# Some remarks on dilating semigroups of completely positive mappings

Gérard G. Emch* and Joseph C. Várilly

Department of Mathematics, University of Rochester, Rochester, NY 14627, U.S.A.

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**Abstract**

We reconsider the problem of embedding a dissipative dynamical system in a conservative one: and we compare some of the partial solutions which have been recently proposed.

## 1 Statement of the problem

We wish to address the following problem, connected with the existence of an equilibrium state in the non-equilibrium Statistical Mechanics of transport phenomena (e.g., [6]).

Given a "dissipative" dynamical system described by $\{\mathfrak{N}, \phi, \gamma(\mathbb{R}_+)\}$, where: $\mathfrak{N}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$; $\gamma(\mathbb{R}_+)$ is a semigroup of completely positive maps from $\mathfrak{N}$ to $\mathfrak{N}$, strongly continuous in the sense that $t \mapsto \gamma_t(N)\,\Phi$ is continuous for each $N \in \mathfrak{N}$ and each $\Phi \in \mathcal{H}$, and such that $\gamma_0 = \mathrm{id}_{\mathfrak{N}}$, $\gamma_t[1] = 1$ for each $t \geqslant 0$; $\phi$ is a faithful normal state of $\mathfrak{N}$ such that $\phi \circ \gamma_t = \phi$ for all $t \geqslant 0$.

Find a "conservative" dynamical system $\{\overline{\mathfrak{N}}, \bar{\phi}, \bar{\alpha}(\mathbb{R})\}$ and a pair of maps $i\colon \mathfrak{N} \to \overline{\mathfrak{N}}$, $\mathcal{E}\colon \overline{\mathfrak{N}} \to \mathfrak{N}$ such that:

(i) $\mathfrak{N}$ is a von Neumann algebra,

(ii) $\bar{\alpha}(\mathbb{R})$ is a strongly continuous group of automorphisms of $\overline{\mathfrak{N}}$,

(iii) $i\colon \mathfrak{N} \to \overline{\mathfrak{N}}$ is an injective $*$-representation,

(iv) $\mathcal{E}\colon \overline{\mathfrak{N}} \to \mathfrak{N}$ s a faithful normal conditional expectation,

(v) $\overline{\mathfrak{N}} = \{\, \bar{\alpha}_t \circ i(\mathfrak{N}) : t \in \mathbb{R} \,\}''$;

(vi) $\mathcal{E} \circ \alpha_t \circ i = \gamma_t$ for all $t \geqslant 0$; and

(vii) $\bar{\phi}$ is a faithful normal state of $\mathfrak{N}$,

(viii) $\phi \circ \mathcal{E} = \bar{\phi}$,

(ix) $\bar{\phi} \circ \bar{\alpha}_t = \bar{\phi}$ for all $t \in \mathbb{R}$.

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This set of conditions is compatible, as established by the existence of complete solutions in some particular cases. The aim of this paper, however, is to direct attention to the fact that a general construction scheme is still lacking, which would provide a solution satisfying simultaneously all of the above conditions.

## 2 Some partial solutions

### 2.a The abelian case

Let $X_t$ (for $t \geqslant 0$) be a stationary Markov process with state space $(\Omega, \Sigma; \mu)$, where the measure $\mu$ defines the equilibrium distribution and $\mathrm{supp}(\mu) = \Omega$. Take $\mathcal{H} := L^2(\Omega, \Sigma; \mu)$, $\mathfrak{N} := L^\infty(\Omega, \Sigma; \mu)$, and let $P_t \in \mathcal{B}(\mathcal{H})$ be the transition operators for the process. With $\Phi\colon \omega \in \Omega \mapsto 1 \in \mathbb{C}$, $\gamma_t[N]\Phi := P_t(N\Phi)$ defines a Markov semigroup $\gamma(\mathbb{R}_+) \subset \mathrm{CP}(\mathfrak{N})$, and $\phi(N) := \int N\,d\mu$ defines a faithful normal state on $\mathfrak{N}$. Let $(\overline{\Omega}, \overline{\Sigma}; \bar{\mu})$ be the underlying probability space obtained from the process by the Kolmogorov construction (see, e.g., [2, Props. 2.26 and 6.5]), where $\overline{\Omega} = \Omega^{\mathbb{R}}$. We then take $\overline{\mathfrak{N}} = L^\infty(\overline{\Omega}, \overline{\Sigma}; \bar{\mu})$; $\bar{\phi}(\overline{N}) = \int \overline{N}\,d\bar{\mu}$ for $\overline{N} \in \overline{\mathfrak{N}}$; $\bar{\alpha}_t$ to be the coordinate shift on $\overline{\Omega} = \Omega^{\mathbb{R}}$, for any $t \in \mathbb{R}$; $i\colon N \in \mathfrak{N} \mapsto N\pi_0 \in \overline{\mathfrak{N}}$ where $\pi_0(\bar{\omega}) := \bar{\omega}(0)$; and $\mathcal{E}\colon \overline{N} \in \overline{\mathfrak{N}} \mapsto E(\overline{N} \mid \mathcal{F}(X_0)) \in \mathfrak{N}$.

Then conditions (i)–(ix) are fulfilled, yielding thus a full solution to the problem in the case that $\mathfrak{N}$ is abelian.

### 2.b The quasi-free case

Our problem has also been completely solved in several particular cases where $\mathfrak{N}$ is not abelian [4–7], but $\gamma(\mathbb{R}_+)$ is quasi-free. In [7] the solution obtained satisfies the following Markov property (Feynman–Kac formula):

$$\begin{aligned}\mathcal{E}\big(\bar{\alpha}_{t_1}[i(N_1)\bar{\alpha}_{t_2}[i(N_2)\cdots\bar{\alpha}_{t_n}[i(N_nM_n)]\cdots i(M_2)]i(M_1)]\big)\\ = \gamma_{t_1}[N_1\gamma_{t_2}[N_2\cdots\gamma_{t_n}[N_nM_n]\cdots M_2]M_1]\end{aligned} \tag{1}$$

for $t_k \geqslant 0$; $N_k, M_k \in \mathfrak{N}$, where $k = 1, 2, \dots, n$.

### 2.c One particular construction

A dilation $\{\overline{\mathfrak{N}}, \bar{\alpha}(\mathbb{R}), i, \mathcal{E}\}$ of $\{\mathfrak{N}, \gamma(\mathbb{R}^+)\}$ satisfying (i)–(vi) has been constructed by Evans and Lewis [9] in the case that $\gamma(\mathbb{R}^+)$ is a norm-continuous semigroup and $\mathfrak{N} = \mathcal{B}(\mathcal{H})$.

### 2.d A general construction

In [3], Davies has suggested a construction which dilates a collection $\{\, \gamma_g \in \mathrm{CP}(\mathfrak{N}) : g \in G \,\}$, indexed by the elements of a locally compact group $G$, where $g \mapsto \gamma_g$ is strongly continuous, to a group of $*$-automorphisms $\{\, \bar{\alpha}_g : g \in G \,\}$ on a larger algebra. We take here $G = \mathbb{R}$, and define $\gamma_t$ for negative $t$ by $\gamma_t := \gamma_{-t}$ for $t < 0$. As outlined in [3] (see also [8]) one proceeds as follows:

**Step 1:**

(a) Let $\mathcal{H}' := L^2(\mathbb{R}; \mathcal{H}) = \{\, \Psi\colon \mathbb{R} \to \mathcal{H} : \int_{\mathbb{R}} \|\Psi(t)\|^2\,dt < +\infty \,\}$.

(b) Define $\gamma\colon \mathfrak{N} \to \mathcal{B}(\mathcal{H}')$ by $(\gamma(N)\Psi)(t) := \gamma_t(N)[\Psi(t)]$.

(c) Let $O_t$ be the (unitary) shift on $\mathcal{H}$: $(O_t\Psi)(s) := \Psi(s-t)$.

(d) Let $\mathfrak{M} := \{\, O_t^*\gamma(\mathfrak{N})O_t : t \in \mathbb{R} \,\}''$.

(e) Let $(f_n)$ be an $L^2$-approximate identity on $\mathbb{R}$:

$$f_n \in L^2(\mathbb{R}), \qquad \|f_n\|_2 = 1, \qquad \mathrm{supp}(f_n) \downarrow \{0\} \text{ as } n \to +\infty.$$

Let $V_n\colon \mathcal{H} \to \mathcal{H}'$ be the isometry defined by $(V_n\Phi)(t) := f_n(t)\,\Phi$.

(f) Let $\mathcal{E}_1\colon \mathcal{B}(\mathcal{H}') \to \mathcal{B}(\mathcal{H})$ be defined by $\mathcal{E}_1(A) := \lim_n V_n^* A V_n$ in the weak operator topology; one verifies that $\mathcal{E}_1$ is defined on $\big(\gamma(\mathfrak{N}) \cup O(\mathbb{R})\big)''$ and that $\mathcal{E}_1(\mathfrak{M}) = \mathfrak{N}$.

**Step 2:**

(a) $\gamma\colon \mathfrak{N} \to \mathcal{B}(\mathcal{H}')$ is completely positive since this is true of each $\gamma_t$. Thus by the Stinespring construction [10], $\gamma(N) = V^* i(\mathfrak{N}) V$, where $i$ is a $*$-representation of $\mathfrak{N}$ on some Hilbert space $\mathcal{K}$, and $V\colon \mathcal{H}' \to \mathcal{K}$ is an isometry.

(b) Let $U_t := VO_tV^* + (1 - VV^*)$ be the "shift" on $\mathcal{K} = V\mathcal{H}' \oplus (V\mathcal{H}')^\perp$.

(c) Let $\overline{\mathfrak{N}} := \{\, U_t^* i(\mathfrak{N}) U_t : t \in \mathbb{R} \,\}''$.

(d) Let $\bar\alpha_t$ be the $*$-automorphism of $\overline{\mathfrak{N}}$ given by $\bar\alpha_t(\overline{N}) := U_t^*\overline{N}U_t$, for all $t \in \mathbb{R}$: $\bar\alpha(\mathbb{R})$ is a strongly continuous subgroup of $\mathrm{Aut}(\overline{\mathfrak{N}})$.

(e) Let $\mathcal{E}_2\colon \overline{\mathfrak{N}} \to \mathfrak{M}$ be defined by $\mathcal{E}_2(\overline{N}) := V^*\overline{N}V$.

(f) Let $\mathcal{E}\colon \overline{\mathfrak{N}} \to \mathfrak{N}$ be defined by $\mathcal{E} := \mathcal{E}_1 \circ \mathcal{E}_2$; one verifies that $\mathcal{E}$ is a conditional expectation from $\overline{\mathfrak{N}}$ onto $\mathfrak{N}$, satisfying condition (vi).

## 3 Comparison of various solutions

**Lemma 1.** *If in Construction 2.d, $N_1, N_2 \in \mathfrak{N}$ and if $t_1 \neq t_2$, we have*

$$\mathcal{E}\big(\bar\alpha_{t_1}[i(N_1)\,\bar\alpha_{t_2-t_1}[i(N_2)]]\big) = \gamma_{t_1}(N_1)\,\gamma_{t_2}(N_2).$$

*Proof.* The LHS is equal to

$$\begin{aligned}\mathcal{E}\big(U_{t_1}^* i(N_1) U_{t_1} U_{t_2}^* i(N_2) U_{t_2}\big) &= \mathcal{E}\big(U_{t_1}^* i(N_1)\,(VO_{t_1}O_{t_2}^*V^* + 1 - VV^*)\,i(N_2)U_{t_2}\big)\\ &= \mathcal{E}_1\big(O_{t_1}^*\gamma(N_1)O_{t_1}O_{t_2}^*\gamma(N_2)O_{t_2}\big) + \mathcal{E}_1\big(O_{t_1}^*\gamma(N_1N_2)O_{t_2}\big) - \mathcal{E}_1\big(O_{t_1}^*\gamma(N_1)\gamma(N_2)O_{t_2}\big).\end{aligned}$$

The second and third terms vanish, since $\mathcal{E}_1(A) = 0$ whenever

$$A = O_{t_1}\gamma(N_1)O_{t_2}\gamma(N_2)\cdots\gamma(N_r)O_{t_{r+1}} \quad \text{and} \quad t_1 + t_2 + \cdots + t_{r+1} \neq 0.$$

On the other hand, if $t_1 + t_2 + \cdots + t_{r+1} = 0$, we may rewrite $A$ as

$$A = O_{s_1}^*\gamma(N_1)O_{s_1}O_{s_2}^*\gamma(N_2)O_{s_2}\cdots O_{s_r}^*\gamma(N_r)O_{s_r},$$

so that in this case $\mathcal{E}_1(A) = \gamma_{s_1}(N_1)\cdots\gamma_{s_r}(N_r)$. The first term of the above expression for the LHS thus reduces indeed to the RHS of the conclusion of the lemma. □

*Remark* 1. Construction 2.d leads to a dilation which is different from that obtained in case 2.b.

*Proof.* The conclusion of Lemma 1 is incompatible with the Feynman–Kac formula (1) above unless

$$s, t > 0 \implies \gamma_t\big(N_1\gamma_s(N_2)\big) = \gamma_t(N_1)\gamma_{t+s}(N_2) \quad \text{for any} \quad N_1, N_2 \in \mathfrak{N},$$

which (by letting $s \downarrow 0$) implies that each $\gamma_t$ is a homomorphism. Since this is in general too strong a restriction on $\gamma(\mathbb{R})$ and is not the case for the examples in [4, 7], Davies' construction leads to a different extension from that of Emch, Albeverio and Eckmann [7]. □

We will now proceed to show that our insistence on the existence of an equilibrium state $\bar{\phi}$, satisfying conditions (vii)–(ix), rules out the possibility that construction 2.d, although devised for more general situations than those treated in subsection 2.b, could support the structure required by a complete solution of our problem, even in the particular cases solved in [7].

**Lemma 2.** *Let $\phi$ be a faithful state on a $C^*$-algebra $\mathfrak{A}$, $T\colon \mathfrak{A} \to \mathfrak{A}$ be a 2-positive linear map such that $T(1)\,T(A) = T(A)$ for any $A \in \mathfrak{A}$, and $\phi(T(AB)) = \phi(T(A)\,T(B))$ for any $A, B \in \mathfrak{A}$; then $T$ is a $*$-homomorphism.*

*Proof.* $T(A^*) = T(A)^*$ for any $A \in \mathfrak{A}$ since $T$ is positive. If $A \in \mathfrak{A}$, $\phi\big(T(A^*A) - T(A^*)\,T(A)\big) = 0$.
But $T(A^*A) \geqslant T(A^*)\,T(A)$ since $T$ is 2-positive and $T(1)\,T(A) = T(A)$: see [1]. Hence

$$T(A^*A) = T(A^*)\,T(A) \quad \text{since } \phi \text{ is faithful.} \tag{2}$$

This implies

$$T((A^* + xB)^*(A^* + xB)) = T((A^* + xB)^*)T(A^* + xB) \quad \text{for any} \quad x \in \mathbb{C}.$$

Thus

$$\begin{aligned} &T(AA^*) + x\,T(AB) + \bar{x}\,T(B^*A^*) + |x|^2\,T(B^*B) \\ &\qquad = T(A)T(A^*) + x\,T(A)\,T(B) + \bar{x}\,T(B^*)\,T(A^*) + |x|^2\,T(B^*)\,T(B). \end{aligned}$$

Using (2) and taking $x = 1$ or $i$, we obtain

$$T(AB) \pm T(B^*A^*) = T(A)\,T(B) \pm T(B^*)\,T(A^*).$$

Hence $T(AB) = T(A)\,T(B)$ for any $A, B \in \mathfrak{A}$. □

*Remark* 2. Construction 2.d yields a solution to our problem only if $\gamma_t$ is a $*$-isomorphism for all $t \geqslant 0$.

*Proof.* Let us indeed suppose that a faithful normal state $\phi$ of $\mathfrak{N}$ is given such that $\phi \circ \gamma_t = \phi$ for all $t \geqslant 0$ (and thus trivially for all $t \in \mathbb{R}$). Let $\bar{\phi}$ be the state of $\overline{\mathfrak{N}}$ given by $\bar{\phi} := \phi \circ \mathcal{E}$, and let us suppose that $\bar{\phi} \circ \bar{\alpha}_t = \bar{\phi}$ for all $t \in \mathbb{R}$. Then in particular

$$\begin{aligned} \bar{\phi}(i(N_1N_2)) &= \bar{\phi}(\bar{\alpha}_t \circ i(N_1N_2)) = \bar{\phi}(\bar{\alpha}_t \circ i(N_1)\,\bar{\alpha}_t \circ i(N_2)) \\ &\qquad = \phi \circ \mathcal{E}\big(U_t^* i(N_1)U_tU_t^* i(N_2)U_t\big) = \phi(\gamma_t(N_1)\gamma_t(N_2)) \end{aligned}$$

for all $t \in \mathbb{R}$, and $N_1, N_2 \in \mathfrak{N}$; for the last equality see the proof of Lemma 1. Since $\bar{\phi} \circ i = \phi \circ \mathcal{E} \circ i = \phi$, this implies

$$\phi(N_1 N_2) = \phi(\gamma_t(N_1)\gamma_t(N_2)) \quad \text{for all} \quad t \geqslant 0. \tag{3}$$

Upon using now $\gamma_t(1) = 1$ and $\phi \circ \gamma_t = \phi$ for any $t \geqslant 0$, we obtain that if (3) holds, $\phi$ and $\gamma_t$ satisfy the hypotheses of Lemma 2, and hence $\gamma_t$ is a $*$-homomorphism.

Furthermore, since (2) holds, $\gamma_t$ is injective because

$$\begin{aligned} \gamma_t(A) = 0 &\implies \gamma_t(A^*A) = \gamma_t(A^*)\,\gamma_t(A) = 0 \\ &\implies \phi(A^*A) = \phi(\gamma_t(A^*A)) = 0 \implies A^*A = 0 \implies A = 0. \end{aligned}$$

□

**Lemma 3.** *If $\mathcal{V}$ is a bounded self-adjoint subset of $\mathfrak{N}$ such that $\mathcal{V}'' = \mathfrak{N}$ and $\gamma_n(V) \to \phi(V)\,1$ strongly as $n \to \infty$ for each $V \in \mathcal{V}$, then* (3) *holds only if $\phi$ is a multiplicative trace on $\mathfrak{N}$.*

*Proof.* Let $V, W \in \mathcal{V}$; then $\text{s-lim}_n\, \gamma_n(V)\gamma_n(W) = \phi(V)\phi(W)\,1$. Let $K := \sup\{\, \|V\| : V \in \mathcal{V}\,\}$; then $\|\gamma_n(V)\gamma_n(W)\| \leqslant K^2$.

Thus since $\phi$ is normal, $\lim_n \phi(\gamma_n(V)\gamma_n(W)) = \phi(V)\phi(W)$. Hence (3) implies $\phi(VW) = \phi(V)\phi(W)$ for all $V, W \in \mathcal{V}$. Since $\phi$ is linear, this relation also holds on the linear span of $\mathcal{V}$, which is strongly dense in $\mathfrak{N}$.

If $M, N \in \mathfrak{N}$, since $\mathcal{H}$ is separable we can find sequences $(M_k),\ (N_k) \subset \text{span}(V)$ so that $\|M_k\| \leqslant \|M\|$, $\|N_k\| \leqslant \|N\|$ for all $k$ and $\text{s-lim}_k\, M_k = M$, $\text{s-lim}_k\, N_k = N$. Hence $\|M_k N_k\| \leqslant \|M\|\,\|N\|$ and $M_k N_k \to MN$ strongly as $k \to \infty$. Thus

$$\phi(MN) = \lim_k \phi(M_k N_k) = \lim_k \phi(M_k)\phi(N_k) = \phi(M)\phi(N).$$

□

*Remark* 3. References [6, 7] provide examples of dynamical systems $\{\mathfrak{N}, \phi, \gamma(\mathbb{R}_+)\}$ where $\mathfrak{N}$ is a factor, $\phi$ is not a trace, and $\gamma_t$ is not a $*$-isomorphism for any $t > 0$; moreover, in each case $\mathfrak{N}$ is generated as a von Neumann algebra by a subset $\mathcal{V}$ satisfying the hypothesis of Lemma 3. The scheme ingeniously devised by Evans [8] and Davies [3], as outlined in subsection 2.d above, will therefore not provide the desired conservative dilation. However, a dilation $\{\overline{\mathfrak{N}}, \bar{\phi}, \bar{\alpha}(\mathbb{R})\}$, satisfying all the requirements imposed in Section 1 above, has been constructed [7] for these examples.

*Remark* 4. In particular, if we restrict [6] to its imaginary part, we recover a Markov process $(X_t)$ which describes the diffusion of a classical particle on $\mathbb{R}$, where $\Omega = \mathcal{S}'(\mathbb{R})$ and $\mu$ is a Gaussian measure; here again, $\gamma_t$ is not an isomorphism for $t > 0$. Hence the Evans–Davies scheme, as outlined in subsection 2.d above, does not lead in this case to the expected (see subsection 2.a) recovery of the Flow of Brownian motion.


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