A nonperturbative form of the spectral action principle in noncommutative geometry

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Abstract

Using the formalism of superconnections, we show the existence of a bosonic action functional for the standard K-cycle in noncommutative geometry, giving rise, through the spectral action principle, only to the Einstein gravity and Standard Model Yang–Mills–Higgs terms.

Introduction

The approaches to fundamental interactions based on noncommutative geometry (NCG) have so far yielded action functionals both for elementary particles (tying together the gauge bosons and the Higgs sector) and for gravity [1–6]. The challenge of unifying the Yang–Mills and gravitational actions was taken up in [7] and subsequently Chamseddine and Connes put forward a model [8, 9] doing so by means of a so-called universal action functional. The NCG method is based on the hypothesis that fundamental interactions are coded in the invariants of a suitably generalized Dirac operator, involving spacetime and internal variables. This bold introduction of spectral geometry in physics has important consequences, even for classical relativity [10]. In the Chamseddine–Connes (CC) Ansatz, particle species are taken as given. Setting the fermionic action is equivalent to fixing a "real K-cycle" comprising the generalized Dirac operator, grading and conjugation for the theory. The bosonic action is constructed out of the K-cycle. Papers [8, 9] start from the K-cycle currently [2, 3] associated to the Standard Model (or standard K-cycle) and concentrate on aspects of that construction depending only weakly on that choice; thus the adjective "universal".

The CC approach has two important merits, namely the possibility of a genuine unification of particle theories and gravity and the introduction of a renormalization process to control the mix of physical scales involved. Nonetheless, there are some difficulties that remain to be addressed. First of all, the particular approach taken by Chamseddine and Connes raises mathematical questions about the information content of the asymptotic developments used [11]. Secondly, their action has a number of extra terms one could perhaps do without. The leading term is a huge cosmological constant that has to be "renormalized away" with fine tuning. The gravity part of the third term contribution, comprising a Weyl gravity term and a term coupling gravity with the Higgs field, is conformally invariant. It is unclear at present if the latter is more an asset or a liability in black hole dynamics and in cosmology [12]. Also, the renormalization scheme proposed in [8,9] exhibits some surprising traits. The CC Lagrangian is neither renormalizable *strictu sensu*, nor unitary within the usual perturbation theory [13, 14]. The first objection is not considered serious in the modern effective field approach to quantum field theory [15]. The second objection is dismissed on the grounds that we expect the product geometry to be replaced by a truly noncommutative geometry at some energy scale lower than the Planck mass. In view of the fact that the most natural coefficients for the boson fields they obtain yield SU(5)-type relations for the chromodynamical and flavourdynamical coupling constants, Chamseddine and Connes chose a cutoff scale of the order 10^{15} GeV; they ran in conflict with the value of Newton's constant. We really do not know the energy scale at which the NCG relations can claim validity, as the theory still lacks a physical unifying mechanism at the 10^{15} GeV or other scale.

Models based on the "universal" functional concept are also aesthetically unappealing to some. One can hardly help being mesmerized by the beauty of the results of [5, 6], in which a particular regularized functional, the Wodzicki residue of the inverse squared (ordinary) Dirac operator, gives directly the Einstein–Hilbert functional for gravity. The idea of then keeping the Wodzicki functional and further modifying the Dirac operator, in such a way that all the action terms of the Standard Model plus gravity – and only them – are obtained, was proposed by Ackermann [16] and spelled out recently by Tolksdorf [17]. The Ackermann–Tolksdorf (AT) formalism – that falls outside NCG – has its own drawbacks, however: their manipulation of the Dirac operator physically amounts to a nonminimal coupling of the fermions and the gauge fields, that would give rise to a direct interaction between two fermions and two bosons, which has never been seen. Moreover, the fermion doubling demonstrated in [18] is compounded.

Underlying Ackermann and Tolksdorf's attempt, there is perhaps the impression that a pure, combined Einstein–SM Lagrangian cannot be obtained within NCG. Nevertheless, using Quillen's superconnection formalism as a tool, in this paper we show that one can obtain the pure Einstein–SM Lagrangian at the tree level from the same standard K-cycle used by Chamseddine and Connes.

Action functionals in NCG

A NCG model is determined by an algebra $\mathcal{A}$ having a representation on a Hilbert space $\mathcal{H}$, on which there also act a grading operator γ, a conjugation J and a selfadjoint operator D, odd with respect to γ and commuting with J, with suitable properties vis-a-vis the algebra; in particular one requires that the operators $[D, a]$ commute with JbJ^{-1}, for a, b in $\mathcal{A}$. This five-term package [2, 3] is called a "spectral triple" or a "real K-cycle".

As stated in [7], a commutative K-cycle is just the spectral version of a Riemannian spin manifold (a compact spacetime, able to uphold fermions, with Euclidean signature). Let M be such a manifold, with dimension n; we take $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$, the space of square-integrable spinors over M, $\gamma = \gamma_5$, J is charge conjugation of the spinors and in this case the Dirac operator is the usual one, including the spin connection: $D = \not{D} = \gamma^a(\partial_a + \omega_a)$, where ω is the spin connection 1-form. The metric tensor on M (and then its functionals) is completely determined by the K-cycle.

At the other extreme, $\mathcal{A} = \mathcal{A}_F$ could be finite-dimensional but noncommutative, $\mathcal{H}_F$ also finite-dimensional and graded, in this case $D = D_F$, an odd matrix; this K-cycle describes a (noncommutative) internal space. In the applications to the Standard Model the entries of D_F are Yukawa–Kobayashi–Maskawa parameters: they are seen as part and parcel of the geometry.

All K-cycles employed in NCG till now are "mildly noncommutative" product K-cycles, where $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\mathcal{H} = L^2(S) \otimes \mathcal{H}_F$ and the "free" Dirac operator is given by $D = D_{\text{free}} = \gamma^a \partial_a \otimes 1 + 1 \otimes D_F$. We call the second piece a Dirac–Yukawa operator. To turn the NCG machinery, one needs to introduce the noncommutative gauge potential A_{nc}, a general selfadjoint element of the form $\sum a[D_{\text{free}}, b]$ corresponding to the "fluctuations" of the internal degrees of freedom. To this one adds the spin connection. For the standard K-cycle, $\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$, and in that way one reproduces the fermionic part of the Standard Model Lagrangian.

According to the spectral action principle, the bosonic action B depends on the whole K-cycle; we shall write $B[D]$ for short. One postulates the Lagrangian density

$$\mathcal{L} = \langle \psi \mid PDP\psi \rangle + B[D],$$

where P projects on the subspace of the physical Weyl fermions. We shall concentrate on $B[D]$. This bosonic part in the original model and its subsequent modifications was fabricated following a "differential" path as follows: given the noncommutative gauge potential A_{nc}, one constructed its curvature $F_{\text{nc}} = [D, A_{\text{nc}}] + A_{\text{nc}}^2$ (a far from straightforward task, due to the ambiguity of the NCG differential structure), and the action was taken to be proportional to $⨍ F^2\, ds^4$ (see further on for the definition of the noncommutative integral $⨍$). In the case of the standard K-cycle, this indeed defines the SM Yang–Mills action and the action for the Higgs field, including the usually ad hoc quartic potential. Thus "low energy" particle interactions were unified in a single term, the square of a geometric object, excluding Einstein gravity.

In this paper, we follow the CC approach in exploring a fully "integral" path for the construction of $B[D]$.

Due to the product structure of the K-cycle, the fermionic states in NCG so far always live in spaces of sections of superbundles. We formalize this last remark. Suppose, for definiteness, that M is an even-dimensional manifold, with a spin structure; let S be the spinor bundle; write $\mathbb{C}\ell\, M$ for the bundle over M whose fibre at x is the complex Clifford algebra $\mathbb{C}\ell(T_x^* M)$; the smooth sections of these bundles form respectively the space of spinors $\Gamma(S)$ and the algebra $\mathcal{C} := \Gamma(\mathbb{C}\ell\, M)$. This algebra acts irreducibly on $\Gamma(S)$, that is, $\mathcal{C} \simeq \operatorname{End} S$; this can be taken as defining the spin structure [19]. If we think of $\mathcal{H}_F$ as the trivial bundle $\mathcal{H}_F \times M$, then $\mathcal{H}$ can be identified to the space of sections $\Gamma(S \otimes \mathcal{H}_F)$ of the tensor product superbundle $S \otimes \mathcal{H}_F$. Any superbundle $E = E^+ \oplus E^-$ on which a graded action of $\mathbb{C}\ell\, M$ is defined (so $\mathcal{C}$ acts on its space of sections) is called a *Clifford module*. Denote by c the action of $\mathcal{C}$ on S; $\alpha \in \mathcal{C}$ acts on $\Gamma(S \otimes \mathcal{H}_F)$ by

$$\psi \otimes \omega \mapsto c(\alpha)\psi \otimes \omega.$$

The passage from S to $S \otimes \mathcal{H}_F$ is an instance of "twisting" of the spinor bundle. On a spin manifold, any Clifford module $\Gamma(E)$ comes from a twisting [20]: by Schur's lemma, any map from $\Gamma(S)$ to $\Gamma(E)$ that commutes with the Clifford action is of the form $\psi \mapsto \psi \otimes \omega$, for ω a section of the (graded) bundle of intertwining maps $W := \operatorname{Hom}_{\mathbb{C}\ell\, M}(S, E)$. Moreover, any endomorphism of $\Gamma(E)$ that commutes with the Clifford action is of the form $\psi \otimes \omega \mapsto \psi \otimes T\omega$ for some bundle map $T\colon W \to W$; in other words, $\operatorname{End}_{\mathbb{C}\ell\, M} E \simeq 1 \otimes \operatorname{End} W$. The whole matrix bundle $\operatorname{End} E$ is generated by the subbundle $\mathbb{C}\ell\, M \simeq \operatorname{End} S \otimes 1$, acting by the spin representation, and by its commutator $1 \otimes \operatorname{End} W$, so we can write $\operatorname{End} E \simeq \mathbb{C}\ell\, M \otimes \operatorname{End} W$.

The analogue of the volume element in noncommutative geometry is the operator $D^{-n} =: ds^n$. And pertinent operators are realized as pseudodifferential operators on the spaces of sections. Extending previous definitions by Connes [1], we introduce a noncommutative integral based on the Wodzicki residue [21]:

$$\fint P\, ds^n := \frac{(\frac{1}{2}n-1)!}{2(2\pi)^{n/2}} \operatorname{Wres} P|D|^{-n} := \frac{(\frac{1}{2}n-1)!}{2(2\pi)^{n/2}} \int_{S^*M} \operatorname{tr} \sigma_{-n}(P|D|^{-n})(x,\xi)\, d\xi\, dx.$$

Here $\sigma_{-n}(A)$ denotes the $(-n)$th-order piece of the complete symbol of A and the numerical coefficient is appropriate for an even dimensional manifold. The Wodzicki residue is known to be the only trace on the space of pseudodifferential operators. The noncommutative integral $\fint$ is a trace on a very large space of operators [22, 23]. The definition is justified by the fact that, for the archetypical commutative K-cycle there holds

$$\fint f\, ds^n = \int f(x)\, d^n x,$$

for $f \in C^\infty(M)$, represented as a multiplication operator on $L^2(S)$. From now on, we take $n = 4$.

It was natural, however, for NCG to be asked about the gravitational interaction, which, after all, is nothing but the manifestation of the commutative geometry of spacetime. But it turns out that to use the operators $D = D\!\!\!\!/ + 1 \otimes D_F$, or $D = \partial\!\!\!/ + 1 \otimes D_F$ i.e., to consider the "free" Dirac operator as comprising the spin connection or not, is immaterial for that purpose, as any reference to the latter vanishes from the noncommutative gauge potential. The first important step in the direction of connecting noncommutative geometry with gravitational physics was carried out independently by Kastler [5] and Kalau and Walze [6] who, following a suggestion by Connes, found that the Einstein–Hilbert action is given by

$$\fint D\!\!\!\!/^2\, ds^4 \propto \operatorname{Wres} D\!\!\!\!/^{-2}.$$

If, in the hope of describing the mix of gravity with the gauge boson interaction, instead of $D\!\!\!\!/$, one uses in this formula the full, gauge covariant operator $D = D\!\!\!\!/ + 1 \otimes D_F + A_{\mathrm{nc}} + J A_{\mathrm{nc}} J^{-1}$, one finds only a term proportional to the square of the Higgs field ϕ [6], in addition to the gravitational curvature term. We were thus stuck in a peculiar situation: one form of the action gave the Yang–Mills term, but not the gravitational part; the situation was inverted for the second form of the action, which only gives the gravitational part.

On the other hand, the Chamseddine–Connes action:

$$f_0 \frac{C_{\mathrm{cosm}}}{l^4} + f_2 \frac{C_{\mathrm{G}} r + C'_{\mathrm{H}}|\phi|^2}{l^2} + f_4 \big[C_{\mathrm{H}}|\phi|^4 + C_{\mathrm{YMH}}|D_\mu\phi||D^\mu\phi| + C_{\mathrm{YM}}(F_{\mu\nu}F^{\mu\nu})_{\mathrm{YM}} + C_{\mathrm{W}} C^2 + C_{\mathrm{GH}} r|\phi|^2\big] + O(l^2) \tag{1}$$

has terms of different orders; the first (cosmological) one is essentially $\operatorname{Wres} D\!\!\!\!/^{-4}$; the second one is again $\operatorname{Wres} D\!\!\!\!/^{-2}$; the third (carrying the Weyl gravity term) and subsequent ones are not Wodzicki residues, but generalized moments [11]. The length scale l is the inverse of the energy scale and the numerical coefficients f_0, f_2, f_4 are indeterminate. The total action contains terms, such as the Riemann scalar curvature r and mass term of the Higgs potential, which are quadratic in the fields (metric-graviton, Higgs and vector bosons), while the higher order terms like the kinetic energy of Yang–Mills fields and the rest of the Higgs potential are quartic or contain derivatives in the fields.

We next demonstrate, on application of Quillen's theory of superconnections [24] to the standard K-cycle, and *provided that the internal and external degrees of freedom can be cleanly separated*, the existence of a functional of the K-cycle containing only the Einstein–Hilbert and Yang–Mills–Higgs terms, on the same footing.

Quillen's superconnections

A key ingredient in our proposed action is that the generalized Dirac operators of product K-cycles arise from superconnections that are compatible with the Clifford action [20]. Superconnections have been already used in NCG in [25], based on earlier work of Ne'eman and Sternberg [26], in a slightly different context and at the Yang–Mills level only. We now briefly describe some key features of superconnections in reference to Dirac operators.

A superconnection on the superbundle E is *any* odd linear operator $\mathbb{A}$ on the module of E-valued differential forms $\mathcal{A}(M,E)$, graded by the sum of the grading on the scalar-valued forms $\mathcal{A}(M)$ and the grading on E, that satisfies the Leibniz rule

$$[\mathbb{A},\beta] = \mathrm{d}\beta \quad \text{for} \quad \beta \in \mathcal{A}(M), \tag{2}$$

where the commutator is graded. If ∇ is any ordinary connection on E, $\mathbb{A} - \nabla$ commutes with exterior products and so is itself an exterior product by an odd matrix-valued form:

$$(\mathbb{A} - \nabla)\,\zeta = \alpha \wedge \zeta \quad \text{for some} \quad \alpha \in \mathcal{A}^-(M, \operatorname{End} E).$$

This yields the general recipe

$$\mathbb{A} = \alpha_0 + \nabla + \alpha_2 + \alpha_3 + \cdots + \alpha_n$$

where $\alpha_{2k} \in \mathcal{A}^{2k}(M, \operatorname{End}^- E)$ and $\alpha_{2k+1} \in \mathcal{A}^{2k+1}(M, \operatorname{End}^+ E)$; we have absorbed the 1-form component α_1 in the connection. In particular, α_0 is just an *odd* matrix-valued bundle map: $\alpha_0 \in \Gamma(\operatorname{End}^- E)$.

The Jacobi identity shows that if θ is a matrix-valued form, then

$$[[\mathbb{A},\theta],\beta] = [\mathbb{A},[\theta,\beta]] + (-1)^{|\theta|\,|\beta|}[\mathrm{d}\beta,\theta] = 0$$

for any β in $\mathcal{A}(M)$, so $[\mathbb{A},\theta]$ is a multiplication operator. In this way the formula $(\mathbb{A}\theta)\wedge\zeta := [\mathbb{A},\theta]\zeta$ serves to define the covariant derivative $\mathbb{A}\theta$ in $\mathcal{A}(M,\operatorname{End} E)$; as operators, $\mathbb{A}\theta = [\mathbb{A},\theta]$. Since $\mathbb{A}$ is odd, we get $[\mathbb{A},\mathbb{A}] = 2\mathbb{A}^2$, and the Jacobi identity yields $2\,[\mathbb{A},[\mathbb{A},T]] = [[\mathbb{A},\mathbb{A}],T] = [2\mathbb{A}^2,T]$ for any operator T. In particular, $[\mathbb{A}^2,\beta] = [\mathbb{A},[\mathbb{A},\beta]] = \mathrm{d}(\mathrm{d}\beta) = 0$ for any β, so $\mathbb{A}^2 = F_{\mathbb{A}}$ in $\mathcal{A}^+(M,\operatorname{End} E)$: this is the *curvature* of the superconnection $\mathbb{A}$, and it satisfies the Bianchi identity $\mathbb{A}F_{\mathbb{A}} = [\mathbb{A},F_{\mathbb{A}}] = [\mathbb{A},\mathbb{A}^2] = 0$.

Following [20], we say that $\mathbb{A}$ is a *Clifford superconnection* if it satisfies a second Leibniz rule, involving the Clifford action:

$$[\mathbb{A}, c(\beta)] = c(\nabla\beta) \quad \text{for each} \quad \beta \in \mathcal{A}(M), \tag{3}$$

where ∇ is the Levi-Civita connection on the cotangent bundle. On a local orthonormal basis of 1-forms θ^a, there holds $\nabla_\mu \theta^a = \partial_\mu \theta^a - \Gamma^a_{\mu b} \theta^b$ (we use throughout Greek indexes for coordinate bases and Latin indexes for vierbeins). The antisymmetric matrices α_μ with entries $-\Gamma^b_{\mu a}$ (defined on a local chart U) make up a Lie-algebra-valued 1-form α in $\mathcal{A}^1(U, \mathfrak{so}(T^*M))$, and $\nabla = d + \alpha$ over U.

The spin connection ∇^S has the property (3). Locally, $\nabla^S = d + \omega = d + \dot\mu(\alpha)$, where $\dot\mu \colon \mathfrak{so}(T^*M) \to \mathbb{C}\ell\, M$ is the infinitesimal spin representation of the Lie algebra of the orthogonal group, $\dot\mu(\alpha_\nu) = -\frac{1}{4}\Gamma^b_{\nu a}\, \gamma^a \gamma^b$. Its curvature is $(\nabla^S)^2 = \dot\mu(\mathrm{d}\alpha + \alpha \wedge \alpha) = \dot\mu(R)$, where $R \in \mathcal{A}^2(M, \mathfrak{so}(T^*M))$ is the Riemann curvature tensor:

$$\dot\mu(R) = \tfrac{1}{4} R_{bav\sigma}\, \gamma^a \gamma^b\, dx^\nu \wedge dx^\sigma \quad \text{with} \quad \gamma^a \equiv c(\theta^a). \tag{4}$$

The basic property of the spin representation [27] is that

$$[\dot\mu(T), c(\beta)] = c(T\beta) \tag{5}$$

when $\beta \in \mathcal{A}^1(M) = \Gamma(T^*M)$ and $T \in \Gamma(\mathfrak{so}(T^*M))$. This can be seen directly, by checking the identity $\frac{1}{4}[\gamma^a \gamma^b, \gamma^c] = \gamma^{[a}\delta^{b]c}$, which entails $[\frac{1}{4}T^b_a\, \gamma^a \gamma^b,\, \beta_c\, \gamma^c] = T^b_a \beta_b\, \gamma^a$ if T is antisymmetric. (For that, notice that the commutator $[\gamma^a \gamma^b, \gamma^c]$ is zero whenever $a = b$ or the three indices are distinct.)

On a twisted bundle $E = S \otimes W$, there is the Clifford connection $\nabla^S \otimes 1$. If $\mathbb{A}$ is any Clifford superconnection, then $\mathbb{A} - \nabla^S \otimes 1$ commutes with the Clifford action, and therefore it is of the form $1 \otimes \mathbb{B}$ where $\mathbb{B}$ is an odd operator on $\mathcal{A}(M, W)$ that satisfies a Leibniz rule like (2). In other words, the most general Clifford superconnection is of the form

$$\mathbb{A} = \nabla^S \otimes 1 + 1 \otimes \mathbb{B}, \tag{6}$$

where $\mathbb{B}$ is any superconnection on the twisting bundle W. That is, the superconnection on a space which is the product of a continuous Riemannian spin geometry times a (noncommutative) internal geometry splits into the usual spin connection which acts trivially on the internal part, plus a superconnection which acts only on the internal part.

The superconnection for the standard K-cycle

We can identify the algebra $\Gamma(\mathbb{C}\ell\, M)$ with the algebra of forms $\mathcal{A}(M)$ by the isomorphism $c(\beta) \mapsto c(\beta)1$; the inverse map $Q \colon \mathcal{A}(M) \to \Gamma(\mathbb{C}\ell\, M)$ – denoted **c** by [20], who call it “quantization” – allows us to Clifford-multiply by forms. For instance, with $\sigma^{\mu\nu} = \frac{1}{2}[c(dx^\mu), c(dx^\nu)] \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, we have $\gamma^\mu \gamma^\nu\, 1 = dx^\mu \wedge dx^\nu + g^{\mu\nu}$, so that

$$Q(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = -g^{\mu\nu} + \gamma^\mu \gamma^\nu = \sigma^{\mu\nu}.$$

Let $\mathbb{B} = \mathbb{B}_0 + \mathbb{B}_{1\mu}\, dx^\mu + \mathbb{B}_{2\mu\nu}\, dx^\mu \wedge dx^\nu + \cdots$ be a superconnection on W. We can now define a Dirac operator associated to the Clifford superconnection $\mathbb{A}$ of (6) by

$$D := \not{D} \otimes 1 + \mathbb{B}_0 + \gamma^\mu\, \mathbb{B}_{1\mu} + \sigma^{\mu\nu}\, \mathbb{B}_{2\mu\nu} + \cdots . \tag{7}$$

It is clear that Dirac operators in this sense are just quantizations of superconnections. There is a one-to-one correspondence between Dirac operators compatible with a given Clifford action and

Clifford superconnections [20]; for example, $D\!\!\!/ = Q(\nabla^S)$. In particular, D and $\mathbb{A}$ have the same information.

All superconnections considered in [24] are of the form $\mathbb{B}_0 + \nabla$. This goes well with Connes' formalism for product K-cycles, as in the present context the superconnection pair $(\mathbb{B}_0, \mathbb{B}_1)$ and noncommutative differential 1-forms are one and the same thing – with the degree zero term corresponding to the Dirac–Yukawa operator. On the other hand, the AT formalism employs superconnections with forms up to degree two.

Now, in view of equation (6) and the relation $[\nabla^S \otimes 1, 1 \otimes \mathbb{B}] = 0$, the curvature of $\mathbb{A}$ splits as

$$\mathbb{A}^2 = \dot{\mu}(R) \otimes 1 + 1 \otimes \mathbb{B}^2 =: \dot{\mu}(R) \otimes 1 + 1 \otimes F_{\mathbb{B}}. \tag{8}$$

One can also remark [20] that, from the Leibniz rule (3):

$$[\mathbb{A}^2, c(\beta)] = [\mathbb{A}, [\mathbb{A}, c(\beta)]] = c(\nabla^2\beta) = c(R\beta),$$

whereas $[\dot{\mu}(R), c(\beta)] = c(R\beta)$ from (5). With regard to the factorization $\operatorname{End} E \simeq \mathbb{C}\ell\, M \otimes \operatorname{End} W$, $\dot{\mu}(R)$ acts by Clifford multiplications and we can write it as $\dot{\mu}(R) \otimes 1$. Thus $\mathbb{A}^2 - \dot{\mu}(R) \otimes 1$ commutes with all $c(\beta)$ and so it lies in $\mathcal{A}^+(M, 1 \otimes \operatorname{End} W)$. In conclusion, the quantity $F_{\mathbb{B}}$ equals $\mathbb{A}^2 - \dot{\mu}(R)$, $F_{\mathbb{B}}$ is an "internal" curvature and a functional of D whenever R is. Now, the Riemann tensor R is a functional of $D\!\!\!/$ [7]. Therefore $F_{\mathbb{B}}$ is a functional of the pair $(D, D\!\!\!/)$. We henceforth write $F[D]$ for short.

It remains to compute $F[D]$ for the standard K-cycle. Recall that there $W = \mathcal{H}_F$ and that $\mathbb{B} = \mathbb{B}_0 + \mathbb{B}_1$, where $\mathbb{B}_0$ holds the Higgs term and $\mathbb{B}_1$ contains the usual Yang–Mills terms. It is not hard to see that $F[D] = \mathbb{B}_0^2 + [\mathbb{B}_1, \mathbb{B}_0] + \mathbb{B}_1^2$, as an orthogonal direct sum of terms.

The representation of $\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ on $\mathcal{H}_F$ decomposes into representations on the lepton, quark, antilepton and antiquark sectors: $\mathcal{H}_F = \mathcal{H}_F^+ \oplus \mathcal{H}_F^- = \mathcal{H}_\ell^+ \oplus \mathcal{H}_q^+ \oplus \mathcal{H}_\ell^- \oplus \mathcal{H}_q^-$, each of which in turn decomposes according to chirality: $\mathcal{H}_\ell^+ = \mathcal{H}_{\ell R}^+ \oplus \mathcal{H}_{\ell L}^+$ and so on. For the quark sector and the lepton sector with massless neutrinos, we have respectively

$$\begin{aligned} \mathcal{H}_q^+ &= (\mathbb{C} \oplus \mathbb{C})_R \otimes \mathbb{C}^N \otimes \mathbb{C}^3_{\mathrm{col}} \ \oplus\ \mathbb{C}^2_L \otimes \mathbb{C}^N \otimes \mathbb{C}^3_{\mathrm{col}}\,, \\ \mathcal{H}_\ell^+ &= \mathcal{H}_{R\ell}^+ \oplus \mathcal{H}_{L\ell}^+ = \mathbb{C}_R \otimes \mathbb{C}^N \ \oplus\ \mathbb{C}^2_L \otimes \mathbb{C}^N. \end{aligned}$$

In this basis, and on applying the "unimodularity condition" [3], the superconnection associated to D corresponds to:

$$\mathbb{B}_{0q} = \begin{pmatrix} 0 & 0 & \bar\phi_2 \otimes M_u^* & -\bar\phi_1 \otimes M_u^* \\ 0 & 0 & \phi_1 \otimes M_d^* & \phi_2 \otimes M_d^* \\ \phi_2 \otimes M_u & \bar\phi_1 \otimes M_d & 0 & 0 \\ -\phi_1 \otimes M_u & \bar\phi_2 \otimes M_d & 0 & 0 \end{pmatrix},$$

$$\mathbb{B}_{0\ell} = \begin{pmatrix} 0 & \phi_1 \otimes M_e^* & \phi_2 \otimes M_e^* \\ \bar\phi_1 \otimes M_e & 0 & 0 \\ \bar\phi_2 \otimes M_e & 0 & 0 \end{pmatrix},$$

$$\mathbb{B}_{1q\mu} = \begin{pmatrix} \partial_\mu - \frac{4}{3}ia_\mu & 0 & 0 & 0 \\ 0 & \partial_\mu + \frac{2}{3}ia_\mu & 0 & 0 \\ 0 & 0 & \partial_\mu - \frac{1}{3}ia_\mu - ib^1_{1\mu} & -ib^1_{2\mu} \\ 0 & 0 & -ib^2_{1\mu} & \partial_\mu - \frac{1}{3}ia_\mu - ib^2_{2\mu} \end{pmatrix} \otimes 1_N \otimes 1_3$$

$$- ic^a_\mu \, 1_4 \otimes 1_N \otimes \frac{\lambda_a}{2},$$

$$\mathbb{B}_{1\ell\mu} = \begin{pmatrix} \partial_\mu + 2ia_\mu & 0 & 0 \\ 0 & \partial_\mu + i(a_\mu - b^1_{1\mu}) & -ib^1_{2\mu} \\ 0 & -ib^2_{1\mu} & \partial_\mu + i(a_\mu - b^2_{2\mu}) \end{pmatrix} \otimes 1_N, \tag{9}$$

where the λ_a are the Gell-Mann matrices and ϕ_1, ϕ_2 denote the (normalized) components of the Higgs field.

The rest is routine; actually what we do is only superficially different from what is done in [28], and we can read off what we need as a subset of their computations. (There are a few misprints in that reference, but they do not affect the final results.) Finally, we obtain

$$\operatorname{tr} F[D]^2 = C_{\mathrm{H}}|\phi|^4 + C_{\mathrm{YMH}}|D_\mu\phi||D^\mu\phi| + C_{\mathrm{YM}}(F_{\mu\nu}F^{\mu\nu})_{\mathrm{YM}}\,,$$

where

$$D_\mu := \partial_\mu - \tfrac{i}{2}g_1 a_\mu - \tfrac{i}{2}g_2\, \tau \cdot b_\mu$$

with an obvious notation. Our nonperturbative approach gives, for the surviving terms, *exactly the same coefficients* C_{H}, C_{YMH}, C_{YM} as the CC Lagrangian. We shall not bother to write them down.

A particular action functional

It should be noted that $F[D] \neq F_{\mathrm{nc}}$. The missing term in $F[D]$ is the mass term in the Higgs sector. (Actually, without fermion families replication, the whole Higgs sector in F_{nc} is simply zero. The present integral formulation eliminates this quirk of the differential one, at the price of withdrawing the tentative claim [3, 22] of a NCG-based explanation for such replication.) That missing term is provided by the already considered $⨍ D^2\, ds^4$ term, that gives us, besides the Einstein–Hilbert Lagrangian, the term in the square of the Higgs field: both pieces of the puzzle fit together!

In conclusion, the bosonic action is schematically written as

$$⨍ (D^2 + F[D]^2)\, ds^4. \tag{10}$$

We must allow in the first summand the length scale l, for dimensional reasons; and an indeterminate numerical coefficient f_4 in the second. Therefore:

$$\begin{aligned} B[D] &= ⨍ (l^{-2}D^2 + f_4F[D]^2)\, ds^4 \\ &= \frac{1}{8\pi^2}\operatorname{Wres}(l^{-2}D^2 + f_4F[D]^2)\, D^{-4} \\ &= \frac{1}{8\pi^2}\operatorname{Wres} l^{-2}D^{-2} + f_4\int_M (\operatorname{tr} F[D]^2). \end{aligned}$$

Upon using D given by

$$D := \not{D} \otimes 1 + \mathbb{B}_0 + \gamma^\mu \, \mathbb{B}_{1\mu}, \tag{11}$$

with $\mathbb{B}_0$, $\mathbb{B}_1$ given by (9), one finds the full bosonic action of the Standard Model plus gravity.

The action (10) is quite simple and has a very familiar look. There is a "kinetic" term given by the square of the derivative (momentum) term. This term provides the action more intrinsically connected with the nature of spacetime. Then, in the presence of an internal structure, there is a "potential" term, which is quadratic, another familiar occurrence. As argued in [29], one can get more freedom by allowing the quark and the lepton sectors to enter with different coefficients. This redefinition of the noncommutative integral is permissible by the existence of a superselection rule. Then one can, perhaps, by adjusting properly the theory to the known Standard Model parameters, indulge in a new round of that favourite pastime of noncommutative geometers, Higgs-particle mass speculation.

The CC formula (1) embodies a *theorem* about the structure of action functionals in Connes' noncommutative geometry. Note that the cosmological term can be disposed of by choosing a particular functional for which $f_0 = 0$. However, the result by Chamseddine and Connes implies that the Yang–Mills terms of the SM are ineluctably accompanied, at the tree level, by the conformal piece that includes the Weyl tensor term and the coupling between the scalar curvature and the Higgs field. How is it, then, that the result of this paper appears to be an exception? The action principle we invoke is not quite the same as the one by Chamseddine and Connes; the conceptual distinction lies in that, whereas the CC functional depends on the Dirac operator (in the occurrence (11)) *taken as a whole*, we use the possibility of splitting the operator into a "spacetime" and an "internal" part. Whether the extra terms present in the CC perturbative development are a necessity or not is to be decided by quantum field theoretical considerations and/or experiment.

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