=1}^N \psi_\rho (\mathfrak{d}b \xi_j^\rho | \mathfrak{d}a \xi_j^\rho).
$$

The right hand side is a finite sum of elements of $\mathcal{A}$, and so belongs to $\mathcal{A}$.

If $b = (\mathfrak{d}a | \mathfrak{d}a)$, then $b(x)$ is the trace of a positive element of $\text{End } S_x$, so $b(x) = 0$ if and only if $\mathfrak{d}a(x) = 0$; thus the pairing is positive definite. Consider the matrix $g_a = [g_{aj}^k] \in M_p(\mathcal{A})$ given by

$$
g_{aj}^k := (\mathfrak{d}a^j_a | \mathfrak{d}a^k_a) = -C_p \text{tr}(\mathfrak{d}a^j_a \mathfrak{d}a^k_a). \quad (4.7)
$$

The matrix $g_a(x) \in M_p(\mathbb{C})$ has the form $C_p \text{tr}(m_a(x)^* m_a(x))$ where $m_a(x) \in (\text{End } S_x)^p$ is the $p$-column with linearly independent entries $\mathfrak{d}a^j_a(x)$. Thus, for $x \in U_a$, each $g_a(x)$ is a positive definite Gram matrix, hence invertible, when $x \in U_a$. Let $g^{-1}_a(x) := [g_{ai,j}] \in \mathcal{A}$ denote the inverse matrix.

We may now invoke Corollary 2.14 – recall that $\mathcal{A}$ is complete – to conclude that $\phi_a g_{a,ij}$ is an element of $\mathcal{A}$ for $i, j = 1, \ldots, p$. Now if $\phi_a \mathfrak{d}a = \sum_i c_{ia} \mathfrak{d}a^i_a$ with supp $c_{ia} \subset U_a$ as in (4.5), we find that

$$
c_{ja} = \sum_{i,k} c_{ia} g_{a,jk} \phi_a \mathfrak{d}a^k = -C_p \sum_{i,k} g_{a,jk} \text{tr}(\mathfrak{d}a^k_a c_{ia} \mathfrak{d}a^i_a)
$$

$$
= -C_p \sum_k g_{a,jk} \text{tr}(\mathfrak{d}a^k_a \phi_a \mathfrak{d}a) = \sum_k \phi_a g_{a,jk} (\mathfrak{d}a^k_a | \mathfrak{d}a),
$$

where each $\phi_a g_{a,jk} \in \mathcal{A}$ and $(\mathfrak{d}a^k_a | \mathfrak{d}a) \in \mathcal{A}$ by previous arguments; we conclude that $c_{ja} \in \mathcal{A}$. 


Finally, for any $a \in A$, we may now write
\[
\delta a = \sum_{a=1}^{n} \phi_a \delta a = \sum_{a=1}^{n} \sum_{j=1}^{p} c_{ja} \delta a^j_a \in \Lambda^1_{\mathcal{D}}A.
\]  
(4.8)

Since the coefficients $c_{ja}$ in this finite sum lie in $A$, the $\delta a^j_a = [\mathcal{D}, a^j_a]$ generate the $A$-module $\Lambda^1_{\mathcal{D}}A$.

To see that $\Lambda^1_{\mathcal{D}}A$ is a projective $A$-module, we rewrite the coefficients in (4.8) as $c_{ja} = \sum_{k=1}^{p} \psi_{jka} (\delta a^k_a \mid \delta a)$, and we get, for $b \in A$,
\[
b \delta a = \sum_{a=1}^{n} \sum_{j,k=1}^{p} \psi_{jka} (\delta a^k_a \mid b \delta a) \delta a^j_a,
\]
and therefore $\Lambda^1_{\mathcal{D}}A \simeq Q A^p$ via standard isomorphisms [30, Prop. 3.9], where $Q \in M_{np}(A)$ is the projector with entries $Q_{ak,bl} := \sum_{m=1}^{p} \psi_{lm\beta} (\delta a^k_a \mid \delta a^l_\beta)$.

We shall frequently need to replace the expansion (4.8) by a “localized” version for a single $\alpha$, as follows.

**Corollary 4.9.** If $a \in A$ is such that $\text{supp} \delta a \subset U_a$, then there exist $c_{1\alpha}, \ldots, c_{p\alpha} \in A$, compactly supported in $U_a$, such that
\[
\delta a = \sum_{j=1}^{p} c_{ja} \delta a^j_a.
\]  
(4.9)

More generally, if $b \in A$, then for any open $V \subset U_a$ there are continuous functions $b_{ja} : V \to \mathbb{C}$ for $j = 1, \ldots, p$, such that
\[
\delta b(x) = \sum_{j=1}^{p} b_{ja}(x) \delta a^j_a(x) \quad \text{for all } x \in V,
\]  
(4.10)

and such that each $c_{b_{ja}} \in A$ whenever $c \in A$ with $\text{supp} c \subset V$.

**Proof.** If $\text{supp} \delta a \subset U_a$, then we may choose the partition of unity of the previous proof such that $\phi_a(x) = 1$ on $\text{supp} \delta a$, by Corollary 2.11. Thus $\phi_a \delta a = \delta a$ and $\phi_\beta \delta a = 0$ for $\beta \neq \alpha$. Thus both (4.5) and (4.8) reduce to (4.9). By construction, $\text{supp} c_{ja} \subseteq \text{supp} \phi_a$.

In the same way, if $c \in A$ with $\text{supp} c \subset V$, we may expand $c \delta b =: \sum_{j=1}^{p} c'_{ja} \delta a^j_a$ with $c'_{ja} \in A$ and $\text{supp} c'_{ja} \subseteq \text{supp} c$. Uniqueness of the coefficients at each $x \in V$ shows that $c'_{ja}(x) = c(x)b_{ja}(x)$, where each function $b_{ja}$ does not depend on $c$; also, $b_{ja}$ is continuous because its restriction to each compact subset of $V$ is continuous.

With the local linear independence and spanning provided by Propositions 4.4 and 4.8, we now obtain a (complex) vector subbundle $E$ of $\text{End} S$, such that $\Lambda^1_{\mathcal{D}}A \subseteq \Gamma(X, E)$. This vector bundle will eventually play the role of the complexified cotangent bundle $T^*_C(X)$, although at this stage we have not yet identified a suitable differential structure on $X$. 

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Proposition 4.10. For each \( x \in U_\alpha \), define a \( p \)-dimensional complex vector space by
\[
E_x := \text{span}\{da_1^\alpha(x), \ldots, da_p^\alpha(x)\} \subseteq \text{End} \, S_x.
\] (4.11)
Then these spaces form the fibres of a complex vector bundle \( E \to X \).

\textbf{Proof.} We prove that \( E \) is a vector bundle by providing transition functions satisfying the usual Čech cocycle condition.

For each pair of indices \( \alpha, \beta \), Corollary 4.9 provides continuous functions \( c_{j_{\alpha\beta}}^k : U_\alpha \cap U_\beta \to \mathbb{C} \) such that
\[
\overline{da}_\alpha^k(x) = \sum_{j=1}^p c_{j_{\alpha\beta}}^k(x) \overline{da}_\beta^j(x) \quad \text{for all} \quad x \in U_\alpha \cap U_\beta.
\] (4.12)
Whenever \( x \in U_\alpha \cap U_\beta \cap U_\gamma \), this entails the additional relation
\[
\overline{da}_\alpha^k(x) = \sum_{j=1}^p c_{j_{\alpha\beta}}^k(x) \overline{da}_\beta^j(x) = \sum_{j,l=1}^p c_{j_{\alpha\beta}}^k(x) c_{l_{\beta\gamma}}^j(x) \overline{da}_\gamma^l(x),
\]
and the linear independence of the \( \overline{da}_\gamma^l(x) \) shows that
\[
c_{l_{\alpha\gamma}}^j(x) = \sum_{j=1}^p c_{j_{\alpha\beta}}^k(x) c_{l_{\beta\gamma}}^j(x) \quad \text{for all} \quad x \in U_\alpha \cap U_\beta \cap U_\gamma.
\]
In particular, the matrix \( c_{\alpha\beta}(x) = [c_{j_{\alpha\beta}}^k(x)] \) is invertible with \( c_{\alpha\beta}^{-1}(x) = c_{\beta\alpha}(x) \) for \( x \in U_\alpha \cap U_\beta \). The relation (4.12) and its analogue with \( \alpha \) and \( \beta \) exchanged show that the vector space \( E_x \) of (4.11) is well defined, independently of \( \alpha \).

Moreover, the cocycle conditions \( c_{\alpha\beta} c_{\beta\gamma} = c_{\alpha\gamma} \) hold over every nonvoid \( U_\alpha \cap U_\beta \cap U_\gamma \), so these are continuous transition matrices for a vector bundle \( E \to X \), whose total space is the disjoint union \( E := \bigcup_{x \in X} E_x \).

\( \square \)

\textbf{Corollary 4.11.} If for each \( x \in U_\alpha \subset X \), we define the real vector space
\[
E_{R,x} = \mathbb{R} \cdot \text{span}\{da_1^\alpha(x), \ldots, da_p^\alpha(x)\},
\]
then \( E_R := \bigcup_{x \in X} E_{R,x} \) is the total space of a real vector bundle over \( X \).

\textbf{Proof.} We need only show that the transition functions are actually real matrices. Since each \( \overline{da}_\beta^j \) is skewadjoint, taking the adjoint (in \( \text{End} \, S_x \)) of (4.12) yields
\[
\overline{da}_\alpha^k(x) = \sum_{j=1}^p \overline{c}_{j_{\alpha\beta}}^k(x) \overline{da}_\beta^j(x).
\]
By uniqueness of the coefficients, we conclude that \( \overline{c}_{j_{\alpha\beta}}^k = c_{j_{\alpha\beta}}^k \) for each \( j, k, \alpha, \beta \).

\( \square \)

We conclude this Section by indicating that the functions \( a_\alpha^j \) are not constant on sets with nonempty interior, and more importantly, that the operator \( \mathcal{D} \) is actually local.
Lemma 4.12. Let $Y \subset U_\alpha$ be a level set of the function $a_\alpha^j$, for some $j = 1, \ldots, p$. Then $Y$ is closed and its interior $\text{Int} Y$ is empty.

Proof. Clearly $Y$ is closed, since $a_\alpha^j \in C(X)$. Suppose that $\text{Int} Y$ were nonempty; then there would be a nonzero element $f \in A$ such that $\text{supp} f \subset \text{Int} Y$. Let $\lambda \in \mathbb{R}$ be the value of $a_\alpha^j$ on the level set $Y$, so that $\lambda f = a_\alpha^j f$. Taking commutators with $\mathcal{D}$ gives

$$\lambda [\mathcal{D}, f] = a_\alpha^j [\mathcal{D}, f] + [\mathcal{D}, a_\alpha^j] f = a_\alpha^j [\mathcal{D}, f] + f [\mathcal{D}, a_\alpha^j],$$

using the first order condition. Therefore, $f(y) \frac{d}{da_\alpha^j}(y) = 0$ for all $y \in Y$. This contradicts $\text{supp} f \subset U_\alpha$, since by definition $U_\alpha \subseteq \{ x \in X : \frac{d}{da_\alpha^j}(x) \neq 0 \}$ for each $j$. \hfill $\square$

Corollary 4.13. For each $\alpha = 1, \ldots, n$, let $a_\alpha : U_\alpha \to \mathbb{R}^p$ be the mapping with components $a_\alpha^j$, $j = 1, \ldots, p$. Then any level set of the mapping $a_\alpha$ is a closed set with empty interior.

Proof. Any such level set for $a_\alpha$ is the intersection of level sets for the several $a_\alpha^j$. \hfill $\square$

Corollary 4.14. If $a \in A$, then $\text{supp}(da) \subseteq \text{supp} a$.

Proof. Suppose that $Y := (X \setminus \text{supp} a)$ is nonempty, otherwise there is nothing to prove. Then $Y$ is a nonvoid open subset of $X$, and the function $a$ vanishes on its closure $\overline{Y}$. Arguing as in the proof of Lemma 4.12, with $\lambda = 0$ and $a_\alpha^j$ replaced by $a$, we see that $\frac{d}{da}(y) = 0$ for all $y \in Y$. \hfill $\square$

Lemma 4.15. If $V \subseteq U_\alpha$ is open, then $a_\alpha(V)$ has nonempty interior in $\mathbb{R}^p$.

Proof. The level sets of each $a_\alpha^j$ on $U_\alpha$ are closed with no interior, so no $a_\alpha^j$ is constant on $V$, or on any open subset of $V$. Then the range of $a_\alpha^j$ contains a nontrivial interval (i.e., not a point), and so it contains an open interval $(s, t) \subset \mathbb{R}$. Let $V_1 = V \cap (a_\alpha^j)^{-1}((s, t))$, an open subset of $V$. Likewise, since $a_\alpha^2$ is not constant on $V_1$, we can find an open $V_2 \subseteq V_1$ which $a_\alpha^1$ maps onto an open subinterval of $a_\alpha^j(V_1)$ for $j = 1, 2$; and so on. After $p$ steps, we obtain an open subset $V_p \subseteq V$ that $a_\alpha$ maps onto an open rectangle in $\mathbb{R}^p$. \hfill $\square$

Corollary 4.16. Let $x \in U_\alpha$ be such that $x$ is neither a local maximum nor minimum of any of the functions $a_\alpha^j$, $j = 1, \ldots, p$. Then there is an open neighbourhood of $x$ on which $a_\alpha$ is an open map.

Proof. The hypothesis says that $a_\alpha(x)$ is not an endpoint of any interval in $a_\alpha^j(U_\alpha)$ for any $j$. Thus we can find an open rectangle $a_\alpha(x) \subset R \subseteq a_\alpha(U_\alpha)$, whereby $a_\alpha^{-1}(R)$ is an open neighbourhood of $x$, such that every point $y \in a_\alpha^{-1}(R)$ also satisfies the hypothesis of the corollary. \hfill $\square$

Corollary 4.17. If $B \subseteq a_\alpha(U_\alpha) \subseteq \mathbb{R}^p$ has empty interior, then $a_\alpha^{-1}(B) \cap U_\alpha$ has empty interior also.

Proof. If $a_\alpha^{-1}(B) \cap U_\alpha$ had an interior point, then so too would $a_\alpha(a_\alpha^{-1}(B) \cap U_\alpha) = B$. \hfill $\square$

5 A Lipschitz functional calculus

Definition 5.1. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ again be a spectral triple whose algebra $\mathcal{A}$ is (unital and) commutative and complete. From now on, we shall say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral manifold of dimension $p$ if it is $QC^\infty$, $p^+$-summable, and satisfies the metric, first order, finiteness, absolute continuity, orientability, irreducibility and closedness conditions, that is, all postulates of subsection 3.1 except perhaps Conditions 8 and 9.
Throughout this section, we shall assume that \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is a spectral manifold of dimension \(p\). We shall use without comment the open cover \(\{U_\alpha\}\) of \(X = \text{sp}(\mathcal{A})\) provided by Proposition 4.4, and the vector bundle \(E \to X\) afforded by Proposition 4.10. As indicated earlier, we shall also assume that the \(a^1_\alpha\) appearing in (3.6), for \(j \neq 0\), are selfadjoint.

Our next task is to develop a Lipschitz version of the functional calculus. For each \(\alpha = 1, \ldots, n\), we shall denote the joint spectrum of \(a^1_\alpha, \ldots, a^p_\alpha\) by \(\Delta_\alpha\), and shall write \(a_\alpha := (a^1_\alpha, \ldots, a^p_\alpha)\) as a continuous mapping from \(U_\alpha\) to \(\mathbb{R}^p\).

We recall Nachbin’s extension [45] of the Stone–Weierstrass approximation theorem to subalgebras of differentiable functions.

**Proposition 5.1** (Nachbin [45]). Let \(U\) be an open subset of \(\mathbb{R}^p\), and let \(\mathcal{B} \subset C^r(U, \mathbb{R})\) for \(r \in \{1, 2, \ldots\}\). A necessary and sufficient condition for the algebra generated by \(\mathcal{B}\) to be dense in \(C^r(U, \mathbb{R})\), in the \(C^r\) topology, is that the following conditions hold:

1. For each \(x \in U\), there exists \(f \in \mathcal{B}\) such that \(f(x) \neq 0\).
2. Whenever \(x, y \in U\) with \(x \neq y\), there exists \(f \in \mathcal{B}\) such that \(f(x) \neq f(y)\).
3. For each \(x \in U\) and each nonzero tangent vector \(\xi_x \in T_xU\), there exists \(f \in \mathcal{B}\) such that \(\xi_x(f) \neq 0\).

In particular, the real polynomials on \(\mathbb{R}^p\), restricted to \(U\), are \(C^r\)-dense in \(C^r(U, \mathbb{R})\); and thus also, the complex-valued polynomials are \(C^r\)-dense in \(C^r(U) = C^r(U, \mathbb{C})\).

**Lemma 5.2.** Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be a spectral manifold of dimension \(p\), and let \(Y\) be a compact subset of \(U_\alpha\) with nonempty interior. Let \(a \in \mathcal{A}\) with \(\text{supp} a \subset Y\), and suppose there is a bounded \(C^1\) function \(f : L \to \mathbb{C}\) such that \(a = f(a^1_\alpha, \ldots, a^p_\alpha)\), where \(L\) is a bounded open subset of \(\mathbb{R}^p\) with \(\Delta_\alpha \subset L\). Then there are positive constants \(C'_a\) and \(C_Y\), independent of \(a\) and \(f\), such that

\[
C'_a \|[[\mathcal{D}, a]]\| \leq \|df\|_\infty \leq C_Y \|[[\mathcal{D}, a]]\| \tag{5.1}
\]

where \(\|df\|_\infty := \sup_{t \in \Delta_\alpha} \sum_j |\partial_j f(t)|^2\).

**Proof.** By Proposition 5.1, we may approximate \(f\) by polynomials \(p_k\) such that \(p_k \to f\) and \(\partial_j p_k \to \partial_j f\) for each \(j\), uniformly on \(\Delta_\alpha\). The first order condition shows that (2.5) holds for each \(p_k\). In the limit, the proof of Proposition 2.8 yields

\[
[\mathcal{D}, a] = [\mathcal{D}, f(a^1_\alpha, \ldots, a^p_\alpha)] = \sum_{j=1}^p \partial_j f(a^1_\alpha, \ldots, a^p_\alpha) [\mathcal{D}, a^j_\alpha]. \tag{5.2}
\]

Since \(\text{supp} a \subset Y\), Corollary 2.11 provides an element \(\phi \in \mathcal{A}\) such that \(0 \leq \phi \leq 1\) and \(\text{supp} \phi \subset U_\alpha\) while \(\phi(x) = 1\) for \(x \in Y\); and consequently, \(\phi a = a\). Moreover, \(\phi [\mathcal{D}, a] = [\mathcal{D}, a]\), on account of Corollary 4.14. The elements \(g^{ij}_\alpha \in \mathcal{A}\) defined by (4.7) form a positive definite matrix of functions on \(U_\alpha\); again let \(g_{\alpha, ij}\) denote the entries of its inverse matrix. The proof of Proposition 4.8 shows that \(\phi g_{\alpha, ij}\) lies in \(\mathcal{A}\), for all \(i, j\).
For $x \in U_\alpha$, we compute that
\[
\text{tr}\left(\sum_j g_{\alpha,j} \frac{\partial}{\partial a^i} da^j\right)(x) = \sum_{j,k=1}^p \partial_k f(a^1_\alpha, \ldots, a^n_\alpha)(x) g_{\alpha,j}(x) \text{tr}(da^i_\alpha(x) \frac{\partial}{\partial a^k}(x))
\]
\[
= -C^{-1}_p \sum_{j,k=1}^p \partial_k f(a^1_\alpha, \ldots, a^n_\alpha)(x) g_{\alpha,j}(x)g^{ik}(x)
\]
\[
= -C^{-1}_p \partial_i f(a^1_\alpha, \ldots, a^n_\alpha)(x).
\]

The trace is defined pointwise in the endomorphism algebra $\Gamma(U_\alpha, \text{End} S)$. Let $\| \cdot \|_x$ be the operator norm in $\text{End} S_x$ induced by the hermitian pairing (4.6) on this algebra. Then
\[
|\partial_if(a^1_\alpha, \ldots, a^n_\alpha)(x)| = C_p \left| \text{tr}\left(\sum_j g_{\alpha,j} \frac{\partial}{\partial a^i} da^j\right)(x) \right| \leq \sum_j \left(\sum_k \left(\phi g_{\alpha,j} \frac{\partial}{\partial a^i} da^j\right)^2 \right)^{1/2}(x) \leq \sum_j \|g_{\alpha,j}(x)\|_x \|\partial_i f(x)\|_x \leq B_i(x) \|\partial_i f(x)\|_x,
\]
where $B_i(x)$ is independent of $\alpha$ and $f$ and is bounded on $Y$. Since $\|\partial_i f(x)\|_x \leq \|[\mathcal{D}, a]\|$ by Lemma 3.16, taking $C_Y := \max_i \sup\{B_i(x) : x \in Y\}$ yields the second inequality in (5.1).

On the other hand, for any $x \in U_\alpha$, the estimate
\[
\|\partial_i f(x)\|_x \leq \sum_j \left|\partial_j f(a^1_\alpha, \ldots, a^n_\alpha)(x)\right| \|da^j_\alpha(x)\|_x \leq \sum_j \|\partial_j f(x)\|_x \leq \sum_j \|\mathcal{D}, a^j_{\alpha}\| \|df\|_\infty,
\]
again by Lemma 3.16, shows that $\|[\mathcal{D}, a]\| \leq (C'_\alpha)^{-1}\|df\|_\infty$ if we take $(C'_\alpha)^{-1} := \sum_j \|[\mathcal{D}, a^j_{\alpha}]\|$; this gives the first inequality in (5.1).

We need to remove the hypothesis that $a \in \mathcal{A}$ has compact support in $Y$. This is possible because the proof of Lemma 5.2 relies on pointwise estimates.

**Corollary 5.3.** Let $a \in \mathcal{A}$ and let $Y \subset U_\alpha$ be any compact subset such that $a|_Y = (f \circ a_\alpha)|_Y$ for some $C^1$ function $f$ defined and bounded in a neighbourhood of $a_\alpha(Y) \subset \mathbb{R}^p$. Then there are positive constants $C'_\alpha$ and $C_Y$, independent of $\alpha$ and $f$, such that
\[
C'_\alpha \|[\mathcal{D}, a]\|_Y \leq \|df\|_{Y,\infty} \leq C_Y \|[\mathcal{D}, a]\|
\]
where $\|df\|_{Y,\infty} := \sup_{y \in a_\alpha(Y)} \sum_j |\partial_j f(y)|$, and $\|[\mathcal{D}, a]\|_Y := \sup_{y \in Y} \|[\mathcal{D}, a](x)\|_x$. 

\[
\text{(5.4)}
\]
Proof. Choose \( \phi \in \mathcal{A} \) with \( \phi(x) = 1 \) for all \( x \in Y \) and \( \text{supp} \ \phi \subseteq U_a \). Then for all \( x \in Y \), the proof of Lemma 5.2 gives us

\[
|\partial_i f \circ a_\alpha(x)| \leq \sum_j p \|g_{a,i}(x)\|_x \|d_{a,i}(x)\|_x = \sum_j p \|\phi(x)g_{a,i}(x)\|_x \|d_{a,i}(x)\|_x.
\]

Taking suprema over \( x \in Y \) yields the second inequality (note that \( \|[D, a]\|_Y \leq \|[D, a]\| \) as a result of Lemma 3.16).

Denote by \( A_D \) the completion of \( \mathcal{A} \) in the norm \( \|a\|_D := \|a\| + \|[D, a]\| \).

**Lemma 5.4.** Let \( Y \subseteq U_a \) be a compact set on which \( a_\alpha : Y \to \mathbb{R}^p \) is one-to-one. Then for all \( b \in A_D \) there exists a unique bounded Lipschitz function \( g : a_\alpha(Y) \to \mathbb{C} \) such that \( b|_Y = g \circ a_\alpha|_Y \).

**Proof.** Since \( a_\alpha \) is one-to-one on \( Y \), the functions \( a_\alpha^1, \ldots, a_\alpha^P \) separate the points of \( Y \), so there is a unique bounded continuous function \( g \in C(a_\alpha(Y)) \) such that \( b|_Y = g(a_\alpha^1, \ldots, a_\alpha^P) \).

Choose \( b_k = g_k(a_\alpha^1, \ldots, a_\alpha^P) \in \mathcal{A} \) where the \( g_k \) are smooth functions defined on a neighbourhood of \( a_\alpha(Y) \), such that

\[
b_k|_Y \to b|_Y \quad \text{and} \quad [D, b_k]|_Y \to [D, b]|_Y.
\]

This is possible since \( A_D \) is the completion of \( \mathcal{A} \) and functions of the form \( f \circ a_\alpha \), with \( f \) smooth, lie in \( \mathcal{A} \) and separate the points of \( Y \). Therefore,

\[
\|[D, b_j] - [D, b_k]\|_Y = \sup_{x \in Y} \|[D, b_j](x) - [D, b_k](x)\|_x \to 0 \quad \text{as} \quad j, k \to \infty,
\]

and \( \{g_k\} \) is a Cauchy sequence in the norm \( f \mapsto \sup_{y \in Y} |f(y)| + \|df\|_{Y, \infty} \), by Corollary 5.3. Hence there is a bounded Lipschitz function \( h : a_\alpha(Y) \to \mathbb{C} \) such that \( h := \lim_k g_k \) in this norm. Thus,

\[
(h \circ a_\alpha)|_Y = \lim_k (g_k \circ a_\alpha)|_Y = \lim_k b_k|_Y = b|_Y.
\]

Since \( b|_Y = g \circ a_\alpha \), it follows that \( (g - h) \circ a_\alpha = 0 \) and we have established that \( g \) is Lipschitz on \( a_\alpha(Y) \), with Lipschitz constant bounded above by \( C_Y \|[D, a]\| \).

**Proposition 5.5.** Suppose that \( B \subseteq U_a \) is such that \( a_\alpha : B \to \mathbb{R}^p \) is one-to-one. Then the map \( a_\alpha^{-1} : a_\alpha(B) \to B \) is continuous for the metric topology of \( B \) (and thus also for its weak* topology).

**Proof.** Choose \( x, y \in B \) and let \( Y \) be any weak* compact subset of \( B \) with \( x, y \in Y \). If \( a \in \mathcal{A} \), let \( f_a \) be the unique Lipschitz function on \( a_\alpha(Y) \) with \( a|_Y = f_a(a_\alpha^1, \ldots, a_\alpha^P) \). Write \( t := a_\alpha(x) \), \( s := a_\alpha(y) \) in \( \mathbb{R}^p \). We now find, using Corollary 5.3 and Lemma 5.4, that

\[
d(x, y) = \sup \left\{ |a(x) - a(y)| : a \in \mathcal{A}, \|[D, a]\| \leq 1 \right\}
\]

\[
= \sup \left\{ \|f_a \circ a_\alpha)(x) - (f_a \circ a_\alpha)(y)\| : a \in \mathcal{A}, \|[D, a]\| \leq 1 \right\}
\]

\[
= \sup \left\{ |f_a(t) - f_a(s)| : a \in \mathcal{A}, \|[D, a]\| \leq 1 \right\}
\]

\[
\leq \sup \{ C_Y \|[D, a]\| \|t - s\| : a \in \mathcal{A}, \|[D, a]\| \leq 1 \}
\]

\[
= C_Y |t - s|,
\]

where \( |t - s| \) is the Euclidean distance between \( t \) and \( s \) in \( \mathbb{R}^p \). \( \square \)
**The metric convergence follows:**

\[ d(x, y) \leq C_Y |a_\alpha(x) - a_\alpha(y)|. \] □

Note that if \( Y, Y' \) are compact subsets of \( U_\alpha \) with \( Y \subseteq Y' \), then \( C_Y \leq C_{Y'} \) by the proof of Lemma 5.2. Thus, in the previous Corollary, the minimal value of \( C_Y \) is \( C_{(x, y)} \).

**Corollary 5.7.** If \( B \subseteq U_\alpha \) is such that \( a_\alpha : B \to \mathbb{R}^p \) is one-to-one, then the map \( a_\alpha|_B \) is a homeomorphism onto its image (for either the metric or the weak\(^*\) topology of \( B \)).

**Proof.** The map \( a_\alpha : B \to a_\alpha(B) \subset \mathbb{R}^p \) is continuous for the weak\(^*\) topology on \( B \) since each \( a_\alpha^j \) lies in \( \mathcal{A} \), and thus it is also continuous for the metric topology on \( B \). The estimate (5.5) shows that the inverse map \( a_\alpha^{-1} : a_\alpha(B) \to B \) is continuous for the metric topology on \( B \), and \( a \) posteriori for the weak\(^*\) topology. □

**Corollary 5.8.** If \( V \subseteq U_\alpha \) is an weak\(^*\)-open subset such that \( a_\alpha : V \to \mathbb{R}^p \) is one-to-one, then \( a_\alpha(V) \) is an open subset of \( \mathbb{R}^p \).

**Proof.** By Lemma 4.15, the set \( a_\alpha(V) \) has nonempty interior in \( \mathbb{R}^p \). Now \( a_\alpha(V) \setminus \text{Int} a_\alpha(V) \) is the boundary of \( \text{Int} a_\alpha(V) \) in the relative topology of \( a_\alpha(V) \). Since \( a_\alpha \) is a homeomorphism from \( V \) onto \( a_\alpha(V) \), it cannot map interior points to this boundary. Thus, since \( V \) is open, this boundary is empty, and hence \( a_\alpha(V) = \text{Int} a_\alpha(V) \) is open in \( \mathbb{R}^p \). □

**Lemma 5.9.** On any subset \( V \subseteq U_\alpha \) on which \( a_\alpha \) is one-to-one, the weak\(^*\) and metric topologies coincide.

**Proof.** By Lemma 5.4, the restriction of any function \( a \in \mathcal{A} \) to a compact subset \( Y \subseteq V \) may be written as \( a|_Y = (f_a \circ a_\alpha)|_Y \), where \( f_a \) is a bounded Lipschitz function on \( a_\alpha(Y) \). We need only show that convergence of a sequence \( V \ni x_m \to y \) for the weak\(^*\) topology implies the convergence \( x_m \to y \) in the metric topology. Choose such a weak\(^*\)-convergent sequence \( \{x_m\} \); without loss of generality, we may suppose that it is contained in a compact subset \( Y \subseteq V \) such that each \( x_m \in Y \) and hence also \( y \in Y \).

Weak\(^*\) convergence of the sequence \( x_m \) means that \( |a(x_m) - a(y)| \to 0 \) for all \( a \in \mathcal{A} \). This implies that

\[
\frac{|a(x_m) - a(y)|}{\|[\mathcal{D}, a]\|} \to 0 \quad \text{as} \quad m \to \infty
\]

for all \( a \in \text{Dom} \mathcal{D} \) with \( \mathcal{D}a = [\mathcal{D}, a] \neq 0 \). By Lemma 5.4, this is equivalent to

\[
\frac{|(f_a \circ a_\alpha)(x_m) - (f_a \circ a_\alpha)(y)|}{\|[\mathcal{D}, a]\|} \to 0 \quad \text{as} \quad m \to \infty
\]

for all such \( a \). Using Corollary 5.3, this is equivalent to

\[
|a_\alpha(x_m) - a_\alpha(y)| \to 0 \quad \text{as} \quad m \to \infty.
\]

The metric convergence follows:

\[
d(x_m, y) = \sup \{ |a(x_m) - a(y)| : \|[\mathcal{D}, a]\| \leq 1 \}
= \sup \{ |(f \circ a_\alpha)(x_m) - (f \circ a_\alpha)(y)| : \|df\|_{\infty} \leq C_Y \}
\leq C_Y |a_\alpha(x_m) - a_\alpha(y)| \to 0 \quad \text{as} \quad m \to \infty,
\]
on invoking Corollary 5.3 once more. □
6 Point-set properties of the local coordinate charts

We must analyze the possible failure of \( a_{\alpha} \) to be one-to-one, by using some point-set topology. Some of this extra effort is due to the arbitrariness of the Hochschild cycle \( e \). For example, consider the manifold \( S^2 \), with volume form \( x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy \). Each \( U_{\alpha} \) (the subset where \( d^0_{\alpha} \) is nonzero, and \( da^1_{\alpha}, da^2_{\alpha} \) are linearly independent) consists of two open hemispheres, missing only an equator. For instance, there is a chart domain consisting of \( S^2 \setminus \{z = 0\} \) with local coordinates \((x, y)\); but these “coordinates” are actually two-to-one on this domain. This rather simple problem can be addressed by restricting the coordinate maps to the connected components of the \( U_{\alpha} \), thereby increasing the number of charts. Ultimately, in the next section, we shall show that this is the worst that can happen.

We begin by considering some set-theoretic properties of the map \( a_{\alpha} : U_{\alpha} \to \mathbb{R}^p \) for a fixed \( \alpha \), with a view to showing that \( a_{\alpha} \) is at worst finitely-many-to-one on an open dense subset of \( U_{\alpha} \). (Eventually, this behaviour will be improved to local injectivity.)

The linear functional

\[
\mu_\mathcal{D}(a) := \int a \langle \mathcal{D} \rangle^{-p}
\]

on \( \mathcal{A} \) extends to a continuous linear functional on \( A \), and by the Riesz representation theorem it is given by a regular Borel measure that we also denote by \( \mu_\mathcal{D} \).

**Lemma 6.1.** The measure \( \mu_\mathcal{D} \) has no atoms.

**Proof.** By [55, Lemma 14, p. 408], we can decompose \( X \) as a disjoint union \( X = X' \cup C \) where \( C \) is countable, its closure \( \overline{C} \) is the support of the atomic part of \( \mu_\mathcal{D} \), and \( X' \) has no isolated points. Recalling that \( \langle \xi \mid \eta \rangle = \mu_\mathcal{D}(\langle \xi \mid \eta \rangle) \) for \( \xi, \eta \in \mathcal{H}_\infty \), we get a corresponding Hilbert-space decomposition

\[
\mathcal{H} = L^2(X, S) = L^2(X', S|_{X'}) \oplus L^2(C, S|_{C}).
\]

If \( \text{supp} \xi = \{x\} \subseteq C \), then \( a\xi = a(x)\xi \) for any \( a \in \mathcal{A} \), so that \( C\xi \) would be a 1-dimensional subrepresentation of \( \pi(A) \) and thus \( \pi(A) \) would contain a nontrivial projector, contradicting Corollary 3.14. \( \square \)

We can write \( \mu_\mathcal{D} \) as a sum of finite measures \( \mu_{\mathcal{D},\alpha} \) concentrated on each \( U_{\alpha} \) by

\[
\int a \langle \mathcal{D} \rangle^{-p} = \int \Gamma^2 a \langle \mathcal{D} \rangle^{-p} = \sum_{a=1}^{n} \int \Gamma a a^0_{\alpha} da^1_{\alpha} \ldots da^p_{\alpha} \langle \mathcal{D} \rangle^{-p} =: \sum_{\alpha=1}^{n} \mu_{\mathcal{D},\alpha}(a)
\]

since the third expression depends only on the skew-symmetrization of the \( da^i_{\alpha} \), by the proof of Proposition 4.4, and each skew-symmetrized \( \Gamma^i_{\alpha} \) vanishes off the respective \( U_{\alpha} \).

We can transfer these measures to the several \( a_{\alpha}(U_{\alpha}) \) by setting

\[
\lambda_{\mathcal{D},\alpha}(f) := \mu_{\mathcal{D},\alpha}(f(a^0_{\alpha}, \ldots, a^p_{\alpha})) \quad \text{for} \quad f \in C_c(a_{\alpha}(U_{\alpha})).
\]

Each \( \lambda_{\mathcal{D},\alpha} \) is a nonatomic regular Borel measure on \( a_{\alpha}(U_{\alpha}) \subset \mathbb{R}^p \). Its Lebesgue decomposition

\[
\lambda_{\mathcal{D},\alpha} = \lambda_{\mathcal{D},\alpha}^s + \lambda_{\mathcal{D},\alpha}^{ac}
\]

provides measures \( \lambda_{\mathcal{D},\alpha}^s \) and \( \lambda_{\mathcal{D},\alpha}^{ac} \) that are respectively singular and absolutely continuous with respect to the Lebesgue measure on \( a_{\alpha}(U_{\alpha}) \). The singular part \( \lambda_{\mathcal{D},\alpha}^s \) is concentrated on a set of Lebesgue measure zero, whereas the absolutely continuous part \( \lambda_{\mathcal{D},\alpha}^{ac} \) has support \( \overline{a_{\alpha}(U_{\alpha})} \).

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Definition 6.1. Define \( n_\alpha : U_\alpha \to \{1, 2, \ldots, \infty\} \) by

\[
n_\alpha(y) := \#(a_\alpha^{-1}(a_\alpha(y)) \cap U_\alpha),
\]

where \# denotes cardinality (all infinite cardinals being treated simply as \( \infty \)).

Lemma 6.2. The set \( n_\alpha^{-1}(\infty) \) of infinite-multiplicity points has empty interior in \( U_\alpha \).

Proof. Let \( \text{sp}_{ac}(a_\alpha) \subseteq \Delta_\alpha \) be the absolutely continuous joint spectrum of \( a_\alpha = (a_\alpha^1, \ldots, a_\alpha^p) \), regarded as \( p \) commuting selfadjoint operators on \( \mathcal{H} \).

Over \( \text{sp}_{ac}(a_\alpha) \), the multiplicity function \( m_\alpha \) of (the representation of) \( a_\alpha \) is \( L^1 \) with respect to Lebesgue measure. Without changing the measure class, we may take \( m_\alpha \) to be precisely the multiplicity of \( a_\alpha \), namely, \( m_\alpha(t) = N n_\alpha(a_\alpha^{-1}(t) \cap U_\alpha) \) for \( t \in \text{sp}_{ac}(a_\alpha) \), where \( N \) is the rank of the bundle \( S \). This is well defined since \( n_\alpha \) is constant on \( a_\alpha^{-1}(t) \cap U_\alpha \).

Suppose that \( V \subseteq a_\alpha(n_\alpha^{-1}(\infty)) \) is a Borel subset of \( a_\alpha(U_\alpha) \). If \( V \) had positive Lebesgue measure, it would follow that

\[
\int_{\text{sp}_{ac}(a_\alpha)} m_\alpha(t) \, dp(t) \geq \int_V m_\alpha(t) \, dp(t) = \infty.
\]

However, by [61], discussed also in [18, Sec. IV.2.5], there is an equality

\[
\int_{\text{sp}_{ac}(a_\alpha)} m_\alpha(t) \, dp(t) = C_p(k_p^{-1}(a_\alpha))^p,
\]

where \( C_p \) is a constant depending only on \( p \), and Voiculescu’s modulus \( k_p^{-1}(a_\alpha) \) is a positive number which measures the size of the joint absolutely continuous spectrum. This number is finite, since by [18, Prop. IV.2.14], see also [10, 30] for similar estimates:

\[
k_p^{-1}(a_\alpha) \leq C_p \max_j \| [\mathcal{D}, a_\alpha^j] \| \left( \int \langle \mathcal{D} \rangle^{-p} \right)^{1/p} < \infty.
\]

This contradicts (6.1); consequently, \( V \) is a Lebesgue nullset. Since any subset of \( a_\alpha(U_\alpha) \) on which \( \lambda_{a_{\alpha}}^{s} \) is concentrated also has Lebesgue measure zero, and elsewhere \( m_\alpha \) is finite, we conclude that \( a_\alpha(n_\alpha^{-1}(\infty)) \) is a Lebesgue nullset, and in particular it has empty interior. Now Lemma 4.15 entails that \( n_\alpha^{-1}(\infty) \) has empty interior in \( U_\alpha \).

The remainder of this section proves that the topological structure of \( U_\alpha \) is sufficiently nice for us to deploy our geometric tools in the following section. These tools will prove the local injectivity of \( a_\alpha : U_\alpha \rightarrow \mathbb{R}^p \).

From now until Theorem 7.20, we fix \( \alpha \) and work solely within \( U_\alpha \) using the relative weak* topology. Thus, all closures, interiors and boundaries are taken in this relative weak* topology, unless specifically noted. Also, for \( E \subseteq \Delta_\alpha \subseteq \mathbb{R}^p \), \( "a_\alpha^{-1}(E)" \) will mean \( a_\alpha^{-1}(E) \cap U_\alpha \). Thus, up until Theorem 7.20, we restrict the universe to \( U_\alpha \).

Definition 6.2. Consider the following subsets of \( U_\alpha \), for \( k = 1, 2, \ldots \):

\[
D_k := n_\alpha^{-1}([1, \ldots, k]),
\]

\[
E_k := \text{Int} D_k,
\]

\[
N_{k+1} := U_\alpha \setminus D_k = n_\alpha^{-1}([k+1, \ldots, \infty]),
\]

\[
W_k := \text{Int}(n_\alpha^{-1}(k)).
\]

(6.2)
Remark 6.3. If $Z \subset U_a$, its closure in the relative (weak\*) topology of $U_a$ is $\overline{Z^X} \cap U_a$. Elements of this closure are limits of sequences in $Z$, since $X$ is metrizable because the $C^*$-algebra $A$ is assumed to be separable.

Lemma 6.4. The set $N_{k+1}$ is open and the set $D_k$ is closed in $U_a$.

Proof. If $D_k$ is finite, there is nothing to prove; otherwise, choose any convergent sequence $\{x_i\} \subset D_k$ with $x_i \to x \in U_a$. Each element $x_i$ has multiplicity $\leq k$, and so $a^{-1}_\alpha(a(\{x_i\}))$ consists of at most $k$ sequences with at most $k$ limit points $y_1, \ldots, y_m$, $m \leq k$, since $U_a$ is Hausdorff. Thus $x$ is one of the $y_j$, say $x = y_1$, and $a_\alpha(y) = a_\alpha(x)$ for $y \in D_k$ if and only if $y \in \{y_1, \ldots, y_m\}$. (Notice that $U_a$ can contain no isolated points, by Corollary 3.14.) Hence the limit point $x$ has multiplicity $m \leq k$, and so $x \in D_k$. \hfill \Box

Notice that $D_k$ is nonempty for some finite $k$, since $n^{-1}_\alpha(\infty) \neq U_a$. The next Proposition shows that the awkward possibility of $n^{-1}_\alpha(\infty)$ being dense cannot occur. We require a preparatory lemma.

Lemma 6.5. If $n^{-1}_\alpha(\infty)$ were dense in $U_a$, then each $N_{k+1}$ would be an open dense subset of $U_a$, every neighbourhood of an infinite-multiplicity point in $n^{-1}_\alpha(\infty)$ would contain elements of $D_k$ for arbitrarily large $k$, and every neighbourhood of a finite-multiplicity point in some $D_k$ would contain infinitely many points in $n^{-1}_\alpha(\infty)$.

Proof. The first statement is clear, since $n^{-1}_\alpha(\infty) \subseteq N_{k+1}$ for each finite $k$.

By Lemma 6.2, the union $\bigcup_{k=1}^\infty D_k$ is dense in $U_a$ and, by the proof of Lemma 6.4, no infinite-multiplicity point can be the limit of a sequence contained in any $D_k$.

The last statement is just the assumed density of $n^{-1}_\alpha(\infty)$. \hfill \Box

Proposition 6.6. The subset $n^{-1}_\alpha(\infty)$ is nowhere dense in $U_a$.

Proof. Let $Y$ be a compact subset of $U_a$ with nonempty interior. Suppose, argiendo, that the set $n^{-1}_\alpha(\infty) \cap \text{Int} Y$ is dense in $\text{Int} Y$. Since $a_\alpha$ cannot then be one-to-one on $Y$, we can choose a finite-multiplicity value $t \in a_\alpha(\text{Int} Y)$ and two distinct points $y, y' \in Y$ such that $a_\alpha(y) = a_\alpha(y') = t$. If necessary, we can add a small compact neighbourhood of $y'$ to $Y$.

Using Lemma 6.5, we can find a sequence $\{t_m\} \subset a_\alpha(Y \cap n^{-1}_\alpha(\infty))$ such that $|t - t_m| < \varepsilon_m$ for all $m$, where $\varepsilon_m \to 0$ with $0 < \varepsilon_m < (2C_Y)^{-1}d(y, y')$ for all $m$, where $C_Y$ is the constant appearing on the right hand side of the estimate (5.4).

Suppose that for all $m$, all points $z \in a^{-1}_\alpha(t_m) \cap Y$ satisfy $d(y, z) \leq C_Y \varepsilon_m$. Then

$$d(y', z) \geq |d(y, y') - d(y, z)| \geq d(y, y') - C_Y \varepsilon_m > C_Y \varepsilon_m.$$  

Consequently, on applying Corollary 5.6 to the pair of points $y', z \in Y$, we obtain

$$C_Y \varepsilon_m < d(y', z) \leq C_Y |a_\alpha(y') - a_\alpha(z)| = C_Y |t - t_m| < C_Y \varepsilon_m,$$

a contradiction. On the other hand, if there is some $m$ and some $z \in a^{-1}_\alpha(t_m) \cap Y$ such that $d(y, z) > C_Y \varepsilon_m$, then we reach the same impasse on replacing $y'$ by $y$ in (6.3).

We conclude that no such sequence $t_m \to t$ can exist, so that in particular $Y$ has a neighbourhood excluding $n^{-1}_\alpha(\infty)$. That is to say, $n^{-1}_\alpha(\infty) \cap \text{Int} Y$ is not dense in $\text{Int} Y$. By the arbitrariness of $Y$, $n^{-1}_\alpha(\infty)$ is nowhere dense in $U_a$. \hfill \Box
Corollary 6.7. The map \( a_\alpha \) is finitely-many-to-one on an open dense subset of \( U_\alpha \). \( \square \)

Lemma 6.8. Within \( U_\alpha \), the following relations hold:

\[
n^{-1}_\alpha (k) \setminus W_k \subset \partial W_k \cup \partial N_{k+1}, \quad n^{-1}_\alpha (k) \setminus \overline{W}_k \subset \partial N_{k+1}, \quad \partial D_k = \partial N_{k+1},
\]

and thus \( U_\alpha = \text{Int} \, D_k \cup \overline{N}_{k+1} \). Moreover,

\[
\partial W_k \subseteq D_k \quad \text{and} \quad n^{-1}_\alpha (\infty) \cap \bigcup_{k=1}^{\infty} \overline{W}_k = \emptyset.
\]

Proof. Consider the set \( H_k := U_\alpha \setminus (D_{k-1} \cup \overline{W}_k \cup \overline{N}_{k+1}) \). It is open in \( U_\alpha \), and \( H_k \subset n^{-1}_\alpha (k) \).

However, \( H_k \cap \text{Int}(n^{-1}_\alpha (k)) = H_k \cap W_k = \emptyset \), entailing \( H_k = \emptyset \). Therefore \( U_\alpha = D_{k-1} \cup \overline{W}_k \cup \overline{N}_{k+1} \), and so

\[
n^{-1}_\alpha (k) \setminus W_k \subset D_{k-1} \cup \partial W_k \cup \overline{N}_{k+1}.
\]

However, both \( D_{k-1} \) and \( \overline{N}_{k+1} \) are disjoint from \( n^{-1}_\alpha (k) \) by definition, so

\[
n^{-1}_\alpha (k) \setminus W_k \subset \partial W_k \cup \partial N_{k+1}.
\]

Taking the closure of \( W_k \) in \( U_\alpha \) gives the second relation. The third relation follows on recalling that \( U_\alpha \setminus N_{k+1} = D_k \).

It is clear that \( \partial W_k \subseteq D_k \), since \( W_k \subset D_k \) and \( D_k \) is closed. The last relation follows from \( \overline{W}_k \subset D_k \), since \( n^{-1}_\alpha (\infty) \) is disjoint from each \( D_k \). \( \square \)

Lemma 6.9. The interiors of \( D_k \) and of \( \bigcup_{j=1}^{k} \overline{W}_j \) coincide: \( E_k = \text{Int} \left( \bigcup_{j=1}^{k} \overline{W}_j \right) \).

Proof. Take \( x \in E_k \). Then every open neighbourhood of \( x \) in \( D_k \) must meet some \( W_j \) with \( j \leq k \), so that \( x \in \bigcup_{j=1}^{k} \overline{W}_j \). Hence \( E_k \subseteq \bigcup_{j=1}^{k} \overline{W}_j \).

On the other hand, since each \( D_j \) is closed in \( U_\alpha \), we get \( \bigcup_{j=1}^{k} \overline{W}_j \subseteq \bigcup_{j=1}^{k} D_j = D_k \). The inclusion \( \text{Int} \left( \bigcup_{j=1}^{k} \overline{W}_j \right) \subseteq E_k \) follows at once. \( \square \)

These topological preliminaries now allow us to prove three basic results about the mapping \( a_\alpha \) and the set \( U_\alpha \).

Proposition 6.10. There exists a weak*-open cover \( \{ Z_j \}_{j \geq 1} \) of \( \bigcup_{k=1}^{\infty} W_k \subseteq U_\alpha \) such that \( a_\alpha : Z_j \to a_\alpha (Z_j) \subset \mathbb{R}^p \) is a homeomorphism and an open map, for each \( j \geq 1 \).

Proof. First observe that \( W_1 \) is an open subset of \( U_\alpha \) such that \( a_\alpha : W_1 \to a_\alpha (W_1) \) is an open homeomorphism. This follows from Corollaries 5.7 and 5.8 and the definition of \( W_1 \).

Choose \( x \in W_k, k > 1 \). Since \( a^{-1}_\alpha (a_\alpha (x)) \) consists of \( k \) points of \( W_k \) and \( W_k \) is Hausdorff, we may choose \( k \) disjoint open subsets of \( W_k \) each containing precisely one of the preimages \( x_1, \ldots, x_k \) of \( a_\alpha (x) \). Call these sets \( V_1, \ldots, V_k \), and suppose that for some \( j = 1, \ldots, k \), the map \( a_\alpha : V_j \to \mathbb{R}^p \) is not one-to-one. Then there exist \( z_1, z_2 \in V_j \) with \( z_1 \neq z_2 \), such that \( a_\alpha (z_1) = a_\alpha (z_2) \). Again we can separate these two points, and thereby suppose that only one of \( z_1, z_2 \) lies in \( V_j \). If for every \( x \in W_k \) we can repeat this process finitely often to obtain neighbourhoods of each of the \( x_j \) on which \( a_\alpha \) is one-to-one, we are done.
On the other hand, if in this manner we always find pairs of distinct points \( z_{1m} \neq z_{2m} \) from successively smaller neighbourhoods of \( x_j \), we thereby obtain two sequences, both converging to \( x_j \). We may alternate the labelling of each pair, if necessary, so that both sequences are infinite. Then for \( i = 1, 2 \), \( a_\alpha(z_{im}) \) converges to \( a_\alpha(x_j) \) for \( j = 1, \ldots, k \), and so in any open neighbourhood \( N \) of \( a_\alpha(x_j) \) there are infinitely many such \( a_\alpha(z_{im}) \). Now take a neighbourhood \( V_l \) of each \( x_l \), for \( l \neq j \), such that \( a_\alpha(V_l) \) maps onto \( N \) (this is possible; we started with disjoint neighbourhoods of the \( x_l \) which map onto a neighbourhood of \( a_\alpha(x_j) \)). Then there is a sequence \( \{y_{lm}\} \subset V_l \) mapping onto \( a_\alpha(z_{im}) \). A quick count now shows that each element of the sequence \( a_\alpha(z_{im}) \) has (at least) \( k + 1 \) preimages in \( W_k \): contradiction.

Hence, for all \( x \in W_k \) we may find a neighbourhood \( V \) of \( x \) in \( W_k \) such that \( a_\alpha : V \to a_\alpha(V) \) is one-to-one, and thus is an open homeomorphism for the weak* topology on \( V \), by Corollary 5.7. In this way we obtain an open cover of the (locally compact, metrizable) set \( \bigcup_{k=1}^{\infty} W_k \) of the desired form; now let \( \{Z_j\}_{j \geq 1} \) be an enumeration of a countable subcover. Notice that we have also shown that each \( Z_j \) is included in some \( W_k \), with \( k = k(j) \). \( \square \)

The next Proposition is the most crucial consequence of the Lipschitz functional calculus. It allows us to extend the homeomorphism \( a_\alpha : Z_j \to a_\alpha(Z_j) \) to the closure \( \overline{Z}_j \) as a homeomorphism. It is essential in the next Proposition that we use the relative topology of \( U_\alpha \).

**Proposition 6.11.** For each open set \( Z_j \) as in Proposition 6.10, the function \( a_\alpha \) extends to a homeomorphism (for both the metric and weak* topologies) \( a_\alpha : \overline{Z}_j \to a_\alpha(\overline{Z}_j) \). The same is true when \( Z_j \) is replaced by any open set \( V \subset U_\alpha \) on which \( a_\alpha \) is one-to-one. In particular, the weak* and metric topologies agree on \( \overline{V} \).

**Proof.** Take any \( b \in \mathcal{A} \). Then by Lemma 5.4, for any compact \( Y \subset Z_j \) there is a unique bounded Lipschitz function \( g : a_\alpha(Y) \to \mathbb{C} \) such that \( b|_Y = g \circ a_\alpha|_Y \). Now we can cover \( Z_j \) by open sets which are interiors of compact subsets, and so obtain many function representations of \( b \) on these sets. By uniqueness, they agree on overlaps, and \( b = g \circ a_\alpha \) for each of these local representations. Hence we arrive at a single function \( g : a_\alpha(Z_j) \to \mathbb{C} \) with \( b|_{Z_j} = g \circ a_\alpha|_{Z_j} \). The function \( g \) is bounded since \( b \) is bounded, but might be only locally Lipschitz (since \( a_\alpha^{-1} \) might only be locally Lipschitz).

To proceed, suppose first that \( g \) is a \( C^1 \)-function on \( a_\alpha(Z_j) \). The proof of Lemma 5.2, in particular (5.3), and Corollary 5.3 guarantee that for any subset \( Y \) of \( Z_j \) with compact closure contained in \( \overline{Z}_j \),

\[
\sup_{x \in Y}|\partial_j g(a_\alpha(x))| \leq \sup_{x \in \overline{Y}} B_j(x) \|[\mathcal{D}, b](x)\|_s < \infty.
\]

The finiteness of the right hand side follows because \( \|[\mathcal{D}, b](x)\| \leq \|[\mathcal{D}, b]\| \), and \( B_j \) is continuous on all of \( U_\alpha \), so that \( B_j \) is bounded on \( \overline{Y} \).

To see that \( g \) extends to a locally Lipschitz function on \( a_\alpha(\overline{Z}_j) \), we argue as follows. If \( t \in a_\alpha(\overline{Z}_j) \), and if \( \{t_n\} \) and \( \{t'_n\} \) are two sequences in \( a_\alpha(Z_j) \) such that \( t_n \to t \) and \( t'_n \to t \) in \( a_\alpha(U_\alpha) \), then

\[
|g(t_n) - g(t'_n)| \leq C |t_n - t'_n|, \quad \text{where} \quad C = \sup_{x \in \overline{Y}} B_j(x) \|[\mathcal{D}, b]\|.
\]

For the estimate, since \( Z_j \subset W_k \) for a suitable \( k \), it is enough take \( Y \) to be the union of the \( k \) preimages of the sequences \( \{t_n\} \) and \( \{t'_n\} \) and their limits, which is compact in \( U_\alpha \). Thus \( \tilde{g}(t) := \lim_{n \to \infty} g(t_n) \)
is well defined, and coincides with \( g(t) \) whenever \( t \in a_\alpha(Z_j) \) already. The upshot is a bounded continuous function \( \tilde{g} : a_\alpha(\overline{Z}_j) \to \mathbb{R} \).

Its Lipschitz norm on any compact subset \( Y \subset \overline{Z}_j \) satisfies \( \|d\tilde{g}\|_Y \leq C_Y \|[\mathcal{D}, b]\| \), and so \( \tilde{g} \) is locally Lipschitz. The continuity of \( \tilde{g} \) and \( b \) yields \( b = \tilde{g} \circ a_\alpha \) over the set \( \overline{Z}_j \).

For an arbitrary \( b \in \mathcal{A} \), we may remove the assumption that \( g \) be \( C^1 \) on \( a_\alpha(Z_j) \) by approximating \( b \) by a sequence \( \{b_r\} \) in the norm \( a \to \|a\| + \|\mathcal{D}, a\| \), where \( b_r|_{Z_j} = g_r \circ a_\alpha|_{Z_j} \) with each \( g_r \) being \( C^1 \). For any subset \( Y \subset Z_j \) with compact closure, we get an estimate

\[
\sup_{x \in \overline{Y}} |\partial_j g_r(a_\alpha(x)) - \partial_j g_s(a_\alpha(x))| \leq \sup_{x \in \overline{Y}} B_j(x) \|\mathcal{D}, b_r - b_s\|,
\]

and thus \( \{g_r\} \) is a Cauchy sequence in the Lipschitz norm of \( a_\alpha(\overline{Y}) \). Each \( g_r \) extends to a locally Lipschitz function \( \tilde{g}_r \) on \( a_\alpha(\overline{Z}_j) \), and these converge uniformly on compact subsets to a locally Lipschitz function \( \tilde{g} \) satisfying \( b = \tilde{g} \circ a_\alpha \) over \( \overline{Z}_j \).

Since we can thereby extend the function representation for all functions \( b \in \mathcal{A} \), we conclude that \( a_\alpha \) separates points of \( \overline{Z}_j \), and thus it is one-to-one on this set. By Corollaries 5.7 and 5.8, \( a_\alpha : \overline{Z}_j \to \mathbb{R}^p \) is an open map, which is a homeomorphism onto its image. By Lemma 5.9, the weak* and metric topologies agree.

Here is a first and critical consequence of Proposition 6.11.

**Lemma 6.12.** For each \( k \geq 1 \), the set \( W_k \) is a disjoint union of \( k \) open subsets \( W_{1k}, \ldots, W_{kk} \), on each of which \( a_\alpha \) is injective, such that \( a_\alpha(W_{jk}) = a_\alpha(W_{j'}) \) for \( j, j' = 1, \ldots, k \).

**Proof.** Choose any point \( x \in W_k \), and choose disjoint open sets \( V_j \subset W_k \), for \( j = 1, \ldots, k \), each containing precisely one of the preimages of \( x \) in \( a_\alpha^{-1}(a_\alpha(x)) \). We may suppose by Proposition 6.10 that \( a_\alpha \) is one-to-one on each \( V_j \), and on replacing the \( V_j \) by the components of \( a_\alpha^{-1}(\cap_j a_\alpha(V_j)) \), we may suppose also that the various \( V_j \) have the same image \( a_\alpha(V_j) \) in \( \mathbb{R}^p \). By Proposition 6.11, \( a_\alpha \) extends to a homeomorphism on each \( \overline{V}_j \).

Suppose first that there exists \( z \in \overline{V}_1 \cap \overline{V}_l \cap W_k \) for some \( l \geq 2 \). Then, since \( a_\alpha \) is a homeomorphism on each \( \overline{V}_j \), the set \( a_\alpha^{-1}(a_\alpha(z)) \cap \bigcup_{j=1}^k \overline{V}_j \) consists of fewer than \( k \) points. So there must be some \( z' \in a_\alpha^{-1}(a_\alpha(z)) \cap W_k \) with \( z' \notin \overline{V}_j \) for \( j = 1, \ldots, k \). The open set \( W_k \setminus \bigcup_{j=1}^k \overline{V}_j \) includes an open neighbourhood \( O \) of \( z' \). Now \( a_\alpha^{-1}(a_\alpha(O)) \) is an open set in \( W_k \) meeting each \( V_j \), since \( z \in \partial V_1 \) and \( a_\alpha(V_1) = a_\alpha(V_j) \) for each \( j \). Consider a sequence \( \{t_m\} \subset a_\alpha(O) \cap a_\alpha(V_j) \) with \( t_m \to a_\alpha(z) = a_\alpha(z') \). Then \( a_\alpha^{-1}(\{t_m\}) \) meets each \( V_j \) and \( O \). Hence for \( m \) sufficiently large, \( a_\alpha^{-1}(t_m) > k \), which is impossible within \( W_k \).

We conclude that the relatively closed subsets \( \overline{V}_j \cap W_k \) of \( W_k \) are disjoint. Since \( W_k \) is metrizable (by Remark 3.5) and is thus a normal topological space, there are disjoint open sets \( U_1, \ldots, U_k \subset W_k \) with \( \overline{V}_j \cap W_k \subset U_j \).

Now choose a point \( z \) on the boundary of \( V_1 \) lying in \( W_k \) (if there is no such point, then \( V_1 \) is a union of connected components of \( W_k \), and so too are the other \( V_j \), and we are done). Choose an open neighbourhood \( U \subset W_k \) of \( z \) on which \( a_\alpha \) is one-to-one. Note that \( a_\alpha \) is one-to-one on \( U \cap V_1 \), on \( U \), and on \( V_1 \). Let \( U' := U_1 \cap U \); we claim that \( a_\alpha : U' \cup V_1 \to \mathbb{R}^p \) is one-to-one. For if not, there would exist \( y \in U' \setminus V_1 \) and \( y' \in V_1 \setminus U' \) such that \( a_\alpha(y) = a_\alpha(y') \). However, \( y \notin V_1 \) and \( a_\alpha^{-1}(a_\alpha(y')) \) consists of \( k \) points already, so this forces \( y \in V_j \cap a_\alpha^{-1}(a_\alpha(y')) \) for some \( j > 1 \); otherwise \( y \) would have multiplicity at least \( k + 1 \), contradicting \( y \in W_k \). However, \( y \in U' \setminus V_1 \subset U_1 \), which forbids
\( y \in V_j \) for \( j > 1 \). The upshot is that \( a_a^{-1}(a_a(U' \cup V_1)) \) is a union of \( k \) disjoint open sets on each of which \( a_a \) is one-to-one. It is clear that \( U \) may be chosen such that their images under \( a_a \) coincide.

By this argument, we may continue this process by taking a boundary point of \( U' \cup V_1 \) within \( W_k \), finding a neighbourhood on which \( a_a \) is one-to-one, and deducing that \( a_a \) is one-to-one on the union. In this way we cover the entire connected component of \( W_k \) in which \( x \) lies. Thus \( W_k \) is a disjoint union of \( k \) open subsets \( W_{1k}, \ldots, W_{kk} \), and by construction we see that \( a_a(W_{jk}) = a_a(W_{j'k}) \) for all \( j, j' \).

**Corollary 6.13.** Each set \( W_{jk} \) is a smooth manifold with the coordinate map \( a_a|_{W_{jk}} \).

**Remark 6.14.** If there is only a single nonempty \( W_k \), and \( U_a = W_k \), Lemma 6.12 shows that we are in the situation of the 2-sphere described at the beginning of this section, where \( k = 2 \).

Consider now \( D_k \), the subset of multiplicity at most \( k \). Clearly, \( \bigcup_{j=1}^k W_j \subseteq E_k = \text{Int} \, D_k \).

If it were true that for all \( k \) one could find some \( m > k \) with \( D_k \subseteq \text{Int} \, D_m \), then one could deduce from Lemmas 6.8 and 6.9 that \( U_a = \text{Int}(\bigcup_k W_k) \cup n_a^{-1}(\infty) \). For lack of such a guarantee (at present), we name the following subsets where the multiplicities may be troublesome. We shall eventually show that these subsets are empty.

**Definition 6.3.** Consider the following subsets of \( U_a \):

\[
B_k := \{ x \in D_k \setminus E_k : x \notin E_m \text{ for all } m > k \}, \quad B_{\infty} := \bigcup_{k \geq 1} B_k, \quad C_k := E_k \setminus \bigcup_{j=1}^k W_j.
\]

The set \( B_k \) consists of (some) boundary points of \( D_k \), while \( C_k \) collects interior points \( x \) of \( D_k \), if any, that lie on the boundary of some \( W_j \) with \( j \leq k \). The points \( C_k \) will be branch points of the “branched manifold” \( \bigcup_{k \geq 1} E_k \subseteq U_a \).

**Lemma 6.15.** If the multiplicity is bounded on \( U_a \), then \( n_a^{-1}(\infty) = B_{\infty} = \emptyset \). If the multiplicity is unbounded, then \( x \in D_k \setminus E_k \) lies in \( B_k \) if and only if \( x = \lim_{n \to \infty} x_n \) for some sequence \( \{x_n\} \) satisfying \( n_a(x_n) \to \infty \).

**Proof.** The first statement is obvious: if the multiplicity is bounded, by \( m \) say, then \( D_m \subseteq U_a \) and every \( D_k \) is included in \( D_m \).

So suppose that the multiplicity is unbounded, and take \( x \in B_k \). Then \( x \in D_m \setminus E_m \) for all \( m > k \). Since \( \partial D_m = \partial N_{m+1} \), for each \( m \) we can find a sequence \( \{y_j^{(m)}\} \subset N_{m+1} \) with \( y_j^{(m)} \to x \). By a diagonal argument, we can now construct a sequence \( \{y_j\} \) converging to \( x \) and such that \( n_a(y_j) \to \infty \).

Conversely, suppose that \( n_a(x) = k \) and that there is a sequence \( \{x_n\} \) with \( x_n \to x \) and \( n_a(x_n) \to \infty \). Then \( x \notin E_k = \text{Int} \, D_k \) since any sequence converging to an \( x \in E_k \) would have multiplicity eventually bounded by \( k \). Thus \( x \in D_k \setminus E_k \) and \( x \notin E_m \) for \( m > k \).

**Lemma 6.16.** Within \( U_a \), the set \( B_{\infty} \) is closed with empty interior.

**Proof.** If \( \{x_j\} \subset B_{\infty} \) is a sequence with a limit \( z \in U_a \), choose for each \( j \) a sequence \( \{x_{jn}\} \) of unbounded multiplicity such that \( x_{jn} \to x_j \). Passing to a subsequence if necessary, the sequence \( \{x_{jj}\} \) converges to \( z \) and \( n_a(x_{jj}) \to \infty \). Hence \( B_{\infty} \) is closed.

By Lemma 6.15, any \( x \in B_{\infty} \) lies in the boundary of \( \bigcup_k E_k \), and thus \( B_{\infty} \) has empty interior.
Corollary 6.17. The set $\bigcup_{k \geq 1} E_k$ is an open dense subset of $U_\alpha$. Moreover,

$$U_\alpha = \bigcup_{k=1}^{\infty} E_k \cup B(\alpha) \cup n^{-1}_\alpha(\infty).$$

Within $E_k = \text{Int} D_k$ we have possible branch points where two or more ‘sheets’ of one or more $W_j$, $j \leq k$, may have common boundary. We characterise these points in $C_k$ next.

Lemma 6.18. If $x \in C_k$, then $n_\alpha(x) < k$.

Proof. Let $x \in C_k = E_k \setminus \bigcup_{j=1}^{k} W_j$ and suppose that $n_\alpha(x) = k$. Since $U_\alpha = E_k \cup \overline{N}_{k+1} = D_k \cup N_{k+1}$ by Lemma 6.8, then not all neighbourhoods of $x$ can meet $\overline{N}_{k+1}$, since $x \in E_k$ entails $x \notin \partial D_k$. (Again we are taking all closures and boundaries in $U_\alpha$.)

Thus there is a neighbourhood of $x$ disjoint from $\overline{N}_{k+1}$. Since $n_\alpha(x) = k$, $x$ cannot lie in $\bigcup_{j=1}^{k-1} W_j \subseteq D_{k-1}$. Therefore, $x \in E_k \setminus \bigcup_{j=1}^{k-1} W_j = \text{Int} \overline{W}_k$. Nor does $x$ lie in any $\overline{W}_k \cap \overline{W}_j$ for $j < k$, since $D_j$ is closed and $n_\alpha(x) = k$. Hence, in the open set $\text{Int} \overline{W}_k \setminus \bigcup_{j=1}^{k-1} (\overline{W}_k \cap \overline{W}_j)$ we can find a neighbourhood $V$ of $x$ consisting of points of multiplicity $k$ only. Then $a^{-1}_\alpha(a_\alpha(V))$ is the union of $k$ neighbourhoods of the $k$ preimages of $x$. But that would imply that $x \in W_k$, contradicting $x \in C_k$. \hfill $\square$

Lemma 6.19. If $x \in C_k$, then either $x \in \text{Int} \overline{W}_j$ for some $j \in \{1, \ldots, k\}$; or else there is some set of (at least two) indices $j_1, \ldots, j_l \in \{n_\alpha(x), \ldots, k\}$ such that $x \in \bigcap_{r=1}^{l} \partial W_{j_r}$ and also $x \in \text{Int} (\bigcup_{r=1}^{l} \overline{W}_{j_r})$.

Proof. Observe that $C_k \subseteq \bigcup_{j=1}^{k} \overline{W}_j \setminus (\bigcup_{j=1}^{k} W_j) \subseteq \bigcup_{j=1}^{k} \partial W_j$. Since $x \in \partial W_j$ for some $j$, every neighbourhood of $x$ intersects $U_\alpha \setminus W_j$.

Since $x \in E_k$, it follows from Lemma 6.9 that $x$ has a neighbourhood $V$ contained in $\bigcup_{j=1}^{k} \overline{W}_j$. Suppose, then, that $x \notin \text{Int} \overline{W}_j$ for any $j$. Then there is some $j_1$, with $n_\alpha(x) \leq j_1 \leq k$ on account of Lemma 6.4, such that $V \setminus \overline{W}_{j_1} \neq \emptyset$. It follows that $V \setminus \overline{W}_{j_1} \subseteq \bigcup_{j \neq j_1} \overline{W}_j$.

Now $x \in \partial W_{j_2}$ for some $j_2 \neq j_1$, with $n_\alpha(x) \leq j_2 \leq k$, too. Moreover, if $V \setminus (\overline{W}_{j_1} \cup \overline{W}_{j_2})$ is nonempty, then there is some other $j_3$ such that $x \in \partial W_{j_3}$, and so on. Eventually we reach $j_l$ such that $x \in \bigcap_{r=1}^{l} \partial W_{j_r}$ and $V$ contained in $\bigcup_{r=1}^{l} \overline{W}_{j_r}$. \hfill $\square$

We now have a fairly detailed picture of the sets $U_\alpha$. There may be two ‘bad’ subsets, both of which are limit sets of sequences of unbounded multiplicity. Away from the ‘bad’ subsets, $X$ looks like a branched manifold, with branchings at points of $C_k$, since for $m > k$ there may be several ‘sheets’ of $W_m$ having boundary with $W_k$, or more generally with $n^{-1}_\alpha(k)$, and this boundary has multiplicity $k$. Ultimately we must both deal with the ‘bad’ subsets and show that there is actually no branching within $E_k$.

7 Reconstruction of a differential manifold

To obtain a clearer view of the chart domain $U_\alpha$, we require the unique continuation properties of (Euclidean) Dirac-type operators. The next subsection analyzes the local properties of the operator $\mathcal{D}$, and in particular shows that $\mathcal{D}$ is locally of Dirac type.

We continue to work within a fixed $U_\alpha$, taking closures and boundaries in the relative topology of $U_\alpha$. 

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7.1 Local structure of the operator $\mathcal{D}$

In order to analyze the operator $\mathcal{D}$ restricted to sections of the bundle $S$ over $W_{jk}$ (recall that $\mathcal{D}$ is local, by Corollary 4.14), we need to identify the smooth sections. The easiest way to accomplish this is to employ our $C^\infty$ functional calculus.

By construction, the vector bundle $E$ is trivialized by the sections $[\mathcal{D}, a_\alpha^p]$ over $W_{jk} \subset U_\alpha$. Any element of $C_\mathcal{D}(A)$ determines a local section of the bundle $\text{End}S|_{W_{jk}} \to W_{jk}$, which is likewise trivialized. Over $W_{jk}$ there is an (involutive) algebra subbundle $C_{jk}$ of $\text{End}S|_{W_{jk}}$ such that $T \in C_\mathcal{D}(A)$ if and only if $T|_{W_{jk}} \in \Gamma(W_{jk}, C_{jk})$. This bundle decomposes over $W_{jk}$ as a Whitney sum of trivial matrix bundles; compare Corollary 3.17:

$$C_{jk} \simeq \bigoplus_{r=1}^s W_{jk} \times M_n(\mathbb{C}).$$  \hspace{1cm} (7.1)

Now define local sections $e_r \in \Gamma(W_{jk}, \text{End} S)$ by $e_r(x) := 1_{n_r}$, for $r = 1, \ldots, s$ and $x \in W_{jk}$, which are the minimal central projectors in this decomposition of $C_{jk}$.

The bundle $S \to X$ is locally trivial, by the Serre–Swan theorem, although a priori the subsets $W_{jk}$ need not be trivializing chart domains for $S$. However, since $X$ is compact each $W_{jk}$ can be covered by finitely many such chart domains. In any case, we may write $S|_{W_{jk}} = \bigoplus e_r S|_{W_{jk}}$.

**Lemma 7.1.** For all $k \geq 1$ and $j = 1, \ldots, k$, the set $W_{jk}$ is a smooth manifold of dimension $p$, by Corollary 6.13, and $E_R|_{W_{jk}}$ is isomorphic to the cotangent bundle of $W_{jk}$. Moreover, the algebra of restrictions of elements of $\mathcal{A}$ to $W_{jk}$ is isomorphic to a subalgebra of $C_b^\infty(a_\alpha(W_{jk}))$.

**Proof.** Define a bundle morphism $\rho: T^*W_{jk} \to E_R|_{W_{jk}}$ by giving the corresponding map $\rho_\ast$ on sections:

$$\rho_\ast\left(\sum_{r=1}^p b_{ra} a_\alpha^r\right) := \sum_{r=1}^p b_{ra} [\mathcal{D}, a_\alpha^r].$$  \hspace{1cm} (7.2)

This is a well-defined bundle isomorphism since the two local bases of sections $\{da_\alpha^1, \ldots, da_\alpha^p\}$ and $\{[\mathcal{D}, a_\alpha^1], \ldots, [\mathcal{D}, a_\alpha^p]\}$ determine the trivial bundles $T^*W_{jk}$ and $E_R|_{W_{jk}}$, respectively.

It suffices to show that each $b \in \mathcal{A}$ can be written as a smooth bounded function of the $a_\alpha$ over $W_{jk}$. We already know that $b|_{W_{jk}} = f \circ a_\alpha$ for a unique bounded locally Lipschitz function $f$. From Corollary 4.9 we get $[\mathcal{D}, b] = \sum_i b_i [\mathcal{D}, a_\alpha^i]$, where each $b_i$ is continuous on $W_{jk}$ and $cb_i \in \mathcal{A}$ for $c \in \mathcal{A}$ compactly supported within $W_{jk}$. If $K \subset W_{jk}$ is compact, by taking $c = \phi_K \in \mathcal{A}$ with supp $\phi_K \subset W_{jk}$ and $\phi_K = 1$ on $K$, we get $b_i|_K = f_i|_K \circ a_\alpha$ from the Lipschitz functional calculus, and the uniqueness of these representatives shows that they agree on overlaps, yielding a single function representation $b_i = f_i \circ a_\alpha$ with $f_i$ a locally Lipschitz function on $a_\alpha(W_{jk})$.

Choosing a sequence of $C^1$ functions $g_k$ converging to $f$ in the Lipschitz norm on every compact subset of $a_\alpha(W_{jk})$, and using the uniqueness of the coefficients $b_i$, we see that $f_i = \partial_i f$. Thus $\partial_i f$ is bounded and locally Lipschitz for $i = 1, \ldots, p$, and so $f$ is in fact $C^1$. A straightforward induction now shows that $f$ is actually $C^\infty$. \hspace{1cm} $\square$

**Remark 7.2.** Since $\mathcal{H}_{\infty} \simeq q\mathcal{A}^m$ is a finitely generated projective module over $\mathcal{A}$, with $q \in M_n(\mathcal{A})$, it follows that over any open subset $U \subset W_{jk}$ for which $S|_U \to U$ is trivial, the local sections $\xi_i \in \Gamma(U, S)$ may be regarded as column vectors with entries $f_i \circ a_\alpha|_U$ for $i = 1, \ldots, N$, where $f_i \in C_b^\infty(a_\alpha(U))$. In other words, the sections in $\mathcal{H}_{\infty}$ have smooth coefficient functions over $W_{jk}$. 38
Lemma 7.3. For each $k \geq 1$ and $j = 1, \ldots, k$, the operator $\mathcal{D}$ is an elliptic first order differential operator on $\mathcal{H}_\infty|_{W_jk}$.

Proof. By Remark 7.2, we can regard $\mathcal{H}_\infty|_{W_jk}$ as a subspace $\Gamma_{\infty,\alpha}(W_{j'k}, S)$ of $\Gamma_{\infty}(W_{j'k}, S)$. In like manner, if $\Omega^1(W_{j'k})$ denotes the $\mathcal{A}$-module of smooth 1-forms on $W_{j'k}$, then

$$\Omega^1(W_{j'k}) \otimes \mathcal{A} \mathcal{H}_\infty|_{W_jk} \cong \Gamma_{\infty,\alpha}(W_{j'k}, T^*_C W_{j'k} \otimes S).$$

Choose any connection $\nabla: \Gamma_{\infty,\alpha}(W_{j'k}, S) \to \Gamma_{\infty,\alpha}(W_{j'k}, T^*_C W_{j'k} \otimes S)$, and define an $\mathcal{A}$-linear map $\hat{\mathcal{D}}: \Gamma_{\infty,\alpha}(W_{j'k}, T^*_C W_{j'k} \otimes S) \to \Gamma_{\infty,\alpha}(W_{j'k}, S)$ by setting $\hat{\mathcal{D}}(da \otimes \xi) := [\mathcal{D}, a] \xi$ for $a \in \mathcal{A}|_{W_jk}$ and $\xi \in \mathcal{H}_\infty|_{W_jk}$, extending by $\mathcal{A}$-linearity. Then, recalling that $\mathcal{D}$ is a local operator so that $\mathcal{D}$ maps $\mathcal{H}_\infty|_{W_jk}$ to $\mathcal{H}_\infty|_{W_jk}$, we get

$$(\mathcal{D} - \hat{\mathcal{D}} \circ \nabla)(a \xi) = [\mathcal{D}, a] \xi + a \mathcal{D} \xi - [\mathcal{D}, a] \xi - a(\hat{\mathcal{D}} \circ \nabla)\xi = a(\mathcal{D} - \hat{\mathcal{D}} \circ \nabla)\xi,$$

and thus $\mathcal{D} = \hat{\mathcal{D}} \circ \nabla + B$ with $B \in \Gamma_{\infty}(W_{j'k}, \text{End } S)$. Since $\hat{\mathcal{D}} \circ \nabla$ is a first-order differential operator, so too is $\mathcal{D}$. Let $\sigma_\mathcal{D}$ denote its principal symbol.

For ellipticity, suppose that $x \in W_{j'k}$ and $v \in T^*_C W_{j'k}$ are such that $\sigma_\mathcal{D}(x, v)$ is not invertible. Since $\mathcal{D}$ is of first order, it may be evaluated as $\sigma_\mathcal{D}(x, v) = [\mathcal{D}, a](x)$, for any $a = a^* \in \mathcal{A}$ such that $da(x) = v$. Since we assume the map $\sigma_\mathcal{D}(x, v) \in \text{End } S_x$ is not invertible, there is some $\xi \in \mathcal{H}_\infty$ with $\xi(x) \neq 0$ such that

$$0 = \sigma_\mathcal{D}(x, v)\xi(x) = \sum_{j=1}^p a_j(x) [\mathcal{D}, a^j_\alpha](x)\xi(x) = \sum_{j=1}^p a_j(x)\xi^j(x),$$

where we have set $\xi^j(x) := [\mathcal{D}, a^j_\alpha](x)\xi(x) := da^j_\alpha(x)\xi(x)$. Now if any $da^j_\alpha(x)$ had a zero eigenvector in $S_x$, then the representative $\Gamma_x'$ in $\text{End } S_x$ of $da^j_\alpha(x) \wedge \cdots \wedge da^p_\alpha(x) \in \Lambda^p E_x$ would have a zero eigenvector. (By Corollary 4.9, this multivector spans $\Lambda^p E_x$.) But in that case $\Gamma_x' = 0$, since $\dim \Lambda^p E_x = 1$, contrary to the proof of Proposition 4.4. Therefore, $da^j_\alpha(x)$, ..., $da^p_\alpha(x)$ are invertible in $\text{End } S_x$, as well as linearly independent. We conclude that the vectors $\xi^j(x) \in S_x$ are linearly independent, which forces $a_1(x) = \cdots = a_p(x) = 0$. This in turn entails $v = da(x) = 0$, since each $a_j(x) = da^j_\alpha(a_\alpha(x))$ when $a = a_\alpha(a(x))$. Hence $\mathcal{D}$ is elliptic.

Lemma 7.4. For all $a, b \in \mathcal{A}|_{W_jk}$, the operator

$$[[\mathcal{D}^2, a], b] = [\mathcal{D}, a] [\mathcal{D}, b] + [\mathcal{D}, b] [\mathcal{D}, a]$$

is a central element of the algebra $\mathcal{C}_\mathcal{D}(\mathcal{A})|_{W_jk}$.

Proof. We need only consider the special case $a = b$, in view of the polarization identity


The operator $[[\mathcal{D}^2, [\mathcal{D}, a], [\mathcal{D}, a]]$ is a differential operator of at most second order over $W_{j'k}$; it is of at most first order if and only if [3]:

$$[[[\mathcal{D}^2, [\mathcal{D}, a], [\mathcal{D}, a]], b], c] = 0, \quad \text{for all } b, c \in \mathcal{A}.$$
Again we simplify to \( b = c \); using the first order condition, we find that

\[
[[[\mathcal{D}^2], [\mathcal{D}, a], [\mathcal{D}, b]], b] = 2[[\mathcal{D}, b], [\mathcal{D}, b], [\mathcal{D}, a], [\mathcal{D}, a]].
\]

Thus \([\mathcal{D}^2], [\mathcal{D}, a], [\mathcal{D}, b]\) is of first order if and only if \([\mathcal{D}, a] [\mathcal{D}, a]\) commutes with all \([\mathcal{D}, b], [\mathcal{D}, b]\) in \(\mathcal{C}_\omega(\mathcal{A})|_{W_{jk}}\). Similarly, the operator \([\mathcal{D}^2], [\mathcal{D}, a]\) is first order if and only if \([\mathcal{D}, a]\) commutes with all \([\mathcal{D}, b], [\mathcal{D}, b]\) in \(\mathcal{C}_\omega(\mathcal{A})|_{W_{jk}}\). To show that \([\mathcal{D}, b] [\mathcal{D}, b]\) is central, it therefore suffices to show that for all \(a \in \mathcal{A}\) the operators \([\mathcal{D}^2], [\mathcal{D}, a], [\mathcal{D}, a]\) and \([\mathcal{D}^2], [\mathcal{D}, a]\) are of first order at most.

For \(T = [\mathcal{D}, a] [\mathcal{D}, a]\) or \([\mathcal{D}, a]\), the operator \(\langle D \rangle^{-1}[\mathcal{D}^2, T]\), regarded now as a pseudodifferential operator on \(W_{jk}\), has order \((-1 + \text{order}[\mathcal{D}^2, T])\). The regularity condition entails that in both cases, the operator \(\langle D \rangle^{-1}[\mathcal{D}^2, T]\) is bounded (see, e.g., [11]) and thus of order at most zero. We conclude that \([\mathcal{D}^2, T]\) is a differential operator at order at most one. Hence the commutators \([\mathcal{D}, b] [\mathcal{D}, b], [\mathcal{D}, a], [\mathcal{D}, a]\]|_{W_{jk}} vanish for all \(a, b \in \mathcal{A}\). \(\square\)

**Lemma 7.5.** For all \(a, b \in \mathcal{A}\) with compact support in \(W_{jk}\), the densely defined operator \([\mathcal{D}, [\mathcal{D}, a] [\mathcal{D}, b] + [\mathcal{D}, b] [\mathcal{D}, a]]\) extends to a bounded operator on \(\mathcal{H}\).

**Proof.** The operator in question maps \(\mathcal{H}_\infty\) to itself, so by Proposition 3.18 we need only show that it is \(\mathcal{A}\)-linear on this domain. This follows from the first order condition together with the centrality of \([\mathcal{D}, a] [\mathcal{D}, b] + [\mathcal{D}, b] [\mathcal{D}, a]\) in \(\mathcal{C}_\omega(\mathcal{A})|_{W_{jk}}\); for if \(c \in \mathcal{A}\), then

\[
\]

since we may assume without loss of generality that \(\text{supp} c \subset W_{jk}\) also. \(\square\)

**Remark 7.6.** Though we do not need it here, it is interesting and perhaps useful that for all \(a, b \in \mathcal{A}\), the operator \([[[\mathcal{D}^2], a], [\mathcal{D}, b]]\) is bounded over \(W_{jk}\). This is proved just as in the previous Lemma. When we later globalize these results, we shall likewise see that this operator is globally bounded.

**Proposition 7.7.** The space \(\mathcal{H}_\infty|_{W_{jk}}\) carries a nondegenerate representation of \(\Gamma_\infty(W_{jk}, C)\), the algebra of smooth sections of an algebra bundle \(C = \bigoplus_{r=1}^{2} \mathbb{C} \ell(T^r W_{jk}, g_r)\), which is the Whitney sum of complex Clifford-algebra bundles defined by finitely many Euclidean metrics \(g_r\) on \(T^r W_{jk}\).

**Proof.** The bundle isomorphism \(\rho: T^r W_{jk} \to E_R|_{W_{jk}}\) determined by (7.2) extends to an isomorphism \(\Lambda^* \rho: \Lambda^* T^r W_{jk} \to \Lambda^* E_R|_{W_{jk}}\). The map (7.2) and the action of \(E|_{W_{jk}}\) on \(\mathcal{H}_\infty|_{W_{jk}}\) together determine an action, also called \(\rho_*\), of the local 1-forms \(\Omega^1(W_{jk})\) on \(\mathcal{H}_\infty|_{W_{jk}}\). If \(\eta = \sum_l b_{l\alpha} da^\alpha_l\) and \(\zeta = \sum_m c_m da^m_a\) are two such 1-forms, then

\[
\rho_*(\eta) \rho_*(\zeta) + \rho_*(\zeta) \rho_*(\eta) = \sum_{l, m} b_{l\alpha} c_m ([\mathcal{D}, a^l_\alpha] [\mathcal{D}, a^m_m] + [\mathcal{D}, a^m_m] [\mathcal{D}, a^l_\alpha])
\]

is central in \(\mathcal{C}_\omega(\mathcal{A})|_{W_{jk}}\), and has bounded commutator with \(\mathcal{D}\) over \(W_{jk}\).

As was noted after (7.1), the algebra \(\mathcal{C}_\omega(\mathcal{A})|_{W_{jk}}\) has minimal central projectors \(e_1, \ldots, e_s\); for all \(a, b \in \mathcal{A}\) and each \(r = 1, \ldots, s\), the maps \(a db \mapsto e_r a [\mathcal{D}, b] e_r = a [\mathcal{D}, b] e_r\) make sense over \(W_{jk}\).

Since \([\mathcal{D}, a] [\mathcal{D}, b] + [\mathcal{D}, b] [\mathcal{D}, a]\) is central in \(\mathcal{C}_\omega(\mathcal{A})|_{W_{jk}}\), over \(W_{jk}\) it decomposes as a sum of scalar matrices:

\[
[[\mathcal{D}, a] [\mathcal{D}, b] + [\mathcal{D}, b] [\mathcal{D}, a]] = \bigoplus_{r=1}^{s} -2g_{r, a}(da, db) 1_{n_r}, \tag{7.3}
\]

where each \(g_{r, a}\) is a symmetric bilinear form on \(\Omega^1(W_{jk})\) with values in \(\mathcal{A}|_{W_{jk}}\).
Note that, for a real, the $g_{r,a}(da, da)$ are nonnegative since the operator $[\mathcal{D}, a]^* [\mathcal{D}, a]$ is positive; and at each $x \in W_{jk}$ the matrix with entries $g_{r,a}(da^i_a, da^k_a)(x)$ is positive definite: compare the discussion after (4.7). Thus, each $g_{r,a}$ is a positive definite Euclidean metric on 1-forms. For each $r$, the map $\rho_r$ defines an action of the Clifford-algebra bundle $\mathbb{C}\ell(T^* W_{jk}, g_{r,a})$.

**Corollary 7.8.** Over the set $W_{jk}$, the operator $\mathcal{D}$ is, up to the addition of an endomorphism of $S|W_{jk}$, a direct sum of Dirac-type operators with respect to the several metrics $g_1, \ldots, g_s$.

**Proof.** The symbol of $\mathcal{D}$ at $(x, v) \in T^* W_{jk}$ is given by $[\mathcal{D}, f](x)$ for any smooth function $f$ such that $df(x) = v$. Also, (7.3) says that $[\mathcal{D}, f]$ is Clifford multiplication by $df$ in each summand of $C_{jk}$. □

### 7.2 Injectivity of the local coordinates

In this section we shall show that the sets $C_k, B_k$ for $k \geq 1$, and $n^{-1}_a(\infty)$ are all empty. For that, we shall use the weak unique continuation property of Dirac-type operators [5], as well as the strong unique continuation property [36]. Both unique continuation properties have long histories, for which we refer the reader to the papers cited. The precise statements we require are as follows.

**Theorem 7.9** (Weak Unique Continuation Property [5, Thm. 2.1]). Let $\mathcal{D}$ be an operator of Dirac type acting on sections of a vector bundle $V$ over a smooth manifold $M$ (that need not be compact), and suppose that $\mathcal{D}\xi = 0$ for some $\xi \in \Gamma(M, V)$. If $\xi$ is zero on an open set $U \subset M$, then $\xi$ vanishes on the whole connected component containing the open set $U$.

**Remark 7.10.** Actually, in [5] the weak unique continuation property is shown to be stable under a large family of possibly nonlinear perturbations of order zero. This allows us to employ unique continuation for nonzero eigenvalues $\lambda$, on replacing $\mathcal{D}$ by $\mathcal{D} - \lambda$.

**Theorem 7.11** (Strong Unique Continuation Property [36, Cor. 2]). Let $U$ be a connected open subset of $\mathbb{R}^p$, with $p \geq 3$, and let $D = \sum_{j=1}^{K} y_j \partial / \partial x^j$ be the constant-coefficient Dirac operator on $U$. Let $V \in L^r(U, M_m(\mathbb{C}))$ where $r = (3p - 2)/2$, and suppose that $(D + V)\xi = 0$ for some $\xi \in L^2(U, \mathbb{C}^m)$ with $D\xi \in L^2(U, \mathbb{C}^m)$. If $\int_{|x - x_0| < \varepsilon} |\xi(x)|^2 d^p x = O(\varepsilon^N)$ for all $N$, for some $x_0 \in U$, then $\xi$ is identically zero on $U$.

**Remark 7.12.** The case $p = 2$ was proved by Carleman [13] for $V \in L^\infty(U)$. The corresponding result for $p = 1$ and $V \in L^1(U)$ can be proved directly from the explicit solution for $(D + V)\xi = 0$. The various integrability constraints on $V$ will not concern us, as we will be working with $V \in L^\infty(U)$, where $U$ is a bounded open set in $\mathbb{R}^p$.

A basic result we require is the smoothness of eigenspinors of the Euclidean Dirac operator, a consequence of its ellipticity only: see, for example, [39, Thm. III.5.4].

**Lemma 7.13.** Let $U \subset \mathbb{R}^p$ be open and bounded, let $\widehat{S} \to U$ be its spinor bundle, and let $\widehat{\mathcal{D}} = \sum c(dx^j) \partial_j + \widehat{V}$ denote the constant coefficient Dirac operator on $U$ perturbed by $\widehat{V} \in L^\infty(U, \text{End} \, \widehat{S})$. If $s \in L^2(U, \widehat{S})$ satisfies $\widehat{D}s = \lambda s$ with $\lambda \in \mathbb{R}$, then $s$ is a smooth section. □

**Lemma 7.14.** Let $V, W \subset U_a$ be open subsets such that $a_{\alpha}$ is injective over $\overline{V}$ and $\overline{W}$ separately. If $a \in \mathcal{A}$ satisfies $a|_V = f \circ a_{\alpha}$ and $a|_W = g \circ a_{\alpha}$ for smooth functions $f$ and $g$, then for each multi-index $K = (k_1, \ldots, k_p)$, the partial derivatives $\partial^K f$ and $\partial^K g$ extend continuously to $a_{\alpha}(\overline{V})$ and
among these Propositions which follow. The techniques and notation in the next proof will be reused when we represented as $A$

**Proof.** Since $a_k \in \mathcal{A}$, the proofs of Proposition 6.11 and Lemma 7.1 show that $\partial_k f$ extends to a continuous, bounded and locally Lipschitz function on $a_\alpha(\overline{V})$. Hence the extension of $f$ is actually $C^1$ on $a_\alpha(\overline{V})$. Replacing $a = f \circ a_\alpha$ by $a_k = \partial_k f \circ a_\alpha$, and the $a_k$ by the coefficients $a_{kl}$ in the expansion of $[\mathcal{D}, a_k]$, and so on, we see that all the $\partial^K f$ extend in like manner to $\overline{V}$, and all these extensions are $C^N$ there. Similar comments apply to the $\partial^K g$.

Now an application of Proposition 6.11 at each stage of the iteration yields $\partial^K f(a_\alpha(x)) = \partial^K g(a_\alpha(x))$ for any $x \in \overline{V} \cap W$.

**Lemma 7.15.** Let $V, W \subset U_\alpha$ be open subsets such that $a_\alpha$ is injective over $\overline{V}$ and $\overline{W}$ separately. Let $U \subset \subset U_\alpha$ be an open subset over which $S\mid_U$ is trivial. For any $\xi \in \mathcal{H}_\infty$, we can simultaneously write
\[
\xi\mid_{\overline{V} \cap U} = (f_1 \circ a_\alpha, \ldots, f_N \circ a_\alpha), \quad \xi\mid_{\overline{W} \cap U} = (g_1 \circ a_\alpha, \ldots, g_N \circ a_\alpha),
\]
for $f_j, g_j \in C^\infty(a_\alpha(U))$, $j = 1, \ldots, N$. Then, at any $x \in \overline{V} \cap \overline{W} \cap U$, these component functions (and their derivatives) agree:
\[
\partial^K f_j(a_\alpha(x)) = \partial^K g_j(a_\alpha(x)), \quad \text{for all } K \text{ and } j = 1, \ldots, N. \tag{7.4}
\]

**Proof.** Since $S\mid_U$ is trivial, we may identify elements of $\mathcal{H}_\infty\mid_U \subset \Gamma_\infty(U, S)$ with column vectors in $\mathcal{A}^N\mid_U$. The restrictions of their component functions to $\overline{V} \cap U$ and $\overline{W} \cap U$ respectively can in turn be represented as $f_j \circ a_\alpha$, respectively $g_j \circ a_\alpha$, using the Lipschitz functional calculus. The uniqueness of the locally Lipschitz functions thereby obtained, together with Lemma 7.14, yields (7.4).

We now begin the process of tidying up $U_\alpha$ by showing that the subsets $C_k$ are empty, in the two Propositions which follow. The techniques and notation in the next proof will be reused when we banish the remaining undesirable subsets.

**Proposition 7.16.** For each $\alpha$ and all $m \geq 1$, the equality $\text{Int} \overline{W}_m = W_m$ holds.

**Proof.** Suppose $W_m$ is not empty, and that $\text{Int} \overline{W}_m \neq W_m$. Then all $x \in \partial W_m$ which lie in the interior of $\overline{W}_m$ have multiplicity $n_\alpha(x) < m$ by Lemma 6.18.

In particular, the result is proved for $m = 1$, and we can now assume that $m \geq 2$. Thus there are at least two ‘sheets’ $W_{mi}$ and $W_{mj}$, for some $i, j \in \{1, \ldots, m\}$, for which some $x \in \overline{W}_m \setminus W_m$ is a common boundary point within $U_\alpha$.

Fix such an $x \in \overline{W}_m \setminus W_m$ and choose an open neighbourhood $U$ of $x$ such that $S\mid_U$ is trivial. Then we may, and shall, regard sections of $S\mid_U$ as functions $\xi: U \to \mathbb{R}^N$.

We choose, once and for all, a numbering of the ‘sheets’ $W_{mj}$ of $W_m$. For $i, j = 1, \ldots, m$, define switching maps $\varphi_{ij}: W_{mj} \to W_{mi}$ by $\varphi_{ij}(x) := a_{\alpha,1}^{-1}(a_\alpha(x)) \cap W_{mi}$; these maps are homeomorphisms among these $m$ subsets of $W_m$, and they permute the $m$-element sets $a_{\alpha,1}^{-1}(a_\alpha(x))$ for each $x$. Write $a_{\alpha,1}^{-1}$ for the homeomorphism $a_\alpha(W_m) \to W_{mj}$ inverse to $a_\alpha$, and note that
\[
\varphi_{ij} \circ a_{\alpha,1}^{-1} = a_{\alpha,1}^{-1}, \quad \text{for } i, j = 1, \ldots, m.
\]
Now consider the bundle \( \hat{S} \to a_\alpha(W_m) \) of rank \( N \), obtained by pulling back \( S \to \overline{W}_m \) via the map \( a_{a,1}^{-1} : a_\alpha(W_m) \to \overline{W}_m \). Here we are using Proposition 6.11. Observe that since \( S|_{U \cap \overline{W}_m} \) is trivial, so too is \( \hat{S}|_{a_\alpha(U \cap \overline{W}_m)} \). Thereby, sections of \( \hat{S}|_{a_\alpha(U \cap \overline{W}_m)} \) are \( N \)-tuples of functions over \( a_{a,1}(U) = a_\alpha(U) = a_{a,j}(U) \).

Given a section \( \xi \in \Gamma(S|_{U \cap \overline{W}_m}) \), the two pullbacks \( \xi \circ a_{a,d}^{-1} \) and \( \xi \circ a_{a,j}^{-1} \) are different sections of \( \hat{S} \) corresponding to the restrictions of \( \xi \) to \( \overline{W}_m \) and \( \overline{W}_m \) respectively. By Lemma 7.15, these two pullback sections agree at points of \( \overline{W}_m \setminus W_m \).

We define an operator \( \hat{D} \) on smooth sections of \( \hat{S}|_{a_\alpha(U \cap \overline{W}_m)} \) by

\[
\hat{D}\xi := (\hat{D}(\xi \circ a_{a,1}|_{\overline{W}_m})) \circ a_{a,1}^{-1}, \quad \text{for} \quad \xi \in \Gamma_\infty(a_\alpha(\overline{W}_m), \hat{S}).
\]

The operator \( \hat{D} \) is well defined, and gives a first order differential operator on \( \Gamma_\infty(a_\alpha(\overline{W}_m), \hat{S}) \). Indeed, by Lemma 7.3,

\[
\hat{D}\xi = \left( \sum_{j=1}^{p} [\hat{D}, a_{a,1}^j] (\partial_j \xi) \circ a_\alpha + V(\xi \circ a_a) \right) \circ a_{a,1}^{-1}
\]

\[
= \sum_{j=1}^{p} (a_{a,1}^{-1})^* c(da_{a,1}^j) \partial_j \xi + ((a_{a,1}^{-1})^* V) \xi,
\]

where \( c \) denotes Clifford multiplication coming from Proposition 7.7 and \( V \in \Gamma(\overline{W}_m, \text{End} S) \) is smooth, because \( V \) maps \( \mathcal{H}_\infty \) to \( \mathcal{H}_\infty \) and is uniformly bounded. That last statement holds because \( V \) is \( \mathcal{A} \)-linear, and the proof of Proposition 3.18 shows that the norm of \( V \) over \( \overline{W}_m \) is determined on a finite generating set of \( \mathcal{H}_\infty \). Thus \( \hat{D} \) is a bounded perturbation of (a possible direct sum of copies of) a constant coefficient Dirac operator on an open subset of \( \mathbb{R}^p \).

Take two sheets \( W_m, W_m \) as above, with \( x \in \overline{W}_m \cap \overline{W}_m \). Observe that \( a_\alpha(W_m) \) is open, by Lemma 6.12 and Corollary 5.8, and that Proposition 6.11 shows that \( a_\alpha(\text{Int} \overline{W}_m) \) is also open. Similarly, the set \( a_\alpha(U \cap \text{Int} \overline{W}_m) \) is open.

Next, let \( \xi \in \mathcal{H}_\infty \) be any eigenvector of \( \mathcal{D} \), with eigenvalue \( \lambda \) say, and define a section over the open set \( a_\alpha(U \cap \text{Int} \overline{W}_m) \) by

\[
\psi_{ij\lambda}(t) := \begin{cases} 0, & \text{if } t \in a_\alpha(U \cap \text{Int} \overline{W}_m \setminus W_m), \\
\xi \circ a_{a,j}^{-1}(t) - \xi \circ a_{a,j}^{-1}(t), & \text{if } t \in a_\alpha(U \cap \text{Int} \overline{W}_m). 
\end{cases}
\]

This section is well defined because at any boundary point \( x \in \partial \overline{W}_m \cap \partial \overline{W}_m \) the two local sections \( \xi \circ a_{a,j}^{-1} \) and \( \xi \circ a_{a,j}^{-1} \) agree, by Lemma 7.15. Since \( D\xi = \lambda \xi \), it is clear that \( \hat{D}\psi_{ij\lambda} = \lambda \psi_{ij\lambda} \) on the given domain. Thus by Lemma 7.13, \( \psi_{ij\lambda} \) is a smooth section of \( \hat{S} \) over \( a_\alpha(U \cap \text{Int} \overline{W}_m) \).

Consider the Taylor expansion of the smooth function \( h_{ij\lambda} := (\psi_{ij\lambda} \mid \psi_{ij\lambda}) \) about some point \( t = a_\alpha(x) \in a_\alpha(U \cap \partial \overline{W}_m \cap \partial \overline{W}_m) \). According to Lemma 7.14, it satisfies

\[
h_{ij\lambda}(s) = \sum_{|J|=0}^N \frac{\partial^J h_{ij\lambda}(t)}{J!} (s-t)^J + R_{N+1}(t, s) = R_{N+1}(t, s),
\]
since the leading terms vanish up to the prescribed order \( N \). Since \( h_{ij\lambda} \) can be taken to have bounded derivatives of order \( N + 1 \), the remainder can be estimated by \( |R_{N+1}(t, s)| \leq C |s - t|^{N+1} \). Thus, on a ball of radius \( \varepsilon \) about \( t = a_\alpha(x) \), the function \( h_{ij\lambda} = (\psi_{ij\lambda} | \psi_{ij\lambda}) \) is bounded by \( C\varepsilon^{N+1} \) for any \( N \).

Hence for each such \( \varepsilon \), we get an estimate

\[
\int_{B(\varepsilon)} (\psi_{ij\lambda} | \psi_{ij\lambda}) \, dt = O(\varepsilon^k)
\]

for every \( k \in \mathbb{N} \). Since we know that \((\widehat{\mathcal{D}} - \lambda)\psi_{ij\lambda} = 0\), and that \(\widehat{\mathcal{D}} - \lambda \) is a bounded perturbation of the constant-coefficient Dirac operator, we may apply the main result of [36], cited here as Theorem 7.11, to deduce that \(\psi_{ij\lambda} \) vanishes on a neighbourhood of \( a_\alpha(x) \) in \( a_\alpha(U \cap \text{Int} \widehat{W}_m) \).

Hence the restrictions of each eigenvector \( \xi \) of \( \mathcal{D} \) to the two sheets \( W_{mi} \) and \( W_{mj} \) yield equal sections of \( \widehat{S} \), namely \( \xi \circ a_\alpha^{-1} = (f_1, \ldots, f_N) \) and \( \xi \circ a_\alpha^{-1} = (g_1, \ldots, g_N) \), over the open set \( a_\alpha(U \cap \text{Int} \widehat{W}_m) \). Thus each \( f_r = g_r \) on this set, and so the sections

\[
\xi|_{W_{mi}} = (f_1, \ldots, f_N) \circ a_\alpha|_{W_{mi}}, \quad \xi|_{W_{mj}} = (g_1, \ldots, g_N) \circ a_\alpha|_{W_{mj}}
\]

satisfy \( \xi|_{W_{mi}} = \xi|_{W_{mj}} \circ \varphi_{ij} \) over \( a_\alpha(U \cap \text{Int} \widehat{W}_m) \). Hence, for any two eigenvectors \( \xi, \eta \) of \( \mathcal{D} \), the function \( (\xi \mid \eta) \) is constant on the fibres \( a_\alpha^{-1}(t) \cap U \cap (W_{mi} \cup W_{mj}) \) for all \( t \in a_\alpha(U \cap W_m) \).

Since the eigenvectors of \( \mathcal{D} \) span \( \mathcal{H} \) and are contained in \( \mathcal{H}_\infty \), and since \( \mathcal{H}_\infty \) is a full right \( \mathcal{A} \)-module, the algebra \( \mathcal{A} \) is densely generated by functions of the form \( (\xi \mid \eta) \). Hence, no function in \( \mathcal{A} \) can distinguish points in \( a_\alpha^{-1}(t) \cap U \cap (W_{mi} \cup W_{mj}) \). Since \( m \geq 2 \) and we have assumed that \( W_m \) is nonempty, we have reached a contradiction \( \Box \).

Next we tackle the other possible “branch points” in \( E_k \). We shall assume, without loss of generality, that \( W_m \neq \emptyset \) for each \( m \geq 1 \). If some \( W_k \) is actually empty, we may omit it and renumber these subsets without affecting the arguments below.

**Proposition 7.17.** For each \( \alpha \) and all \( k, l = 1, \ldots, m \) with \( k \neq l \), the sets \( \overline{W}_k \cap E_m \) and \( \overline{W}_l \cap E_m \) are disjoint. Hence \( E_m \) is the disjoint union of its subsets \( W_j \), that is,

\[
E_m = W_1 \cup \cdots \cup W_m.
\]

**Proof.** We proceed by induction. First of all, \( E_1 = \text{Int}(n_\alpha^{-1}(1)) = W_1 \). Suppose then that \( E_k = \biguplus_{j=1}^{k} W_j \) for \( k < m \). Using Proposition 7.16 and Lemma 6.19, we obtain

\[
C_m = \bigcup_{k=1}^{m-1} \partial W_k \cap \partial W_m \cap E_m,
\]

and \( \bigcup_{k=1}^{m-1} \partial W_k \cap \partial W_m \cap E_m \subseteq \text{Int}(\bigcup_{k=1}^{m-1} \overline{W}_k \cap \overline{W}_m \cap E_m) \). It follows that

\[
\partial \overline{W}_m \cap E_m = \partial W_m \cap E_m \subseteq \bigcup_{j=1}^{m-1}(\partial W_j \cap E_m).
\]

(Notice that \( \partial \overline{W}_m = \overline{W}_m \setminus \text{Int} \overline{W}_m = \overline{W}_m \setminus W_m = \partial W_m \) on account of Proposition 7.16.)
Since \( E_{m-1} = \bigcup_{j=1}^{m-1} W_j \) by the inductive hypothesis, we obtain \( E_{m-1} \cap \overline{W}_j \cap \overline{W}_k = \emptyset \) for \( j \neq k \in \{1, \ldots, m-1\} \). Thus,

\[
\partial E_{m-1} \cap E_m \subseteq \partial W_m \cap E_m \subseteq \bigcup_{j=1}^{m-1} \partial W_j \cap E_m.
\]

We require an open image of an open neighbourhood of \( x \in C_m \). Such an \( x \) lies in \( \text{Int}(\bigcup_{r=1}^{l} \overline{W}_{j_r}) \) for some indices \( j_1, \ldots, j_l \). We shall show that \( a_\alpha(\text{Int}(\bigcup_{r=1}^{l} (W_{j_r} \cup W_m) \cap E_m)) \) is open in \( \mathbb{R}^p \). Since we deal with finite unions, we may suppose here that actually \( \bigcup_{r=1}^{l} \overline{W}_{j_r} = \overline{W}_k \) for a single \( k \).

The reader may replace \( \overline{W}_k \) by \( W_{kl} \) below by \( \bigcup_{r=1}^{l} \overline{W}_{j_r} \) and \( \bigcup_{r=1}^{l} \overline{W}_{j_r,i_r} \) and check that the result still holds for these more general unions.

Within \( \text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m \), choose a sheet \( W_{kl} \) and a sheet \( W_{mn} \). Then \( a_\alpha(\text{Int}(\bigcup_{r=1}^{l} W_{j_r} \cup W_m) \cap E_m) \) is a homeomorphism onto its image for the topology of \( \mathbb{R}^p \). This homeomorphism extends to the closure in \( U_\alpha \), and \textit{a fortiori} to the closure in \( E_m \). Thus \( a_\alpha : E_m \cap (\overline{W}_k \cup \overline{W}_m) \to a_\alpha(\text{Int}(\bigcup_{r=1}^{l} W_{j_r} \cup W_m) \cap E_m) \) is a homeomorphism.

Next choose any other pair of sheets \( W_{kl'} \) and \( W_{mn'} \) within \( \text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m \). Then, taking closures in \( E_m \), we get

\[
a_\alpha(\overline{W}_k \cup \overline{W}_m) = a_\alpha(W_{kl} \cup W_{mn}) = a_\alpha(W_{kl'} \cup W_{mn'}) = a_\alpha(W_{kl'} \cup \overline{W}_{mn'}),
\]
on account of Lemma 6.12. In fine, the image \( a_\alpha((\overline{W}_k \cup \overline{W}_m) \cap E_m) \) does not depend on the choices of sheets in \( \overline{W}_k \) and \( W_m \).

Consequently, the set \( a_\alpha(\text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m) \) is open and is independent of the choice of sheets. Since each pair of sheets yields the same image, we see that \( a_\alpha(\text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m) \) is open.

Now let \( x \in \partial W_k \cap \partial W_m \cap E_m \). Choose a neighbourhood \( U \) of \( x \) with \( U \subseteq \text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m \) such that \( S|_U \) is trivial. As in Proposition 7.16, we choose sheets \( W_{mi} \), \( W_{mj} \) of \( W_m \) such that \( x \in \partial W_k \cap \partial W_{mi} \cap \partial W_{mj} \). This is possible since \( m > 1 \). [In the case of a more general union \( \bigcup_{r=1}^{l} \overline{W}_{j_r} \), each sheet \( W_{mi} \) of \( W_m \) with \( x \in \partial W_{mi} \cap \bigcap_{r=1}^{l} \partial W_{j_r} \) has \( x \in \partial W_{mi} \cap \partial W_{j_r,i_r} \) for all sheets \( W_{j_r,i_r} \) of \( W_{j_r} \) with \( x \in \partial W_{j_r,i_r} \). Hence we can find two sheets of \( W_m \) meeting all such sheets of \( W_{j_r} \). This happens because \( n_\alpha(x) \leq j_1 < \cdots < j_r < m \).]

We pull back the bundle \( S|_U \) to a bundle \( \overline{S} \) over \( a_\alpha(\text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m \cap U) \). Let \( \xi \in \mathcal{H}_\infty \) any eigenvector of \( \overline{D} \) of eigenvalue \( \lambda \) and define a section of \( \overline{S} \) by the formula

\[
\psi_{ij,\lambda}(t) := \begin{cases} 0, & \text{if } t \in a_\alpha(\overline{W}_k \cap E_m \cap U), \\ \xi \circ a_{ij,\lambda}^{-1}(t) - \xi \circ a_{ij,\lambda}(t), & \text{if } t \in a_\alpha(W_{mi} \cap U). \end{cases}
\]

Just as in Proposition 7.16, this section is well defined since at any point of \( (\overline{W}_k \cup \overline{W}_m) \cap E_m \cap U \), the local sections \( \xi \circ a_{ij,\lambda}^{-1} \) and \( \xi \circ a_{ij,\lambda}^{-1} \) agree, and \( (\overline{D} - \lambda)\psi_{ij,\lambda} = 0 \).

Now \( \psi_{ij,\lambda} \) vanishes on the set \( a_\alpha(\text{Int}(\overline{W}_k \cup E_m \cup U)) \) which has nonempty interior, so the weak unique continuation property for Dirac operators, Theorem 7.9, says that either \( \psi_{ij,\lambda} \) is identically zero on \( a_\alpha(\text{Int}(\overline{W}_k \cup \overline{W}_m) \cap E_m \cap U), \) or \( a_\alpha(\overline{W}_k \cap E_m \cap U) \) is disconnected from \( a_\alpha(W_{mi} \cap U) \).

If there is at least one eigenvector \( \xi \) such that this \( \psi_{ij,\lambda} \) is not identically zero, then there is no boundary, and we are done for this possible intersection (i.e., \( \overline{W}_k \cap E_m = W_k \cap E_m \)). Otherwise,
for all eigenvectors $\xi$, the corresponding $\psi_{ij,\lambda}$ vanishes identically. Therefore, for all such $\xi$, the equality $\xi|_{W_m} = \xi|_{W_m} \cdot \varphi_{ij}$ holds and moreover, given two eigenvectors $\xi, \eta$, the function $(\xi | \eta)$ cannot distinguish points of the fibre $a_\alpha^{-1}(t)$ for any $t \in a_\alpha(W_m \cap U)$.

Again, since $\mathcal{A}$ is generated by such functions and they separate the points of $X$, we have reached a contradiction.

Hence, $\partial W_k \cap \partial W_m \cap E_m = \emptyset$. Repeating the argument for any other $\partial W_j \cap \partial W_m \cap E_m$, or more generally for $\bigcap_{r=1}^l \partial W_{j_r} \cap \partial W_m \cap E_m$, completes the inductive step. The conclusion follows. \qed

At this point, if the multiplicity is bounded, $U_\alpha$ is the disjoint union of finitely many $W_k$.

**Proposition 7.18.** The equality $n_\alpha^{-1}(\infty) \cup B_{(\alpha)} = B_1$ holds.

**Proof.** Let $x \in n_\alpha^{-1}(\infty) \cup B_{(\alpha)}$. Then there is a sequence $\{x_m\}_{m \geq 1} \subset U_\alpha$ such that $x_m \to x$ and $n_\alpha(x_m) = m \to \infty$. Here we suppose, without loss of generality, that each $W_m$ is nonempty.

Fix, for now, any compact set $K \subset U_\alpha$ such that $x \in K$ and $x_m \in K$ for each $m$. Put

$\epsilon_m := |a_\alpha(x_m) - a_\alpha(x)| \to 0,$

$e_m := \max\{d(x_m, x_m') : x_m' \in a_\alpha^{-1}(a_\alpha(x_m)) \cap K\}$.

Choose $x_m' \in a_\alpha^{-1}(a_\alpha(x_m)) \cap K$ such that $d(x_m, x_m') = e_m$. Then

$e_m = d(x_m, x_m') \leq d(x_m, x) + d(x_m', x) \leq C_{Y_m} \epsilon_m + C_{Y_m'} \epsilon_m$,

where the last estimate comes from Corollary 5.6 – note that (eventually) $a_\alpha(x_m) \neq a_\alpha(x)$ since $n_\alpha(x_m) \neq n_\alpha(x)$ – with $Y_m = \{x_m, x\}$ and $Y_m' = \{x_m', x\}$. Then, after perhaps passing to a subsequence, we are faced with two possibilities.

1. There is some $\delta > 0$ such that $e_m \geq \delta$ for all $m$. Then, since $e_m/\epsilon_m \leq C_{Y_m} + C_{Y_m'}$, we see that $C_{Y_m} + C_{Y_m'} \to \infty$. However, this cannot happen since $C_{Y_m} + C_{Y_m'}$ is bounded by $2C_K$.

2. Otherwise, $e_m \to 0$. In this case, for each $m$ we choose any $x_m'' \in a_\alpha^{-1}(a_\alpha(x_m)) \cap K$. Then the sequence $\{x_m''\}$ converges to $x$ since

$d(x_m'', x) \leq d(x_m', x_m) + d(x_m, x) \leq e_m + C_{Y_m} \epsilon_m \leq e_m + C_K \epsilon_m \to 0$.

Thus, any preimage of $\{a_\alpha(x_m)\}_{m \geq 1}$ in $K$ converges to $x$.

Continuing with the second case, observe now that $\{x_m\}_{m \geq 1}$ determines a subset $U_{I,\alpha} \subset U_\alpha$ as follows. First fix a numbering $W_{k1}, \ldots, W_{kk}$ of the $k$ sheets of each $W_k$. For each $k \geq 1$, choose $i(k) \in \{1, \ldots, k\}$, write $I := \{i(k)\}_{k \geq 1}$ and put

$U_{I,\alpha} := \bigcup_{k \geq 1} W_{k,i(k)}$.

The set $U_{I,\alpha}$ determined by $\{x_m\}_{m \geq 1}$ is given by taking, for each $m$, the sheet of $W_m$ in which the point $x_m$ lies.

Then $a_\alpha : U_{I,\alpha} \to \mathbb{R}^p$ is one-to-one and open and is a homeomorphism onto its image. This image is the same for each $U_{I,\alpha}$, by Lemma 6.12.
Each \( y \in n_{a}^{-1}(\infty) \cup B(a) \) is the limit of a sequence of unbounded multiplicity, and thus is contained in the boundary of some \( U_{j,a} \). In particular, let \( y \in a_{a}^{-1}(a_{a}(x)) \), and suppose that \( y \in \partial U_{j,a} \). Then the sequence \( \{y_{m}\}_{m \geq 1} := a_{a}^{-1}(a_{a}(\{x_{m}\}_{m \geq 1})) \cap U_{j,a} \) converges to \( y \).

Define a new compact subset \( K' \) of \( U_{a} \) by \( K' := K \cup \{y\} \cup \{y_{m}\}_{m \geq 1} \). On running our initial argument again, and observing that \( \{y_{m}\}_{m \geq 1} \) is a preimage of \( \{a_{a}(x_{m})\}_{m \geq 1} \) contained in \( K' \), we deduce that \( y_{m} \to x \). Therefore, \( y = x \) since \( K' \) is a Hausdorff space.

Hence, the point \( x \in n_{a}^{-1}(\infty) \cup B(a) \) is such that \( a_{a}(x) \) has only one preimage in \( U_{a} \), so that \( x \in B_{1} \).

It is ironic that our final task is to remove a (possible) set of points of multiplicity one.

**Proposition 7.19.** The set \( B_{1} \) is empty. Hence

\[
U_{a} = \bigcup_{k=1}^{\infty} W_{k} = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{k} W_{k,j}.
\]

*Proof.* The previous Proposition shows that

\[
U_{a} = W_{1} \cup B_{1} \cup \bigcup_{k=2}^{\infty} W_{k}.
\]

The set \( W_{1} \cup B_{1} = D_{1} \) is closed and its boundary is \( B_{1} \). Thus elements of \( B_{1} \) are limits of sequences of multiplicity 1 and simultaneously limits of sequences of unbounded multiplicity, other possibilities being excluded.

Any open neighbourhood \( V \) in \( U_{a} \) of the closed set \( D_{1} \) will thus contain points of arbitrarily high multiplicity, if \( B_{1} \) is nonempty. Suppose, then, that \( B_{1} \neq \emptyset \).

Choose any \( U_{l,a} \) as described in the proof of Proposition 7.18. Then \( \partial U_{l,a} = B_{1} \) and \( a_{a} : U_{l,a} \to \mathbb{R}^{p} \) is one-to-one, open, and a homeomorphism onto its image. By Proposition 6.11, \( a_{a} : \overline{U}_{l,a} \to \mathbb{R}^{p} \) is also a homeomorphism onto its image.

Take \( V := W_{1} \cup B_{1} \cup \bigcup_{k \geq r} W_{k,i(k)} \subseteq \overline{U}_{l,a} \) for some \( r \geq 2 \). Then \( V \) is open in \( \overline{U}_{l,a} \), since \( V = \overline{U}_{l,a} \cap (W_{1} \cup B_{1} \cup \bigcup_{k \geq r} W_{k}) \) and the complement of \( W_{1} \cup B_{1} \cup \bigcup_{k \geq r} W_{k} \) is \( \bigcup_{k=1}^{r} W_{k} \) which is closed in \( U_{a} \) (its boundary is empty). Actually, since \( \bigcup_{k=1}^{r} W_{k} \) is also open in \( U_{a} \), it is a union of open connected components. Thus \( \overline{U}_{l,a} = V \cup \bigcup_{k=1}^{r} W_{k,i(k)} \) expresses \( \overline{U}_{l,a} \) as a union of two mutually disconnected pieces. In the relative topology of \( \overline{U}_{l,a} \), we get \( B_{1} \subseteq \overline{V} \setminus B_{1} \subseteq \overline{V} = V \) (since \( V \) is a component), so that \( B_{1} \subseteq \text{Int} \overline{V} \setminus B_{1} \subseteq V \). We conclude that \( B_{1} \subseteq \text{Int} \overline{U}_{l,a} \).

Now we choose \( x \in B_{1} \) and a neighbourhood \( U \) of \( x \) such that \( S|_{U} \) is trivial. Choose two sets of sheets \( U_{l,a} \) and \( U_{j,a} \) with \( x \in \partial U_{l,a} \cap \partial U_{j,a} \).

Observe that, by the above argument and Corollary 5.8, the set \( a_{a}(U \cap \overline{U}_{l,a}) \) is open. Let \( \xi \) be any eigenvector of \( \mathcal{D} \) with eigenvalue \( \lambda \); define a section of \( \tilde{S} \to a_{a}(U) = a_{a}(U \cap \overline{U}_{l,a}) \) by

\[
\psi_{l,a}(t) := \begin{cases} 0, & \text{if } t \in a_{a}(W_{1} \cup B_{1}), \\ \xi \circ a_{a,1}^{-1}(t) - \xi \circ a_{a,1}^{-1}(t), & \text{if } t \in a_{a}(\overline{U}_{l,a} \setminus (W_{1} \cup B_{1})), \end{cases}
\]

where \( a_{a,1}^{-1} : a_{a}(\overline{U}_{l,a}) \to \overline{U}_{l,a} \) and similarly for \( a_{a,j}^{-1} \). As before, \( \psi_{l,a} \) is a well-defined eigenvector for \( \tilde{\mathcal{D}} \) on \( a_{a}(U) \).
The weak unique continuation property for Dirac operators now says that any such $\psi_{IJ\lambda}$ is identically zero, or $W_1$ is disconnected from $a_a(U_{I,\alpha} \setminus W_1)$. If there is any $\psi_{IJ\lambda}$ which is not identically zero, for each $x \in B_1 \subseteq U$, we are done.

Otherwise every $\psi_{IJ\lambda}$ vanishes identically, and thus $\xi|_{U \cap U_{I,\alpha}} = \xi|_{U \cap U_{I,\alpha}} \circ a_a^{-1} \circ a_a$. In that case, no function of the form $(\xi | \eta)$, with $\xi, \eta$ being eigenvectors of $\mathcal{D}$, can separate points of the fibre $a_a^{-1}(t)$ for $t \in a_a(U \cap U_{I,\alpha})$. Since these functions generate $\mathcal{A}$, we have reached a contradiction. □

**Theorem 7.20.** The space $X$ is a compact topological manifold.

**Proof.** By Propositions 7.16, 7.17, 7.18 and 7.19, we now know that

$$X = \bigcup_{\alpha=1}^n U_\alpha = \bigcup_{\alpha=1}^n \bigcup_{j,k} W_{j\kappa,\alpha}.$$  

This is a weak$^*$-open cover, and since $X$ is compact, it has a finite subcover. Since $a_\alpha$ is one-to-one on each $W_{j\kappa}$, now renamed $W_{j\kappa,\alpha}$, Corollary 5.7 says that $a_\alpha: W_{j\kappa,\alpha} \to a_\alpha(W_{j\kappa}) \subset \mathbb{R}^p$ is a homeomorphism, and is an open map to $\mathbb{R}^p$ by Corollary 5.8. On any overlap $V = W_{j\kappa,\alpha} \cap W_{j\kappa',\beta}$ of our finite subcover, there exist locally Lipschitz functions $g^j: a_\alpha(V) \to \mathbb{R}$ for $j = 1, \ldots, p$, such that $a_{\beta}^j|_V = g^j \circ a_\alpha|_V$. This follows from Lemma 5.4 (and the selfadjointness of each $a_\beta^j$). Thus, the transition functions $a_\beta \circ a_\alpha^{-1}: a_\alpha(V) \to a_\beta(V)$ are given by

$$a_\beta(a_\alpha^{-1}(\xi)) = (g^1(\xi), \ldots, g^p(\xi)), \quad \text{for all } \xi \in a_\alpha(V).$$  

Thus they are all locally Lipschitz, and in particular continuous, which is what we need. □

**Remark 7.21.** Since we now know that we can obtain suitable charts for the coordinate functions $a_\alpha$, we may suppose henceforth that the $a_\alpha$ are actually one-to-one on each $U_\alpha$. This can be achieved by relabelling the charts, whereby each former $W_{j\kappa,\alpha}$ participating in the finite open cover of $X$ is relabelled as some $U_\beta$. (We have not imposed any requirement that the elements $a_\alpha^j$ of $\mathcal{A}$ be distinct, as $\alpha$ varies.)

**Proposition 7.22.** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral manifold of dimension $p$. Then the metric and weak$^*$ topologies on $X = \text{sp}(\mathcal{A})$ agree.

**Proof.** This follows from Theorem 7.20 and Lemma 5.9 since each $x \in X$ has a neighbourhood on which some $a_\alpha$ is one-to-one. □

**Proposition 7.23.** Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral manifold of dimension $p$, and let $b \in \mathcal{A}$. Then there is a $C^\infty$ function $g: a_\alpha(U_\alpha) \to \mathbb{C}$ such that $g(a_\alpha(x)) = b(x)$ for all $x \in U_\alpha$.

**Proof.** Since $a_\alpha: U_\alpha \to \mathbb{R}^p$ is (now) one-to-one and open, and so a homeomorphism onto its image, Lemma 5.4 says that there is a unique locally Lipschitz function $g: a_\alpha(U_\alpha) \to \mathbb{C}$ such that $b|_{U_\alpha} = g \circ a_\alpha$. Since $b \in \mathcal{A}$, it is also true that

$$\phi [\mathcal{D}, b] = \phi \sum_{j=1}^p b_j [\mathcal{D}, a_\alpha^j],$$  

where $\phi \in \mathcal{A}$ is any function with supp $\phi \subseteq U_\alpha$, and $\phi b_j \in \mathcal{A}$.  

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Now by choosing a sequence \( \{g \langle k \rangle \} \) of \( C^1 \) functions converging in the Lipschitz norm to \( g \) on \( \text{supp} \phi \), we find that
\[
\phi [\mathcal{D}, b] = \phi \sum_{j=1}^{p} (g_j \circ a_{\alpha}) [\mathcal{D}, a_{\alpha}^j],
\]
where \( g_j(\xi) := \lim_k \partial_j g_{\langle k \rangle}(\xi) \). The linear independence of the \([\mathcal{D}, a_{\alpha}^j]\) over \( U_\alpha \) gives us the uniqueness of the coefficients in this expansion of \([\mathcal{D}, b]\); hence \( \phi b_j = \phi g_j \circ a_{\alpha} \). Since this holds for any \( \phi \in \mathcal{A} \) supported in \( U_\alpha \), we see that \( g_j \) is a continuous function defined on all of \( a_{\alpha}(U_\alpha) \) that does not depend on \( \phi \), and that \( b_j|_{U_\alpha} = g_j \circ a_{\alpha} \). Since \( b_j \in \mathcal{A} \), our Lipschitz functional shows that each \( g_j \) is locally Lipschitz, and therefore that \( g \) is actually \( C^1 \), with \( \partial_j g = g_j \) on \( a_{\alpha}(U_\alpha) \). Repeating this argument with \( b \) replaced by any \( b_j \) then shows that \( g \) is \( C^2 \), and so on. Thus \( g \) is \( C^\infty \) by induction. \( \square \)

All elements of \( \mathcal{A} \) can now be written as smooth functions of finitely many elements \( a_{\alpha}^j \), and the smooth manifold structure of \( X \) is easier to describe.

**Proposition 7.24.** Let \((\mathcal{A}, \mathcal{K}, \mathcal{D})\) be a spectral manifold of dimension \( p \). The unital algebra generated by the \( np \) functions \( a_{\alpha}^j \) is dense in the unital Fréchet algebra \( \mathcal{A} \). In the norm topology, this algebra is dense in \( C^* \)-algebra \( \mathcal{A} \).

**Proof.** Let \( \{\phi_{\alpha}\} \) be a partition of unity subordinate to the finite open cover \( \{U_\alpha\} \), and let \( b \in \mathcal{A} \). For each \( \alpha \), define \( g_{\alpha} : a_{\alpha}(U_\alpha) \rightarrow \mathbb{C} \), as in Proposition 7.23, so that \( \phi_{\alpha} b = g_{\alpha}(a_{\alpha}^1, \ldots, a_{\alpha}^p) \) on \( U_\alpha \). Then
\[
b = \sum_{\alpha=1}^{n} \phi_{\alpha} b = \sum_{\alpha=1}^{n} g_{\alpha}(a_{\alpha}^1, \ldots, a_{\alpha}^p).
\]

(7.5)

Since \( X \) is compact and Hausdorff, and the algebra \( \mathcal{A} \) separates points of \( X = \text{sp}(\mathcal{A}) \), for any two points \( x, y \in X \) we can find \( b \in \mathcal{A} \) such that \( b(x) = 1 \) and \( b(y) = 0 \). Since \( b \) is of the form (7.5), it follows that the \( np \) functions \( a_{\alpha}^j \) themselves separate the points of \( X \). Since the unital algebra they generate (over \( \mathbb{C} \)) is closed under complex conjugation, the Stone–Weierstrass theorem shows that it is dense in the \( C^* \)-algebra \( C(X) = \mathcal{A} \).

Now consider the closure \( \mathcal{A}_0 \) in the Fréchet algebra \( \mathcal{A} \) of this subalgebra \( \mathbb{C}[1, a_{\alpha}^1, \ldots, a_{\alpha}^p] \). By Proposition (2.8), \( \mathcal{A}_0 \) contains all functions of the form \( h(a_{\alpha}^1, \ldots, a_{\alpha}^p) \) where \( h \) is \( C^\infty \), and in particular it contains all elements of the form (7.5). Thus \( \mathcal{A}_0 = \mathcal{A} \), as required. \( \square \)

**Lemma 7.25.** Each mapping \( a_{\alpha} \circ a_{\beta}^{-1} : a_{\beta}(U_\alpha \cap U_\beta) \rightarrow a_{\alpha}(U_\alpha \cap U_\beta) \) is \( C^\infty \).

**Proof.** Proposition 7.23 shows that for each \( \alpha \) and each \( j = 1, \ldots, p \), there exists a \( C^\infty \) function \( f^j : a_{\beta}(U_\alpha \cap U_\beta) \rightarrow \mathbb{R} \) such that
\[
a_{\alpha}^j(x) = f^j(a_{\beta}^1, \ldots, a_{\beta}^p)(x) \text{ for } x \in U_\alpha \cap U_\beta.
\]

(7.6)

For \( \xi = a_{\beta}(x) \) with \( x \in U_\alpha \cap U_\beta \), it follows that \( a_{\alpha} \circ a_{\beta}^{-1}(\xi) = a_{\alpha}(x) = (f^1(\xi), \ldots, f^p(\xi)) \), and so \( a_{\alpha} \circ a_{\beta}^{-1} \) is \( C^\infty \) on \( U_\alpha \cap U_\beta \). \( \square \)

**Theorem 7.26.** Let \((\mathcal{A}, \mathcal{K}, \mathcal{D})\) be a spectral manifold of dimension \( p \). Then \( X = \text{sp}(\mathcal{A}) \) is a smooth manifold and \( \mathcal{A} = C^\infty(X) \). Moreover, \( X \) is orientable.
Proof. Theorem 7.20 and Lemma 7.25 together establish that $X$ is a smooth manifold. If $f \in C^\infty(X)$ with $\text{supp } f \subset U_a$, then by definition $g_a = f \circ a_a^{-1} : a_a(U_a) \to \mathbb{C}$ is a smooth mapping. Now $f = g_a \circ a_a$ lies in $\mathcal{A}$ by the multivariate $C^\infty$-functional calculus of Proposition 2.8. More generally, if $f \in C^\infty(X)$ we can write $f = \sum_a f_a$ where $\{h_a\}$ is a finite partition of unity in $C^\infty(X)$ subordinate to $\{U_a\}$. Therefore, $C^\infty(X) \subseteq \mathcal{A}$.

Conversely, if $b \in \mathcal{A}$, then by Lemma 2.10 we can choose a finite partition of unity $\{\phi_a\}$ in $\mathcal{A}$, also subordinate to $\{U_a\}$, and by Proposition 7.23 we can write $b \phi_a = g(a_a^1, \ldots, a_a^p)$ where $g$ is a smooth function defined on $a_a(U_a)$. Thus, $b = \sum_a b \phi_a$ lies in $C^\infty(X)$. In fine, $\mathcal{A} = C^\infty(X)$.

The real vector bundle $E_\mathbb{R} \to X$ of Corollary 4.11 is now seen to have smooth transition functions. The local formula (7.2) extends to a map on sections

$$
\rho_a \left( \sum_\alpha \phi_\alpha \sum_j b_{j\alpha} \, d a^j_a \right) := \sum_\alpha \phi_\alpha \sum_j b_{j\alpha} \, [\mathcal{D}, a^j_a],
$$

(7.7)

that determines a bundle morphism $\rho : T^* X \to E_{\mathbb{R}}$. Since two local coordinate bases of 1-forms $\{d a^1_a, \ldots, d a^p_a\}$ and $\{d a^1_\beta, \ldots, d a^p_\beta\}$ are related on $U_a \cap U_\beta$, according to (7.6), by

$$
da^j_a = \sum_{k=1}^p \partial_k f^j(a^1_\beta, \ldots, a^p_\beta) \, d a^k_\beta
$$

and $[\mathcal{D}, a^j_\beta]$ is expressed in terms of the $[\mathcal{D}, a^k_\beta]$ with the same coefficients, by (5.2), it follows that the map (7.7) is well-defined and that $\rho$ is an isomorphism of vector bundles over $X$. We also get the corresponding isomorphism $\Lambda^* \rho : \Lambda^* T^* X \to \Lambda^* E_{\mathbb{R}}$.

Recall the skewsymmetrization $\Gamma = \sum_\alpha \Gamma_\alpha'$ of $\Gamma$ introduced in (4.2). We have established in Section 4 that $\Gamma'$ is a nowhere vanishing section of the complex vector bundle $\Lambda^p E \to X$. A small but necessary adjustment now yields a nowhere vanishing section of $\Lambda^p E_{\mathbb{R}} \to X$. Since each $[\mathcal{D}, a^j_\beta]$ is skewadjoint, we see that, with $\wedge$ denoting skewsymmetrization,

$$
([\mathcal{D}, a^1_\alpha] \wedge \cdots \wedge [\mathcal{D}, a^p_\alpha])^* = (-1)^{\binom{p+1}{2}} \left[ [\mathcal{D}, a^1_\alpha] \wedge \cdots \wedge [\mathcal{D}, a^p_\alpha] \right].
$$

Note that the sign can also be written as $(-1)^{\binom{p+1}{2}}$. Now $\Gamma'$ differs from $\Gamma$ by adding selfadjoint junk terms, see Lemma 4.1, so that $\Gamma'$ is also selfadjoint. We may therefore rewrite each coefficient $a^0_\alpha$ in (4.2) as $a^0_\alpha = \eta(\binom{p+1}{2}) d^0_\alpha$, where $d^0_\alpha$ is a selfadjoint element in $\mathcal{A}$ and is thus a real function on $X$. Therefore, $\eta(\binom{p+1}{2}) \Gamma'$ is a nonvanishing smooth section of $\Lambda^p E_{\mathbb{R}} \to X$, so it equals $\Lambda^* \rho_{\ast}(\nu)$, where $\nu$ is a nonvanishing $p$-form on $X$. The de Rham cohomology class $[\nu]$ of this volume form confers the desired orientation on $X$.

The following Corollary is true because it holds locally, as proved earlier in Lemmas 7.3 and 7.4.

**Corollary 7.27.** The operator $\mathcal{D}$ is a first order elliptic differential operator on $\Gamma_\infty(X, S)$. For all $a, b \in \mathcal{A}$, the operator

$$
[[\mathcal{D}^2, a], b] = [\mathcal{D}, a] [\mathcal{D}, b] + [\mathcal{D}, b] [\mathcal{D}, a]
$$

is a central element of the algebra $\mathbb{C}_\mathcal{D}(\mathcal{A})$. \(\Box\)
7.3 Riemannian structure of the spectral manifold

Next, we show that the algebra \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \), acting as operators on \( \mathcal{H}_\infty \), is the carrier of a Clifford action for a unique Riemannian metric on \( X \).

Similar techniques to those used to construct \( E \) allow us to show that the representation of the algebra \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \) is irreducible, according to the following Proposition.

Proposition 7.28. Let \( e \) be a projector in \( \Gamma(X, \text{End} \mathcal{S}) \) such that \( e \mathcal{H}_\infty \subseteq \mathcal{H}_\infty \). If \( e \) commutes with \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \), then \( e = 0 \) or \( 1 \).

Proof. Note first that \([\mathcal{D}, a] = e [\mathcal{D}, a] e + (1 - e) [\mathcal{D}, a] (1 - e)\) for any \( a \in \mathcal{A} \), since \( e \) commutes with \([\mathcal{D}, a] \). Consider

\[ B := e\mathcal{D}(1 - e) + (1 - e)\mathcal{D}e \]

as a linear map of \( \mathcal{H}_\infty \) to itself. Since \( ea = ae \) for \( a \in \mathcal{A} \), we find that

\[ [B, a] = e [\mathcal{D}, a] (1 - e) + (1 - e) [\mathcal{D}, a] e = 0, \]

so that \( B \) is \( \mathcal{A} \)-linear. By Proposition 3.18, the operator \( B \) extends to a bounded operator on \( \mathcal{H} \), which is selfadjoint. The same is also true of \( e \), which is \( \mathcal{A} \)-linear too.

As a bounded selfadjoint perturbation of \( \mathcal{D} \), the operator \( \mathcal{D} - B \) is also selfadjoint and has \( \text{Dom}(\mathcal{D} - B) = \text{Dom} \mathcal{D} \); see, for instance, [8, App. A]. If \( \xi \in \mathcal{H}_\infty \), then

\[ (\mathcal{D} - B)e\xi = \mathcal{D}(e\xi) - (1 - e)\mathcal{D}e\xi = e\mathcal{D}(e\xi) = e\mathcal{D}\xi - e\mathcal{D}(1 - e)\xi = e(\mathcal{D} - B)\xi, \]

so that \([\mathcal{D} - B, e] = 0\) on \( \mathcal{H}_\infty \), or equivalently, \([\mathcal{D}, e]\xi = [B, e]\xi\) for \( \xi \in \mathcal{H}_\infty \). Thus \([\mathcal{D}, e]\) extends to the bounded operator \([B, e]\), and therefore [7] we get \( e \in \text{Dom}_\mathcal{D} \) with \( de = [\mathcal{D}, e] = [B, e] \).

Since \( e \) commutes with \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \), we may now apply Corollary 3.21 to obtain

\[ \Gamma [\mathcal{D}, e] = \frac{1}{2} (-1)^{p-1} \sum_{\alpha=1}^{n} a^0_{\alpha} \sum_{j=1}^{p} (-1)^{j-1} da^1_{j} \ldots (de da^1_{j} + da^1_{j} de) \ldots da^p_{a}. \]

Skewsymmetrizing in \( de, da^1_{j}, \ldots, da^p_{a} \) gives zero, so for all \( x \in X \) we can find \( c_{ja}(x) \) with

\[ de(x) = \sum_{j, a} c_{ja}(x) da^1_{j}(x). \]

The \( c_{ja} \) define bounded functions on \( X \), and since \( de \) and each \( da^1_{j} \) are bounded operators on the Hilbert space \( \mathcal{H} \) of \( L^2 \)-sections of \( S \) with respect to the measure \( \mu_{\mathcal{D}} \), these functions are measurable and preserve \( L^2 \)-sections. The endomorphism \( e \) commutes with all such functions and \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \), and thus \( e \) commutes with \([\mathcal{D}, e]\); therefore, \( e [\mathcal{D}, e] = e^2 [\mathcal{D}, e] = e [\mathcal{D}, e] e = 0 \), and similarly \( (1 - e)[\mathcal{D}, e] = -(1 - e)[\mathcal{D}, (1 - e)] = 0 \), since \( e(\delta e)e = 0 \) for any idempotent \( e \) and derivation \( \delta \). Thus \([\mathcal{D}, e] = 0 \). By irreducibility, \( e \) is a scalar, and so it equals 0 or 1. \( \square \)

Corollary 7.29. There is no proper subbundle \( \tilde{\mathcal{S}} \subset S \) such that \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \Gamma_\infty(X, \tilde{\mathcal{S}}) \subseteq \Gamma_\infty(X, \tilde{\mathcal{S}}) \). That is, \( \mathcal{H}_\infty = \Gamma_\infty(X, S) \) is irreducible as a \( \mathfrak{C}_{\mathcal{D}}(\mathcal{A}) \)-module.

Proof. If \( \tilde{\mathcal{S}} \) were reducible, we could find a projector \( e \in \Gamma(X, \text{End} \mathcal{S}) \) with \( e \mathcal{H}_\infty = \Gamma_\infty(X, \tilde{\mathcal{S}}) \) and \([e, \mathfrak{C}_{\mathcal{D}}(\mathcal{A})] = 0 \). By the previous Proposition, such an \( e \) is either 0 or 1. \( \square \)
Proposition 7.30. The Hilbert space $\mathcal{H}$ carries a nondegenerate representation of the algebra $\Gamma_{\infty}(X, C)$ of smooth sections of an algebra bundle $C = C\ell(T^*X, g)$, which is the complex Clifford-algebra bundle defined by a Riemannian metric $g$ on $X$.

Proof. If $\eta = \sum_{\alpha,j} \phi_{\alpha} b_{j\alpha} \, da_{\alpha}$ and $\zeta = \sum_{\beta,k} \phi_{\beta} c_{k\beta} \, da_{\beta}^k$ are two 1-forms in $\Omega^1(X)$, then (7.7) yields

$$\rho_*(\eta) \rho_*(\zeta) + \rho_*(\zeta) \rho_*(\eta) = \sum_{\alpha, \beta, j, k} \phi_{\alpha} \phi_{\beta} b_{j\alpha} c_{k\beta} \left( [\mathcal{D}, a_{\alpha}^j] \, [\mathcal{D}, a_{\beta}^k] + [\mathcal{D}, a_{\beta}^k] \, [\mathcal{D}, a_{\alpha}^j] \right),$$

(7.8)

which is central in $\mathcal{C}_T(A)$ and has bounded commutator with $\mathcal{D}$.

By Corollary 4.9, the finite products of the local sections $\underline{a}_{\alpha}^j = [\mathcal{D}, a_{\alpha}^j]$, which are restrictions to $U_{\alpha}$ of elements of $\mathcal{C}_T(A)$, determine a trivialization over $U_{\alpha}$ of the bundle $\text{End} \, S$. Thus there are algebra subbundles $C_{\alpha} \rightarrow U_{\alpha}$ of $\text{End} \, S|_{U_{\alpha}}$ for each $\alpha$, such that $T \in \mathcal{C}_T(A)$ if and only if $T|_{U_{\alpha}} \in \Gamma(U_{\alpha}, C_{\alpha})$ for all $\alpha$. Over $U_{\alpha}$, the bundle $C_{\alpha}$ decomposes as a Whitney sum of trivial matrix bundles, compare Corollary 3.17:

$$C_{\alpha} \simeq \bigoplus_{i=1}^r U_{\alpha} \times M_{k_i}(\mathbb{C}).$$

(7.9)

Just as in the proof of Proposition 7.7, the local sections $\bar{e}_i \in \Gamma_{\infty}(U_{\alpha}, \text{End} \, S)$ given by $\bar{e}_i(x) := 1_{k_i}$, for $i = 1, \ldots, r$, are the minimal central projectors in this decomposition. For $a, b \in \mathcal{A}$ compactly supported in $U_{\alpha}$, the map $a \, db \mapsto \bar{e}_i \, \bar{a} \, \bar{d} \, \bar{b} \, \bar{e}_i = a \, db \, \bar{e}_i$ makes sense over $U_{\alpha}$.

Since $\underline{a} \, \underline{d} \, \underline{b} + \underline{d} \, \underline{b} \, \underline{d} \, \underline{a}$ is central in $\mathcal{C}_T(A)$, it decomposes over each $U_{\alpha}$ as a sum of scalar matrices:

$$\underline{a} \, \underline{d} \, \underline{b} + \underline{d} \, \underline{b} \, \underline{d} \, \underline{a} =: \bigoplus_{i=1}^r -2g_{ia}(da, db) \, 1_{k_i},$$

where each $g_{ia}$ is again a positive definite symmetric bilinear form whose values this time are bounded smooth functions on $U_{\alpha}$, i.e., restrictions to $U_{\alpha}$ of elements of $\mathcal{A}$. On any overlap $U_{\alpha} \cap U_{\beta}$, we can write

$$\underline{a} \, \underline{d} \, \underline{b} + \underline{d} \, \underline{b} \, \underline{d} \, \underline{a} = \sum_{j,k} a_{j\alpha} b_{k\alpha} \left( \underline{a}_{j\alpha}^i \underline{d}_{\alpha}^k + \underline{d}_{\alpha}^k \underline{a}_{j\alpha}^i \right)$$

$$= -2 \bigoplus_{i=1}^r \sum_{j,k} a_{j\alpha} b_{k\alpha} g_{ia}(da_{\alpha}^j, da_{\alpha}^k) \, 1_{k_i}$$

$$= -2 \bigoplus_{i=1}^r \sum_{j,k,n,m} a_{j\alpha} b_{k\alpha} c_{m,\alpha,\beta}^{j,k} g_{\alpha\beta}(da_{\beta}^m, da_{\beta}^n) \, 1_{k_i}$$

$$= -2 \bigoplus_{i=1}^r \sum_{n,m} a_{\alpha\beta} b_{\alpha\beta} g_{\alpha\beta}(da_{\alpha}^m, da_{\alpha}^n) \, 1_{k_i}.$$ 

(7.10)

Here the $c_{m,\alpha,\beta}^{j,k}$ are the transition functions for $E_{\mathbb{R}}$, defined in (4.12).

Some of the scalar components of $\underline{a} \, \underline{d} \, \underline{b} + \underline{d} \, \underline{b} \, \underline{d} \, \underline{a}$ in the block-matrix decomposition (7.10) might coincide, even when $a, b$ run over all elements of $\mathcal{A}$. To consolidate such blocks, we relabel the decomposition (7.9) as

$$C_{\alpha} = C_{\alpha,1} \oplus \cdots \oplus C_{\alpha,s},$$

(7.11)
where each $C_{a,j}$ is a Whitney sum of matrix-algebra bundles over $U_a$, in which the sections $da db + db da$ have distinct components $-2g_{ia}(da, db)$ in general. Let $N_j$ be the rank of $C_{a,j}$, so that $\sum_{j=1}^s N_j = \sum_{i=1}^r k_i$. On comparing the scalar components of $da db + db da$ on any overlap $U_a \cap U_b$, we see that the number $s$ and the ranks $N_1, \ldots, N_s$ do not depend on $U_a$, so the block decomposition (7.11) is global. Let $e_j \in C_2(A)$ denote the central projector given by the identity element of $C_{a,j}$. The relation (7.10) shows that each corresponding $g_j := g_{ja}$ is a globally defined symmetric 2-tensor.

Applying Corollary 7.29, we find that there can only be one such 2-tensor $g := g_1$, and only one such global block in (7.11). Thus $g$ is a positive definite Euclidean metric on $T^*X$, that is to say (after transposing to the tangent bundle $TX$, if one prefers), a Riemannian metric on $X$. In view of (7.8), the map $\rho$ defines an action of $C := \mathbb{C}\ell(T^*X, g)$ on $\mathcal{H}_c$, whose algebra of smooth sections is precisely $\mathcal{C}_D(A)$.

Comparing now with (4.6), we see that

$$\langle da | db \rangle = C_p \text{tr}((da)^* db) = C_p \text{tr}((da)^* db) = g(da, db)$$

for selfadjoint $a, b \in A$. Consequently we set $C_p := N^{-1}$ where $N = \text{rank } S$. □

**Corollary 7.31.** The operator $\mathcal{D}$ is, up to the addition of an endomorphism of $S$, a Dirac-type operator with respect to the metric $g$. □

**Remark 7.32.** The orientation on (the cotangent bundle of) the manifold $X$ is fixed by $\Gamma$, according to Theorem 7.26. Recall that the chirality element $\gamma$ for the Clifford algebra $\mathbb{C}\ell(T^*X, g)$ depends on a choice of orientation of $T^*X$ [30, 39]; reversal of this orientation replaces $\gamma$ by $-\gamma$. Having chosen the orientation of $T^*X$, let $\gamma$ be the chirality element in $C$.

**Lemma 7.33.** The chirality element $\gamma$ of $C$ may be chosen so that $\rho_*(\gamma) = \Gamma$.

**Proof.** In the representation $C$ of $\mathbb{C}\ell(T^*X, g)$, the chirality element is given by

$$\rho_*(\gamma) := i^{[(p+1)/2]} \sum_a \sqrt{\text{det } g} \rho_*(da_a^1) \ldots \rho_*(da_a^p) = i^{[(p+1)/2]} \sum_a \sqrt{\text{det } g} [\mathcal{D}, a_a^1] \ldots [\mathcal{D}, a_a^p].$$

In $C$, skewsymmetrization of $i^{-(p+1)/2} \Gamma$ and of $i^{-(p+1)/2} \rho_*(\gamma)$ yields nonvanishing sections of $\Lambda^p T^*X$. Consequently, $\Gamma = f \rho_*(\gamma)$ for some nonvanishing real function $f$ on $X$. Squaring gives $1 = f^2 \rho_*(\gamma^2) = f^2$, and therefore $f = \pm 1$ since $X$ is connected. We now fix the orientation on $T^*X$ for $g$ so that $f = +1$, and thereby $\rho_*(\gamma) = \Gamma$. □

**8 Poincaré duality and spin*c structures**

This Section uses Poincaré duality in $K$-theory to identify the manifold $X$ as a spin*c manifold.

We shall use the Kasparov intersection product [34], but in fact only require its functorial properties and some results for products with particular classes. For an executive summary of its properties, see [18, IV.A].

In order to identify the spectral triple $(A, \mathcal{H}, \mathcal{D})$ with that of the Dirac operator on an irreducible bundle of spinors, we must in particular show that the manifold $X = \text{sp}(A)$ is spin*c. By the work of [33] and [48], when $\dim X$ is even this amounts to the existence of a spinor bundle
$S \to X$, carrying a (pointwise) irreducible representation of the Clifford algebra bundle $\mathbb{C}\ell(X)$; and likewise in the odd-dimensional case, using instead irreducible representations of the algebra subbundle $\mathbb{C}\ell^+(X) := \frac{1+\gamma}{2} \mathbb{C}\ell(X)$.

The spin$^c$ condition can also be rephrased as the existence of a Morita equivalence bimodule between $\Gamma(X, \mathbb{C}\ell(X))$ and $C(X)$ if $\dim X$ is even, respectively between $\Gamma(X, \mathbb{C}\ell^+(X))$ and $C(X)$ if $\dim X$ is odd; this bimodule is provided by the sections of an irreducible spinor bundle. Proposition 8.4 below shows that the existence of a class $\mu \in K^*(A \otimes A)$ satisfying Poincaré duality in $K$-theory, Condition 8, implies that the manifold is spin$^c$. We do this by comparing the Poincaré duality isomorphism coming from $\mu$ and the “Riemannian” Poincaré duality isomorphism described next, which holds for any compact oriented manifold, spin$^c$ or not.

The algebra $\lambda \in KK(\Gamma(\mathbb{C}\ell(X)) \otimes C(X), \mathbb{C})$ for a compact oriented manifold $X$ is defined as follows. Choose any Riemannian metric $g$, and consider $d + d^*$ on $\mathcal{H} = L^2(\Lambda^\bullet T^*_C X, g)$. The algebra $C(X)$ acts by multiplication operators $m(f)$ on $\mathcal{H}$, and with the phase $V$ of $d + d^*$ gives a Fredholm module $(\mathcal{H}, V)$ for $C(X)$. This module is even, since $d + d^*$ anticommutes with the grading $e$ of differential forms by degree mod 2.

The Clifford algebra bundle, which is isomorphic as a vector bundle to the exterior bundle $\Lambda^\bullet T^*X$, is likewise $\mathbb{Z}_2$-graded as $\mathbb{C}\ell(X) = \mathbb{C}\ell^0(X) \oplus \mathbb{C}\ell^1(X)$. We need sections of $\mathbb{C}\ell(X)$ to graded-commute with $d + d^*$ (respectively, with $V$) up to bounded (respectively, compact) operators and to graded-commute with $e$. For that [35, Defn. 4.2], we define the action of covectors $v \in T^*X$, as usual in physics, by

$$\tilde{c}(v) \omega := v \wedge \omega + i_g(v) \omega, \quad \text{for } \omega \in \Lambda^\bullet T^*_C X,$$

where $i_g$ denotes contraction with respect to $g$. We find that

$$\tilde{c}(v) \tilde{c}(w) + \tilde{c}(w) \tilde{c}(v) = +2g(v, w).$$

This differs from the action of covectors arising from the symbol of $d + d^*$, since $c(v) := \sigma_{d+d^*} (x, v)$ yields $c(v) \omega = v \wedge \omega - i_g(v) \omega$. That action satisfies $c(v) c(w) + c(w) c(v) = -2g(v, w)$. As is pointed out in [30], $\tilde{c}(v)$ comes from the action of the symbol of $i(d - d^*)$. While these actions may generate non-isomorphic real algebras, their complexifications are isomorphic.

The pair $(\mathcal{H}, V)$ is thus a $\mathbb{Z}_2$-graded Fredholm module carrying the left action $\tilde{c} \otimes m$ of $\Gamma(\mathbb{C}\ell(X)) \otimes C(X)$ on $\mathcal{H}$ and the trivial right action of $\mathbb{C}$; we denote its (operator homotopy) class by $\lambda \in KK(\Gamma(\mathbb{C}\ell(X)) \otimes C(X), \mathbb{C})$. We use the $KK$ notation to distinguish $\lambda$ from a $K$-homology class in $K_0(\Gamma(\mathbb{C}\ell(X)) \otimes C(X))$ in order to stress that we are dealing here with $\mathbb{Z}_2$-graded algebras.

It turns out that $KK(\mathbb{C}, \Gamma(\mathbb{C}\ell(X)))$ is the Grothendieck group of equivalence classes of $\mathbb{Z}_2$-graded $\Gamma(\mathbb{C}\ell(X))$-modules. By [39], there is a canonical isomorphism with the group of ungraded modules for $\Gamma(\mathbb{C}\ell^0(X))$, which is $K_0(\Gamma(\mathbb{C}\ell^0(X)))$.

The interest in the class $\lambda$ is the following special case of [35, Thm. 4.10].

**Theorem 8.1.** Let $X$ be a compact boundaryless oriented Riemannian manifold. Then

$$\bullet \otimes \lambda : KK^i(C, \Gamma(\mathbb{C}\ell(X))) \to KK^i(C(X), \mathbb{C}) \cong K^i(C(X))$$

is an isomorphism. □
We shall suppose in what follows that $X$ is a compact boundaryless manifold on which a Riemannian metric $g$ is given, and we write, somewhat sloppily, $\mathcal{C}(X) \coloneqq \mathcal{C}(T^*X, g)$ for the corresponding Clifford algebra bundle over $X$. We also write $\mathcal{B} = \Gamma_\infty(X, \mathcal{C}(X))$ for its algebra of smooth sections, $B$ for the norm completion of $\mathcal{B}$ (the continuous sections), and $B^0$ for the even part of $B$ in the natural $\mathbb{Z}_2$-grading. We also write $B^p = \frac{1 + p}{2} B$ in the odd case, and note that while $B^+$ is isomorphic to $B^0$, $B^+$ contains both odd and even elements of $B$.

There are two useful representations of $B$ on $L^2(\Lambda^*T^*X)$. We denote by $\theta_-$ the representation coming from the symbol of $d + d^*$, since $c(v)^2 = -g(v, v)$ for each $v \in T^*X$; and by $\theta_+$ the representation coming from the symbol of $i(d - d^*)$, since therein $\tilde{c}(v)^2 = +g(v, v)$ for all covectors. These two representations graded-commute.

Let $\lambda \in KK(B \otimes A, \mathbb{C})$ be the class described above, so the representation of the $\mathbb{Z}_2$-graded algebra $B$ making this a graded Kasparov module is $\theta_+$. We set

$$\lambda_A := i_B \lambda \in K^0(A),$$

where $i_B : A \to B \otimes A : a \mapsto 1_B \otimes a$. This $\lambda_A$ is the class (see [31, Sec. 10.9]) of the spectral triple $(C^\infty(X), L^2(\Lambda^\bullet T^*_C X), d + d^*)$. Thus if $(\mathcal{E}, F)$ is any Fredholm module representing $\lambda_A$, there is a unitary operator $U : L^2(\Lambda^\bullet T^*_C X) \to \mathcal{E}$ that preserves gradings and an operator homotopy $\{F_t\}$ such that, modulo degenerate Fredholm modules, $F_0 = F$ and

$$(U^* \mathcal{E}, U^* F_1 U) = (L^2(\Lambda^\bullet T^*_C X), V), \quad \text{where} \quad d + d^* =: V \cdot (d + d^*). \quad (8.1)$$

Using $U$, we can transport both graded-commuting representations of $B$ on $L^2(\Lambda^\bullet T^*_C X)$ to $\mathcal{E}$, and $\theta_+(B)$ will give $(\mathcal{E}, F_1)$ the structure of a $(B \otimes A)$-Kasparov module.

**Lemma 8.2.** Let $(\mathcal{A}, \mathcal{H}, \mathbb{D})$ be a spectral manifold of dimension $p$. Then if $p$ is even, there is no faithful $\mathbb{Z}_2$-graded (by $\Gamma$) representation of $B$ on $\mathcal{H}$ graded-commuting with $\mathcal{C}_D(A) = \pi_D(\Omega^\bullet A)$. If $p$ is odd, there is no faithful representation of $B^+$ on $\mathcal{H}$ graded-commuting with $\mathcal{C}_D(A)$.

**Proof.** The algebra $B$ (and also the representation $\mathcal{C}_D(A)$ of $\mathcal{B}$) is $\mathbb{Z}_2$-graded by the parity of the number of 1-form components in a product $\omega = b_0 c(db_1) \cdots c(db_k)$, which coincides with the grading given by $\Gamma = \pi_D(c)$. In the even-dimensional case, we observe that a representation of $B$ on $\mathcal{H}$ graded-commuting with $\mathcal{C}_D(A)$ must commute with $\Gamma$ since $\Gamma$ is an even element of $\mathcal{C}_D(A)$, whereas any odd element of $B$ must be represented by an operator anticommuting with $\Gamma$. Any such representation must kill the odd elements of $B$ and thus cannot be faithful.

In the odd-dimensional case, since $\Gamma = 1$ is an odd element of $\mathcal{C}_D(A)$, because it is a sum of products of an odd number of 1-forms, such a representation of $B$ must graded-commute with $\Gamma = 1$, and so its 1-form elements would anticommute with 1, which is absurd. The same remains true for $B^+$ since it contains elements with nonzero odd components. \hfill \Box

**Remark 8.3.** In fact, the proof shows that even if we can (globally) split the cotangent bundle into $r$- and $s$-dimensional subspaces with $r + s = p$, and so write $\Gamma(\mathcal{C}(X)) \cong \Gamma(\mathcal{C}_{r}(X)) \times_A \Gamma(\mathcal{C}_{s}(X))$ (graded tensor product) then neither tensor factor can act faithfully on $\mathcal{H}$ in such a way as to graded-commute with $\mathcal{C}_D(A)$ and, in the even case, be graded by $\Gamma$.

In all of what follows, the group $KK(C, B)$ consists of formal differences of $\mathbb{Z}_2$-graded right $B$-modules, while $KK(C, A) = K_0(A)$ consists of formal differences of right $A$-modules: we consider only even representations of $A = C(X)$. By [34, Thm. 5.4], we may regard elements of $K_1(A)$ as
elements of $KK(\mathcal{C}L_1, A)$, where $\mathcal{C}L_1$ is the (2-dimensional) graded complex algebra generated by a single odd element $\gamma$ with $\gamma^2 = 1$.

**Proposition 8.4.** If $X$ is a compact boundaryless oriented manifold and $A = C(X)$, then $X$ has a $\text{spin}^c$ structure if and only if there is a class $\mu \in K^\bullet (A \otimes A)$, represented by a spectral manifold $(C^\infty (X), \mathcal{H}, \mathcal{D})$, for which

$$x \mapsto x \otimes_A \mu : K_\bullet (A) \to K^\bullet (A)$$

(8.2)
is an isomorphism.

**Proof.** Suppose first that there exists a $\mu \in K^\bullet (A \otimes A)$ such that (8.2) is an isomorphism, and that $\mu$ is represented by a spectral manifold (regarded as an unbounded Fredholm module [31]). Write $\mu_A := i_A^* \mu \in K^\bullet (A)$, where $i_A : A \to A \otimes A : a \mapsto 1 \otimes a$. Since $\bullet \otimes \lambda : KK^i (\mathbb{C}, B) \to K^i (A)$ is an isomorphism by Theorem 8.1, and (8.2) is an isomorphism by hypothesis, there exist classes $y \in KK^p (\mathbb{C}, B)$ and $x \in K_p (A)$ such that

$$\mu_A = y \otimes_B \lambda \quad \text{and} \quad \lambda_A = x \otimes_A \mu.$$  

We shall treat explicitly below only the even case, employing the suspension isomorphism $s : K^{p+1} (\mathbb{C}_0 (\mathbb{R}) \otimes A) \to K^p (A)$ to handle the odd case. By Theorem 10.8.7 and Proposition 11.2.5 of [31], for any Dirac type operator $\mathcal{D}$ on a manifold there are identifications

$$[[\mathcal{D}_R] \otimes_C [\mathcal{D}] = [[\mathcal{D}_R \otimes \mathcal{D}] \quad \text{and} \quad s([[\mathcal{D}_R \otimes \mathcal{D}]) = [\mathcal{D}],$$

where $\mathcal{D}_R = -i \partial/dx$ is the usual Dirac operator on $\mathbb{R}$. The analogous result for the $K$-theory suspension can also be found in [31].

Suppose, then, that $p$ is even. Let $x$ be represented by $(\mathcal{E}, 0)$, where $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is a finitely generated projective graded $A$-module, and denote the grading by $\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that the corresponding $K$-theory class is $x = [\mathcal{E}_1] - [\mathcal{E}_2] \in K_0 (A)$.

We shall represent the product $x \otimes_A \mu$ using the unbounded formalism of [38]. Using Lemma 2.5, we suppose without loss of generality that $\mathcal{E}$ is a finitely generated projective $A$-module (rather than $A$-module). Let $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1 (X)$ be a connection, and define

$$\hat{\epsilon} : \mathcal{E} \otimes_A \Omega^1 (X) \otimes_A \mathcal{E} \to \mathcal{E} \otimes_A \mathcal{H} : s \otimes a db \otimes \xi \mapsto \epsilon s \otimes a [\mathcal{D}, b] \xi,$$

for $s \in \mathcal{E}, a, b \in A, \xi \in \mathcal{H}$. Then $\hat{\mathcal{D}} := \hat{\epsilon} \circ (\nabla \otimes 1) + \epsilon \otimes \mathcal{D}$ is well defined and essentially selfadjoint on $\mathcal{E} \otimes_A \mathcal{H}_\infty$, and the pair $(\mathcal{E} \otimes_A \mathcal{H}_\infty, \hat{\mathcal{D}})$ is an unbounded representative of the Kasparov product $x \otimes_A \mu$, by [38, Thm. 13]. Moreover, $\hat{\mathcal{D}}$ is a first-order Dirac-type operator since

$$[\hat{\mathcal{D}}, a] (s \otimes \xi) = \epsilon s \otimes [\mathcal{D}, a] \xi, \quad \text{for all} \quad s \in \mathcal{E}, \xi \in \mathcal{H}_\infty, a \in A.$$

Now modulo degenerate Kasparov modules, there is a unitary $U$ such that

$$(U (\mathcal{E} \otimes_A \mathcal{H}), U \hat{\mathcal{D}} U^*) = (L^2 (\Lambda^* T^*_C X), \hat{d} + \hat{d}^*),$$

(8.3)

where the phase of $\hat{d} + \hat{d}^*$ is operator homotopic to the phase of $d + d^*$. Using this unitary we may transport the two graded-commuting representations $\theta_-, \theta_+ B$ to $\mathcal{E} \otimes_A \mathcal{H}$.  

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Now any (unbounded) operator \( \widetilde{D} \) representing the product \( x \otimes_A \mu \) on the module \( \mathcal{E} \otimes_A \mathcal{H}_\infty \) has principal symbol homotopic to

\[
\tilde{\sigma}_D : da \mapsto e \otimes [D, a] : \Omega^1(X) \to \text{End}_A(\mathcal{E} \otimes_A \mathcal{H}).
\]

This makes sense, since we may work on smooth sections and can define the principal symbol. By Lemma 8.2 and Remark 8.3, no nonzero 1-form on \( X \) can act nontrivially on \( \mathcal{H} \) so that it graded-commutes with \( \mathcal{C}_D(A) \) and anticommutes with \( \Gamma = \pi_D(c) \). Hence the representation \( \text{Ad} U^* \circ \theta_+ \) acts effectively on the first tensor factor \( \mathcal{E} \) of \( \mathcal{E} \otimes_A \mathcal{H} \); in this way, \( \mathcal{E} \) becomes a left \( B \)-module. This action is \( A \)-linear (since \( A \) is central in \( B \)) and moreover is adjointable. Indeed, for \( s, t \in \mathcal{E}, \xi, \eta \in \mathcal{H} \) and \( b \in B \), we find that

\[
\langle \xi \mid (bs \mid t)\eta \rangle = \langle bs \otimes \xi \mid t \otimes \eta \rangle = \langle b(s \otimes \xi) \mid t \otimes \eta \rangle = \langle s \otimes \xi \mid b^* (t \otimes \eta) \rangle = \langle s \otimes \xi \mid b^* t \otimes \eta \rangle = \langle \xi \mid (s \mid b^* t)\eta \rangle
\]

which implies \( (bs \mid t) = (s \mid b^* t) \) for the action of \( B \) on \( \mathcal{E} \). Thus \( B \) acts by endomorphisms of the \( \mathbb{Z}_2 \)-graded vector bundle \( E \) for which \( \mathcal{E} = \Gamma_\infty(X, E) \).

Now we let \( V \subseteq U_a \subseteq X \) be an open set over which the bundle \( E \) is trivial, so that \( \mathcal{E} \cdot C_0(V) \cong C_0(V)^{\text{rank} E} \). Let \( j : C_0(V) \to A \) be the homomorphism obtained by extending functions by zero, and let \( E_+, E_- \) be the subbundles of \( E \) whose sections are \( \mathcal{E}_\pm := \frac{1}{2}(1 \pm \varepsilon)\mathcal{E} \). We claim that

\[
j^*(x \otimes_A \mu) = (\text{rank } E_+ - \text{rank } E_-) j^* \mu. \tag{8.4}
\]

Recall that the map \( j^* \) just restricts the representation of \( A \) to \( C_0(V) \). To prove (8.4), we begin by splitting the Hilbert space \( \mathcal{E} \otimes_A \mathcal{H} \) as

\[
\mathcal{E} \otimes_A \mathcal{H} = L^2(V, E \otimes S) \oplus L^2(X \setminus V, E \otimes S).
\]

If we let \( P \) be the projector whose range is the first summand, then we can replace \( F_{\widetilde{D}} := \widetilde{D}(1 + \widetilde{D}^2)^{-1/2} \) by \( PF_{\widetilde{D}}P + (1 - P)F_{\widetilde{D}}(1 - P) \). This is valid because for \( a \in C_0(V) \) the relation

\[
aPF_{\widetilde{D}}(1 - P) = P[a, F_{\widetilde{D}}](1 - P)
\]

holds and defines a compact operator. Now the representation of \( C_0(V) \) on the second summand is zero, and so the straight line path

\[
t \mapsto (1 - P)(1 - t)F_{\widetilde{D}} + t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(1 - P),
\]

where the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is defined relative to the splitting \( \pi_D(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), gives an operator homotopy from the second summand to a degenerate Kasparov module.

For the first factor we now observe that

\[
L^2(V, E \otimes S) = L^2(V, (X \times C^{\text{rank} E}) \otimes S) = C_0(V)^{\text{rank} E} \otimes_{C_0(V)} L^2(V, S).
\]
Similarly, by choosing the trivial connection over $V$, we get $\widehat{\mathcal{D}}|_V = \hat{c} \circ (d|_V \otimes 1) + \varepsilon|_V \otimes \mathcal{D}|_V$, where $\varepsilon^2 = 1$ on $C_0(V)^{\text{rank} E}$. Thus,

$$[j^*(\mathcal{E} \otimes_A \mathcal{H}, \widehat{\mathcal{D}})] = [(C_0(V)^{\text{rank} E}, 0)] \otimes_{C_0(V)} [(P\mathcal{K}, PF_D P)]
= [(C_0(V)^{\text{rank} E}, 0)] \otimes_{C_0(V)} j^*[\mathcal{H}, F_D]
= [(C_0(V)^{\text{rank} E}, 0)] \otimes_{C_0(V)} j^*\mu
= (\text{rank } E_+ - \text{rank } E_-) j^*\mu.$$

Now if we choose $V \subset U_\alpha$ so that the closure of $V$ is contractible, then every bundle on $X$ restricts to a trivial bundle on $\overline{V}$. Then $V$ is (trivially) a spin$^c$ manifold, and so

$$j^*\lambda_A = \lambda_A|_V = [S^*_V] \otimes_A [\mathcal{D}]_V = 2^{p/2} [\mathcal{D}]_V$$

where $[\mathcal{D}]_V$ is the class of any Dirac operator on $V$, and $S_V \to V$ is the (complex) spinor bundle over $V$. Likewise, from (8.4) we get

$$j^*\lambda_A = j^*(x \otimes_A \mu) = (\text{rank } E_+ - \text{rank } E_-) j^*\mu = r(\text{rank } E_+ - \text{rank } E_-) [\mathcal{D}]_V,$$

where $r$ is the number of (pointwise) irreducible components of the representation $\mathcal{C}_2(A)$, obtained in Proposition 7.7.

Comparing these two expressions for $j^*\lambda_A$, we see that rank $E_+ - \text{rank } E_- > 0$. Write $E|_V = E_1 \oplus E_2 \oplus E_3$, where $\varepsilon = +1$ on $E_1 \oplus E_2$, $\varepsilon = -1$ on $E_3$, and $E_2 \approx E_3$ (since these are trivial bundles over $V$, this just means that $E_2$ and $E_3$ have equal rank). Then the explicit formula for the Kasparov product shows that on $\mathcal{E}_2 \oplus \mathcal{E}_3$, $\widehat{\mathcal{D}}$ decomposes as

$$\widehat{\mathcal{D}} = \left( \begin{array}{cc} \hat{c}_+(d \otimes 1) + 1 \otimes \mathcal{D} & 0 \\ 0 & -\hat{c}_+(d \otimes 1) - 1 \otimes \mathcal{D} \end{array} \right),$$

graded by $\left( \begin{array}{cc} 1 \otimes \Gamma & 0 \\ 0 & -1 \otimes \Gamma \end{array} \right)$,

where $\hat{c}_+(s \otimes a db \otimes \xi) = s \otimes a [\mathcal{D}, b]\xi$, i.e., just like $\hat{c}$ but with $\varepsilon$ acting by 1. This displays the restriction of the product to $\mathcal{E}_2 \oplus \mathcal{E}_3$ as the sum of a Fredholm module for $A$ and its negative. Hence it is homotopic to a degenerate module, and therefore

$$j^*(x \otimes_A \mu) = [(\mathcal{E}_1 \otimes_A \mathcal{K}, \hat{c}_+(d \otimes 1) + 1 \otimes \mathcal{D})] = r(\text{rank } E_1) [D]|_V.$$

Now $j^*\lambda_A = \lambda_{\mathcal{C}_2(V)}$, by [31, Prop. 10.8.8], and since the $B$-module structure of $\lambda$ is locally defined, our previous arguments apply to show that $\mathcal{E}_1$ carries a faithful representation of $B$. This immediately implies that rank $E_1 \geq 2^{p/2}$, and we conclude that

$$\text{rank } E_1 = 2^{p/2} \quad \text{and} \quad r = 1.$$

Therefore, the bundle $S$ underlying the module $\mathcal{H}_\infty \approx \Gamma_\infty(X, S)$ is not only globally irreducible as a $B$-module, but is also fibrewise irreducible. Thus, as shown in [48], $S$ is the spinor bundle for a spin$^c$ structure on $X$. Our Propositions 7.7 and 7.30 and Corollaries 7.8 and 7.31 have established that $\mathcal{D}$ is a Dirac operator for this spin$^c$ structure and a suitable metric, up to the addition of an endomorphism of the spinor bundle. Hence $\mu = [(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ is the fundamental class of the spin$^c$ manifold $X$.

Conversely, if $X$ carries a spin$^c$ structure, we may construct the spectral triple of any Dirac operator on its complex spinor bundle. By [2, 18, 31], this class satisfies Poincaré duality in $K$-theory. \hfill \Box

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Remark 8.5. The above proof uses only a little of the structure of the spectral manifold representing $\mu$: we need the first-order property, we require $\Gamma$ to provide the grading on $\mathcal{H}$, and we need the equality $\Gamma = \pi_D(c)$.

We summarize our results by stating the following theorem.

**Theorem 8.6.** Let $(A, \mathcal{H}, D)$ be a spectral manifold of dimension $p$, and suppose that $\mu := [(A \otimes A, \mathcal{H}, D)] \in K^*(A \otimes A)$ satisfies Condition 8. Then $X$ is a spin$^c$ manifold, $\mathcal{H}_\infty$ is the module of smooth sections of its complex spinor bundle, the representation $\mathcal{C}_D(A)$ of the associated Clifford algebra bundle is irreducible, and thus $D$ is, up to adding an endomorphism, a Dirac operator on this spinor bundle. The class of $(A, \mathcal{H}, D)$ is thus the fundamental class of the spin$^c$ manifold $X = \text{sp}(A)$ with the spin$^c$ structure defined by the irreducible representation $\mathcal{C}_D(A)$.

**Proof.** The assertion that $X$ is spin$^c$ follows from Proposition 8.4, under the assumption that the $K$-homology class of $(A, \mathcal{H}, D)$ in $K^{\bullet}(A \otimes A)$ gives a Poincaré duality isomorphism.

Moreover, Proposition 7.30 establishes that $\mathcal{C}_D(A)$ is a representation on $\mathcal{H}_\infty = \Gamma_\infty(\mathcal{X}, S)$ of the algebra $\Gamma_\infty(X, C)$ for the complex Clifford-algebra bundle $C = \mathbb{C} \ell(T^*X, g)$, for a specific Riemannian metric $g$ on $X$.

Thus $\mathcal{H}_\infty$ is a spinor module, $S$ is the corresponding complex spinor bundle, and up to the addition of an $\mathcal{A}$-linear endomorphism of $\mathcal{H}_\infty$, $D$ is the Dirac operator for this metric and spinor bundle. \qed

**Corollary 8.7.** The functional $\mu_D$ is given by the Wodzicki residue:

$$
\mu_D(a) = \int a \langle D \rangle^{-p} = \frac{N \text{Vol}(\mathbb{S}^{n-1})}{p(2\pi)^p} \int_X a(x) \, d\text{vol}_g.
$$

Remark 8.8. The only condition not used at all so far is the “reality” requirement, Condition 9. Since we have now at our disposal a spin$^c$ structure, we only need to refine it to a spin structure. For that, we refer to [48] and [30], wherein it is shown that the spinor module for a spin structure is just the spinor module for a spin$^c$ structure equipped with compatible charge conjugation that is none other than the real structure operator $J$ (acting on $\mathcal{H}_\infty$); and to [50], where the spin structure is extracted, using $J$, from a representation of the real Clifford algebra of $T^*X$. Thus, by additionally invoking Condition 9, we may replace each mention of ‘spin$^c$’ by ‘spin’ in the statement of Theorem 8.6.

9 Conclusion and outlook

We close with several remarks on the hypotheses of the reconstruction theorem, and some possible variants.

In [50] and in [30], under the additional assumption that $\mathcal{A} = C^\infty(X)$, which is now redundant, the Dirac operator for the spinor bundle $S$ and the metric $g$ is also recovered from the given spectral triple, by minimizing the action functional described in [21]: see also [60] for that. Thus most of the statements of [21] can be recovered, if need be.

Poincaré duality in $K$-theory, expressed here as Condition 8, enters our proof only to show that the manifold carries a (preferred) spin$^c$ structure, and identify the fundamental class of that spin$^c$ structure as the $K$-theory class of a spectral triple. We can only make this identification once we know that $\text{sp}(A)$ is a manifold.

It would be more economical to replace Condition 8 with the following one.
Condition 12 (Poincaré duality II: Morita equivalence). The $C^\ast$-module completion of $\mathcal{H}_\infty$ is a Morita equivalence bimodule between $A$ and the norm completion of $\mathcal{C}_D(A)$.

There are good reasons to prefer this Morita equivalence condition. First, we could have employed it from the beginning to simplify the structure of $\mathcal{C}_D(A)$. This would slightly simplify the task of building the manifold.

Also, if we wished to propose axiomatics for different kinds of manifold: Riemannian, complex, Kähler, and so on, the Morita equivalence axiom can be easily replaced with an appropriate characterization of the behaviour of the Clifford algebra. Together with specifying the behaviour of the volume form, $\pi_D(c)$, and adapting the reality condition, these axiomatics should be flexible enough to cope with different types of manifolds. A framework for this has been suggested in [26].

Our proof may be adapted to deal with less smooth cases. If we start from a $QC^k$ spectral triple, $k \in \mathbb{N}$ (see [10] for instance), the completion of $A$ in the $QC^k$-topology is a Banach algebra stable under holomorphic functional calculus. The $C^\infty$ functional calculus will still work, as would a $C^k$ functional calculus. The Lipschitz functional calculus so crucial to our results only requires $QC^0$.

Additional smoothness (beyond $QC^0$) is needed for the following items. To employ the Hochschild class of the Chern character to deduce measurability, $QC^{\text{max}(2,p-2)}$ is needed, although we do not actually use this in our proof. To deduce that the noncommutative integral given by the Dixmier trace defines a hypertrace on $A$ needs only $QC^0$, but to obtain a trace on $\mathcal{C}_D(A)$ requires $QC^2$ [14]; the question is further explored in [12, Sec. 6]. Finally, Lemma 7.5 requires more than $QC^0$. Observe that the smoothness used in the unique continuation arguments is a manifestation of ellipticity on $\mathbb{R}^p$; we do not need to assume regularity of $A$ for that, the smoothness we use is that of the eigenvectors of an elliptic operator.

Reformulating the proof to deal with less smoothness would be somewhat inelegant, as the metric $g(da, db)$ need only take values in bounded measurable functions, so we would need to work with more algebras, such as the bicommutant $A'\prime$. Nevertheless, it appears feasible. In this vein, one should replace $\mathcal{H}_\infty$ with $\text{Dom} \mathcal{D}$, and similarly modify the finiteness condition.

It is fascinating to reflect on just what input gave us a manifold. While of course we needed everything, the role of the Dirac operator is crucial in producing the coordinate charts. It would be of some interest to understand this in the language of geometric topology.

Geometrically, the weak unique continuation property may be regarded as a consequence of the local splitting $\mathcal{D} = n(\partial_\xi + A)$ for any direction $\xi$ [4]. Thus Dirac operators know how to split any neighbourhood into a product of a hypersurface and its normal.

The strong unique continuation property is better understood by analogy with holomorphic functions and Cauchy–Riemann operators. Essentially, Dirac type operators have “sufficiently rigid” eigenvectors to determine the local geometry in the neighbourhood of a point.

The closedness property, our Condition 11, does not appear explicitly in the axiom scheme proposed by Connes in [21]. However, it played a critical role in earlier formulations of Poincaré duality at the level of chains, needed to address colour symmetry in the Standard Model [18, VI.4.γ] and we hope to have exemplified that it is still a useful tool.

The relation of this closedness condition with Condition 8, that of Poincaré duality in $K$-theory, deserves further scrutiny. We leave open the question of whether a manifold could be reconstructed if one chooses to drop Condition 11 in favour of $K$-theoretic Poincaré duality alone. We suspect not; on the other hand, neither do we have a counterexample.
A Hermitian pairings on finite projective modules

The finiteness condition calls for a Hermitian inner product on a finite projective module with specific properties. We investigate the nature of such Hermitian pairings here.

There is a small question of starting points to deal with. If we take as given the scalar product on the Hilbert space, and try to find a Hermitian pairing to satisfy the finiteness condition, we must tackle a difficult existence problem. On the other hand, if we suppose that there is a Hermitian pairing related to the scalar product as in Condition 4 above, then we face a question of characterisation. Here we adopt this latter point of view.

Lemma A.1. Let $A$ be a unital $C^*$-algebra and let $q \in M_m(A)$ be a projector, $q = q^* = q^2$. Suppose that $M \in qM_m(A)q$. Then $M$ is invertible in $qM_m(A)q$ if and only if $M + 1 - q$ is invertible in $M_m(A)$.

Proof. If $M$ is invertible in $qM_m(A)q$ with inverse $M^{-1}$ satisfying $MM^{-1} = M^{-1}M = q$, then $M^{-1} + 1 - q$ is inverse to $M + 1 - q$ in $M_m(A)$.

Conversely, suppose that $M + 1 - q$ is invertible, with inverse $N$ in $M_m(A)$. Then $Nq$ is inverse to $M$ in $qM_m(A)q$, since $N(M + 1 - q) = 1$ implies $q = NM = qNMq = qNqM$, and similarly $(M + 1 - q)N = 1$ implies $q = MN = qMNq = MqNq$. □

Lemma A.2. Let $A$ be a unital $C^*$-algebra, and $E = qA^m$ a finite projective right $A$-module. Then every Hermitian pairing on $E$ is of the form

$$(e \mid f) = \sum_{j,k} e_j^* M_{jk} f_k$$

where $e = (e_1, \ldots, e_m)^T$ with each $e_j \in A$, $qe = e$; similarly for $f$; and $M \in qM_m(A)q$ is positive.

Proof. Denote by $\varepsilon \in A^m$ the column vector with 1 in the $j$-th spot, and zeroes elsewhere. We let $x_j = q\varepsilon_j = \sum_k q_{kj} \varepsilon_k$, so we can write all $e \in E$ as $e = \sum_j x_j e_j$.

If $(\cdot \mid \cdot)$ is a Hermitian pairing on $E$, then for any $e \in E$,

$$0 \leq (e \mid e) = \sum_{j,k} e_j^* (x_j \mid x_k) e_k.$$ 

Hence the matrix $M = [(x_j \mid x_k)]_{j,k} \in M_m(A)$ is positive, by [30, Proposition 1.20]. Next

$$(qM)_{mk} = \sum_j q_{mj}(x_j \mid x_k) = \sum_j q_{mj}(\sum_l q_{lj}\varepsilon_l \mid x_k) = \sum_{j,l} q_{mj}(\varepsilon_l q_{lj} \mid x_k)$$

$$= \sum_{j,l} q_{mj} q_{lj} (\varepsilon_l \mid x_k) = \sum_l q_{ml} (\varepsilon_l \mid x_k) = (x_m \mid x_k) = M_{mk},$$

and similarly $Mq = M$. □

Lemma A.3. Let $A$ be a unital $C^*$-algebra, and $E = qA^m$ a finite projective right $A$-module. Let $M \in qM_m(A)q$ be positive, so that

$$(e \mid f)_M := \sum_{j,k} e_j^* M_{jk} f_k, \quad \text{for all} \quad e, f \in E,$$ \hspace{1cm} (A.1)

defines a Hermitian pairing on $E$. Then the right $C^*$ $A$-module $(E, (\cdot \mid \cdot)_M)$ is full if and only if $M$ is invertible in $qM_m(A)q$. 

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Proof. First suppose that the $C^*$ $A$-module $(E, \langle \cdot | \cdot \rangle_M)$ is full. By [49], the compact endomorphisms of the full right $A$-module $E$ fulfil $\text{End}_A^0(E) = \text{End}_A(E) = qM_m(A)q$. This algebra is generated by the rank-one operators $\Theta^q_{e,f}: E \to E$ defined by

$$\Theta^q_{e,f}(g) = e(f | g)_M = \Theta^q_{e,f}(Mg),$$

where $(e | f)_q := \sum_{j,k} e_j^* q_{jk} f_k$. The operators $\Theta^q_{e,f}$ generate $qM_m(A)q$ as an algebra. If $M$ is not invertible in $qM_m(A)q$, the operators $\Theta^q_{e,f} M$ and their adjoints generate a proper two-sided ideal of it, contradicting fullness.

Conversely, suppose that $M$ is invertible but that $E$ is not full. Let $I$ be the closure of $(E | E)_M$ in $A$, a two-sided ideal, so that $E$ is a full right $I$-module, and thus $\text{End}_A^0(E) = \text{End}_I^0(E) = qM_m(I)q$. Then the algebra generated by all $\Theta^q_{e,f} M = \Theta^q_{e,f} M_q$ is $qM_m(I)q$. But the invertibility of $M$ entails that the $\Theta^q_{e,f} M$ generate all of $qM_m(A)q$: contradiction. □

Lemma A.4. Let $A$ be a dense pre-$C^*$-subalgebra of the unital $C^*$-algebra $A$ with $1 \in A$. If $\langle \cdot | \cdot \rangle$ is an $A$-valued Hermitian pairing on $E = qA^m$ making $E$ full, with $q = q^* = q^2 \in M_m(A)$, then it is given by $M \in qM_m(A)q$ as in Lemma A.2.

Proof. By [49, Lemma 2.16], the Hermitian form $(\cdot | \cdot)$ on $E$ has a canonical extension to the completion $E = qA^m$. By Lemma A.2, this extension is determined by a positive invertible $M \in qM_m(A)q$. Since $(x_j | x_k) \in A$ for all $j, k$, we obtain $M \in qM_m(A)q$.

Observe also that by using the stability under the holomorphic functional calculus of $A$, we find $M^{-1} \in qM_m(A)q$ also. □

Proposition A.5. Let $(A, \mathcal{H}, \mathcal{D})$ be a spectral triple satisfying Condition 1, Conditions 3–6 and Condition 10 of subsection 3.1 (dimension, regularity, finiteness, absolute continuity, first order, irreducibility). Then its Hermitian pairing $(\cdot | \cdot)$ on $\mathcal{H}_\infty = qA^m$ is (a positive multiple of) the standard one.

Proof. Let $M \in qM_m(A)q$ be the positive invertible element such that $(\xi | \eta) = (\xi | \eta)_M$ for $\xi, \eta \in \mathcal{H}_\infty$. Then, for each $a \in A$,

$$\langle \xi | a\eta \rangle = \langle a^* \xi | \eta \rangle = \int (a^* \xi | M^{-1} a M \eta)_M \langle \mathcal{D}^{-p} = \int (\xi | M^{-1} a M \eta)_M \langle \mathcal{D}^{-p} = \langle \xi | M^{-1} a M \eta \rangle.$$

Hence $[M, a] = 0$ for all $a \in A$.

Now since $\mathcal{D}$ is a selfadjoint operator on $\mathcal{H}$, we obtain, for $\xi, \eta \in \mathcal{H}_\infty$:

$$0 = \langle \mathcal{D} \xi | \eta \rangle - \langle \xi | \mathcal{D} \eta \rangle = \int ((\mathcal{D} \xi | \eta)_M - (\xi | \mathcal{D} \eta)_M) \langle \mathcal{D}^{-p} = \int ((\mathcal{D} \xi | M \eta)_q - (\xi | M \mathcal{D} \eta)_q) \langle \mathcal{D}^{-p} =: \langle \langle \mathcal{D} \xi | M \eta \rangle - \langle M \xi | \mathcal{D} \eta \rangle \rangle \quad (A.2)$$

where $\langle \langle \cdot | \cdot \rangle := \langle M^{-1} \cdot | \cdot \rangle$ defines a new Hilbert space scalar product. Since $M^{-1}$ is bounded with bounded inverse, this new scalar product $\langle \cdot | \cdot \rangle$ is topologically equivalent to the old one $\langle \cdot | \cdot \rangle$, and so $\mathcal{H}$ is the completion of $\mathcal{H}_\infty$ with respect to either scalar product.

Now the right hand side of (A.2) is the quadratic form defining the commutator $[\mathcal{D}, M]$ with respect to the scalar product $\langle \cdot | \cdot \rangle$. It vanishes on $\mathcal{H}_\infty$ and thus $[\mathcal{D}, M] = 0$. Now the irreducibility condition implies that $M$ is (a positive multiple of) the identity $q$ in $qM_m(A)q$, represented by a scalar operator on $\mathcal{H}$. □
**B  Another look at the geometric conditions**

In this Appendix we consider the potential redundancy of, and the relations between, our metric and irreducibility postulates, Conditions 2 and 10.

Our Condition 2 is unnecessarily strong as stated, since the boundedness of the set \{ \eta(a) \in \mathcal{A}/\mathbb{C} \setminus 1 : \|[\mathcal{D}, a]\| \leq 1 \} in the Banach space \mathcal{A}/\mathbb{C} 1 ensures that the distance formula (3.2) not only defines a metric, but one for which \( X \) has finite diameter. However, the reconstruction of a manifold requires merely a metric, with its corresponding topology.

Indeed, with no additional assumptions, it is easy to see that the formula (3.2) defines a function \( d \) satisfying all the properties of a metric distance, except that there might exist states \( \phi, \psi \) for which \( d(\phi, \psi) = \infty \).

Thus we could (provisionally) adopt the following condition:

For all states \( \phi, \psi \) of \( \mathcal{A} \), the set \{ \( |\phi(a) - \psi(a)| : \|[\mathcal{D}, a]\| \leq 1 \) \} is bounded.

Our original Condition 2 asks for a uniform bound for all \( \phi, \psi \). As it turns out, this condition can be dispensed with altogether.

**Proposition B.1.** Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be a spectral triple such that \( \mathcal{A} \) is unital, \( \mathcal{A}\mathcal{H} \) is dense in \( \mathcal{H} \), and \( \mathcal{A} \) has separable norm closure \( \mathcal{A} \). Assume that \([\mathcal{D}, a] = 0\) if and only if \( a = \lambda 1 \) for some \( \lambda \in \mathbb{C} \). Then the formula

\[
    d_\mathcal{D}(\phi, \psi) := \sup \{ |\phi(a) - \psi(a)| : \|[\mathcal{D}, a]\| \leq 1 \}
\]

defines a metric on the state space of \( \mathcal{A} \).

**Proof.** We need only show that \( d(\phi, \psi) = \infty \) cannot occur for any pair of states \( \phi, \psi \).

Without loss of generality, we replace \( \mathcal{A} \) by its completion in the norm \( a \mapsto \|a\| + \|[\mathcal{D}, a]\| \), since this will not change the definition of the metric or the norm closure.

Suppose, then, that there is a pair of states \( \phi, \psi \) with \( d(\phi, \psi) = \infty \). There exists (using the separability of \( \mathcal{A} \)) a sequence \( \{a_N\}_{N \geq 1} \subset \mathcal{A} \) such that

\[
    |\phi(a_N) - \psi(a_N)| > N \quad \text{and} \quad \|[\mathcal{D}, a_N]\| \leq 1 \quad \text{for all } N.
\]

On replacing \( a_N \) by \( a_N - \psi(a_N) 1 \) if necessary, we may assume that \( \psi(a_N) = 0 \), and therefore

\[
    \|a_N\| \geq |\phi(a_N)| > N \quad \text{for all } N \in \mathbb{N}.
\]

Set \( u_N := a_N/\|a_N\| \), so that \( \|u_N\| = 1 \) for all \( N \).

Let \( \xi \in \mathcal{H} \) and observe that \( \mathcal{D} \) has compact resolvent: the definition of spectral triple assumes \( a(\mathcal{D} - \lambda)^{-1} \) is compact for all \( a \in \mathcal{A} \) and \( \lambda \notin \text{sp(\mathcal{D})} \), and since the representation of \( \mathcal{A} \) is nondegenerate, \( 1 \in \mathcal{A} \) acts as the identity on \( \mathcal{H} \). Choose and fix \( \lambda \in \mathbb{C} \setminus \text{sp(\mathcal{D})} \). The sequence \( \{(\mathcal{D} - \lambda)^{-1}u_N\xi\} \) has a Cauchy subsequence, by compactness of \( (\mathcal{D} - \lambda)^{-1} \) and boundedness of the sequence \( \{u_N\xi\} \). Each \( u_N \) lies in \( \text{Dom } \mathcal{D} \), so

\[
    (\mathcal{D} - \lambda)^{-1}u_N\xi = - (\mathcal{D} - \lambda)^{-1}[\mathcal{D}, u_N](\mathcal{D} - \lambda)^{-1}\xi + u_N(\mathcal{D} - \lambda)^{-1}\xi.
\]

Now \([\mathcal{D}, u_N] \rightarrow 0\) in norm, so there is a subsequence \( \{u_{N_j}\} \) such that \( \{u_{N_j}(\mathcal{D} - \lambda)^{-1}\xi\} \) is Cauchy. Since every \( \xi \in \text{Dom } \mathcal{D} \) is of the form \( \xi = (\mathcal{D} - \lambda)^{-1}\xi \) for some \( \xi \in \mathcal{H} \), we see that for every \( \zeta \in \text{Dom } \mathcal{D} \) there is a Cauchy subsequence \( \{u_{N_j}\} \).

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Let \( \{ \zeta_k \}_{k \geq 1} \) be an orthonormal basis of \( \mathcal{H} \) consisting of eigenvectors of \( D \). By applying the above argument successively to \( \xi_k := (D - \lambda) \zeta_k \) for \( k = 1, \ldots, m \), we can extract from \( \{ u_N \} \) a subsequence \( \{ u_{j_N}^{(m)} \} \) such that \( u_{j_N}^{(m)} \) converges, for \( k = 1, \ldots, m \). Inductively, we can choose a subsequence \( \{ u_{j_N}^{(m+1)} \} \) of \( \{ u_{j_N}^{(m)} \} \) so that \( u_{j_N}^{(m+1)} = u_{j_N}^{(m)} \) for \( j \leq m \) and \( \{ u_{j_N}^{(m+1)} \} \) converges, too. In the end, we obtain a subsequence \( \{ u_{j_N}^{(\infty)} \} \) of \( \{ u_N \} \) such that \( u_{j_N}^{(\infty)} \zeta_k \) converges in \( \mathcal{H} \) as \( j \to \infty \), for each basic eigenvector \( \zeta_k \). We rename this subsequence to \( \{ u_N \} \), noting that (B.1) still holds.

Note that the resulting subsequence cannot terminate, since if \( u_n = u_m \) for all \( m, n \geq N_0 \), then \( a_m = \lambda_m a_{N_0} \) for \( n \geq N_0 \), and \( \| [D, a_m] \| = |\lambda_m| \| [D, a_{N_0}] \| \leq 1 \), implying \( |\lambda_m| \leq \| [D, a_{N_0}] \|^{-1} \). But then

\[
|\phi(a_m)| = |\lambda_m| |\phi(a_{N_0})| \leq \frac{|\phi(a_{N_0})|}{\| [D, a_{N_0}] \|},
\]

contradicting the unboundedness (B.1) of the sequence \( |\phi(a_m)| \).

If \( \zeta \) is an eigenvector of \( (D - \lambda)^{-1} \) with eigenvalue \( \rho \), then

\[
(D - \lambda)u_N \zeta = [D, u_N] \zeta + \rho^{-1}u_N \zeta. \tag{B.2}
\]

The right hand side converges in \( \mathcal{H} \) and \( [D, u_N] \to 0 \) in norm, so \( \lim_N u_N \zeta \) lies in \( \text{Dom} \, D \) and is an eigenvector of \( D - \lambda \) with eigenvalue \( 1/\rho \).

Let \( P_\rho \) denote the (finite rank) projector whose range is the eigenspace of \( D \) for the eigenvalue \( \lambda + \rho \). Then

\[
\lim_{N \to \infty} \| P_\rho u_N P_\rho - u_N P_\rho \| = 0,
\]

the norm limit being appropriate since \( P_\rho \mathcal{H} \) is finite-dimensional. Let \( V_\rho := \lim_N P_\rho u_N P_\rho \), a finite-rank operator. Now (B.2) entails

\[
\rho^{-1} V_\rho \zeta = \rho^{-1} \lim_N u_N \zeta = (D - \lambda) \lim_N [D, u_N] \zeta + \rho^{-1} \lim_N u_N \zeta,
\]

and thus \( V_\rho \zeta \to u_N \zeta \) for \( \zeta \in P_\rho \mathcal{H} \). Therefore, if \( \nu := \bigoplus_{\rho \in \text{sp}(D - \lambda)} V_\rho \), then

\[
\lim_N \| u_N - \nu \| = \lim_N \left\| \bigoplus_{\rho \in \text{sp}(D - \lambda)} \rho [D, u_N] P_\rho \right\| = 0,
\]

thus the sequence \( \{ u_N \} \) is actually norm-convergent in \( \mathcal{L}(\mathcal{H}) \).

In particular, since \( [D, u_N] \to 0 \), the sequence \( \{ u_N \} \) is Cauchy in the norm \( a \mapsto \| a \| + \| [D, a] \| \). Hence \( \nu = \lim_N u_N \) lies in \( \mathcal{A} \). Again using (B.2), we see that \( [D, \nu] = \lim_N [D, u_N] = 0 \). Our hypothesis that \( [D, a] = 0 \) only for \( a \in \mathcal{C} \) now implies that \( \nu =: \sigma \) is a scalar.

We have already chosen the several \( a_N \) (and so also the \( u_N \)) so that \( \psi(a_N) = \psi(u_N) = 0 \). Therefore, \( \sigma = \lim_N \psi(u_N) = 0 \); we conclude that \( u_N \to 0 \) in norm. However, each \( u_N \) has norm 1, so we have arrived at a contradiction.

In fine, \( d_D(\phi, \psi) < \infty \) holds for all \( \phi, \psi \); thus \( d_D \) is a metric on the state space of \( A \). \( \square \)

Consider now the irreducibility, as given by Condition 10. This condition is imposed to guarantee a connected spectrum (in the commutative case), and to ensure that the Hilbert space is not too large.
However, already in the commutative case this irreducibility condition has two aspects. The first is the connectedness of the spectrum: see Lemma 3.13 and Corollary 3.14. The second is the irreducibility of $\mathcal{H}_\infty$ as a $\mathbb{C}_D(\mathcal{A})$-module, best controlled using our proposed Condition 12 (the Morita-equivalence version of the spin$^c$ condition). Inspection of the proof shows that in the reconstruction of a manifold we invoke irreducibility both to specify the Hermitian pairing and to deduce that the spectrum is connected, with no nontrivial projectors existing in $\mathcal{A}$ or $\mathcal{A}$, in the cited Lemma and Corollary. (Once the manifold has been found, we invoke it once more to get irreducibility of $\mathcal{H}_\infty$ via Proposition 7.28.)

A closer look shows that, without the irreducibility condition, the rest of the proof holds with occasional adjustments to take account of the possible nonconstancy of the rank of the bundle $S \to X$ (which of course remains locally constant), provided we can guarantee the existence of the metric defined by $D$.

A disconnected spectrum would be a disjoint union of closed components $X = \bigcup_j X_j$. If there are only finitely many components $X_j$, then they are clopen (i.e., both closed and open). However, if there be infinitely many components, they would all be clopen if and only if the space $X$ is locally connected. When $X$ is compact, this of course forces the number of components to be finite.

If some component is not clopen, then its characteristic function is not continuous and so does not lie in the $C^*$-algebra $C(X)$. In the given (faithful, nondegenerate) representation $\pi$ of $C(X)$, we can decompose $\text{Id}_{\mathcal{H}} = \pi(1_{C(X)}) = \sum_j \pi(p_j)$ for a sum of projectors $p_j \in C(X)$. Then this sum is finite by the separability of $C(X)$. Hence there can only be finitely many clopen components of $X$, say $X_1, \ldots, X_n$, and $\bigcup_{j>n} X_j$ is also clopen.

Since $D$ is a local operator, by Corollary 4.14, the (continuous) characteristic functions of the clopen components lie in the domain of $d = \text{ad} D$ in $\mathcal{A}$. A little more thought shows that these continuous characteristic functions actually lie in $\mathcal{A}$. Therefore, provided we can ensure the existence of the metric defined by $D$ on each clopen piece of the decomposition of $X$, we may proceed as follows:

1. On each clopen subset of $X$ with continuous characteristic function, run the existing argument (with modifications for nonconstant rank of $S$) to deduce that this clopen piece is a topological manifold;

2. observe that topological manifolds are locally connected (indeed, locally path connected);

3. deduce that all connected components of $X$ are clopen;

4. conclude that there are only finitely many components, and that $X$ is a disjoint union of finitely many connected compact topological manifolds.

Since we should be able to deal with disconnected manifolds as well as connected ones, we propose the following alternative to Condition 10.

**Condition 13 (Connectivity).** There is an orthogonal family of projectors $p_j \in \mathcal{A}$ such that $1_{\mathcal{A}} = \sum_j p_j$ and

$$ (a \in \mathcal{A} \quad \text{with} \quad [D, a] = 0) \iff a = \sum_j \lambda_j p_j \quad \text{for some} \{\lambda_j\} \subset \mathbb{C}. $$

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The separability of $A$ guarantees that these sums are finite, so convergence questions are moot. We may assume the family $\{p_j\}$ to be maximal. The original spectral triple breaks up as a finite direct sum of spectral triples:

$$(\mathcal{A}, \mathcal{H}, \mathcal{D}) = \bigoplus_j (p_j A p_j, p_j \mathcal{H}, p_j \mathcal{D} p_j),$$

each of which satisfies the remaining conditions. By Proposition B.1 and Condition 13, each $p_j A p_j$ has a connected spectrum carrying a metric defined by $p_j \mathcal{D} p_j$. The procedure above allows us to deduce that $X = \text{sp}(A)$ is a disjoint union of finitely many compact connected manifolds $X_j$.

We may then introduce Morita equivalence, as in Condition 12, to control the irreducibility of the vector bundle $\mathcal{S} \to X$ as a $\mathcal{C}_D(\mathcal{A})$-module. Write $A_j := p_j A p_j$, $\mathcal{H}_{\infty,j} := p_j \mathcal{H}_{\infty}$ and $\mathcal{D}_j := p_j \mathcal{D} p_j$, and note that $A$ acts block-diagonally, that is, $\mathcal{H}_{\infty,j} \cdot A = \mathcal{H}_{\infty,j} \cdot A_j$. Then, since $\text{End}_A(\mathcal{H}_{\infty})$ consists of all adjointable linear operators on $\mathcal{H}_{\infty}$ commuting with the right action of $A$, we see that

$$\text{End}_A(\mathcal{H}_{\infty}) = \text{End}_A \left( \bigoplus_j \mathcal{H}_{\infty,j} \right) = \bigoplus_{i,j} \text{Hom}_A(\mathcal{H}_{\infty,i}, \mathcal{H}_{\infty,j}) \supseteq \bigoplus_j \text{End}_{A_j}(\mathcal{H}_{\infty,j}).$$

(If preferred, one may restate these relations using the norm completions.)

The Morita equivalence condition amounts to $\mathcal{C}_D(A)$ being generated by rank-one endomorphisms $\Theta_\xi, \eta$, for $\xi, \eta \in \mathcal{H}_{\infty}$; and $\Theta_\xi, \eta = 0$ if $\xi \in \mathcal{H}_{\infty,j}$ and $\eta \in \mathcal{H}_{\infty,k}$ with $j \neq k$. Hence the algebra generated by the rank-one operators is block diagonal with respect to the direct sum decomposition, and therefore

$$\mathcal{C}_D(A) = \bigoplus_j \text{End}_{A_j}(\mathcal{H}_{\infty,j}) = \bigoplus_j \mathcal{C}_{D_j}(A_j).$$

Thus the Morita equivalence condition for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ implies the same one for each $(A_j, \mathcal{H}_j, \mathcal{D}_j)$.

Let $T \in \mathcal{B}(\mathcal{H}_j)$ (strongly) commute with $\mathcal{D}_j$ and $A_j$, for some $j$. Then $T$ preserves the domain of $\mathcal{D}_j$ (since it lies in the commutant of the von Neumann algebra generated by $A_j$ and the spectral projectors of $\mathcal{D}_j$), and so maps $\mathcal{H}_{\infty,j}$ to itself. Since $T$ commutes with the functions in $A_j$, it is local and so is an endomorphism of the bundle $S|_{X_j}$, and from Morita equivalence these endomorphisms are precisely the elements of the norm closure of $p_j \mathcal{C}_D(A)p_j$. Hence $T$ is a central element of this norm closure, and since $p_j \mathcal{C}_D(A)p_j$ is an irreducible representation of a complex Clifford algebra, $T$ is a function in $C(X_j)$. Since $T$ preserves $\mathcal{H}_{\infty}$, it is smooth and thus $T \in \mathcal{A}_j$; and $[\mathcal{D}_j, T] = 0$ now implies that $T$ is constant on $X_j$.

Hence over each connected component of $X$ the irreducibility condition holds automatically when we assume both the connectivity and Morita equivalence conditions.

References


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