Improved Epstein–Glaser renormalization in x-space versus differential renormalization

José M. Gracia-Bondía,[1,2,3] Heidy Gutiérrez[3] and Joseph C. Várilly[4]

[1] Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain

[2] BIFI Research Center, Universidad de Zaragoza, 50018 Zaragoza, Spain

[3] Escuela de Física, Universidad de Costa Rica, 11501 San José, Costa Rica

[4] Escuela de Matemática, Universidad de Costa Rica, 11501 San José, Costa Rica

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Abstract

Renormalization of massless Feynman amplitudes in x-space is reexamined here, using almost exclusively real-variable methods. We compute a wealth of concrete examples by means of recursive extension of distributions. This allows us to show perturbative expansions for the four-point and two-point functions at several loop order. To deal with internal vertices, we expound and expand on convolution theory for log-homogeneous distributions. The approach has much in common with differential renormalization as given by Freedman, Johnson and Latorre; but differs in important details.

1 Introduction

The long-awaited publication of [1] has again brought to the fore renormalization of Feynman amplitudes in x-space. The method in that reference is distribution-theoretical, in the spirit of Epstein and Glaser [2]. That means cutoff- and counterterm-free. That infinities never be met is something devoutly to be wished, as regards the logical and mathematical status of quantum field theory [3].

Twenty-odd years ago, an equally impressive paper [4] with the same general aim introduced to physicists a version of differential renormalization in x-space. From the beginning, it must have been obvious to the cognoscenti that this version and Epstein–Glaser renormalization were two sides of the same coin. The main aim of this article is to formalize this relation, to the advantage of x-space renormalization in general.

Both [1] and [4] grant pride of place to the massless Euclidean ϕ^4_4 model, and it suits us to follow them in that. Namely, we show how to compute amputated diagrams which are proper (without cutlines), contributing to the four-point functions for this model, up to the fourth order in the coupling constant.

In [1] a recursive process to deal with "subdivergences" seeks to demonstrate the renormalization process as a sequence of extensions of distributions. Since in the present paper we are concerned with the two-point and four-point functions needed to obtain the effective action, we furthermore need to integrate over internal vertices of the graphs. Indeed, in [4], internal vertices are integrated over, yielding convolution-like integrals. However, some bogus justifications for this essentially sound procedure are put forward there. Also, computations in [4] lack the natural algebraic rules set forth by one of us in [5, 6] – that [1] also adopts and generalizes as the "causal factorization property" [1, Thm. 2.1].

For the Euclidean quantum amplitudes with which we deal in this paper, we borrow the language of (divergent) subgraphs and cographs [7]. Let $\Gamma(\mathcal{V},\mathcal{L})$ denote a graph one is working with: $\mathcal{V}$ is the set of its vertices and $\mathcal{L}$ the set of its internal lines. A subgraph $\gamma \subseteq \Gamma$ is a set of vertices $\mathcal{V}(\gamma) \subseteq \mathcal{V}$ and the set of *all* lines in $\mathcal{L}$ joining any two elements of $\mathcal{V}(\gamma)$ – which is to say, a full subgraph in the usual mathematical parlance. Let γ be any of the subgraphs. A lodestone is the rule contained in the (rigorous as well as illuminating) treatment in [8, Sect. 11.2]. It is written:

$$\langle R[\Gamma], \varphi\rangle = \langle R[\gamma], (\Gamma/\gamma)\, \varphi\rangle, \tag{1.1}$$

where $R[\Gamma]$, $R[\gamma]$ and Γ/γ denote corresponding amplitudes, and φ is supported outside the singular points of Γ/γ.

As long as we need not integrate over internal vertices in Γ, this is all we require for the recursive treatment of the hierarchy of cographs in the diagrams. This rule implies the Euclidean version of the causal factorization property of [1], as will be thoroughly checked in the upcoming examples.

The treatment of diagrams with internal vertices calls for a convolution-like machine. Thus we streamline the framework of [4], and in passing we correct some minor mistakes there. While proceeding largely by way of example, along the way we tune up that machine by (invoking and/or) proving a few rigorous results.

The reader will notice how easy these computations in x-space are, once the right methods are found. The outcome, we expect, is a convincing case study for gathering the ways of [1] and [4] together.

The plan of the paper is as follows. In Section 2 we lay out the basic renormalization procedure, for divergent graphs which are primitive (that is, without divergent subgraphs). The mathematical task is to extend a function or distribution defined away from the origin of $\mathbb{R}^d$ to the whole of $\mathbb{R}^d$. This is a simplification of a more general extension problem of distributions defined off a diagonal, calling on the translation invariance of the distributions involved.

In Section 3, adapting work by Horváth, and later by Ortner and Wagner, we adjust our convolution-like engine. Horváth's results have not found their way into textbooks, and seem little known to physicists: it falls to us here to report on, and complete them, in some detail.

The long Sections 4 and 5 deliver detailed and fully explicit calculations of concrete (tadpole-part-free) graphs. The "integration by parts" method of [4] is put on a surer theoretical footing here by linking it to the successive extensions in [1] and the "locality" rule in [8]. In every case, differential renormalization yields the leading term; here we do find the correcting terms necessary to fulfil the algebraic strictures in [1, 5]. The dilation behaviour of the renormalized graphs is examined. We trust that these two sections give a clear picture of the perturbative expansions for the four-point function $\mathcal{G}^4(x_1, x_2, x_3, x_4)$.

In Section 6 we turn towards conceptual matters: the renormalization group (RG) and the β-function, leading to the "main theorem of renormalization" [9] and Bogoliubov's recurrence relation at the level of the coupling constant in this context. These are briefly discussed in the concluding Section 7.

In Appendix A, we collect for easy reference explicit formulas for the distributional extensions in x-space employed throughout. Graphs contributing to the two-point function $\mathcal{G}^2(x_1, x_2)$ are solved in Appendix B. Calculations of p-amplitudes are dealt with briefly in Appendix C.

The relation between our scheme and dimensional regularization in x-space was investigated in [5, 6] – as "analytical prolongation" – and has recently been exhaustively researched [10, 11]. Reasons of spacetime prevent us from going into that, for now; nor do we take up attending issues of the Hopf algebra approach to the combinatorics of renormalization [12].

2 Primitive extensions of distributions

The reader is supposed familiar with the basics of distribution theory; especially homogeneous distributions. Apart from this, the article is self-contained, in that extension of homogeneous distributions is performed from scratch. In that respect, outstanding work in the eighties by Estrada and Kanwal [13, 14] has been very helpful to us.

Ref. [1] uses a complex-variable method for extension of homogeneous distributions, following in the main [15]. Basically, this exploits that Riesz's normalized radial powers on $\mathbb{R}^d$, defined by

$$R_\lambda(r) := A_d(\lambda)\, r^\lambda, \quad \text{where}$$
$$A_d(\lambda) := \frac{2}{\Omega_d\, \Gamma\big(\frac{\lambda+d}{2}\big)} \quad \text{and} \quad \Omega_d = \mathrm{Vol}\big(\mathbb{S}^{d-1}\big) = \frac{2\pi^{d/2}}{\Gamma(d/2)}\,,$$

constitute an entire function of λ. There is much to be said in favour of such methods; the interested reader should consult also [16] and [17]. Nevertheless, we choose to recruit and popularize here real-variable methods. They are more in the spirit both of the original Epstein–Glaser procedure [2] and differential renormalization itself [4].

Let us call a homogeneous distribution T on $\mathbb{R}^d$ *regular* if it is smooth away from the origin; the smooth function on $\mathbb{R}^d \setminus \{0\}$ associated to it is homogeneous of the same degree. Homogeneous distributions of all kinds are tempered (see the discussion in Section 2.2), and thus possess Fourier transforms.

Consider first spaces of homogeneous distributions on the real half-line. The function r^{-1} defines a distribution on the space[1] of Schwartz functions vanishing at the origin, $r\,\mathcal{S}(\mathbb{R}^+)$. It seems entirely natural to extend it to a functional on the whole space $\mathcal{S}(\mathbb{R}^+)$ by defining

$$r_1[r^{-1}] := \Big(\log \frac{r}{l}\Big)' \tag{2.1}$$

where l is a convenient scale. Note that $\log(r/l)$ is a well defined distribution, and so is its distributional derivative. The difference between two versions of this recipe, with different scales, lies in the kernel of the map $f \mapsto rf$ on distributions, i.e., it is a multiple of the delta function. Of course $r_1[r^{-1}]$ is no longer homogeneous, since

$$r_1[(\lambda r)^{-1}] = \lambda^{-1}\, r_1[r^{-1}] + \lambda^{-1} \log \lambda\, \delta(r). \tag{2.2}$$

[1]It is not satisfactory for us to consider extensions merely from $\mathcal{S}(\mathbb{R}^+ \setminus \{0\})$ to $\mathcal{S}(\mathbb{R}^+)$.

Now, for z not a negative integer, the property

$$r\, r^z = r^{z+1} \quad \text{holds; and also} \tag{2.3}$$

$$\frac{d}{dr}(r^z) = z\, r^{z-1}. \tag{2.4}$$

For the homogeneous functions r^{-n} with $n = 2, 3, \dots$ which are not locally integrable, one might adopt the recipe:

$$r_1[r^{-n}] := \frac{(-)^{n-1}}{(n-1)!}\Big(\log\frac{r}{l}\Big)^{(n)},$$

generalizing (2.4) by definition. This is *differential renormalization on* $\mathbb{R}^+$ in a nutshell. That, however, loses property (2.3).

Thus we look for a recipe respecting (2.3) instead. Let f denote a smooth function on $\mathbb{R}^+ \setminus \{0\}$ with $f(r) = O(r^{-k-1})$ as $r \downarrow 0$. Epstein and Glaser introduce a general subtraction projection W_w from $\mathcal{S}(\mathbb{R}^+)$ to the space of test functions vanishing at order k at the origin, whereby the whole k-jet of a test function φ on $\mathbb{R}^d$ at the origin

$$j_0^k(\varphi)(x) := \varphi(0) + \sum_{|\alpha|=1} \frac{x^\alpha}{\alpha!}\,\varphi^{(\alpha)}(0) + \cdots + \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!}\,\varphi^{(\alpha)}(0)$$

is weighted by an infrared regulator w, satisfying $w(0) = 1$ and $w^{(\alpha)}(0) = 0$ for $1 \leqslant |\alpha| \leqslant k$:

$$W_w\varphi(x) := \varphi(x) - w(x)\, j_0^k(\varphi)(x).$$

One may use instead [5, 6] the simpler subtraction projection:

$$P_w\varphi(x) := \varphi(x) - j_0^{k-1}(\varphi)(x) - w(x) \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!}\,\varphi^{(\alpha)}(0). \tag{2.5}$$

Just $w(0) = 1$ is now required from w for the projection property $P_w(P_w\varphi) = P_w\varphi$ to hold.

Define now the operations W_w and P_w on $\mathcal{S}'(\mathbb{R}^n)$ by duality:

$$\langle W_w f, \varphi\rangle := \langle f, W_w\varphi\rangle \quad \text{and likewise} \quad \langle P_w f, \varphi\rangle := \langle f, P_w\varphi\rangle.$$

On the half-line, by use of Lagrange's expression for MacLaurin remainders, for $k = 0$ (logarithmic divergence) we arrive in a short step [5] at the dual integral formula:

$$W_w f(r) = P_w f(r) = -\frac{d}{dr}\bigg[r \int_0^1 \frac{dt}{t^2}\, f\Big(\frac{r}{t}\Big) w\Big(\frac{r}{t}\Big)\bigg].$$

We choose (and always use henceforth) the simple regulator

$$w(r) := \theta(l - r), \quad \text{for some fixed } l > 0, \tag{2.6}$$

with θ being the Heaviside function. Actually, for $k = 0$ this yields the general result, since the difference between two extensions with acceptable dilation behaviour is a multiple of the delta function. In the homogeneous case, one immediately recovers (2.1):

$$R_1[r^{-1}] := P_{\theta(l-\cdot)}[r^{-1}] = -\bigg[\int_{r/l}^1 \frac{dt}{t}\bigg]' = \Big(\log\frac{r}{l}\Big)' =: r_1[r^{-1}].$$

For any positive integer k,

$$P_w f(r) = (-)^k\, k\left[\frac{r^k}{k!}\int_0^1 \frac{(1-t)^{k-1}}{t^{k+1}} f\Big(\frac{r}{t}\Big)\Big(1 - w\Big(\frac{r}{t}\Big)\Big)\, dt\right]^{(k)} + (-)^{k+1}(k+1)\left[\frac{r^{k+1}}{(k+1)!}\int_0^1 \frac{(1-t)^k}{t^{k+2}} f\Big(\frac{r}{t}\Big) w\Big(\frac{r}{t}\Big)\, dt\right]^{(k+1)} \tag{2.7}$$

which yields

$$R_1[r^{-k-1}] := P_{\theta(l-\cdot)}[r^{-k-1}] = \frac{(-)^k}{k!}\left[\Big(\log\frac{r}{l}\Big)^{(k+1)} + H_k\, \delta^{(k)}(r)\right]. \tag{2.8}$$

Here H_k is the k-th harmonic number:

$$H_k := \sum_{j=1}^{k} \frac{(-)^{j+1}}{j}\binom{k}{j} = \sum_{j=1}^{k} \frac{1}{j}\,; \quad \text{and} \quad H_0 := 0.$$

(See [18, p. 267] for the equality of the two sums.) Note that $R_1[r^{-k-1}] \neq W_{\theta(l-\cdot)}[r^{-k-1}]$ for $k > 0$, as well as $R_1[r^{-k-1}] \neq r_1[r^{-k-1}]$.

One of us in [5] proved that:

- R_1 coincides with (a straightforward generalization of) Hadamard's finite part extension and the meromorphic continuation extension of [1, 15–17].
- R_1 (but not in general the W_w subtraction) fulfils the *algebra property*

  $$r^m\, R_1[r^{-k-1}] = R_1[r^{-k+m-1}],$$

  extending (2.3) to the realm of renormalized distributions.[2]

2.1 Dimensional reduction

The task is now to extend radial-power distributions, that is, to compute $\langle f(r), \varphi(x)\rangle$, where $f(r)$ denotes a scalar, radially symmetric distribution defined on $\mathbb{R}^d$. We keep borrowing from Estrada and Kanwal [13, 14]. First sum over all angles, by defining

$$\Pi\varphi(r) := \int_{|\omega|=1} \varphi(r\omega)\, d\sigma(\omega).$$

The resulting function $\Pi\varphi$ is to be regarded either as defined on $\mathbb{R}^+$, or as an even function on the whole real line. Its derivatives of odd order with respect to r at 0 vanish, and those of even order satisfy:

$$(\Pi\varphi)^{(2l)}(0) = \Omega_{d,l}\, \Delta^l\varphi(0) := \Big(\int_{|\omega|=1} x_i^{2l}\Big)\Delta^l\varphi(0) = \frac{2\,\Gamma(l+\frac{1}{2})\,\pi^{(d-1)/2}}{\Gamma(l+\frac{d}{2})}\,\Delta^l\varphi(0).$$

[2] The algebra property can be in contradiction with arbitrary "renormalization prescriptions" [19]; but this does not detract from its utility.

With that in mind, one can write the *dimensional reduction* formula:

$$\langle R_d[f(r)], \varphi(x)\rangle_{\mathbb{R}^d} = \big\langle R_1[f(r)r^{d-1}], \Pi\varphi(r)\big\rangle_{\mathbb{R}^+}. \tag{2.9}$$

The notation $R_d[f(r)]$ for the renormalized object handily keeps track of the space dimension. The formula can be taken as a *definition* of $R_d[f(r)]$, and so for radially symmetric distributions the simple R_1 method, as well as Hadamard's and meromorphic continuation on the real line, are lifted in tandem to higher dimensions by (2.9). As a bonus, the multiplicativity condition for radial functions is automatically preserved.

Keep also in mind, however, that Epstein–Glaser-type subtraction works in any number of dimensions. In particular, our modified Epstein–Glaser method for $f(r) = O(r^{-k-d})$ leads to the integral form, generalizing (2.7),

$$\begin{aligned} P_w f(x) = (-)^k k \sum_{|\alpha|=k} \partial^\alpha \biggl[\frac{x^\alpha}{\alpha!} \int_0^1 dt\, \frac{(1-t)^{k-1}}{t^{k+d}} f\Bigl(\frac{x}{t}\Bigr)\Bigl(1 - w\Bigl(\frac{x}{t}\Bigr)\Bigr)\biggr] \\ + (-)^{k+1}(k+1) \sum_{|\beta|=k+1} \partial^\beta \biggl[\frac{x^\beta}{\beta!} \int_0^1 dt\, \frac{(1-t)^k}{t^{k+d+1}} f\Bigl(\frac{x}{t}\Bigr) w\Bigl(\frac{x}{t}\Bigr)\biggr]; \end{aligned} \tag{2.10}$$

and, as it turns out, $P_{\theta(l-\cdot)} f(r) = R_d[f(r)]$, when using the regulator (2.6). All this was clarified in [5]. The operator $\partial_\alpha x^\alpha = E + d$, with $E := x^\alpha\,\partial_\alpha$ denoting the Euler operator, figures prominently there.

Note, moreover, when both the distribution f and the regulator w enjoy rotational symmetry, employing the MacLaurin–Lagrange expansion for φ and summation over the angles, the last displayed formula amounts to the computation:

$$\langle P_w f(r), \varphi(x)\rangle \equiv \biggl\langle f(r), \varphi(x) - \varphi(0) - \frac{\Delta\varphi(0)}{2!\,d}\, r^2 - \cdots - w(r) \frac{\Omega_{d,l}\, \Delta^l \varphi(0)}{(2l)!\,\Omega_d}\, r^{2l}\biggr\rangle, \tag{2.11}$$

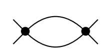
up to the highest l such that $2l \leqslant k$. Rotational symmetry of extensions in general can be studied like Lorentz covariance was in [6, Sect. 4] and in [20, Sect. 3.3].

We remark finally that the MacLaurin expansion for $\Pi\varphi$ is written

$$\Pi\varphi(r) = 2^{d/2-1}\Gamma(d/2)\bigl(r\sqrt{-\Delta}\,\bigr)^{1-d/2} J_{d/2-1}\bigl(r\sqrt{-\Delta}\,\bigr)\varphi(0),$$

for J_α the Bessel function of the first kind and order α. This makes sense for complex α. That is the nub of dimensional regularization in position space, as found by Bollini and Giambiagi themselves [21] – with Euclidean signature, in the present case.

2.2 Log-homogeneous distributions

We are interested in the amplitude $R_4[r^{-4}]$, corresponding to the "fish" graph  in the ϕ^4_4 model. For clarity, it is useful to work in any dimension $d \geqslant 3$. Note the following:

$$\begin{aligned} R_d[r^{-d}] &= r^{-d+1}\Bigl(\log\frac{r}{l}\Bigr)' = r^{-d} E\Bigl(\log\frac{r}{l}\Bigr) = \partial_\alpha\Bigl(x^\alpha r^{-d} \log\frac{r}{l}\Bigr) \\ &= -\partial_\alpha\partial^\alpha\biggl(\frac{r^{-d+2}}{d-2}\log\frac{r}{l} + \frac{r^{-d+2}}{(d-2)^2}\biggr) = -\frac{1}{d-2}\biggl[\Delta\biggl(r^{-d+2}\log\frac{r}{l}\biggr) - \Omega_d\,\delta(r)\biggr]. \end{aligned} \tag{2.12}$$

The last term appears because $r^{-d+2}/(-d+2)\Omega_d$ is the fundamental solution for the Laplacian on $\mathbb{R}^d$. An advantage of this form is that the corresponding momentum space amplitudes are easily computed – as will be exploited later: see Appendix C.

In calculation of graphs on $\mathbb{R}^4$ with subdivergences, extensions of $r^{-4}\log^m(r/l)$, with growing powers of logarithms, crop up again and again. It is best to grasp them all together. One can introduce different scales, but for simplicity we stick with just one scale. Dimensional reduction gives

$$R_d\Big[r^{-d}\log^m\frac{r}{l}\Big] = \frac{1}{m+1}\,r^{-d+1}\,\frac{d}{dr}\bigg(\log^{m+1}\frac{r}{l}\bigg) \tag{2.13}$$

for any $m = 0, 1, 2, \dots$

A distribution f on $\mathbb{R}^d$ is called *log-homogeneous* of *bidegree* (a, m) if

$$(E-a)^{m+1}f = 0, \quad \text{but} \quad (E-a)^m f \neq 0. \tag{2.14}$$

Here m is a nonnegative integer but a can be any complex number; the case $m = 0$ obviously corresponds to homogeneous distributions. For example, the distribution $\log r \in \mathcal{S}'(R^d)$ is log-homogeneous of bidegree $(0, 1)$. Essentially the same definition is found in [22, Sect. 4.1.6]. See also [16, Sect. I.4], [5, Sect. 2.4] and [1, Prop. 4.4], where the nomenclature used is "associate homogeneous of degree a and order m".

Log-homogeneous distributions are tempered. Indeed, if f is homogeneous of bidegree $(a, 0)$, then [23] one can find $g_0 \in \mathcal{D}'(\mathbb{S}^{n-1})$ so that $f(x) = r^a\, g_0(\omega)$ for $x = r\omega \in \mathbb{R}^d \setminus \{0\}$. More generally, for f of bidegree (a, m), one can inductively construct $g_0, \dots, g_m \in \mathcal{D}'(\mathbb{S}^{n-1})$ such that $f(r\omega) = \sum_{k=0}^m r^a \log^{m-k} r\, g_k(\omega)$ for $r > 0$. It follows that $f \in \mathcal{S}'(\mathbb{R}^d)$.

A related issue is whether a log-homogeneous distribution on $\mathbb{R}^d \setminus \{0\}$ can be extended to one on $\mathbb{R}^d$. As we shall immediately exemplify, this can always be achieved although the bidegree may change: if the bidegree off the origin is (a, m), that of the extension may be (a, n) with $n \geqslant m$. For a general proof of that, showing also that rotational (or Lorentz) invariance may be kept in the extension, see Lemma 6 of [20].

The dilation behaviour of $R_d[r^{-d}]$ is immediate from formula (2.12):

$$R_d[(\lambda r)^{-d}] = \partial_\alpha\bigg(x^\alpha(\lambda r)^{-d}\Big(\log\frac{r}{l} + \log\lambda\Big)\bigg) = \lambda^{-d}R_d[r^{-d}] + \lambda^{-d}\log\lambda\,\Omega_d\,\delta(r),$$

generalizing (2.2). Note that Ω_d is simply minus the coefficient of $\log l$. In infinitesimal terms,

$$E R_d[r^{-d}] = -d\,R_d[r^{-d}] + \Omega_d\,\delta(r), \quad \text{so that} \quad \operatorname{Res}[r^{-d}] := [E, R_d](r^{-d}) = \Omega_d\,\delta(r).$$

Hence $R_d[r^{-d}]$ is log-homogeneous of bidegree $(-d, 1)$. The functional Res coincides with the Wodzicki residue [24, Chap. 7.3]. It coincides as well with the "analytic residue" in [1].

For our own purposes (RG calculations), we prefer to invoke the logarithmic derivative of the amplitudes with respect to the length scale l:

$$\frac{\partial}{\partial\log l}\,R_d[r^{-d}] = l\frac{\partial}{\partial l}\,R_d[r^{-d}] = -\Omega_d\,\delta(r); \tag{2.15}$$

which for primitive diagrams like the fish yields the residue yet again. As was shown in [5], this is actually a functional derivative with respect to the regulator w; and so it can be widely generalized.

Lemma 1. *For $d \geqslant 3$, $m = 0, 1, 2, \dots$ and $\lambda > 0$, the following relation holds:*

$$R_d\Big[(\lambda r)^{-d} \log^m \frac{\lambda r}{l}\Big] = \sum_{k=0}^{m} \lambda^{-d} \log^k \lambda \binom{m}{k} R_d\Big[r^{-d} \log^{m-k} \frac{r}{l}\Big] + \lambda^{-d} \log^{m+1} \lambda \, \frac{\Omega_d}{m+1}\, \delta(r). \qquad (2.16)$$

Therefore, $R_d\big[r^{-d} \log^m (r/l)\big]$ is log-homogeneous of bidegree $(-d, m+1)$.

Proof. This is a direct verification:

$$\begin{aligned} R_d\Big[(\lambda r)^{-d} \log^m \frac{\lambda r}{l}\Big] &= \frac{\lambda^{-d}}{m+1}\, \partial_\alpha\Big(x^\alpha r^{-d}\Big(\log \frac{r}{l} + \log \lambda\Big)^{m+1}\Big) \\ &= \frac{\lambda^{-d}}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} \log^k \lambda \, \partial_\alpha\Big(x^\alpha r^{-d} \log^{m-k+1} \frac{r}{l}\Big) \\ &= \sum_{k=0}^{m+1} \lambda^{-d} \log^k \lambda \, \frac{m!}{k!(m-k+1)!}\, \partial_\alpha\Big(x^\alpha r^{-d} \log^{m-k+1} \frac{r}{l}\Big) \\ &= \sum_{k=0}^{m} \lambda^{-d} \log^k \lambda \binom{m}{k} R_d\Big[r^{-d} \log^{m-k} \frac{r}{l}\Big] + \lambda^{-d} \log^{m+1} \lambda \, \frac{1}{m+1}\, \partial_\alpha(x^\alpha r^{-d}), \end{aligned}$$

and the result follows from the relation $\partial_\alpha(x^\alpha r^{-d}) = -\Delta((d-2) r^{-d+2}) = \Omega_d\, \delta(r)$. □

As an immediately corollary, we get the effect of the Euler operator, when $m \geqslant 1$:

$$E R_d\Big[r^{-d} \log^m \frac{r}{l}\Big] = -d\, R_d\Big[r^{-d} \log^m \frac{r}{l}\Big] + m\, R_d\Big[r^{-d} \log^{m-1} \frac{r}{l}\Big].$$

On the other hand, an elementary calculation for $r > 0$ gives

$$E\Big[r^{-d} \log^m \frac{r}{l}\Big] = r\, \frac{d}{dr}\Big[r^{-d} \log^m \frac{r}{l}\Big] = -d\, r^{-d} \log^m \frac{r}{l} + m\, r^{-d} \log^{m-1} \frac{r}{l}\,,$$

so that $R_d E\big[r^{-d} \log^m (r/l)\big] = E R_d\big[r^{-d} \log^m (r/l)\big]$; consequently, higher residues all vanish:

$$\operatorname{Res}\Big[r^{-d} \log^m \frac{r}{l}\Big] := [E, R_d]\Big(r^{-d} \log^m \frac{r}{l}\Big) = 0 \quad \text{for } m = 1, 2, 3, \dots$$

We summarize. First, by the same trick of (2.12),

$$R_d\Big[r^{-d} \log^m \frac{r}{l}\Big] = \frac{E+d}{m+1}\Big(r^{-d} \log^{m+1} \frac{r}{l}\Big),$$

which makes obvious much of the above. This formula also shows that the aforementioned algebra property applies to logarithms as well as polynomials:

$$\log \frac{r}{l}\, R_d\Big[r^{-d} \log^m \frac{r}{l}\Big] = R_d\Big[r^{-d} \log^{m+1} \frac{r}{l}\Big].$$

Second, we can use this operator to amplify a well-known property of homogeneous distributions: the *Fourier transform* $\mathcal{F} f$ of a log-homogeneous distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ of bidegree (a, m)

is itself log-homogeneous of bidegree $(-d-a, m)$. Indeed, since $\mathcal{F}(x^\alpha\,\partial_\alpha) = -(\partial_\alpha x^\alpha)\mathcal{F}$, i.e., $\mathcal{F}E = -(E+d)\mathcal{F}$ as operators on $\mathcal{S}'(\mathbb{R}^d)$, the relations (2.14) are equivalent to

$$(E+d+a)^{m+1}\mathcal{F}f = 0, \quad \text{but} \quad (E+d+a)^m\mathcal{F}f \neq 0.$$

The Fourier transforms of the considered regular distributions are also regular [24, Chap. 7.3]. Moreover, $\mathcal{F}$ is an isomorphism of the indicated spaces [25].

Third, one can rewrite the result of Lemma 1 to show that it exhibits a representation of the dilation group. Indeed, on multiplying both sides of (2.16) by $\lambda^d/m!$, we obtain

$$\frac{\lambda^d}{m!} R_d\Big[(\lambda r)^{-d}\log^m\frac{\lambda r}{l}\Big] = \sum_{k=0}^{m}\frac{\log^k\lambda}{k!}\,\frac{1}{(m-k)!}\,R_d\Big[r^{-d}\log^{m-k}\frac{r}{l}\Big] + \frac{\log^{m+1}\lambda}{(m+1)!}\,\Omega_d\,\delta(r).$$

This shows that the distributions $\frac{1}{k!}\,R_d[r^{-d}\log^k(r/l)]$, for $k = 0, 1, \dots$, plus the special case $\Omega_d\,\delta(r)$ for $k=-1$, form an eigenvector (with eigenvalue λ^d) for a certain unipotent matrix $\exp(A\log\lambda)$, yielding an action of the dilation group – this is just Proposition 3.2 of [1].

Fourth, the obvious relation

$$l\frac{\partial}{\partial l}\,R_d\Big[r^{-d}\log^m\frac{r}{l}\Big] = -mR_d\Big[r^{-d}\log^{m-1}\frac{r}{l}\Big], \quad \text{for } m \geqslant 1, \tag{2.17}$$

will be most useful in the sequel.

Fifth, formulas involving the Laplacian, like (2.12), do exist for all the log-homogeneous distributions (2.13), and thus for the graphs. We develop these formulas in Appendix A.

2.3 Here comes the sun

One of us introduced in [5, Sect. 4.2], on the basis of related expressions by Estrada and Kanwal [13, 14], the powerful formula

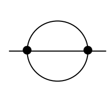
$$\Delta^n R_d\big[r^{-d-2m}\big] = \frac{(d+2m+2n-2)!!}{(d+2m-2)!!}\,\frac{(2m+2n)!!}{(2m)!!}\,R_d\big[r^{-d-2m-2n}\big] \\ - \frac{\Omega_{d,m}}{(2m)!}\sum_{l=1}^{n}\frac{(4m+4l+d-2)}{2(m+l)(2m+2l+d-2)}\,\Delta^{n+m}\delta(r). \tag{2.18}$$

The first term on the right hand side just corresponds to the naïve derivation formula. Once the case $n=1$ is established, the general formula follows by a straightforward iteration, using the relation $\Omega_{d,m+1}/\Omega_{d,m} = (2m+1)/(2k+d)$. This provides explicit expressions for divergences higher than logarithmic.

Consider thus the case: $d=4$, $m=0$, $n=1$, which yields

$$\Delta R_4[r^{-4}] = 8R_4[r^{-6}] - \frac{3\pi^2}{2}\,\Delta\delta(r).$$

Without further ado, we get the renormalization of the quadratically divergent "sunset" graph of the ϕ^4_4 model:

 which in x-space is *primitive* (subdivergence-free) :

$$R_4[r^{-6}] = \frac{1}{8}\,\Delta R_4[r^{-4}] + \frac{3\pi^2}{16}\,\Delta\delta(r) = -\frac{1}{16}\,\Delta^2\Big(r^{-2}\log\frac{r}{l}\Big) + \frac{5\pi^2}{16}\,\Delta\delta(r). \tag{2.19}$$

The same result can be retrieved directly from formula (2.10), see [5]. Its first term is log-homogeneous of bidegree $(-6, 1)$. It is worth noting here that in the paper by Freedman, Johnson and Latorre two different extensions [4, Eq. (A.1)] and [4, Eq. (2.8)] are given for this graph,

$$r_{\mathrm{FJL}}[r^{-6}] = -\frac{1}{16}\,\Delta^2\Big(r^{-2}\log\frac{r}{l}\Big), \quad \text{respectively} \quad r_{\mathrm{FJL}}[r^{-6}] = -\frac{1}{16}\,\Delta^2\Big(r^{-2}\log\frac{r}{l}\Big) - \frac{16\pi^2\delta(r)}{l^2}\,.$$

The first one does not fulfil the algebra property, the second one moreover brings in an unwelcome type of dependence on l. Note as well that rotational symmetry allows *two* arbitrary constants in the renormalization of this graph; the algebra property reduces the ambiguity to one.

The scale derivative gives

$$l\frac{\partial}{\partial l}\,R_4[r^{-6}] = -\frac{\Omega_4}{8}\,\Delta\delta(r). \tag{2.20}$$

The reader may renormalize straightforwardly from (2.18) the simplest vacuum graph of the model.

We compute the following commutation relations:

$$[\Delta, E + d] = [\Delta, E] = 2\Delta; \quad [\Delta, r^2] = 2d + 4E; \quad [E, r^2] = 2r^2, \tag{2.21}$$

valid for radial functions or distributions. This allows us to run an indirect check of (2.19), highlighting the algebra property:

$$\begin{aligned} r^2 R_4[r^{-6}] &= \frac{1}{8}\,\Delta(r^{-2}) - R_4[r^{-4}] - \frac{1}{2}\,E R_4[r^{-4}] + \frac{3\pi^2}{2}\,\delta(r) \\ &= -\frac{\pi^2}{2}\,\delta(r) - R_4[r^{-4}] + 2R_4[r^{-4}] - \pi^2\,\delta(r) + \frac{3\pi^2}{2}\,\delta(r) = R_4[r^{-4}]; \end{aligned}$$

where we have used $r^2\Delta\delta(r) = 2d\,\delta(r)$ and the third identity in (2.21).

More generally, the distribution $R_d[r^{-d-2k}]$ is log-homogeneous of bidegree $(-d-2k, 1)$, since one finds [5] that

$$R_d\big[(\lambda r)^{-d-2k}\big] = \lambda^{-d-2k}\,R_d[r^{-d-2k}] + \lambda^{-d-2k}\log\lambda\,\frac{\Omega_{d,k}}{k!}\,\Delta^k\delta. \tag{2.22}$$

A few explicit expressions for $R_4\big[r^{-6}\log^m(r/l)\big]$ terms, which we shall need later on, are given in Appendix A.

2.4 Trouble with the formulas for derivatives

We remind the reader that there are no extensions of r^{-k-1} for which the generalization of both requirements (2.3) and (2.4) hold simultaneously. One finds [14] that

$$r^m\,r_1[r^{-k-1}] = r_1[r^{-k+m-1}] + [H_{k-m} - H_k]\,\delta^{(k-m)}(r).$$

This also means that differential renormalization in the sense of Freedman, Johnson and Latorre is inconsistent with dimensional reduction; which early on drew justified criticism [26] towards heuristic prescriptions on $\mathbb{R}^4$ in [4], such as

$$r_{\mathrm{FJL}}[r^{-4}] = -\frac{1}{2}\,\Delta\Big(r^{-2}\log\frac{r}{l}\Big).$$

For instance, with a glance at (2.9) and (2.12), we immediately see that the implied renormalization of r^{-1} on the half-line would be $\log'(r/l) - \frac{1}{2}\delta(r)$, instead of $\log'(r/l)$. This makes too much of a break with the rules of calculus.

In general, the distributional derivative of a natural extension of a singular function will not coincide with the natural extension of its derivative. An instructive discussion of this point is given in [27].

3 Convolution-like composition of distributions

The convolution of two integrable functions defined on a Euclidean space $\mathbb{R}^d$ is given by the well-known formula

$$f * g(x) = \int f(y)g(x-y)\,dy,$$

the integral being taken over $\mathbb{R}^d$. To convolve two distributions, one starts with the equivalent formula

$$\langle f * g, \varphi\rangle := \iint f(x)g(y)\varphi(x+y)\,dx\,dy \tag{3.1}$$

where $\varphi \in \mathcal{D}(\mathbb{R}^d)$ here and always denotes a test function. The right hand side of (3.1) may be regarded as a duality formula:

$$\langle f * g, \varphi\rangle := \langle f \otimes g, \varphi^\Delta\rangle, \tag{3.2}$$

where $\varphi^\Delta(x, y) := \varphi(x + y)$ and the pairing on the right hand side takes place over $\mathbb{R}^{2d}$.

Notice that $\varphi^\Delta \in C^\infty(\mathbb{R}^{2d})$ is smooth but no longer has compact support, so that (3.2) only makes sense for certain pairs of distributions f and g. If, say, one of the distributions f or g has compact support, then $f * g$ is well defined as a distribution by (3.2), and associativity formulas like $(f * g) * h = f * (g * h)$ are meaningful and valid if at least two of the three factors have compact support. Also if, for instance, a distribution is tempered, $f \in \mathcal{S}'(\mathbb{R}^d)$, then one can take $g \in \mathcal{O}'_c(\mathbb{R}^d)$, the space of distributions "with rapid decrease at infinity". This variant is dealt with in the standard references, see [28, p. 246] or [29, p. 423].

However, these decay conditions are not met in quantum field practice, so we must amplify the definition of convolution.

One can alternatively interpret the integral in (3.1) as pairing by duality the distribution $f(x)g(y)\varphi(x + y)$ with the constant function 1:

$$\langle f * g, \varphi\rangle := \langle \varphi^\Delta(f \otimes g), 1\rangle. \tag{3.3}$$

For that, one must determine conditions on f and g so that the pairing on the right hand side – again over $\mathbb{R}^{2d}$ – makes sense.

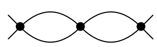
Consider the space $\mathcal{B}_0(\mathbb{R}^d)$ of smooth functions on $\mathbb{R}^d$ that vanish at infinity together with all their derivatives. Its dual space $\mathcal{B}'_0(\mathbb{R}^d)$ is the space of *integrable distributions*. (The notation follows [29]; the space of integrable distributions is called $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ by Schwartz [28].) The dual space of $\mathcal{B}'_0(\mathbb{R}^d)$ itself is larger than $\mathcal{B}_0(\mathbb{R}^d)$: it is $\mathcal{B}''_0(\mathbb{R}^d) \equiv \mathcal{B}(\mathbb{R}^d)$, the space of smooth functions all of whose derivatives are merely bounded on $\mathbb{R}^d$.

It is known [29, Sect. 4.5] that a distribution f is integrable if and only if it can be written as $f = \sum_\alpha \partial^\alpha \mu_\alpha$, a finite sum of derivatives of finite measures μ_α. A particularly useful class of integrable distributions are those of the form $f = h + k$ where h has compact support and k is a function which is integrable (in the usual sense) and vanishes on the support of h.

Definition. Two distributions $f, g \in \mathcal{D}'(\mathbb{R}^d)$ are called *convolvable* if $\varphi^\Delta(f \otimes g) \in \mathcal{B}'_0(\mathbb{R}^{2d})$ for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Since $1 \in \mathcal{B}(\mathbb{R}^{2d})$, the right hand side of (3.3) then makes sense as the evaluation of a (separately continuous) bilinear form; and hence it defines $f * g \in \mathcal{D}'(\mathbb{R}^d)$.

This definition of convolvability was introduced in [30] by Horváth, under the name "condition (Γ)"; and he showed there that it subsumes previous convolvability conditions, such as the aforementioned one between $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{O}'_c(\mathbb{R}^d)$. It is known that $\mathcal{B}'_0(\mathbb{R}^d)$ is an (associative) convolution algebra: such a result already appears in [28] and the proof has been adapted to the above notion of convolvability by Ortner and Wagner [31, Prop. 9].

Now, how can one tell when two given (say, log-homogeneous) distributions are convolvable or not? Consider, for instance, the log-homogeneous distribution $R_d[r^{-4}]$ on $\mathbb{R}^4$, defined in the previous section. It yields the renormalization of the "fish" graph in the massless ϕ^4_4 model; and its convolution with itself amounts to the correct definition of a chain (articulated, one-vertex reducible) diagram, the "spectacles" or "bikini" graph  . The following result allows us to attack the calculation of several graphs in the next section.

Proposition 2. *The convolution of log-homogeneous distributions*

$$R_d\Big[r^\lambda \log^m \frac{r}{l}\Big] * R_d\Big[r^\mu \log^k \frac{r}{l}\Big] \tag{3.4}$$

is well defined whenever $\Re(\lambda + \mu) < -d$*, for any* $m, k = 0, 1, 2, \dots$

Proof. The convolution algebra property takes care of the case where $\lambda < -d$ and $\mu < -d$. The weaker condition $\Re(\lambda + \mu) < -d$ allows us to incorporate also the borderline cases $\lambda = -d$, which we shall need.

Consider first the case where $m = k = 0$. Theorem 3 in [32] shows that a distribution f on $\mathbb{R}^d$ is convolvable with $R_d[r^\mu]$ if $(1 + r^2)^{\Re\mu/2} f$ lies in $\mathcal{B}'_0(\mathbb{R}^d)$; this uses our earlier remark that $R_d[r^\mu]$ coincides with the meromorphic continuation extension of the function r^μ. This sufficient condition on f is guaranteed in turn if $f = f_0 + f_1$ where f_0 has compact support and f_1 is locally integrable with $|f_1(x)| \leqslant C\,|x|^{\Re\mu}$ for large $|x|$.

The last statement is not obvious; the crucial lemma of [32] shows that integrability follows from the boundedness of the following functions:

$$h_{s,c,p}(y) := \int_{A_c} |x|^s\, \partial_y^p \big((1 + |y|^2)^{-s/2}\big)\, dx, \quad \text{where} \quad A_c = \{\, x : |x| \geqslant 1,\ |x + y| \leqslant c \,\},$$

which holds for any real s, any $c > 0$ and derivatives ∂_y^p of all orders. Consequently, in the previous argument, $R_d[r^\mu]$ itself may be replaced by any distribution g of the form $g = g_0 + g_1$ where g_0 has compact support and g_1 is locally integrable with $|g_1(x)| \leqslant C\,|x|^{\Re\mu}$ for large $|x|$.

In particular, taking $f = R_d[r^\lambda]$ and letting f_0 be its restriction to a ball centred at the origin, it follows that $R_d[r^\lambda]$ and $R_d[r^\mu]$ are convolvable whenever $r^{\Re\lambda}(1 + r^2)^{\Re\mu/2}$ is integrable for $r \geqslant 1$, which is true if $\Re(\lambda + \mu) < -d$.

For the general case, a similar decomposition may be applied to both convolution factors. Subtracting off their restrictions to balls centred at the origin, we can bound the remainders thus:

$$r^{\Re\lambda}\log^m\frac{r}{l}\leqslant r^{\alpha},\quad r^{\Re\mu}\log^k\frac{r}{l}\leqslant r^{\beta},\quad \text{for all}\quad r\geqslant r_0.$$

Since $\Re(\lambda+\mu)<-d$, we can still choose α,β so that $\alpha+\beta<-d$ if r_0 is large enough. The convolvability then follows from integrability of $(1+r^2)^{(\alpha+\beta)/2}$ over $r\geqslant r_0$. □

The actual calculation of $R_d[r^{-d}]*R_d[r^{-d}]$ was performed by Wagner in [33], by a simple meromorphic continuation argument. In dimensions $d\geqslant 3$, and for scale $l=1$, the result is:

$$R_d[r^{-d}]*R_d[r^{-d}]=2\,\Omega_d\,R_d[r^{-d}\log r]+\frac{\Omega_d^2}{4}\Big(\psi'(d/2)-\frac{\pi^2}{6}\Big)\,\delta(r), \tag{3.5}$$

with ψ denoting the digamma function. By the same token, convolutions of (renormalized) logarithmically divergent graphs, and in particular chain graphs *of any length*, are rigorously defined and computable in massless ϕ^4_4 theory in x-space. We shall perform a few of these convolutions later on.

The calculation may be transferred to p-space. Now – contrary to an implied assertion in [4] – the product of two tempered distributions is not defined in general. Even so, in the present case, the product of the Fourier transforms may be defined as the Fourier transform of the convolution of their preimages in x-space, under the same condition $\Re(\lambda+\mu)<-d$. In Appendix C, we calculate these regular p-space representatives for $d=4$.

To conclude: maybe because the relevant information [17, 30–34] is scattered in several different languages, the powerful mathematical framework for this, by Horváth, Ortner and Wagner, appears to be little known. So we felt justified in giving a detailed treatment here.

In the computation of graphs of the massless ϕ^4_4 model up to fourth order, one moreover finds slightly more complicated convolution-like integrals. They will be tackled here by easy generalizations of Horváth's theory of convolution. We do not claim that every infrared problem lurking in higher-order graphs can be solved by these methods.

4 Graphs

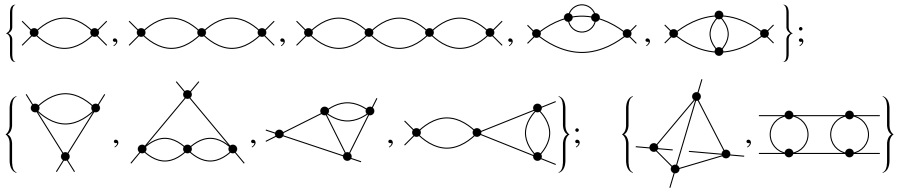
The renormalization of multiloop graphs may be accomplished in position space with the tools developed in Sections 2 and 3. For a model $g\phi^4/4!$ scalar field theory on $\mathbb{R}^4$ – widely used, e.g., in the theory of critical exponents [35, 36] – we perform here the detailed comparison of Epstein–Glaser renormalization with the differential renormalization approach, which was the subject of extensive calculation in [4].

We go about this as follows: first we compute the graphs of the second and third order in the coupling constant for the (one-particle irreducible) four-point function (respectively corresponding to one and two loops), seemingly by brute force. Along the way we find the scale derivatives for these graphs. Next, we exhibit the perturbation expansion up to that order.

After that, we solve the more involved graphs of the fourth order in the coupling constant, corresponding to three loops, for the four-point function. We trust that the procedure to construct the perturbation expansion up to fourth order is by then clear.

In Appendix B, we perform similar calculations for the two-point function, up to the same order in the coupling.

There are three groups of Feynman diagrams involved in the four-point function:



(4.1)

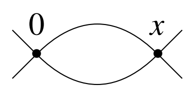
depending on the external leg configurations.

We begin with the one-loop fish graph. Since the Euclidean "propagator" is given by $(-4\pi^2)^{-1}r^{-2}$, its bare amplitude is of the form:

$$\frac{g^2}{(4\pi^2)^2}\big[\delta(x_1-x_2)\,(x_2-x_3)^{-4}\,\delta(x_3-x_4)+\delta(x_1-x_3)\,(x_3-x_4)^{-4}\,\delta(x_4-x_2)$$
$$+\,\delta(x_1-x_4)\,(x_4-x_2)^{-4}\,\delta(x_2-x_3)\big].$$

We write three terms because there are three topologically distinct configurations of the vertices. Moreover, one must divide their contribution by the "symmetry factor", which counts the order of the permutation group of the lines, with the vertices fixed. Here this number is equal to 2. This gives a total *weight* of $3/2$ for the fish graph in the perturbation expansion. In this paper we do not use these variations, so we simply compute the weights of all the graphs we deal with by the direct method in [35, Chap. 14].

Let us moreover leave aside weights and $(4\pi^2)^{-1}$ factors until the moment when we sum the perturbation expansions. Taking advantage of translation invariance to label the vertices as:

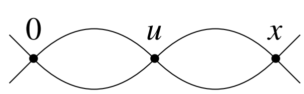


we are left with just our old acquaintance $R_4[r^{-4}]$, with $r = |x|$.

In what follows, we write $x^2 = x_\alpha x^\alpha$ for $x \in \mathbb{R}^4$, $x^4 = (x^2)^2$ and so on; all integrals are taken over $\mathbb{R}^4$.

4.1 A third-order graph by convolution

The next simplest case is the "bikini" graph, which can be labelled thus, with u denoting the internal vertex:



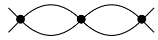
The rules of quantum mechanics prescribe integration over the internal vertices in x-space. From the unrenormalized amplitude

$$\int \frac{1}{u^4}\,\frac{1}{(x-u)^4}\,du$$

(which looks formally like a convolution, though the factors are not actually convolvable) we obtain, on replacing these factors by their extensions, the renormalized version

$$\int R_4[u^{-4}]\, R_4[(x-u)^{-4}]\, du, \tag{4.2}$$

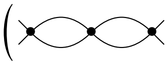
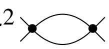
a *bona fide* convolution, since, as shown in the previous section, $R_4[r^{-4}]$ is convolvable with itself. Specializing (3.5) to $d = 4$ and using $\psi'(2) - \pi^2/6 = -1$, one may conclude that

 $$= 4\pi^2 R_4\Big[x^{-4} \log \frac{|x|}{l}\Big] - \pi^4\, \delta(x). \tag{4.3}$$

This looks like a log-homogeneous amplitude of bidegree $(-4, 2)$. More precisely, in the three difference variables, say $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_4$, with the obvious relabeling of the indices, it is quasi-log-homogeneous of bidegree $(-4, 0; -4, 2; -4, 0)$.

A peek at Eq. (A.4) now shows that the coefficient of $\log^2 l$ in the result (4.3) is equal to $4\pi^4$. Here we observe that "predicting" the coefficients of $\log^k l$ for $k > 1$ is fairly easy. With the help of [37], which determines the primitive elements in bialgebras of graphs,[3] a method recommended by Kreimer [38] was applied in [39]. To wit, primitive elements should have *vanishing* coefficients of $\log^k l$ for $k > 1$. In the present case, the bikini graph minus the square of the fish is primitive, and so for the coefficient of $\log^2 l$ the value Ω_4^2, equal to the obtained $4\pi^4$, was predicted.

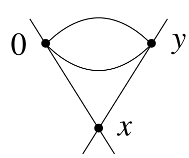
For later use, we obtain the scale derivative:

$$l \frac{\partial}{\partial l} \Big($$  $$\Big) = -4\pi^2$$  $$, \tag{4.4}$$

directly from (2.17) and (4.3).

4.2 A third-order ladder graph: the winecup

Next comes the winecup or ice-cream ladder graph, with vertices labelled as follows:



We denote it  (x, y) for future use. The corresponding bare amplitude is given by

$$f(x, y) = \frac{1}{x^2 (x-y)^2 y^4}\,.$$

Consider a "partially regularized" version of it, for which the known formulas yield:

$$\overline{R}_8\big[x^{-2}(x-y)^{-2} y^{-4}\big] = -\frac{1}{2}\, x^{-2}(x-y)^{-2}\, \Delta\Big(y^{-2} \log \frac{|y|}{l}\Big) + \pi^2 x^{-4}\, \delta(y). \tag{4.5}$$

[3] Actually that reference deals with the bialgebra of rooted trees, but *cela fait rien à l'affaire*.

The last expression indeed makes sense for all $(x, y) \neq (0, 0)$. To proceed, we largely follow [4],[4] which invokes Green's integration-by-parts formula,

$$(\Delta B)\, A = (\Delta A)\, B + \partial^\beta (A \partial_\beta B - B \partial_\beta A), \tag{4.6}$$

that will be rigorously justified soon, in the present context. Thus

$$\begin{aligned} -\frac{1}{2}\, x^{-2}(x-y)^{-2}\, \Delta_y\Big(y^{-2}\log\frac{|y|}{l}\Big) &= -\frac{1}{2}\, x^{-2} y^{-2}\log\frac{|y|}{l}\, \Delta_y\big((x-y)^{-2}\big) + \frac{1}{2}\, x^{-2}\, \partial_y^\beta L_\beta(y; x-y) \\ &= 2\pi^2 x^{-4}\log\frac{|x|}{l}\, \delta(x-y) + \frac{1}{2} x^{-2}\, \partial_y^\beta L_\beta(y; x-y), \end{aligned}$$

where

$$L_\beta(y; x-y) := y^{-2}\log\frac{|y|}{l}\, \partial_\beta^y((x-y)^{-2}) - (x-y)^{-2}\, \partial_\beta^y\Big(y^{-2}\log\frac{|y|}{l}\Big) \tag{4.7}$$

deserves a name, since it is going to reappear often. The presence of the $\delta(x-y)$ factor is rewarding. Now it is evident that renormalized forms of x^{-4} and $x^{-4}\log|x|$ should be used. The only treatment required by the last term is that the derivative be understood in the distributional sense. Thus, in the end, we have computed:

$$\begin{aligned} (x, y) &= R_8\big[x^{-2}(x-y)^{-2}y^{-4}\big] \\ &= 2\pi^2 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big]\, \delta(x-y) + \pi^2 R_4[x^{-4}]\, \delta(y) + \frac{1}{2} x^{-2}\, \partial_y^\beta L_\beta(y; x-y). \end{aligned} \tag{4.8}$$

We now carefully justify Eq. (4.6) for this case. Under the hypothesis $\varphi(0,0) = 0$, substitute $A(y) = (x-y)^{-2}$ and $B(y) = y^{-2}\log(|y|/l)$ there, and compute:

$$\begin{aligned} &\Big\langle x^{-2}\Delta\Big(y^{-2}\log\frac{|y|}{l}\Big), (x-y)^{-2}\varphi(x,y)\Big\rangle = \Big\langle x^{-2}y^{-2}\log\frac{|y|}{l}, \Delta_y\big((x-y)^{-2}\varphi(x,y)\big)\Big\rangle \\ &= \Big\langle x^{-2}y^{-2}\log\frac{|y|}{l}\, \Delta_y((x-y)^{-2}), \varphi(x,y)\Big\rangle + 2\Big\langle x^{-2}y^{-2}\log\frac{|y|}{l}\, \partial_\beta^y((x-y)^{-2}), \partial_y^\beta\varphi(x,y)\Big\rangle \\ &\qquad - \Big\langle x^{-2}\, \partial_\beta^y\Big((x-y)^{-2}y^{-2}\log\frac{|y|}{l}\Big), \partial_y^\beta\varphi(x,y)\Big\rangle \\ &= \Big\langle x^{-2}y^{-2}\log\frac{|y|}{l}\, \Delta_y((x-y)^{-2}), \varphi(x,y)\Big\rangle + \Big\langle x^{-2}y^{-2}\log\frac{|y|}{l}\partial_\beta^y((x-y)^{-2}), \partial_y^\beta\varphi(x,y)\Big\rangle \\ &\qquad - \Big\langle x^{-2}(x-y)^{-2}\, \partial_\beta^y\Big(y^{-2}\log\frac{|y|}{l}\Big), \partial_y^\beta\varphi(x,y)\Big\rangle \\ &= \Big\langle x^{-2}y^{-2}\log\frac{|y|}{l}\Delta_y((x-y)^{-2}), \varphi(x,y)\Big\rangle \\ &\quad - \Big\langle x^{-2}\, \partial_\beta^y\Big(y^{-2}\log\frac{|y|}{l}\, \partial_y^\beta((x-y)^{-2}) - (x-y)^{-2}\, \partial_y^\beta\Big(y^{-2}\log\frac{|y|}{l}\Big)\Big), \varphi(x,y)\Big\rangle. \end{aligned}$$

Observe again that the coefficient of $\log^2 l$ was foreordained: the two-vertex tree minus half of the square of the one-vertex tree (here the fish) is primitive, and so for the numerical coefficient of $\log^2 l$ in (4.8) we were bound to obtain $\frac{1}{2}\Omega_4^2 = 2\pi^4$, which is correct: see (A.4).

[4]Since $\overline{R}_4[y^{-4}]x^{-2}(x-y)^{-2}$ is undefined only at the origin, the procedure (2.10) assuredly works, giving rise to alternative expressions, very much like the ones proposed by Smirnov and Zav'yalov some time ago [40]. This was the path taken in [39]. We find those, however, somewhat unwieldy.

We turn to the scale derivative for the winecup. *Prima facie* it yields:

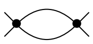
$$l\frac{\partial}{\partial l}$$  $$(x,y) = -2\pi^2 R_4[x^{-4}]\,\delta(x-y) - 2\pi^4\,\delta(x)\,\delta(y) + 2\pi^2 x^{-4}\,\delta(x-y) - 2\pi^2 x^{-4}\,\delta(y). \quad (4.9)$$

The difference between the third and fourth terms above *is* a well-defined distribution, since

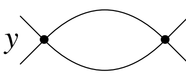
$$\int \frac{\varphi(x,x)-\varphi(x,0)}{x^4}\,d^4x$$

converges for any test function φ. We may reinterpret the x^{-4} in (4.9) as $R_4[x^{-4}]$, since the corresponding difference is still the same unique extension. So the scale derivative becomes

$$-2\pi^2 R_4[x^{-4}]\,\delta(x-y) - 2\pi^4\,\delta(x)\,\delta(y) + 2\pi^2 R_4[x^{-4}]\,\delta(x-y) - 2\pi^2 R_4[x^{-4}]\,\delta(y)$$

$$= -2\pi^2$$  $$(x)\,\delta(y) - 2\pi^4\,\delta(x)\,\delta(y). \quad (4.10)$$

Let us now reflect on what we have just done, in order to set forth our methods. Understanding and fulfilment of the fundamental equation (1.1) is essential. The winecup graph exemplifies it well. There formula (4.5) expresses $R[\gamma]$ for $\mathcal{V}(\gamma) = \{0, y\}$. The cograph Γ/γ is a fish:

$0 = y$  x

Now, the test function φ in (1.1) is assumed to *vanish* on the thin diagonal $x = y = 0$. Then (1.1) simply means

$$\begin{aligned}\langle R_8\big[x^{-2}(x-y)^{-2}y^{-4}\big], \varphi(x,y)\rangle &= \langle \overline{R}_8\big[x^{-2}(x-y)^{-2}y^{-4}\big], \varphi(x,y)\rangle \\ &= \langle \overline{R}_4\big[y^{-4}\big], x^{-2}(x-y)^{-2}\varphi(x,y)\rangle\end{aligned}$$

when $\varphi(0,0) = 0$; and this is all we need to ask.[5]

As we shall see next, similar procedures obeying the fundamental formula (1.1), canvassing help from Section 3 when necessary, allow one to compute all the fourth-order contributions to the four-point function. It will become clear that the scale derivative is related to the hierarchy of cographs.

4.3 Empirical remarks on the main theorem of renormalization

For the four-point function $\mathcal{G}^4$, we are able to consider already the contributions at orders g, g^2, g^3. From now on we adopt a standard redefinition of the coupling constant:

$$\bar{g} = \frac{g}{16\pi^2},$$

which eliminates many π factors. Thus, for the first-order contribution, introducing a global minus sign as a matter of convention:

$$\mathcal{G}^4_{(1)}(x_1,x_2,x_3,x_4;\bar{g}) \equiv \mathcal{G}^4_{\times}(x_1,x_2,x_3,x_4) = 16\pi^2\bar{g}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4).$$

[5]As also noted in [41, Remark 6.1], our use of renormalized expressions for divergent subgraphs avoids appeal to the forest formula entirely.

Contributions of graphs of different orders to the four-point function come with alternating signs [35, Chap. 13]. Thus the fish diagram contribution $\mathcal{G}^4_{\asymp}$ is given by

$$\begin{aligned}\mathcal{G}^4_{\asymp}(x_1,x_2,x_3,x_4;\bar{g};l) &= -\frac{g^2}{32\pi^4}\big(\delta(x_1-x_2)\,R_4[(x_2-x_3)^{-4};l]\,\delta(x_3-x_4)+2\text{ permutations}\big)\\ &= -8\bar{g}^2\big(\delta(x_1-x_2)\,R_4[(x_2-x_3)^{-4};l]\,\delta(x_3-x_4)+\delta(x_1-x_3)\,R_4[(x_3-x_4)^{-4};l]\,\delta(x_4-x_2)\\ &\qquad +\delta(x_1-x_4)\,R_4[(x_4-x_2)^{-4};l]\,\delta(x_2-x_3)\big).\end{aligned}$$

The practical rule to go from the scale derivatives of the graphs as we have calculated them to their actual contributions to the four-point function is simple: multiply the coefficient of the scale derivative by $-\bar{g}/\pi^2$ raised to a power equal to the difference in the number of vertices, and also by the relative weight, for the diagrams in question.

A key point here, harking back to (2.15) and recalling that the fish has weight $\frac{3}{2}$, is that

$$l\frac{\partial}{\partial l}\,\mathcal{G}^4_{\asymp} = 3\bar{g}\,\mathcal{G}^4_{\times}. \tag{4.11}$$

If we now define the second-order approximation $\mathcal{G}^4_{(2)} := \mathcal{G}^4_{\times} + \mathcal{G}^4_{\asymp}$, we find that

$$\begin{aligned}&\mathcal{G}^4_{(2)}(x_1,x_2,x_3,x_4;\bar{g};l) - \mathcal{G}^4_{(2)}(x_1,x_2,x_3,x_4;\bar{g};l')\\ &\qquad = 48\pi^2\bar{g}^2\,\log\frac{l}{l'}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4).\end{aligned}$$

This of course means that

$$\begin{aligned}\mathcal{G}^4_{(2)}(x_1,x_2,x_3,x_4;\bar{g};l) &= \mathcal{G}^4_{(2)}\big(x_1,x_2,x_3,x_4;\overline{G}^{\,ll'}_{(2)}(\bar{g});l'\big),\\ \text{where}\quad \overline{G}^{\,ll'}_{(2)}(\bar{g}) &= \bar{g} + 3\log\frac{l}{l'}\,\bar{g}^2 + O(\bar{g}^3).\end{aligned}$$

The bikini graph has a weight of $3/4$ in the perturbation expansion. We obtain

$$\begin{aligned}\mathcal{G}^4_{\asymp\asymp}(x_1,x_2,x_3,x_4;\bar{g};l) = 16\,\bar{g}^3\,\delta(x_1-x_2)\bigg(&R_4\bigg[(x_2-x_3)^{-4}\log\frac{|x_2-x_3|}{l}\bigg]\\ &-\frac{\pi^2}{4}\,\delta(x_2-x_3)\bigg)\,\delta(x_3-x_4)+2\text{ permutations}.\end{aligned}$$

Here the two permutations have the same structure as those of the fish graph. Therefore,

$$l\frac{\partial}{\partial l}\,\mathcal{G}^4_{\asymp\asymp} = 2\bar{g}\,\mathcal{G}^4_{\asymp}, \tag{4.12}$$

coming from (4.4) when all factors have been taken into account, according to the rule explained above.

In the third-order approximation,

$$\mathcal{G}^4_{(3)} := \mathcal{G}^4_{\times} + \mathcal{G}^4_{\asymp} + \mathcal{G}^4_{\asymp\asymp} + \mathcal{G}^4_{\triangledown},$$

we examine first the difference

$$\mathcal{G}^4_{\asymp\asymp}(x_1,x_2,x_3,x_4;\bar{g};l) - \mathcal{G}^4_{\asymp\asymp}(x_1,x_2,x_3,x_4;\bar{g};l').$$

From (A.4) one sees that

$$R_4\Big[r^{-4}\log\frac{r}{l}\Big] - R_4\Big[r^{-4}\log\frac{r}{l'}\Big] = \pi^2(\log^2 l - \log^2 l')\,\delta(r) - \log\frac{l}{l'}\,R_4[r^{-4};1],$$

so the difference may be rewritten as

$$\begin{aligned}&48\,\bar{g}^3\pi^2\big(\log^2 l - \log^2 l'\big)\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4)\\ &\quad - 16\,\bar{g}^3\log\frac{l}{l'}\big(\delta(x_1-x_2)\,R_4[(x_2-x_3)^{-4};1]\,\delta(x_3-x_4) + 2\text{ permutations}\big).\end{aligned} \tag{4.13}$$

The winecup graph enters with weight 3 in the perturbation expansion. In detail:

$$\begin{aligned}&\mathcal{G}^{(4)}_{\triangledown}(x_1,x_2,x_3,x_4;\bar{g};l) = 16\bar{g}^3\,\delta(x_1-x_2)\bigg(R_4\bigg[(x_2-x_4)^{-4}\log\frac{|x_2-x_4|}{l}\bigg]\,\delta(x_2-x_3)\\ &+\frac{1}{2}R_4\big[(x_2-x_4)^{-4};l\big]\delta(x_3-x_4) + \frac{(x_2-x_4)^{-2}}{4\pi^2}\,\partial^\beta_{x_3}\bigg((x_3-x_4)^{-2}\log\frac{|x_3-x_4|}{l}\,\partial^\beta_{x_3}(x_2-x_3)^{-2}\\ &-(x_2-x_3)^{-2}\,\partial^\beta_{x_3}\Big((x_3-x_4)^{-2}\log\frac{|x_3-x_4|}{l}\Big)\bigg) + R_4\bigg[(x_2-x_3)^{-4}\log\frac{|x_2-x_3|}{l}\bigg]\,\delta(x_2-x_4)\\ &+\frac{1}{2}R_4\big[(x_2-x_3)^{-4};l\big]\delta(x_3-x_4) + \frac{(x_2-x_3)^{-2}}{4\pi^2}\,\partial^\beta_{x_4}\bigg((x_3-x_4)^{-2}\log\frac{|x_3-x_4|}{l}\,\partial^\beta_{x_4}(x_2-x_4)^{-2}\\ &-(x_2-x_4)^{-2}\,\partial^\beta_{x_4}\Big((x_3-x_4)^{-2}\log\frac{|x_3-x_4|}{l}\Big)\bigg)\bigg) + 2\text{ permutations of each.}\end{aligned} \tag{4.14}$$

The scale derivative can be obtained either from (4.10) by applying the conversion rule, or now directly from (4.14):

$$l\frac{\partial}{\partial l}\,\mathcal{G}^4_{\triangledown} = 4\bar{g}\,\mathcal{G}^4_{\asymp} - 6\bar{g}^2\,\mathcal{G}^4_{\times}. \tag{4.15}$$

The difference between the winecup expansion (4.14) at two scales has several contributions. From the $R_4\big[r^{-4}\log(r/l)\big]$ terms we collect

$$-16\,\bar{g}^3\log\frac{l}{l'}\big(\delta(x_1-x_2)\,R_4[(x_2-x_4)^{-4};1]\,\delta(x_2-x_3) + \delta(x_1-x_2)\,R_4[(x_2-x_3)^{-4};1]\,\delta(x_2-x_4)\big),$$

plus its two permutations. None of those is of the same type as those coming from (4.13). Nevertheless, from the divergence part of (4.14) we recover

$$16\,\bar{g}^3\log\frac{l}{l'}\Big(\delta(x_1-x_2)R_4\big[(x_2-x_4)^{-4};1\big]\delta(x_2-x_3)\Big),$$

and similar terms, *cancelling all* of the previous terms. The divergence part also produces terms like the second line of (4.13), with a doubled coefficient. The remaining summand $8\bar{g}^3\,\delta(x_1-x_2)R_4\big[(x_2-x_4)^{-4};l\big]\,\delta(x_3-x_4)$, and its permutations, generate

$$-96\pi^2\bar{g}^3\log\frac{l}{l'}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4).$$

Putting together all the contributions up to third order, we arrive at

$$\begin{aligned}
&\mathcal{G}^4_{(3)}(x_1,x_2,x_3,x_4;\bar g;l) - \mathcal{G}^4_{(3)}(x_1,x_2,x_3,x_4;\bar g;l')\\
&\quad = 48\pi^2\bar g^2\,\log\frac{l}{l'}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4)\\
&\qquad - 144\bar g^3\log\frac{l}{l'}\Big(\delta(x_1-x_2)R_4\big[(x_2-x_4)^{-4};1\big]\,\delta(x_2-x_3)\Big)\\
&\qquad - 96\pi^2\bar g^3\log\frac{l}{l'}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4)\\
&\qquad + 144\,\bar g^3\pi^2\big(\log^2 l-\log^2 l'\big)\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4).
\end{aligned}$$

Now, we look for a coefficient α in

$$\overline{G}^{ll'}_{(3)}(\bar g) = \bar g + 3\log\frac{l}{l'}\,\bar g^2 + \alpha\,\bar g^3 + O(\bar g^4),$$

such that

$$\mathcal{G}^4_{(3)}(x_1,x_2,x_3,x_4;\bar g;l) = \mathcal{G}^4_{(3)}\big(x_1,x_2,x_3,x_4;\overline{G}^{ll'}_{(3)}(\bar g);l'\big) + O(\bar g^4). \tag{4.16}$$

We see that the term in $\bar g^2$ does not need revisiting. Of course, general theory ensures that. Next,

$$\begin{aligned}
&\mathcal{G}^4_{(3)}\big(x_1,x_2,x_3,x_4;\overline{G}^{ll'}_{(3)}(\bar g);l'\big)\\
&\quad = \mathcal{G}^4_{(3)}(x_1,x_2,x_3,x_4;\bar g;l) + 16\pi^2\alpha\bar g^3\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4)\\
&\qquad - 144\,\bar g^3\pi^2(\log^2 l-\log^2 l')\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4)\\
&\qquad + 96\pi^2\bar g^3\log\frac{l}{l'}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4) + O(\bar g^4).
\end{aligned}$$

Therefore, the relation (4.16) is verified by taking

$$\overline{G}^{ll'}_{(3)}(\bar g) = \bar g + 3\log\frac{l}{l'}\,\bar g^2 + \left(9\,(\log^2 l-\log^2 l') - 6\log\frac{l}{l'}\right)\bar g^3 + O(\bar g^4).$$

At this very humble level, this illustrates the Popineau–Stora "main theorem of renormalization" [9] – as applied to the effective action. To wit, there exists a formal power series $\overline{G}(\bar g)$, tangent to the identity, that effects the change between any two renormalization recipes.[6] Actually, there is more to the theorem than was allowed in [9]. One can write

$$\mathcal{G}^4_{(n)}(x_1,x_2,x_3,x_4;\bar g;l) = \mathcal{G}^4_{(n)}\big(x_1,x_2,x_3,x_4;\overline{G}^{ll'}_{(N-1)}(\bar g);l'\big) + O(\bar g^{n+1}),$$

for any $n \leqslant N-1$. Here $\overline{G}^{ll'}_{(N-1)}$ need only be taken up to order n. Let

$$\overline{G}^{ll'}_{(N-1)}(\bar g) = \bar g + H_2^{ll'}(\bar g) + H_3^{ll'}(\bar g) + \cdots + H_{N-1}^{ll'}(\bar g) =: \bar g + H_{(N-1)}^{ll'}(\bar g).$$

[6] When one allows $\bar g$ to become a test function $g(x)$, that series is local in the coordinates, although it may depend on the derivatives of $g(x)$.

Here each $H_n^{ll'}$ comes from a distribution with support on the corresponding thin diagonal *exclusively*. Then

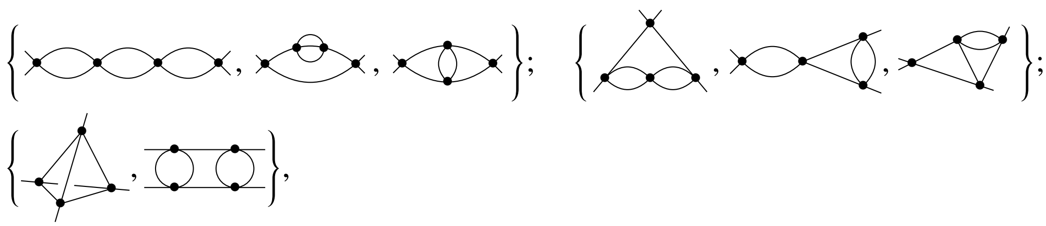
$$\overline{G}^{ll'}_{(N)} = \overline{G}^{ll'}_{(N-1)} + H^{ll'}_{(N)}\big(\overline{G}^{ll'}_{(N-1)}\big),$$

where only terms up to order N in the last expansion need be taken. One may let $N \uparrow \infty$, obtaining the "tautological" identity

$$\overline{G}^{ll'} = \mathrm{id} + H\big(\overline{G}^{ll'}\big).$$

This is what Stora understands by the Bogoliubov recursion relation for the coupling constant – apparently a deeper fact than the Bogoliubov recursion for the graphs [42, 43].

For future reference, we report here the weight factors of the graphs contributing to $\mathcal{G}^4$ at order g^4. For the sets:



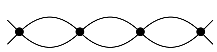
these numbers are respectively given by $\{\frac{3}{8}, \frac{1}{2}, \frac{3}{4}\}$; $\{\frac{3}{2}, \frac{3}{2}, 6\}$; $\{1, \frac{3}{2}\}$.

5 More graphs

In this section, we consider each one of the eight graphs with four vertices required to compute the four-point function. We discuss first the three graphs with two external vertices, then those three with three external vertices, and finally those two with four external vertices. The last of these do not require convolution-like operations.

5.1 The trikini

The "trikini" is a fourth-order, three-loop chain graph, that is a convolution cube. The quickest method is to pass to multiplication in p-space, using (C.4) from Appendix C:



$$= \big(R_4[x^{-4}]\big)^{*3}$$
$$= 12\pi^4 R_4\Big[x^{-4}\log^2\frac{|x|}{l}\Big] - 3\pi^4 R_4[r^{-4}] + (4\zeta(3) - 2)\pi^6\,\delta(x). \qquad (5.1)$$

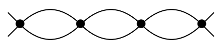
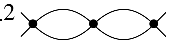
In general, the amplitude of a chain graph with $(n+1)$ vertices in p-space is given by

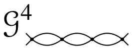
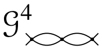
$$\pi^{2n}\bigg(1 - 2\log\frac{|p|}{\Lambda}\bigg)^n \qquad (5.2)$$

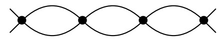
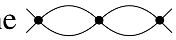
where $\Lambda = 2e^{-\gamma}/l$. The result is readily transferable to x-space, using the formulas of Appendix C to invert the Fourier transforms.

We note that with the differential renormalization method of [4] only $\big(2\log(|p|/\Lambda)\big)^n$ is computed. The problem is compounded by a mistake in their Fourier transform formula, whose origin is dealt with in the Appendix.

We easily compute as usual the scale derivative:

$$l\frac{\partial}{\partial l}$$  $$= -6\pi^2$$  , translating into

$$l\frac{\partial}{\partial l}\,\mathcal{G}^4$$  $$= 3\bar{g}\,\mathcal{G}^4$$  ,

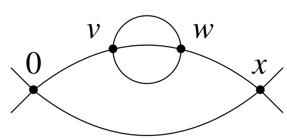
since the weight of the  graph is half the weight of the  graph. In general, the scale derivative of the contribution to $\mathcal{G}^4$ of a chain graph with n bubbles equals $n\bar{g}$ times that of the graph with $(n-1)$ bubbles. Indeed, it follows at once from (2.15) or alternatively from (5.2) that

$$l\frac{\partial}{\partial l}\left(R_4[x^{-4}]\right)^{*n} = -2n\pi^2\left(R_4[x^{-4}]\right)^{*(n-1)},$$

and the aforementioned practical rule gives the result for the contributions to $\mathcal{G}^4$.

5.2 The stye

The "partially renormalized" amplitude for the stye diagram, labelled as follows:



is of the form

$$x^{-2}\iint \overline{R}_{12}\left[v^{-2}(v-w)^{-6}(w-x)^{-2}\right]dv\,dw = x^{-2}\iint \overline{R}_{12}\left[v^{-2}u^{-6}(v-u-x)^{-2}\right]du\,dv.$$

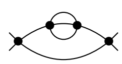
Notice that this is a nested convolution; the inner integral is of the form $R_4[r^{-6}] * r^{-2}$, which *exists* by the theory of Section 3. On account of (2.19), the integral becomes

$$-\frac{1}{16}x^{-2}\iint v^{-2}(v-u-x)^{-2}\left(\Delta^2\left(u^{-2}\log\frac{|u|}{l}\right) - 5\pi^2\,\Delta\delta(u)\right)dv\,du.$$

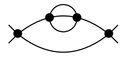
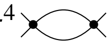
On integrating by parts with (4.6) and dropping total derivatives in the integrals over internal vertices, we then obtain

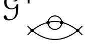
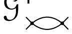
$$\begin{aligned}
&\frac{\pi^2}{4}x^{-2}\iint v^{-2}\left(\Delta\left(u^{-2}\log\frac{|u|}{l}\right) - 5\pi^2\,\delta(u)\right)\delta(v-u-x)\,dv\,du \\
&= \frac{\pi^2}{4}x^{-2}\int (u+x)^{-2}\left(\Delta\left(u^{-2}\log\frac{|u|}{l}\right) - 5\pi^2\,\delta(u)\right)du \\
&= -\pi^4x^{-2}\int u^{-2}\log\frac{|u|}{l}\,\delta(u+x)\,du - \frac{5\pi^4}{4}x^{-4} = -\pi^4x^{-4}\log\frac{|x|}{l} - \frac{5\pi^4}{4}x^{-4}.
\end{aligned}$$

The fully renormalized amplitude for the stye graph is then simply given by

 $$= -\pi^4 R_4\left[x^{-4}\log\frac{|x|}{l}\right] - \frac{5\pi^4}{4}R_4[x^{-4}].$$

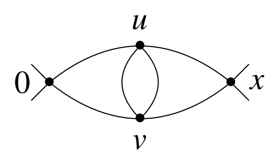
Therefore

$l\dfrac{\partial}{\partial l}$  $= \pi^4$  $+ \dfrac{5\pi^6}{2}$ , translating into

$l\dfrac{\partial}{\partial l}\,\mathcal{G}^4$  $= \dfrac{1}{3}\,\bar{g}^2\,\mathcal{G}^4$  $- \dfrac{5}{4}\,\bar{g}^3\,\mathcal{G}^4$  .

5.3 The cat's eye

The "cat's eye" graph, which we label as follows:



is sometimes counted as an "overlapping divergence" in p-space. But for renormalization on configuration space, this problem is more apparent than real: "the external points can be kept separated until the regularization of subdivergences is accomplished" [4, Sect. 3.3]. For the same reasons, when dealing with this graph we find it unnecessary to bring in partitions of unity for overlapping divergences [11, Example 4.16].

There are two internal vertices; its "bare" amplitude is of the form

$$f(x) = \iint u^{-2}v^{-2}(u-x)^{-2}(v-x)^{-2}(u-v)^{-4}\,du\,dv.$$

A first natural rewriting is

$$\begin{aligned} f(x) \overset{R}{\longmapsto} -\frac{1}{2}\iint u^{-2}v^{-2}(u-x)^{-2}(v-x)^{-2}\,\Delta_u\Big((u-v)^{-2}\log\frac{|u-v|}{l}\Big)\,du\,dv \\ + \pi^2\iint u^{-2}v^{-2}(u-x)^{-2}(v-x)^{-2}\,\delta(u-v)\,du\,dv. \end{aligned}$$

From now on, we shall use the notation $\overset{R}{\longmapsto}$ to denote a single step in a sequence of one of more partial renormalizations, by replacements $r^{-d-2m}\log^k(r/l) \overset{R}{\longmapsto} R_d\big[r^{-d-2m}\log^k(r/l)\big]$. That is *not yet* $\overline{R}_4[f(x)]$, since there are other untreated subdivergences in the diagram. The second term in this expression, however, is just the bikini convolution integral, which becomes

$$\pi^2\int v^{-4}(v-x)^{-4}\,dv \overset{R}{\longmapsto} 4\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big] - \pi^6\,\delta(x).$$

The first term can be simplified by an integration by parts, to get

$$\begin{aligned} 2\pi^2x^{-2}\int(v-x)^{-2}v^{-4}\log\frac{|v|}{l}\,dv + 2\pi^2x^{-2}\int v^{-2}(v-x)^{-4}\log\frac{|v-x|}{l}\,dv \\ - \iint v^{-2}(v-x)^{-2}\,\partial_u^\beta(u^{-2})\,\partial_\beta^u((u-x)^{-2})\,(u-v)^{-2}\log\frac{|u-v|}{l}\,du\,dv. \end{aligned} \tag{5.3}$$

The first two of these integrals are equal. To deal with the $v \sim 0$ region, we proceed as before:

$$\begin{aligned}
&4\pi^2 x^{-2} \int (v-x)^{-2} v^{-4} \log\frac{|v|}{l}\, dv \\
&\quad\overset{R}{\longmapsto} -\pi^2 x^{-2} \int (v-x)^{-2} \Big(\Delta\Big(v^{-2}\log^2\frac{|v|}{l}\Big) + \Delta\Big(v^{-2}\log\frac{|v|}{l}\Big) - 2\pi^2\,\delta(v) \Big)\, dv \\
&\quad\overset{R}{\longmapsto} 4\pi^4 R_4\Big[x^{-4}\log^2\frac{|x|}{l}\Big] + 4\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big] + 2\pi^4 R_4[x^{-4}].
\end{aligned}$$

The third term in (5.3) is only divergent *overall*, at $x = 0$. After rescaling the integrand by $u \mapsto |x|s$, $v \mapsto |x|t$, and setting $x = |x|\omega$, this term takes the form

$$-c_1\pi^4 x^{-4}\log\frac{|x|}{l} - c_2\pi^4 x^{-4},$$

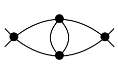
where

$$\begin{aligned}
c_1 &= \frac{1}{\pi^4} \iint t^{-2}(t-\omega)^{-2}\, \partial_s^\beta(s^{-2})\, \partial_\beta^s((s-\omega)^{-2})\, (s-t)^{-2}\, ds\, dt = 4, \\
c_2 &= \frac{1}{\pi^4} \iint t^{-2}(t-\omega)^{-2}\, \partial_s^\beta(s^{-2})\, \partial_\beta^s((s-\omega)^{-2})\, (s-t)^{-2} \log\frac{|s-t|}{l}\, ds\, dt = 4.
\end{aligned}$$

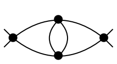
These are computed straightforwardly, if tediously, by use of Gegenbauer polynomials [4]. Thus, this third term yields

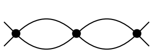
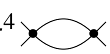
$$-4\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big] - 4\pi^4 R_4[x^{-4}].$$

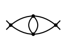
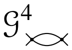
Putting it all together, we arrive at

 $= 4\pi^4 R_4\Big[x^{-4}\log^2\frac{|x|}{l}\Big] + 4\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big] - 2\pi^4 R_4[x^{-4}] - \pi^6\,\delta(x).$

With that, we obtain

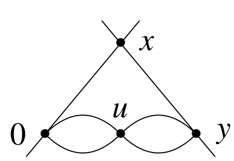
$l\dfrac{\partial}{\partial l}$  $= -8\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big] - 4\pi^4 R_4[x^{-4}] + 4\pi^6\,\delta(x)$

$= -2\pi^2$  $- 4\pi^4$  $+ 2\pi^6\,\delta(x)$; translating into

$l\dfrac{\partial}{\partial l}\, \mathcal{G}^4$  $= 2\bar{g}\,\mathcal{G}^4$  $- 2\bar{g}^2\,\mathcal{G}^4$  $- \dfrac{3}{2}\bar{g}^3\,\mathcal{G}^4$  .

5.4 The duncecap

Consider next the "duncecap", which contains a bikini subgraph:



The unrenormalized amplitude is given by

$$f(x,y) = x^{-2}(x-y)^{-2}(y^{-4})^{*2}.$$

Once again, we may partially regularize this, using (4.3), to get

$$\overline{R}_8\big[x^{-2}(x-y)^{-2}(y^{-4})^{*2}\big] = \pi^2 x^{-2}(x-y)^{-2}\biggl(4R_4\Bigl[y^{-4}\log\frac{|y|}{l}\Bigr] - \pi^2\,\delta(y)\biggr),$$

which is well defined for $(x,y) \neq (0,0)$. We want to apply the integration by parts formula (4.6) here, and we invoke (A.4) for the purpose:

$$\begin{aligned}\overline{R}_8\big[x^{-2}(x-y)^{-2}(y^{-4})^{*2}\big] &= -\pi^2 x^{-2}(x-y)^{-2}\,\Delta\Bigl(y^{-2}\log^2\frac{|y|}{l}\Bigr)\\ &\qquad - \pi^2 x^{-2}(x-y)^{-2}\,\Delta\Bigl(y^{-2}\log\frac{|y|}{l}\Bigr) + \pi^4 x^{-4}\,\delta(y),\end{aligned}$$

which leads easily to the renormalized version:

$$\begin{aligned}R_8\big[x^{-2}(x-y)^{-2}(y^{-4})^{*2}\big] &= 4\pi^4\,R_4\Bigl[x^{-4}\log^2\frac{|x|}{l} + x^{-4}\log\frac{|x|}{l}\Bigr]\,\delta(x-y) + \pi^4 R_4[x^{-4}]\,\delta(y)\\ &\qquad + \pi^2 x^{-2}\partial_y^\beta\big(L_\beta(y;x-y) + M_\beta(y;x-y)\big),\end{aligned}$$

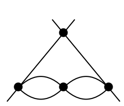
where we call upon (4.7) and a companion formula:

$$M_\beta(y;x-y) := y^{-2}\log^2\frac{|y|}{l}\,\partial_\beta^y\big((x-y)^{-2}\big) - (x-y)^{-2}\,\partial_\beta^y\Bigl(y^{-2}\log^2\frac{|y|}{l}\Bigr). \tag{5.4}$$

The Green formula (4.6) easily yields

$$l\frac{\partial}{\partial l}\,\big[\partial_y^\beta L_\beta(y;x-y)\big] = 4\pi^2 x^{-2}\big(\delta(x-y) - \delta(y)\big),$$

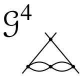
and clearly $l\frac{\partial}{\partial l}\,\big[\partial_y^\beta M_\beta(y;x-y)\big] = -2\,\partial_y^\beta L_\beta(y;x-y)$. From this, we obtain for the scale derivative, in the same way as for the winecup:

$l\frac{\partial}{\partial l}$ 

$$\begin{aligned}&= -8\pi^4 R_4\Bigl[x^{-4}\log\frac{|x|}{l}\Bigr]\,\delta(x-y) - 4\pi^4 R_4[x^{-4}]\,\delta(x-y) - 2\pi^6\,\delta(x)\,\delta(y)\\ &\qquad + 4\pi^4 R[x^{-4}]\,\delta(x-y) - 4\pi^4 R[x^{-4}]\,\delta(y) - 2\pi^2\,x^{-2}\,\partial_y^\beta L_\beta(y;x-y)\end{aligned}$$

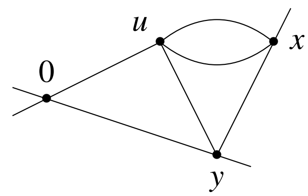
$= -4\pi^2$  $(x,y) - 2\pi^6\,\delta(x)\,\delta(y);$

yielding, after the usual manipulation,

$l\frac{\partial}{\partial l}\,\mathcal{G}^4$  $= 2\bar{g}\,\mathcal{G}^4$  $+ 3\bar{g}^3\,\mathcal{G}^4_{\times}.$

5·5 The kite

The *kite* graph has the following structure, showing an internal winecup:



The labelling is that recommended by [4].

One may anticipate the coefficients of logarithmic degrees 3 and 2 in the final result, by invoking again [37], particularly its Section 4.2, and [38, 39]. We focus on the third-degree coefficients. Notice that the kite is a rooted-tree graph, actually a "stick", with the fish as unique "decoration". Subtracting from it the disconnected juxtaposition of the winecup and the fish graph, and adding one third of the product of three fishes, one obtains a primitive graph in the cocommutative bialgebra of sticks. For this combination the dilation coefficient of $\log^3 l$ must vanish. We obtain then for this graph the coefficient $2\pi^4 \times 2\pi^2 - 8\pi^6/3 = 4\pi^6/3$.

Starting from the bare amplitude

$$f(x,y) = y^{-2}(x-y)^{-2} \int u^{-2}(u-y)^{-2}(u-x)^{-4}\,du,$$

partial renormalization gives

$$\begin{aligned} f(x,y) &\overset{R}{\longmapsto} y^{-2}(x-y)^{-2} \int u^{-2}(u-y)^{-2} R_4\big[(u-x)^{-4}\big]\,du \\ &= -\frac{1}{2} y^{-2}(x-y)^{-2} \int u^{-2}(u-y)^{-2}\,\Delta_x\Big((x-u)^{-2}\log\frac{|x-u|}{l}\Big)\,du + \pi^2 x^{-2} y^{-2}(x-y)^{-4}. \end{aligned}$$

The second summand is a winecup: namely, the cograph obtained on contracting the u–x fish. It contributes

$$\begin{aligned} \pi^2 x^{-2} y^{-2} R_4\big[(x-y)^{-4}\big] &= -\frac{\pi^2}{2} x^{-2} y^{-2}\,\Delta_y\Big((x-y)^{-2}\log\frac{|x-y|}{l}\Big) + \pi^4 x^{-4}\,\delta(x-y) \\ &= 2\pi^4 x^{-4} \log\frac{|x|}{l}\,\delta(y) + \pi^4 x^{-4}\,\delta(x-y) - \frac{\pi^2}{2}\,x^{-2}\,\partial_y^\beta L_\beta(x-y;y) \\ &\overset{R}{\longmapsto} 2\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big]\,\delta(y) + \pi^4 R_4[x^{-4}]\,\delta(x-y) - \frac{\pi^2}{2}\,x^{-2}\,\partial_y^\beta L_\beta(x-y;y). \end{aligned} \qquad (5.5)$$

There remains the term $-\frac{1}{2} y^{-2}(x-y)^{-2}\,\Delta_x \int u^{-2}(u-y)^{-2}(x-u)^{-2}\log\frac{|x-u|}{l}\,du$. Following a suggestion of [4], we may write

$$\log\frac{|x-u|}{l} = \log\frac{|x-u|}{|y-u|} + \log\frac{|y-u|}{l}\,. \qquad (5.6)$$

The second summand above contributes

$$
\begin{aligned}
&-\frac{1}{2}\,y^{-2}(x-y)^{-2}\int u^{-2}(u-y)^{-2}\,\Delta_x\big((x-u)^{-2}\big)\log\frac{|y-u|}{l}\,du \\
&\qquad = 2\pi^2x^{-2}y^{-2}(x-y)^{-4}\log\frac{|x-y|}{l} \\
&\overset{R}{\longmapsto}\ -\frac{\pi^2}{2}x^{-2}y^{-2}\,\Delta_x\Big((x-y)^{-2}\log^2\frac{|x-y|}{l}+(x-y)^{-2}\log\frac{|x-y|}{l}\Big)+\pi^4x^{-4}\,\delta(x-y) \\
&\qquad = 2\pi^4\Big(y^{-4}\log^2\frac{|y|}{l}+y^{-4}\log\frac{|y|}{l}\Big)\,\delta(x)+\pi^4x^{-4}\,\delta(x-y) \\
&\qquad\qquad -\frac{\pi^2}{2}\,y^{-2}\,\partial_x^\beta\big(L_\beta(x-y;x)+M_\beta(x-y;x)\big) \\
&\overset{R}{\longmapsto}\ 2\pi^4R_4\Big[y^{-4}\log^2\frac{|y|}{l}+y^{-4}\log\frac{|y|}{l}\Big]\,\delta(x)+\pi^4R_4[x^{-4}]\,\delta(x-y) \\
&\qquad\qquad -\frac{\pi^2}{2}\,y^{-2}\,\partial_x^\beta\big(L_\beta(x-y;x)+M_\beta(x-y;x)\big).
\end{aligned}
\tag{5.7}
$$

It helps to introduce the integral:

$$
\begin{aligned}
\int u^{-2}(x-u)^{-2}(y-u)^{-2}\log\frac{|x-u|}{|y-u|}\,du &= \frac{1}{2}\,\log\frac{|x|}{|y|}\int u^{-2}(x-u)^{-2}(y-u)^{-2}\,du \\
&=: \frac{1}{2}\,\log\frac{|x|}{|y|}\,K(x,y).
\end{aligned}
$$

The first equality above is obtained by easy symmetry arguments. Thus the remaining term of the partially renormalized expression for the kite is of the form

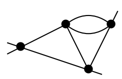
$$
-\frac{1}{4}y^{-2}(x-y)^{-2}\,\Delta_x\Big(K(x,y)\log\frac{|x|}{|y|}\Big).
$$

Integration by parts once more expands this to

$$
\begin{aligned}
&\pi^2y^{-2}K(x,y)\log\frac{|x|}{|y|}\,\delta(x-y) \\
&\quad+\frac{1}{4}\,y^{-2}\,\partial_x^\beta\Big(\partial^x_\beta((x-y)^{-2})K(x,y)\log\frac{|x|}{|y|}-(x-y)^{-2}\partial^x_\beta\Big(K(x,y)\log\frac{|x|}{|y|}\Big)\Big).
\end{aligned}
$$

The first term vanishes since $K(x,y)\log\big(|x|/|y|\big)$ is skewsymmetric; and only the total derivative part remains.

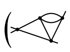
Combining this total derivative with the other contributions (5.5) and (5.7), we arrive at the renormalized amplitude for the kite:



$$
\begin{aligned}
&= 2\pi^4R_4\Big[y^{-4}\log^2\frac{|y|}{l}\Big]\,\delta(x)+2\pi^4R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\,\delta(x)+2\pi^4R_4\Big[x^{-4}\log\frac{|x|}{l}\Big]\,\delta(y) \\
&\quad+2\pi^4R_4[x^{-4}]\,\delta(x-y)-\frac{\pi^2}{2}\,x^{-2}\,\partial_y^\beta L_\beta(x-y;y) \\
&\quad-\frac{\pi^2}{2}\,y^{-2}\,\partial_x^\beta\big(L_\beta(x-y;x)+M_\beta(x-y;x)\big) \\
&\quad+\frac{1}{4}\partial_x^\beta\Big(y^{-2}\partial^x_\beta((x-y)^{-2})K(x,y)\log\frac{|x|}{|y|}-y^{-2}(x-y)^{-2}\partial^x_\beta\Big(K(x,y)\log\frac{|x|}{|y|}\Big)\Big).
\end{aligned}
$$

Note now, using (A.5), that the coefficient $-\pi^4/3$ of $y^{-2}\log^3(|y|/l)$ agrees with our expectation. To wit, $-4\pi^2 \times (-\pi^4/3) = 4\pi^6/3$.

The scale derivative of the kite now follows readily; note that the last line in the above display for the amplitude will not contribute. We obtain

$l\dfrac{\partial}{\partial l}\Big($  $\Big) = -4\pi^4 R_4\Big[y^{-4}\log\dfrac{|y|}{l}\Big]\,\delta(x) - 2\pi^4 R_4[y^{-4}]\,\delta(x) - 2\pi^4 R_4[x^{-4}]\,\delta(y)$

$\qquad - 4\pi^6\,\delta(x)\,\delta(y) - 2\pi^4 R_4[x^{-4}]\big(\delta(x-y) - \delta(y)\big)$

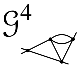
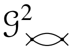
$\qquad - 2\pi^4 R_4[y^{-4}]\big(\delta(x-y) - \delta(x)\big) + \pi^2 y^{-2}\,\partial_x^\beta L_\beta(x-y;x)$

$\qquad = -4\pi^4 R_4\Big[y^{-4}\log\dfrac{|y|}{l}\Big]\,\delta(x) - 4\pi^4 R_4[y^{-4}]\,\delta(x-y) - 4\pi^6\,\delta(x)\,\delta(y)$

$\qquad - \pi^2 y^{-2}\,\partial_x^\beta L_\beta(y-x;x)$

$\qquad = -2\pi^2$  $(y, y-x) - 2\pi^4$  $(x)\,\delta(x-y) - 4\pi^6\,\delta(x)\,\delta(y).$

(The first cograph on the right hand side has a different labelling of the vertices from that of Section 4.2.) We end up with

$l\dfrac{\partial}{\partial l}\,\mathcal{G}^4_{\,}$  $= 4\bar g\,\mathcal{G}^4_{\,}$  $- 8\bar g^2\,\mathcal{G}^2_{\,}$  $+ 24\bar g^3\,\mathcal{G}^4_{\times}.$

To derive an explicit expression for $K(x,y)$, note first that $K(tx,ty) = t^{-2}K(x,y)$, so we can assume for now that $|y| = 1$. Writing $x = r\omega$, $u = s\sigma$, $y = \eta$ in polar form, we can expand the integrand in Gegenbauer polynomials. For instance, in the region $s < r < 1$ of the (r,s) positive quadrant, we get

$$(y-u)^{-2} = \sum_{n=0}^{\infty} s^n C_n^1(\eta\cdot\sigma); \qquad (x-u)^{-2} = \frac{1}{r^2}\sum_{m=0}^{\infty}\frac{s^m}{r^m}\,C_m^1(\sigma\cdot\omega).$$

Following [44], we take advantage of the underlying conformal symmetry to introduce a complex variable z determined by

$$|z| = r = |x| \quad\text{and}\quad (y-x)^2 = 1 - 2r\eta\cdot\omega + r^2 = |1-z|^2.$$

The Gegenbauer orthogonality relations and the formula $C_m^1(\cos\theta) = \sin(m+1)\theta/\sin\theta$ show that

$$\int_{\mathbb{S}^3} C_n^1(\eta\cdot\sigma)C_m^1(\sigma\cdot\omega)\,d^3\sigma = \frac{\Omega_4\,\delta_{mn}}{n+1}\,C_n^1(\eta\cdot\omega) = \frac{2\pi^2\,\delta_{mn}}{n+1}\,C_n^1\bigg(\frac{z+\bar z}{2|z|}\bigg)$$
$$= \frac{2\pi^2\,\delta_{mn}}{(n+1)|z|^n}\,\frac{z^{n+1}-\bar z^{n+1}}{z-\bar z}\,.$$

In the given (r,s)-region, each such term is multiplied by $r^{-n-2}\int_0^r s^{2n+1}\,ds = |z|^n/2(n+1)$, yielding a contribution to $K(x,y)$ of

$$\frac{\pi^2}{z-\bar z}\sum_{n=0}^{\infty}\frac{z^{n+1}-\bar z^{n+1}}{(n+1)^2} = \frac{\pi^2}{z-\bar z}\big(\mathbb{L}_2(z) - \mathbb{L}_2(\bar z)\big),$$

where $\mathbb{L}_2(\zeta) = \sum_{m=1}^{\infty} \zeta^n/n^2$ is the Euler dilogarithm.

Similar expressions are found for the other regions of the positive quadrant: see [44] for more detail. The full result, always assuming that $|y| = 1$, is

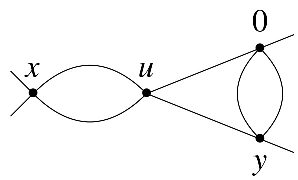
$$K(x,y) = \frac{\pi^2}{z-\bar{z}}\biggl(2\mathbb{L}_2(z) - 2\mathbb{L}_2(\bar{z}) + \log|z|^2 \log\frac{1-z}{1-\bar{z}}\biggr),$$

which turns out to be a single-valued complex function, the so-called Bloch–Wigner dilogarithm [45]. For general $|y|$, this yields the expression of [4] for $K(x,y)$:

$$\frac{\pi^2}{2i\sqrt{x^2y^2-(x\cdot y)^2}}\Biggl(2\mathbb{L}_2\biggl(\frac{(x\cdot y)^2 + i\sqrt{x^2y^2-(x\cdot y)^2}}{y^2}\biggr) - 2\mathbb{L}_2\biggl(\frac{(x\cdot y)^2 - i\sqrt{x^2y^2-(x\cdot y)^2}}{y^2}\biggr) \\ + \log\frac{x^2}{y^2}\,\log\frac{y^2-(x\cdot y)^2 - i\sqrt{x^2y^2-(x\cdot y)^2}}{y^2-(x\cdot y)^2 + i\sqrt{x^2y^2-(x\cdot y)^2}}\Biggr).$$

5.6 The shark

Last of its class, we consider the *shark* graph, which has the structure of a convolution of two renormalized diagrams:



One-vertex reducible diagrams of this type do not require overall renormalization: once the convolution is effected, the task is over. However, the matter is not as simple as implied in [4, Sect. 3.3]: "Since each factor in the convolution has a finite Fourier transform, so does the full result". Of course not: rather, it is in view of our Proposition 2 that the convolution product makes sense.

For the winecup subgraph, (4.8) gives:

$$\overline{R}_8[u^{-2}(u-y)^{-2}y^{-4}] = 2\pi^2 R_4\Bigl[u^{-4}\log\frac{|u|}{l}\Bigr]\,\delta(u-y) + \pi^2 R_4[u^{-4}]\,\delta(y) + \frac{1}{2}\,u^{-2}\,\partial_y^\beta L_\beta(y; u-y).$$

The fish subgraph is just $R_4[(x-u)^{-4}]$.

Several contributions are immediately computable. First, a (well-defined) product of distributions,

$$2\pi^2 \int R_4\Bigl[u^{-4}\log\frac{|u|}{l}\Bigr]\,R_4[(x-u)^{-4}]\,\delta(u-y)\,du = 2\pi^2 R_4\Bigl[y^{-4}\log\frac{|y|}{l}\Bigr]\,R_4[(x-y)^{-4}].$$

Next, the straightforward convolution,

$$\pi^2\,\delta(y)\int R_4[u^{-4}]\,R_4[(x-u)^{-4}]\,du = 4\pi^4 R_4\Bigl[x^{-4}\log\frac{|x|}{l}\Bigr]\,\delta(y) - \pi^6\,\delta(x)\,\delta(y).$$

Thirdly, using $R_4[(x-u)^{-4}] = -\frac{1}{2}\,\Delta\bigl((x-u)^{-2}\log(|x-u|/l)\bigr) + \pi^2\,\delta(x-u)$, we extract

$$\frac{\pi^2}{2}\int u^{-2}\,\partial_y^\beta L_\beta(y; u-y)\,\delta(x-u)\,du = \frac{\pi^2}{2}\,x^{-2}\,\partial_y^\beta L_\beta(y; x-y).$$

Lastly, we have to add the term

$$-\frac{1}{4}\,\partial_y^\beta\,\Delta_x \int u^{-2} L_\beta(y; u-y)\,(x-u)^{-2} \log\frac{|x-u|}{l}\,du.$$

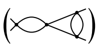
Using (5.6) once more to expand $\log(|x-u|/l)$, this can be rewritten in terms of $K(x,y)$ as defined in the previous case; but this is hardly worthwhile.

For the scale derivative, we look first at $l\frac{\partial}{\partial l}\int \frac{1}{2}u^{-2}\,\partial_y^\beta L_\beta(y;u-y) R_4[(x-u)^{-4}]\,du$, yielding:

$$\begin{aligned} &-\pi^2 \int u^{-2}\,\partial_y^\beta L_\beta(y; u-y)\,\delta(x-u)\,du \\ &\qquad + 2\pi^2 \int \big(R_4[u^{-4}]\,\delta(u-y) - R_4[u^{-4}]\,\delta(y)\big) R_4[(x-u)^{-4}]\,du \\ &= -\pi^2 x^{-2}\,\partial_y^\beta L_\beta(y; x-y) + 2\pi^2 R_4[y^{-4}]\,R_4[(x-y)^{-4}] - 2\pi^2 \end{aligned}$$

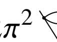
$$(x)\,\delta(y).$$

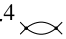
Therefore,

$$l\frac{\partial}{\partial l}\Big($$

$$\Big) = -2\pi^2 R_4[y^{-4}]\,R_4[(x-y)^{-4}] - 4\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big]\,\delta(x-y) - 4\pi^4 R_4[x^{-4}]\,\delta(y)$$

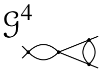
$$- \pi^2 x^{-2}\,\partial_y^\beta L_\beta(y; x-y) + 2\pi^2 R_4[y^{-4}]\,R_4[(x-y)^{-4}] - 2\pi^2$$

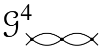
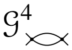
$$(x)\,\delta(y)$$

$$= -2\pi^2$$

$$(x,y) - 2\pi^2$$

$$(x)\,\delta(y) - 2\pi^4$$

$$(x)\,\delta(y).$$

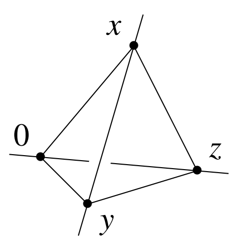
This translates into:

$$l\frac{\partial}{\partial l}\,\mathcal{G}^4_{}$$

$$= \bar g\,\mathcal{G}^4_{}$$

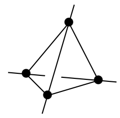
$$+ 4\bar g\,\mathcal{G}^4_{}$$

$$- 2\bar g^2\,\mathcal{G}^4_{}$$

$$.$$

5·7 The tetrahedron diagram

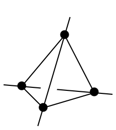
The tetrahedron graph is *primitive* and already understood [5]. We report the results. On labelling the vertices thus:



we arrive at


$$= (E+12)\Big[x^{-2}y^{-2}z^{-2}(x-y)^{-2}(y-z)^{-2}(x-z)^{-2}\log\frac{|(x,y,z)|}{l}\Big];$$

$$\text{with}\quad E+12 = \partial_\alpha^x x^\alpha + \partial_\beta^y y^\beta + \partial_\rho^z z^\rho;$$

$$l\frac{\partial}{\partial l}$$

$$= -12\pi^6\zeta(3)$$

$$,\quad \text{leading to}\quad l\frac{\partial}{\partial l}\,\mathcal{G}^4_{}$$

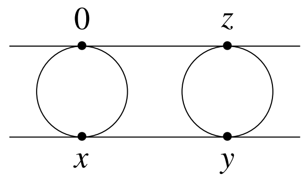
$$= 12\bar g^3\zeta(3)\,\mathcal{G}^4_{}$$

$$.$$

Notice that the scale derivative for this graph, coincident with the residue for this case, is numerically large.

5.8 The roll

The unrenormalized amplitude for the roll diagram, with vertices labelled as follows:



is of the form

$$f(x, y, z) = (x - y)^{-2} x^{-4} z^{-2} (y - z)^{-4}.$$

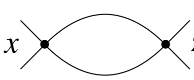
Before plunging into calculation, this very interesting graph without internal vertices prompts a couple of comments. It exemplifies well the "causal factorization property". Consider the relevant partition $\{0, x\}, \{z, y\}$ of its set of vertices. Partial renormalization adapted to it will yield a valid distribution outside the diagonals $0 = z$, $x = y$. Formula (1.1) here means:

$$\langle R[\Gamma], \varphi\rangle = \langle R[\gamma_1 \uplus \gamma_2], \big(\Gamma/(\gamma_1 \uplus \gamma_2)\big)\varphi\rangle = \langle R[\gamma_1]R[\gamma_2], \big(\Gamma/(\gamma_1 \uplus \gamma_2)\big)\varphi\rangle,$$

for φ vanishing on those diagonals; where the rule for disconnected graphs, also found in [8, Sect. 11.2]:

$$R[\gamma_1 \uplus \gamma_2] = R[\gamma_1]\, R[\gamma_2],$$

holds; and

$$\Gamma/(\gamma_1 \uplus \gamma_2) = \Bigg(0 = x$$  $$z = y \Bigg).$$

Second, one may anticipate the coefficients of logarithmic degrees 3 and 2 in the final result, by using again [37] and [38]. Note that the roll is a rooted-tree graph with the fish as unique "decoration"; subtracting from it the juxtaposition of the winecup and the fish graph, and adding one sixth of the product of three fishes, one obtains a primitive graph in the bialgebra of rooted trees. For this combination the coefficient of $\log^3 l$ must vanish. We obtain for such a graph $2\pi^4 \times 2\pi^2 - 8\pi^6/6 = 8\pi^6/3$.

Partial renormalization leads at once to

$$\begin{aligned} \overline{R}_{12}[f(x, y, z)] &= (x - y)^{-2} R_4[x^{-4}]\, z^{-2} R_4[(y - z)^{-4}] \\ &= \frac{1}{4}\bigg((x - y)^{-2}\, \Delta\Big(x^{-2} \log \frac{|x|}{l}\Big) - 2\pi^2 y^{-2}\, \delta(x)\bigg) \\ &\qquad \times \bigg(z^{-2}\, \Delta_z\Big((y - z)^{-2} \log \frac{|y - z|}{l}\Big) - 2\pi^2 y^{-2}\, \delta(y - z)\bigg). \end{aligned} \tag{5.8}$$

The expression (5.8) is a sum of four terms. Three of these yield, respectively:

$$\pi^4 y^{-4}\,\delta(x)\,\delta(y-z) \overset{R}{\longmapsto} \pi^4 R_4[y^{-4}]\,\delta(x)\,\delta(y-z);$$

$$-\frac{\pi^2}{2}\,y^{-2}(x-y)^{-2}\,\Delta\Big(x^{-2}\log\frac{|x|}{l}\Big)\,\delta(y-z)$$
$$\overset{R}{\longmapsto} 2\pi^4 R_4\Big[x^{-4}\log\frac{|x|}{l}\Big]\,\delta(x-y)\,\delta(x-z) + \frac{\pi^2}{2}\,y^{-2}\,\partial_x^\alpha L_\alpha(x;y-x)\,\delta(y-z);$$

$$-\frac{\pi^2}{2}\,y^{-2}z^{-2}\,\Delta_z\Big((y-z)^{-2}\log\frac{|y-z|}{l}\Big)\,\delta(x)$$
$$\overset{R}{\longmapsto} 2\pi^4 R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\,\delta(x)\,\delta(z) + \frac{\pi^2}{2}\,y^{-2}\,\partial_z^\beta L_\beta(y-z;z)\,\delta(x).$$

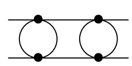
The remaining contribution from (5.8), after integrating by parts twice with (4.6), yields

$$4\pi^4 x^{-4}\log^2\frac{|x|}{l}\,\delta(x-y)\,\delta(z) + \pi^2 y^{-2}\log\frac{|y|}{l}\,\partial_x^\alpha L_\alpha(x;y-x)\,\delta(z)$$
$$+\pi^2 y^{-2}\log\frac{|y|}{l}\,\partial_z^\beta L_\beta(y-z;z)\,\delta(x-y) + \frac{1}{4}\,\partial_x^\alpha L_\alpha(x;y-x)\,\partial_z^\beta L_\beta(y-z;z).$$

Only the first of these terms requires renormalization:

$$4\pi^4 x^{-4}\log^2\frac{|x|}{l}\,\delta(x-y)\,\delta(z) \overset{R}{\longmapsto} 4\pi^4 R_4\Big[x^{-4}\log^2\frac{|x|}{l}\Big]\,\delta(x-y)\,\delta(z).$$

Summing up, we arrive at



$$= 4\pi^4 R_4\Big[y^{-4}\log^2\frac{|y|}{l}\Big]\,\delta(x-y)\,\delta(z)$$
$$+ 2\pi^4 R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\big(\delta(x)\,\delta(z) + \delta(x-y)\,\delta(x-z)\big)$$
$$+ \pi^4 R_4[y^{-4}]\,\delta(x)\,\delta(y-z) + \text{(total derivative terms)}. \qquad (5.9)$$

A peek at (A.5) confirms that the coefficient for $\log^3 l$ in this expression is

$$-\frac{2\pi^4}{3}\times(-4\pi^2) = \frac{8\pi^6}{3},$$

as predicted by Kreimer's argument.

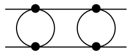
For the scale derivative of the roll amplitude, the first three terms in (5.9) contribute

$$-8\pi^4 R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\delta(x-y)\,\delta(z) - 2\pi^4 R_4[y^{-4}]\big(\delta(x)\,\delta(z) + \delta(x-y)\,\delta(y-z)\big)$$
$$-2\pi^6\,\delta(x)\,\delta(y)\,\delta(z),$$

while the other (total derivative) terms contribute

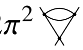
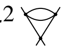
$$-4\pi^4 R_4[y^{-4}]\,\delta(x)\,\delta(y-z) + 2\pi^4 R_4[y^{-4}]\big(\delta(x)\,\delta(z) + \delta(x-y)\,\delta(x-z)\big)$$
$$+8\pi^4 R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\,\delta(x-y)\,\delta(z) - 4\pi^4 R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\big(\delta(x)\,\delta(z) + \delta(x-y)\,\delta(y-z)\big)$$
$$-\pi^2 y^{-2}\,\partial_x^\alpha L_\alpha(x;y-x)\,\delta(y-z) - \pi^2 y^{-2}\,\partial_z^\beta L_\beta(y-z;z)\,\delta(x).$$

Putting them together, we arrive at

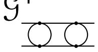
$l\frac{\partial}{\partial l}$  $= -4\pi^4 R_4\Big[y^{-4}\log\frac{|y|}{l}\Big]\big(\delta(x)\,\delta(z) + \delta(x-y)\,\delta(y-z)\big)$

$- 4\pi^4 R_4[y^{-4}]\,\delta(x)\,\delta(y-z) - 2\pi^6\,\delta(x)\,\delta(y)\,\delta(z)$

$- \pi^2 y^{-2}\,\partial_x^{\alpha} L_{\alpha}(x; y-x)\,\delta(y-z) - \pi^2 y^{-2}\,\partial_z^{\beta} L_{\beta}(y-z; z)\,\delta(x)$

$= -2\pi^2$  $(y,x)\,\delta(y-z) - 2\pi^2$  $(y, y-z)\,\delta(x) - 2\pi^6$  .

Taking into account the weight factors, we then conclude that

$l\frac{\partial}{\partial l}\,\mathcal{G}^4$  $= 2\bar{g}\,\mathcal{G}^4$  $+ 3\bar{g}^3\,\mathcal{G}^4$  .

6 The renormalization group γ- and β-functions

Enter the Callan–Symanzik differential equations with zero mass:

$$\left[\frac{\partial}{\partial \log l} - \beta(\bar{g})\frac{\partial}{\partial \bar{g}} + 2\gamma(\bar{g})\right]\mathcal{G}^2_p(x_1, x_2) = 0 \tag{6.1}$$

and
$$\left[\frac{\partial}{\partial \log l} - \beta(\bar{g})\frac{\partial}{\partial \bar{g}} + 4\gamma(\bar{g})\right]\mathcal{G}^4(x_1, x_2, x_3, x_4) = 0. \tag{6.2}$$

Here $\mathcal{G}^2$ starts at order $\bar{g}^0$ and $\mathcal{G}^4$ at order $\bar{g}$; the subindex in $\mathcal{G}^2_p$ recalls that the equations refer to proper parts. (The detailed calculations for the two-point function $\mathcal{G}^2$ and its scale derivative are outlined in Appendix B.) The scale derivative in both cases starts at order $\bar{g}^2$. Therefore we may assume that:

$$\beta(\bar{g}) = \bar{g}^2\beta_1 + \bar{g}^3\beta_2 + \bar{g}^4\beta_3 + \cdots; \qquad \gamma(\bar{g}) = \bar{g}^2\gamma_2 + \bar{g}^3\gamma_3 + \bar{g}^4\gamma_4 + \cdots; \tag{6.3}$$

where γ does not contribute to (6.2) at the first significant order, and similarly for β and (6.1); and we try to compute then the γ_i and β_i. Following [4], our labelling of the expansion coefficients differs for the β and γ functions.

Order by order, we find:

$$\bar g^2: l\frac{\partial}{\partial l}\,\mathcal{G}^4_{\text{fish}} = \beta_1\,\bar g^2\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\times}; \qquad l\frac{\partial}{\partial l}\,\mathcal{G}^2_{\ominus} = -2\gamma_2\,\bar g^2\,\mathcal{G}^2_{\text{—}};$$

$$\bar g^3: l\frac{\partial}{\partial l}\left[\mathcal{G}^4_{\text{double fish}} + \mathcal{G}^4_{\text{triangle}}\right] = (\beta_2 - 4\gamma_2)\,\bar g^3\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\times} + \beta_1\,\bar g^2\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\text{fish}}; \tag{6.4}$$

$$l\frac{\partial}{\partial l}\,\mathcal{G}^2_{\text{3-loop}} = \beta_1\,\bar g^2\frac{\partial}{\partial \bar g}\,\mathcal{G}^2_{\ominus} - 2\gamma_3\,\bar g^2\,\mathcal{G}^2_{\text{—}}; \tag{6.5}$$

$$\bar g^4: l\frac{\partial}{\partial l}\Big[\mathcal{G}^4_{\text{triple fish}} + \mathcal{G}^4 + \mathcal{G}^4 + \mathcal{G}^4 + \mathcal{G}^4 + \mathcal{G}^4 + \mathcal{G}^4 + \mathcal{G}^4\Big]$$

$$= (\beta_3 - 4\gamma_3)\bar g^4\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\times} + (\beta_2 - 2\gamma_2)\bar g^3\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\text{fish}} + \beta_1\bar g^2\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\text{double fish}} + \beta_1\bar g^2\frac{\partial}{\partial \bar g}\,\mathcal{G}^4_{\text{triangle}}; \tag{6.6}$$

$$l\frac{\partial}{\partial l}\Big[\mathcal{G}^2 + \mathcal{G}^2 + \mathcal{G}^2 + \mathcal{G}^2\Big]$$

$$= \beta_1\,\bar g^2\frac{\partial}{\partial \bar g}\,\mathcal{G}^2_{\text{3-loop}} + \beta_2\,\bar g^3\frac{\partial}{\partial \bar g}\,\mathcal{G}^2_{\ominus} - 2\gamma_2\,\bar g^2\,\mathcal{G}^2_{\ominus} - 2\gamma_4\,\bar g^4\,\mathcal{G}^2_{\text{—}}. \tag{6.7}$$

The first equality above, in view of (4.11), yields $\beta_1 = 3$. This is the standard result. Off the same line, with the help of (B.8), we read $\gamma_2 = 1/12$. This is also the standard result.

Then, on consulting (4.12) and (4.15), equality (6.4) is seen to yield:

$$6\bar g\,\mathcal{G}^4_{\text{fish}} - 6\bar g^2\,\mathcal{G}^4_{\times} = 6\bar g\,\mathcal{G}^4_{\text{fish}} + \left(\beta_2 - \tfrac{1}{3}\right)\bar g^2\,\mathcal{G}^4_{\times};$$

that is, $\beta_2 = -17/3$. This is the standard result. Note the automatic cancellation of the terms in $\mathcal{G}^4_{\text{fish}}$.

We need the value of γ_3 in order to compute β_3. Now, from (6.5) we observe that

$$6\bar g\,\mathcal{G}^2_{\ominus} = 6\bar g\,\mathcal{G}^2_{\ominus} + 2\gamma_3\,\bar g^2\,\Delta\delta;$$

so that $\gamma_3 = 0$ obtains, at variance with both [4] and [35] (differing between them); but in agreement with [26].

We turn to the computation of β_3, noting beforehand that knowledge of γ_4 is not necessary for it. First we check the automatic cancellation of the terms in $\mathcal{G}^4_{\text{fish}}$ in (6.6):

$$\left(\frac{1}{3} - 2 - 2 - 8\right)\bar g^2\,\mathcal{G}^4_{\text{fish}} = 2\left(-\frac{17}{3} - \frac{1}{6}\right)\bar g^2\,\mathcal{G}^4_{\text{fish}};$$

as well as in $\mathcal{G}^4_{\text{double fish}}$ and $\mathcal{G}^4_{\text{triangle}}$:

$$(3 + 2 + 4)\bar g\,\mathcal{G}^4_{\text{double fish}} = 9\bar g\,\mathcal{G}^4_{\text{double fish}};$$

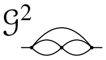
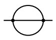
$$(2 + 4 + 1 + 2)\bar g\,\mathcal{G}^4_{\text{triangle}} = 9\bar g\,\mathcal{G}^4_{\text{triangle}};$$

which of course vouches for the soundness of our method. We should also notice that, since the chain graphs do not yield $\mathcal{G}^4_\times$ terms, they contribute nothing to the renormalization group functions. The same is true of the shark graph.

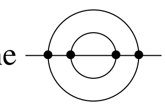
Thus we read off β_3 from the terms in $\mathcal{G}^4_\times$ on the left hand-side of (6.6), with the result:

$$\beta_3 = \frac{109}{4} + 12\,\zeta(3),$$

numerically intermediate between the results in [4] and [35]. The discrepancy with the former is due to different results for the $\mathcal{G}^4_\times$ terms in the stye, cat's eye, duncecap and roll graphs; for any others our scale derivatives reproduce the results in the seminal paper on differential renormalization. This number has, at any rate, no fundamental significance; in fact β_3 and all subsequent coefficients of the β-function can be made to vanish in an appropriate renormalization scheme.

Before computing γ_4 in our scheme, with an eye on the formulas (B.9), we check the cancellation of terms in $\mathcal{G}^2$  and $\mathcal{G}^2$  (coming from proper graphs only) in (6.7):

$$(3+2+4) = 3\beta_1 = 9; \qquad \Big(\frac{1}{2} - 6 - 6\Big) = 2(\beta_2 - \gamma_2) = -\frac{34}{3} - \frac{1}{6}\,.$$

Finally, for γ_4 only the contribution from the  graph survives in (6.7), and we obtain $\gamma_4 = -5/96$, again in agreement with [26].

7 Conclusion

We reckon to have shown that, with a small amount of ingenuity, plus eventual recourse to Gegenbauer polynomial techniques and polylogarithms, renormalization of proper "divergent" graphs in coordinate space by Epstein–Glaser methods is quite feasible.

As the perturbation order grows, this often demands integration over the internal vertices. We have proved here the basic proposition underpinning this latter technique, and provided quite a few examples. We trust that, all along, the logical advantage of distribution-theoretic methods for the recursive treatment of divergences shines through.

The Popineau–Stora theorem was mentioned in Section 4.3. A refinement of this theorem [42,43] provides an expression for $\overline{G}^{ll'}(\bar g)$ of the form

$$\overline{G}^{ll'}(\bar g) = \bar g + H\big(\overline{G}^{ll'}(\bar g)\big),$$

where the series H is made up of successive contributions at each perturbation step, always supported exclusively on the corresponding main diagonal. We expect to take up in a future paper this interesting combinatorial aspect of the procedure; at any rate, our methods guarantee that no combinatorial *difficulties* worth mentioning appear.

The attentive reader will not have failed to notice the connection between the apparently fortuite cancellations signaled in the previous section and those in Section 4.3. Those cancellations look to be just an infinitesimal aspect of the Popineau–Stora theorem. Here we have done no more than to illustrate the workings of that theorem and the Callan–Symanzik equations in the Epstein–Glaser paradigm. A befitting derivation of those equations within the distribution-theoretic approach is also left for the future.

7.1 The roads not taken

We have taken some pains to point out the shortcomings of differential renormalization; however, in keeping with its spirit, our approach to cograph parts is unabashedly "low-tech": some of the standard tools of renormalization in x-space are not used.

- Even for dealing with overlapping divergences, we have had no recourse to partitions of unity.
- We do not use here *meromorphic continuation*: we wanted to illustrate the fact that real-variable methods *à la* Epstein and Glaser are enough to deal with the problems at hand. This is of course a net loss in practice, since the analytic continuation tools [15–17] borrowed in [1, 10, 11] are quite powerful, although not always available. The wisest course is to employ both real- and complex-variable methods.
- The calculus of *wave front sets* was not required.
- Steinmann's *scaling degree* for distributions was never invoked. There is no point in using it for massless diagrams [43], for which the log-homogeneous classification is finer: all distributions of bidegree (a, m) have the same scaling degree irrespectively of the value of m. Recent work [41] adapts the latter classification to the case of massive particles, casting doubt on the future usefulness of the scaling degree in quantum field theory. Even for general distributions, Meyer's concept of weakly homogeneous distributions, exploited in [47], appears more seductive.

A Formulas for extensions of distributions in x-space

Recall the definition of $R_d\big[r^{-d}\log^m(r/l)\big]$: from (2.13) we immediately obtain

$$R_d\Big[r^{-d}\log^m\frac{r}{l}\Big] = \frac{1}{m+1}\,\partial_\alpha\Big(x^\alpha r^{-d}\log^{m+1}\frac{r}{l}\Big) = \sum_{k=0}^{m+1} c_{m+1,k}\,\Delta\Big(r^{-d+2}\log^k\frac{r}{l}\Big) \tag{A.1}$$

for suitable constants $c_{m+1,k}$. These are computed as follows.

Lemma 3. *For any* $m = 0, 1, 2, \dots,$

$$R_d\Big[r^{-d}\log^m\frac{r}{l}\Big] = -\sum_{k=1}^{m+1}\frac{m!}{k!}\,(d-2)^{-m+k-2}\,\Delta\Big(r^{-d+2}\log^k\frac{r}{l}\Big) + \frac{m!}{(d-2)^{m+1}}\,\Omega_d\,\delta(r);$$

and in particular,

$$R_4\Big[r^{-4}\log^m\frac{r}{l}\Big] = -\sum_{k=1}^{m+1}\frac{m!}{k!}\,\frac{1}{2^{m-k+2}}\,\Delta\Big(r^{-2}\log^k\frac{r}{l}\Big) + \frac{m!}{2^m}\,\pi^2\,\delta(r). \tag{A.2}$$

Proof. Taking derivatives, we get

$$\partial_\alpha\Big(r^{-d+2}\log^k\frac{r}{l}\Big) = x^\alpha r^{-d}\Big((2-d)\log^k\frac{r}{l} + k\log^{k-1}\frac{r}{l}\Big).$$

The coefficients $c_{m+1,k}$ are determined by the defining relation

$$x^\alpha r^{-d} \log^{m+1} \frac{r}{l} = (m+1) \sum_{k=0}^{m+1} c_{m+1,k}\, \partial_\alpha \Big(r^{-d+2} \log^k \frac{r}{l} \Big);$$

so we get the recurrence, for $k \leqslant m$:

$$(k+1) c_{m+1,k+1} - (d-2) c_{m+1,k} = 0.$$

Clearly $c_{m+1,m+1} = -1/(d-2)(m+1)$. Thus $c_{m+1,m} = -1/(d-2)^2$. The remaining $c_{m+1,k}$ terms follow at once. The last summand is $c_{m+1,0}\, \Delta(r^{-d+2}) = m!(d-2)^{-m-1} \Omega_d\, \delta(r)$. □

We expand out the cases $m = 0, 1, 2$ of $d = 4$ for ready reference:

$$R_4[r^{-4}] = -\frac{1}{2} \Delta\Big(r^{-2} \log \frac{r}{l} \Big) + \pi^2 \delta(r). \tag{A.3}$$

$$R_4\Big[r^{-4} \log \frac{r}{l} \Big] = -\frac{1}{4} \Delta\Big(r^{-2} \log^2 \frac{r}{l} \Big) - \frac{1}{4} \Delta\Big(r^{-2} \log \frac{r}{l} \Big) + \frac{\pi^2}{2} \delta(r); \tag{A.4}$$

$$R_4\Big[r^{-4} \log^2 \frac{r}{l} \Big] = -\frac{1}{6} \Delta\Big(r^{-2} \log^3 \frac{r}{l} \Big) - \frac{1}{4} \Delta\Big(r^{-2} \log^2 \frac{r}{l} \Big) - \frac{1}{4} \Delta\Big(r^{-2} \log \frac{r}{l} \Big) + \frac{\pi^2}{2} \delta(r). \tag{A.5}$$

Explicit expressions for log-homogeneous distributions of higher bidegrees could in principle be computed from (2.10) and (2.11). We have already met $R_4[r^{-6}]$ in (2.19). We also need:

$$R_4\Big[r^{-6} \log \frac{r}{l} \Big] = \frac{1}{8} \Delta R_4\Big[r^{-4} \log \frac{r}{l} \Big] + \frac{3}{32} \Delta R_4[r^{-4}] + \frac{7\pi^2}{64} \Delta\delta(r), \tag{A.6}$$

$$R_4\Big[r^{-6} \log^2 \frac{r}{l} \Big] = \frac{1}{8} \Delta R_4\Big[r^{-4} \log^2 \frac{r}{l} \Big] + \frac{3}{16} \Delta R_4\Big[r^{-4} \log \frac{r}{l} \Big] + \frac{7}{64} \Delta R_4[r^{-4}] + \frac{15\pi^2}{128} \Delta\delta(r);$$

$$l \frac{\partial}{\partial l} R_4\Big[r^{-6} \log^m \frac{r}{l} \Big] = -m\, R_4\Big[r^{-6} \log^{m-1} \frac{r}{l} \Big] \quad \text{for} \quad m \geqslant 1. \tag{A.7}$$

The last equation follows from (2.17) and the algebra property. It is worth mentioning that an expression equivalent to (A.6) was obtained by Jones on extending $r^{-6} \log r$ by meromorphic continuation: see Eq. (34) on page 255 of [48]. The one on the following line appears to be new.

The reader might wish to run the algebra checks:

$$r^2 R_4\Big[r^{-6} \log^m \frac{r}{l} \Big] = R_4\Big[r^{-4} \log^m \frac{r}{l} \Big].$$

B On the two-point function

The free Green function $G_{\text{free}}(x_1, x_2)$ is the same as the Euclidean "propagator" but for the sign:

$$\frac{1}{4\pi^2 (x_1 - x_2)^2}\,.$$

Perturbatively, the corrected or dressed propagator is:

$$
\begin{aligned}
G(x_1 - x_2) = G_{\text{free}}(x_1 - x_2) + \iint G_{\text{free}}(x_1 - x_a)\, \Sigma(x_a - x_b)\, G_{\text{free}}(x_b - x_2) \\
+ \iiiint G_{\text{free}}(x_1 - x_2)\, \Sigma(x_1 - x_b)\, G_{\text{free}}(x_b - x_c)\, \Sigma(x_c - x_d)\, G_{\text{free}}(x_d - x_2) + \cdots
\end{aligned}
$$

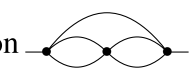
Here Σ is the proper (one-particle irreducible) self-energy. The solution of the above convolution equation is given by the convolution inverse

$$(G_{\text{free}} - \Sigma)^{*-1}(x) =: \mathcal{G}^2(x);$$

this is what we call the *two-point function*. Thus in first approximation $\mathcal{G}^2(x)$ is given by

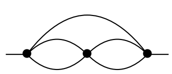
$$\left(\frac{1}{4\pi^2 x^2}\right)^{*-1} = -\Delta\delta(x).$$

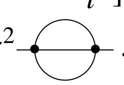
We gather:

- At order $\bar{g}^0$ the two-point function is simply given by $-\Delta\delta(x) \equiv \mathcal{G}^2_{\bullet}$.
- The sunset graph comprises the first correction, at order $\bar{g}^2$, already computed in (2.19).
- The unique third-order graph for the two-point function  is partially renormalized as

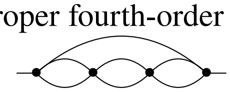
$$\overline{R}_8\big[x^{-2}(x^{-4})^{*2}\big] = \pi^2 x^{-2}\left(4R_4\left[x^{-4}\log\frac{|x|}{l}\right] - \pi^2\,\delta(x)\right).$$

  The homogeneous distribution $x^{-2}\,\delta(x)$ is renormalized by the standard formula (2.11), yielding $\frac{1}{8}\,\Delta\delta(x)$; note that the algebra rule is fulfilled. Thus from (3.5),

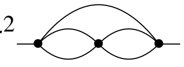
 $= R_4\big[x^{-2}(x^{-4})^{*2}\big] = 4\pi^2 R_4\left[x^{-6}\log\frac{|x|}{l}\right] - \frac{\pi^4}{8}\,\Delta\delta(x),$

with scale derivative $-4\pi^2$  . (B.1)

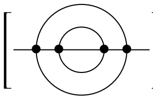
  Here and for subsequent graphs, the reader can consult (A.6) for totally explicit forms in terms of Laplacians.

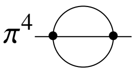
- There are four proper fourth-order graphs for the two-point function. A very simple one, of the chain type, is  whose renormalized form we write from (5.1) at once,

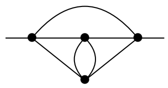
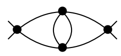
$$R_4\big[x^{-2}(x^{-4})^{*3}\big] = 12\pi^4 R_4\left[x^{-6}\log^2\frac{|x|}{l}\right] - 3\pi^4 R_4[x^{-6}] + \frac{2\zeta(3)-1}{4}\,\pi^6\,\Delta\delta(x),$$

with scale derivative $-6\pi^2$  . (B.2)

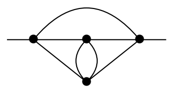
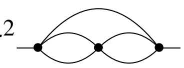
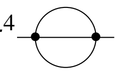
- Next we consider the *saturn* graph. Analogously, the ground work was already done for the stye diagram, and we may write at once:

$R_4\Big[$  $\Big] = -\pi^4 R_4\Big[x^{-6}\log\frac{|x|}{l}\Big] - \frac{5\pi^4}{4}\,R_4[x^{-6}]\,,$

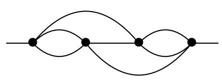
with scale derivative $\quad \pi^4$  $+\ \frac{5\pi^6}{16}\,\Delta\delta\,.$ (B.3)

- The *roach* graph  has the bare form x^{-2} , renormalized at once by

$$4\pi^4 R_4\Big[x^{-6}\log^2\frac{|x|}{l}\Big] + 4\pi^4 R_4\Big[x^{-6}\log\frac{|x|}{l}\Big] - 2\pi^4 R_4[x^{-6}] - \frac{\pi^6}{8}\,\Delta\delta(x),$$

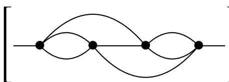
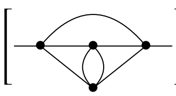
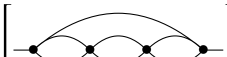
with $\quad l\frac{\partial}{\partial l}$  $= -2\pi^2$  $-\ 4\pi^4$  $+\ \frac{\pi^6}{4}\,\Delta\delta\,.$ (B.4)

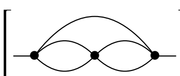
The simplicity of our treatment for this graph stands in stark contrast with the "combinatorial monstrosity of the forest formula" [46], patent in [11, Example 3.2].

- Finally, there is the *snail* graph  whose amplitude is of the bare form

$$f(x) = \iint \frac{du\,dv}{u^4(v-u)^2(x-u)^2v^2(x-v)^4}\,.$$

As indicated in [26], this can be related to previously computed amplitudes:

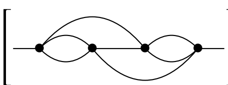
$R_4\Big[$  $\Big] + R_4\Big[$  $\Big] - R_4\Big[$  $\Big]$

$= 2\pi^2 R_4\Big[$  $\Big] + \int du\,\partial^u_\mu Q^\mu(u;v,x),$

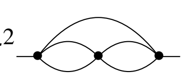
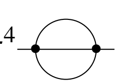
where

$$Q^\mu(u;v,x) := \big((u^\mu - x^\mu)u^{-2}(u-v)^{-2}(u-x)^{-2} - u^\mu u^{-4}(u-v)^{-2}\big)v^{-2}(v-x)^{-4}x^{-2}.$$

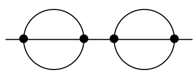
This yields:

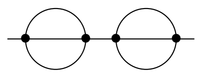
$R_4\Big[$  $\Big]$

$$= 8\pi^4 R_4\Big[x^{-6}\log^2\frac{|x|}{l}\Big] + 4\pi^4 R_4\Big[x^{-6}\log\frac{|x|}{l}\Big] - \pi^4 R_4[x^{-6}] + \frac{4\zeta(3)-3}{8}\pi^6\Delta\delta(x)\,,$$

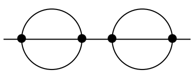
with scale derivative $\quad -\ 4\pi^2$  $-\ 4\pi^4$  $-\ \frac{\pi^6}{4}\,\Delta\delta\,.$ (B.5)

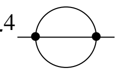
- At order four there is an improper contribution to the propagator, by the double sunset:

$$\mathcal{F}R_4\Big[$$  $$\Big] = -p^2\left(\frac{\pi^2}{4}\log\frac{|p|}{\Lambda} - \frac{5\pi^2}{16}\right)^2 = -p^2\,\frac{\pi^4}{16}\left(\log^2\frac{|p|}{\Lambda} - \frac{5}{2}\log\frac{|p|}{\Lambda} + \frac{25}{16}\right),$$

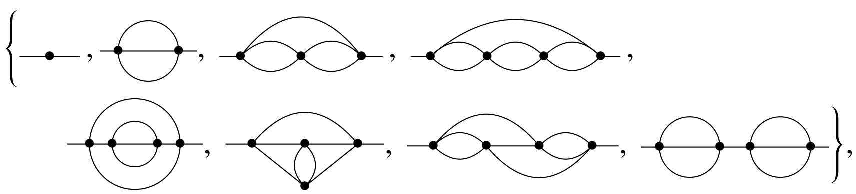
implying  $$= \frac{\pi^2}{2}\,R_4\Big[x^{-6}\log\frac{|x|}{l}\Big] - \frac{9\pi^4}{256}\,\Delta\delta(x).$$

This is just $R_4[r^{-6}] * r^{-2} * R_4[r^{-6}]$, yielding:

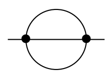
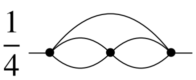
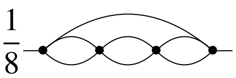
$$R_4\Big[$$  $$\Big] = -2\pi^4\,R_4\Big[x^{-6}\log\frac{|x|}{l}\Big] + \frac{9\pi^6}{64}\,\Delta\delta(x);$$

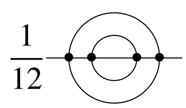
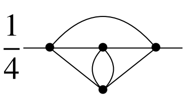
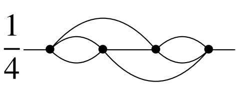
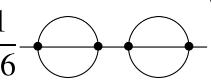
with scale derivative: $2\pi^4$  . (B.6)

For the set:



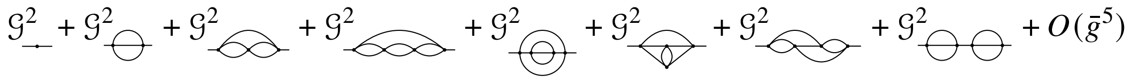
the weights are respectively given by $\{1, \frac{1}{6}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{36}\}$. Therefore, up to four loops, with $r = |x_1 - x_2|$, the two-point function $\mathcal{G}^2(x_1, x_2)$ is of the form:

$$-\Delta\delta(r) - \frac{(16\pi^2\bar{g})^2}{(4\pi^2)^3}\frac{1}{6}$$  $$+ \frac{(16\pi^2\bar{g})^3}{(4\pi^2)^5}\frac{1}{4}$$  $$- \frac{(16\pi^2\bar{g})^4}{(4\pi^2)^7}\Bigg(\frac{1}{8}$$ 

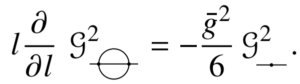
$$+ \frac{1}{12}$$  $$+ \frac{1}{4}$$  $$+ \frac{1}{4}$$  $$+ \frac{1}{36}$$  $$\Bigg)$$

$$\begin{aligned}
&= -\Delta\delta(r) - \bar{g}^2\frac{2}{3\pi^2}\,R_4[r^{-6}] + \bar{g}^3\left(\frac{4}{\pi^2}\,R_4\Big[r^{-6}\log\frac{r}{l}\Big] - \frac{1}{8}\,\Delta\delta(r)\right) \\
&\quad - \bar{g}^4\Bigg[\left(\frac{6}{\pi^2}\,R_4\Big[r^{-6}\log^2\frac{r}{l}\Big] - \frac{3}{2\pi^2}\,R_4[r^{-6}] + \frac{2\zeta(3)-1}{8}\,\Delta\delta(r)\right) \\
&\qquad - \left(\frac{1}{3\pi^2}\,R_4\Big[r^{-6}\log\frac{r}{l}\Big] + \frac{5}{12\pi^2}\,R_4[r^{-6}]\right) \\
&\qquad + \left(\frac{4}{\pi^2}\,R_4\Big[r^{-6}\log^2\frac{r}{l}\Big] + \frac{4}{\pi^2}\,R_4\Big[r^{-6}\log\frac{r}{l}\Big] - \frac{2}{\pi^2}\,R_4[r^{-6}] - \frac{1}{8}\,\Delta(r)\right) \\
&\qquad + \left(\frac{8}{\pi^2}\,R_4\Big[r^{-6}\log^2\frac{r}{l}\Big] + \frac{4}{\pi^2}\,R_4\Big[r^{-6}\log\frac{r}{l}\Big] - \frac{1}{\pi^2}\,R_4[r^{-6}] + \frac{4\zeta(3)-3}{8}\,\Delta\delta(r)\right) \\
&\qquad + \left(-\frac{2}{9\pi^2}\,R_4\Big[r^{-6}\log\frac{r}{l}\Big] + \frac{1}{64}\,\Delta(x)\right)\Bigg] + O(\bar{g}^5).
\end{aligned} \tag{B.7}$$

The right hand side of (B.7) may be abbreviated as

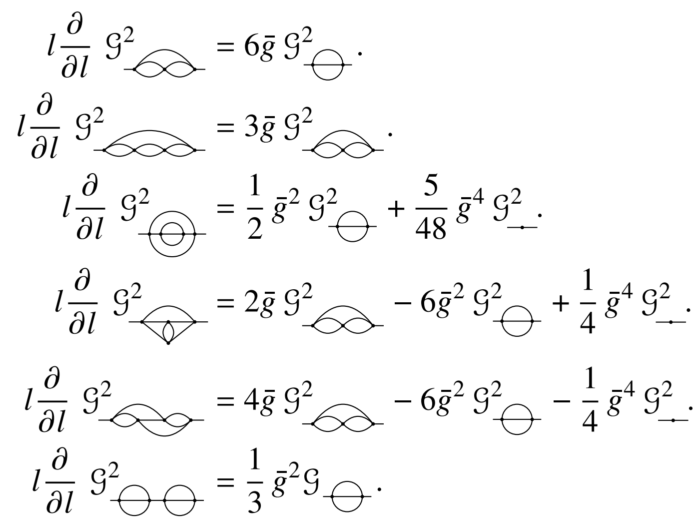


We see that the practical rule to go from the calculated scale derivatives of the graphs to their contributions to the two-point function is as before: multiply the coefficient of the scale derivative by $-\bar{g}/\pi^2$ raised to a power equal to the difference in the number of vertices, and also by the relative weight, for each diagram in question. An exception is the case of the sunset graph, which, on account of (B.7) and (2.20), fulfils:



(B.8)

The tableau of scale derivatives for the two-point function, on account of (B.1)–(B.6) and (B.8) is as follows.



(B.9)

C Radial extensions in p-space and momentum amplitudes

C.1 Fourier transforms

Let now $\mathcal{F}$ denote the Fourier transformation on $\mathcal{S}'(\mathbb{R}^d)$, with a standard convention

$$\mathcal{F}\varphi(p) = \int e^{-ip\cdot x}\varphi(x)\,dx, \quad \text{so that, e.g.,} \quad \mathcal{F}[e^{-r^2/2}] = (2\pi)^{d/2}e^{-p^2/2}.$$

A standard calculation [16] gives

$$\mathcal{F}[r^\lambda] = 2^{\lambda+d}\pi^{d/2}\,\frac{\Gamma(\frac{1}{2}(\lambda+d))}{\Gamma(-\frac{1}{2}\lambda)}\,|p|^{-d-\lambda},$$

valid for $-d < \Re\lambda < 0$, where both sides are locally integrable functions. In particular, $\mathcal{F}[r^{-2}] = 4\pi^2\,p^{-2}$ when $d = 4$.

The Fourier transforms of the radial distributions $R_d\big[r^{-d}\log^m(r/l)\big]$ are found as follows. Since $\mathcal{F}(\delta) = 1$ and $\mathcal{F}(\Delta h) = -p^2\,\mathcal{F}h$, it is enough to compute $\mathcal{F}\big[r^{-d+2}\log^k(r/l)\big]$ for $k \leqslant m$, on account of Lemma 3. For simplicity, we do it here only for $d = 4$.

Lemma 4. *Write $\Lambda := 2/le^{\gamma}$, where γ is Euler's constant. Then, for $d = 4$, we get*

$$\mathcal{F}\Big[r^{-2}\log^k\frac{r}{l}\Big] = 4\pi^2\sum_{j=0}^{k}(-)^k C_{jk}\,p^{-2}\log^j\frac{|p|}{\Lambda}\,;$$
$$\mathcal{F}\Big[\Delta\Big(r^{-2}\log^k\frac{r}{l}\Big)\Big] = 4\pi^2\sum_{j=0}^{k}(-)^{k+1} C_{jk}\,\log^j\frac{|p|}{\Lambda}\,, \tag{C.1}$$

for suitable nonnegative constants C_{jk}.

Proof. If $|t| < 1$, we obtain

$$\mathcal{F}[r^{-2}(r/l)^{2t}] = \frac{4\pi^2}{p^2}\,\frac{\Gamma(1+t)}{\Gamma(1-t)}\left(\frac{l|p|}{2}\right)^{-2t}.$$

Then the known Taylor series expansion [49, Thm. 10.6.1], valid for $|t| < 1$:

$$\log\Gamma(1+t) = -\gamma t + \sum_{k=2}^{\infty}(-)^k\frac{\zeta(k)}{k}\,t^k \tag{C.2}$$

suggests rewriting the previous equality as

$$\begin{aligned}\mathcal{F}[r^{-2}(r/l)^{2t}] &= \frac{4\pi^2}{p^2}\left(\frac{|p|}{\Lambda}\right)^{-2t} e^{2\gamma t}\,\frac{\Gamma(1+t)}{\Gamma(1-t)} \\ &= \frac{4\pi^2}{p^2}\exp\Big\{-2t\log\frac{|p|}{\Lambda} - 2\sum_{m=1}^{\infty}\frac{\zeta(2m+1)}{2m+1}\,t^{2m+1}\Big\}.\end{aligned} \tag{C.3}$$

Differentiation at $t = 0$ then yields $\mathcal{F}[r^{-2}] = 4\pi^2\,p^{-2}$, already noted, as well as

$$\begin{aligned}\mathcal{F}\Big[r^{-2}\log\frac{r}{l}\Big] &= -4\pi^2 p^{-2}\log\frac{|p|}{\Lambda}\,, \\ \mathcal{F}\Big[r^{-2}\log^2\frac{r}{l}\Big] &= 4\pi^2 p^{-2}\log^2\frac{|p|}{\Lambda}\,, \\ \mathcal{F}\Big[r^{-2}\log^3\frac{r}{l}\Big] &= -4\pi^2 p^{-2}\Big(\log^3\frac{|p|}{\Lambda} + \frac{1}{2}\,\zeta(3)\Big), \\ \mathcal{F}\Big[r^{-2}\log^4\frac{r}{l}\Big] &= 4\pi^2 p^{-2}\Big(\log^4\frac{|p|}{\Lambda} + 2\zeta(3)\log\frac{|p|}{\Lambda}\Big), \\ \mathcal{F}\Big[r^{-2}\log^5\frac{r}{l}\Big] &= -4\pi^2 p^{-2}\Big(\log^5\frac{|p|}{\Lambda} + 5\,\zeta(3)\log^2\frac{|p|}{\Lambda} + \frac{3}{2}\,\zeta(5)\Big),\end{aligned} \tag{C.4}$$

and so on, using the Faà di Bruno formula. The corresponding formula to (C.4) in [4] is in error. In [26] the correct value does appear. □

Conversely, the first few renormalized log-homogeneous distributions of Section 2.2 have the following Fourier transforms in $\mathcal{S}'(\mathbb{R}^4)$:

$$
\begin{aligned}
\mathcal{F}R_4\big[r^{-4}\big] &= -2\pi^2 \log \frac{|p|}{\Lambda} + \pi^2, \\
\mathcal{F}R_4\Big[r^{-4} \log \frac{r}{l}\Big] &= \pi^2 \log^2 \frac{|p|}{\Lambda} - \pi^2 \log \frac{|p|}{\Lambda} + \frac{\pi^2}{2}, \\
\mathcal{F}R_4\Big[r^{-4} \log^2 \frac{r}{l}\Big] &= -\frac{2\pi^2}{3} \log^3 \frac{|p|}{\Lambda} + \pi^2 \log^2 \frac{|p|}{\Lambda} - \pi^2 \log \frac{|p|}{\Lambda} + \pi^2\Big(\frac{1}{2} - \frac{\zeta(3)}{3}\Big), \\
\mathcal{F}R_4\Big[r^{-4} \log^3 \frac{r}{l}\Big] &= \frac{\pi^2}{2} \log^4 \frac{|p|}{\Lambda} - \pi^2 \log^3 \frac{|p|}{\Lambda} + \frac{3\pi^2}{2} \log^2 \frac{|p|}{\Lambda} \\
&\qquad - \pi^2\Big(\frac{3}{2} - \zeta(3)\Big) \log \frac{|p|}{\Lambda} + \pi^2\Big(\frac{3}{4} - \frac{\zeta(3)}{2}\Big).
\end{aligned}
$$

The quadratically divergent graphs for the two-point function required here possess the transforms:

$$
\begin{aligned}
\mathcal{F}R_4[r^{-6}] &= \frac{\pi^2}{4} p^2 \log \frac{|p|}{\Lambda} - \frac{5\pi^2}{16} p^2, \\
\mathcal{F}R_4\Big[r^{-6} \log \frac{r}{l}\Big] &= -\frac{\pi^2}{8} p^2 \log^2 \frac{|p|}{\Lambda} + \frac{5\pi^2}{16} p^2 \log \frac{|p|}{\Lambda} - \frac{17\pi^2}{64} p^2; \qquad (C.5) \\
\mathcal{F}R_4\Big[r^{-6} \log^2 \frac{r}{l}\Big] &= \frac{\pi^2}{12} p^2 \log^3 \frac{|p|}{\Lambda} - \frac{5\pi^2}{16} p^2 \log^2 \frac{|p|}{\Lambda} + \frac{17\pi^2}{32} p^2 \log \frac{|p|}{\Lambda} - \pi^2\Big(\frac{49}{128} - \frac{\zeta(3)}{24}\Big) p^2,
\end{aligned}
$$

in view of (2.19) and (A.6). The first identity here gives the sunset graph in momentum space. The second one gives essentially the "goggles" graph (B.1) of Appendix B.

It follows by induction from (2.18) that $\mathcal{F}R_4[r^{-4-2m}]$ is a linear combination of p^{2m} and $p^{2m} \log(|p|/\Lambda)$. It can be written as

$$
\mathcal{F}R_4[r^{-4-2m}] = \frac{1}{4^m m!(m+1)!}\Big(a_m\, p^{2m} \log \frac{|p|}{\Lambda} + b_m\, p^{2m}\Big).
$$

On Fourier-transforming the case $d = 4$, $n = 1$ of (2.18), one finds the recurrence relation

$$
-p^2\Big(a_m\, p^{2m} \log \frac{|p|}{\Lambda} + b_m\, p^{2m}\Big) = a_{m+1}\, p^{2m+2} \log \frac{|p|}{\Lambda} + \Big(b_{m+1} + (-)^m \frac{(2m+3)\pi^2}{(m+1)(m+2)}\Big) p^{2m+2}.
$$

Thus $a_m = (-)^m a_0 = (-)^{m-1} 2\pi^2$ and $b_{m+1} + b_m = (-)^m \pi^2\big(1/(m+1) + 1/(m+2)\big)$. Since $b_0 = \pi^2$, that yields $b_m = (-)^m \pi^2 (H_m + H_{m+1})$. We recover a result already found in [5]:

$$
\mathcal{F}R_4[r^{-4-2m}] = \frac{(-)^{m-1}\pi^2}{4^m m!(m+1)!}\Big(2\, p^{2m} \log \frac{|p|}{\Lambda} - (H_m + H_{m+1}) p^{2m}\Big).
$$

Analogous formulas for $\mathcal{F}R_d[r^{-d-2m}]$ can be derived from (2.18) in the same way.

C.2 On the amplitudes in p-space

It is now straightforward to perform the conversion to momentum space graph by graph; but the details are hardly worthwhile for us, since our renormalization scheme and consequent treatments of the RG and the β-function for the model do not require that conversion.

C.2.1 Two-point amplitudes in p-space

The free Green function has Fourier transform

$$(2\pi)^4 \, \frac{\delta(p_1 + p_2)}{p^2} =: (2\pi)^4 \, \delta(p_1 + p_2) G_{\text{free}}(|p|),$$

where $|p| := |p_1| = |p_2|$ corresponds to the difference variable on x-space. There is a series of corrections to the free propagator:

$$G(|p|) = \frac{1}{p^2} + \frac{1}{p^2}\,\Sigma(|p|)\,\frac{1}{p^2} + \frac{1}{p^2}\,\Sigma(|p|)\,\frac{1}{p^2}\,\Sigma(|p|)\,\frac{1}{p^2} + \cdots$$

Just as on x-space, the above equation is solved by

$$G(|p|) = \big(p^2 - \Sigma(|p|)\big)^{-1}.$$

What we call the momentum-space two-point function $\mathcal{G}^2(|p|)$ is $G(|p|)^{-1} = p^2 - \Sigma(|p|)$; this is the Fourier transform of $\mathcal{G}^2(r)$, by the definition of the latter.

One can now obtain from the list (C.5) of Fourier transforms of quadratically divergent graphs, together with Eqs. (2.19) and (B.1)–(B.4), the corrections to the propagator associated to these graphs in momentum space. We omit this trivial conversion; taking account of different conventions, this coincides in a few cases with results in [26]. It is remarkable that the Apéry constant $\zeta(3)$ disappears in the computation of the chain graph $-\!\bigcirc\!\!\bigcirc\!\!\bigcirc\!-$.

C.2.2 Four-point amplitudes in p-space

The graphs in momentum space relevant for the four-point function are more complicated. We just exemplify for the fish graph. Still omitting the permutations of the vertices, the amplitude in x-space is of the form

$$\begin{aligned}
\mathcal{G}_{\succ\!\bigcirc\!\prec}(x_1, x_2, x_3, x_4) &= \frac{g^2}{(4\pi^2)^2}\bigg[\delta(x_1 - x_2)\,\delta(x_2 - x_3)\,\delta(x_3 - x_4) \\
&\qquad - \frac{1}{2\pi^2}\,\delta(x_1 - x_2)\,\Delta\Big((x_2 - x_3)^{-2}\log\frac{|x_2 - x_3|}{l}\Big)\,\delta(x_3 - x_4)\bigg] \\
&= \frac{g^2}{(4\pi^2)^2}\bigg[\delta(\xi_1)\,\delta(\xi_2)\,\delta(\xi_3) - \frac{1}{2\pi^2}\,\delta(\xi_1)\,\Delta\Big(\xi_2^{-2}\log\frac{|\xi_2|}{l}\Big)\,\delta(\xi_3)\bigg] \\
&=: \widehat{\mathcal{G}}_{\succ\!\bigcirc\!\prec}(\xi_1, \xi_2, \xi_3);
\end{aligned}$$

where we have introduced the difference variables $\xi_1 = x_1 - x_2$, $\xi_2 = x_2 - x_3$, $\xi_3 = x_3 - x_4$. The *reduced* Fourier transform $\widehat{\mathcal{G}}_{\succ\!\bigcirc\!\prec}(p_1, p_2, p_3)$, defined by

$$\begin{aligned}
\widehat{\mathcal{G}}_{\succ\!\bigcirc\!\prec}(p_1, p_2, p_3) := \int &\widehat{\mathcal{G}}_{\succ\!\bigcirc\!\prec}(\xi_1, \xi_2, \xi_3) \\
&\times \exp\big[-i(p_1\xi_1 + (p_1 + p_2)\xi_2 + (p_1 + p_2 + p_3)\xi_3)\big]\, d\xi_1\, d\xi_2\, d\xi_3
\end{aligned}$$

yields $\mathcal{G}_{\bowtie}(p_1, p_2, p_3, p_4) = (2\pi)^4\, \delta(p_1 + p_2 + p_3 + p_4)\, \widehat{\mathcal{G}}_{\bowtie}(p_1, p_2, p_3)$, where $\mathcal{G}_{\bowtie}$ is the *ordinary* Fourier transform, defined by:

$$\mathcal{G}_{\bowtie}(p_1, p_2, p_3, p_4) := \int \mathcal{G}_{\bowtie}(x_1, x_2, x_3, x_4) \exp[-i(p_1x_1 + \cdots + p_4x_4)]\, dx_1\, dx_2\, dx_3\, dx_4.$$

This is general for functions of the difference variables. In our present case, we obtain

$$\begin{aligned} \mathcal{G}_{\bowtie}(p_1, p_2, p_3, p_4) &= g^2\, \delta(p_1 + \cdots + p_4) \left[1 - 2\log\frac{|p_1 + p_2|}{\Lambda}\right] \\ &= g^2\, \delta(p_1 + \cdots + p_4) \left[1 - 2\log\frac{|p_3 + p_4|}{\Lambda}\right]. \end{aligned}$$

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