Sparse Bounds for the Discrete Spherical Maximal Function

Presented by
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2021 Virtual Fall Western Sectional Meeting
October, 2021

Some background on sparse bounds

Introduction

A collection of cubes $\mathcal{S}$ is c-sparse if for each $S \in \mathcal{S}$ there is $E_S \subseteq S$ such that

1. $|E_S| > c|S|$,
2. $\| \sum_{S \in \mathcal{S}} \mathbb{1}_{E_S} \|_\infty \leq c^{-1}$.

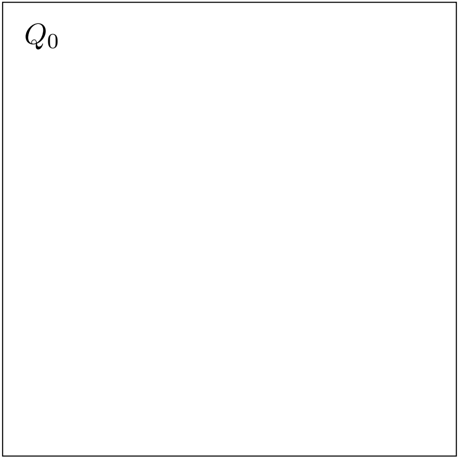
Sometimes the second condition is replaced by requiring that the sets E_S are disjoint.

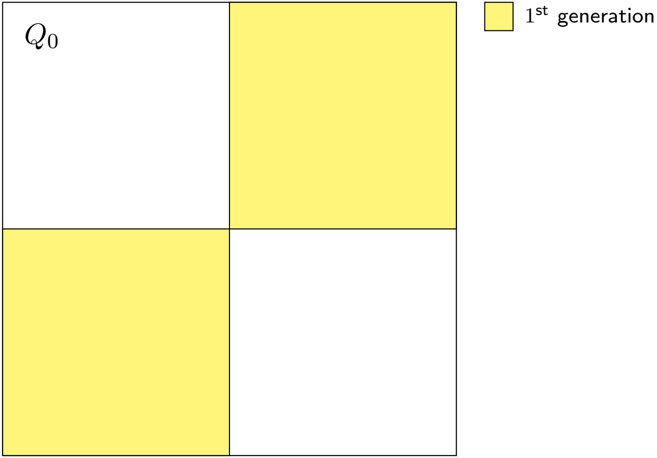
Usually we take $\mathcal{S}$ a subcollection of a dyadic grid such that

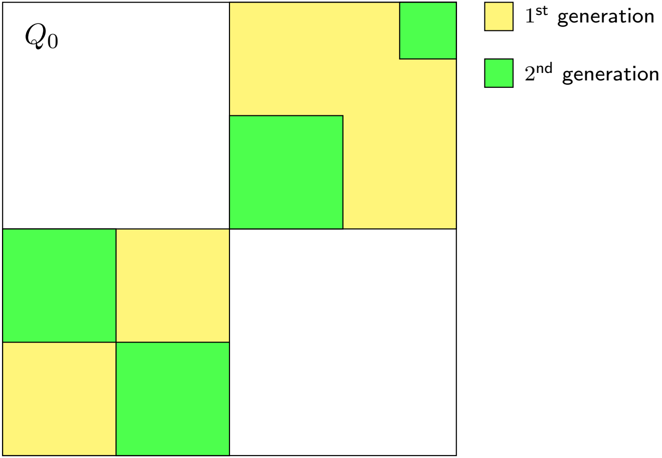
$$\sum_{S' \in \mathrm{Ch}_{\mathcal{S}}(S)} |S'| \leq \frac{1}{2}|S|.$$

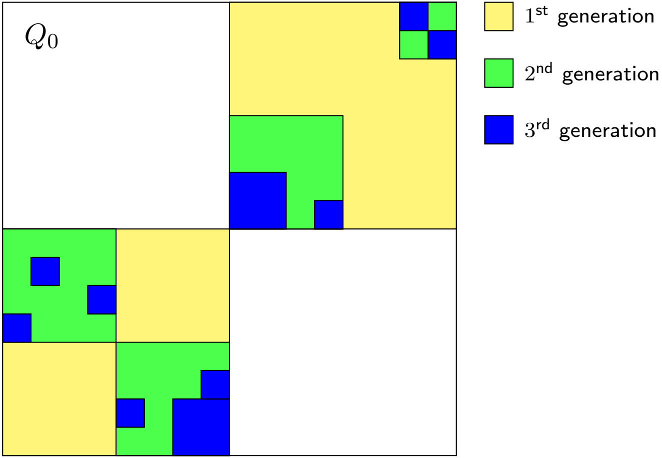
Here, $\mathrm{Ch}_{\mathcal{S}}(S) = \{\, S' \in \mathcal{S}$ maximal : $S' \subsetneq S \,\}$.

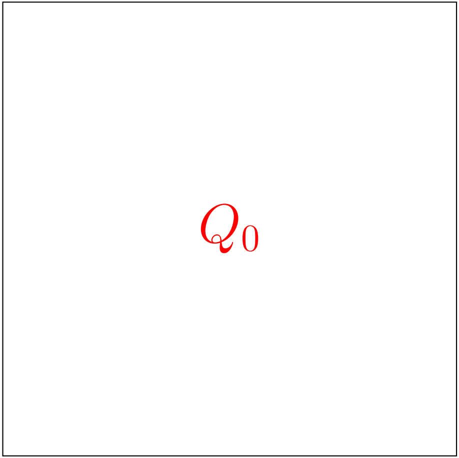
Then take $E_S = S \backslash \bigcup_{S' \in \mathrm{Ch}_{\mathcal{S}}\, S}$ S'

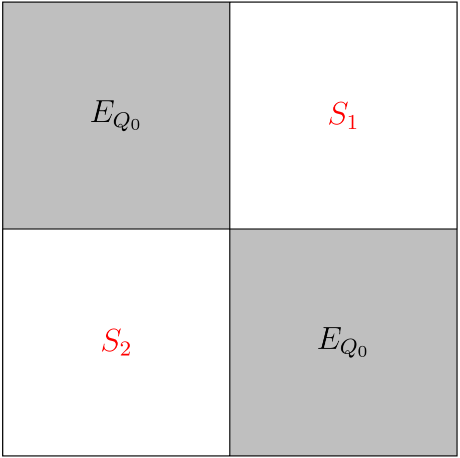

Q_0

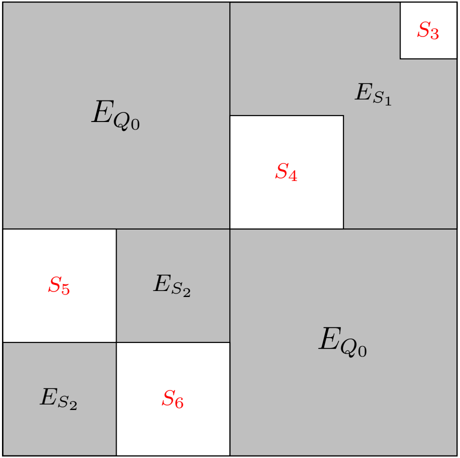

Q_0
1^{st} generation

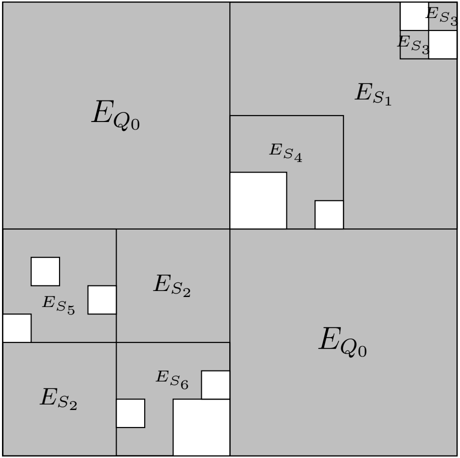

Q_0
1st generation
2nd generation


Q_0
1st generation
2nd generation
3rd generation


Q_0


E_{Q_0}
S_1
S_2
E_{Q_0}


S_3
E_{S_1}
E_{Q_0}
S_4
S_5
E_{S_2}
E_{Q_0}
E_{S_2}
S_6


E_{Q_0}
E_{S_3}
E_{S_3}
E_{S_1}
E_{S_4}
E_{S_2}
E_{S_5}
E_{Q_0}
E_{S_2}
E_{S_6}

Given a sparse collection $\mathcal{S}$, a sparse operator is defined by

$$\Lambda_{\mathcal{S}} f(x) = \sum_{S \in \mathcal{S}} \langle f \rangle_S \, \mathbb{1}_S(x).$$

These operators satisfy a weak 1-1 bound and are bounded (strongly) on L^p for $p > 1$. They are also bounded on $L^p(w)$ for an A_2 weight w.

To exemplify the convenience of working with these operators we look at the proof of the boundedness on L^p.

If $f \in L^p$ and $g \in L^q$ with $\|g\| = 1$,

$$\langle \Lambda f, g \rangle = \left\langle \sum_{S \in \mathcal{S}} \langle f \rangle_S \mathbb{1}_S, g \right\rangle = \sum_{S \in \mathcal{S}} \langle f \rangle_S \langle \mathbb{1}_S, g \rangle = \sum_{S \in \mathcal{S}} \langle f \rangle_S \langle g \rangle_S |S|$$

$$\leq c \sum_{S \in \mathcal{S}} \langle f \rangle_S \langle g \rangle_S |E_S| = c \sum_{S \in \mathcal{S}} \int \langle f \rangle_S \langle g \rangle_S \mathbb{1}_{E_s}(x)\, dx$$

$$\lesssim \int \sum_{S \in \mathcal{S}} \langle f \rangle_S \mathbb{1}_{E_S} \langle g \rangle_S \mathbb{1}_S(x)\, dx \leq \int \mathcal{M}g(x) \sum_{S \in \mathcal{S}} \langle f \rangle_S \mathbb{1}_{E_S}\, dx$$

$$\leq \int \mathcal{M}g(x)\mathcal{M}f(x)\, dx \leq \|\mathcal{M}f\|_{L^p} \|\mathcal{M}g\|_{L^q}$$

Let $\langle f \rangle_{S,r} = \left(\frac{1}{|S|} \int_S |f(x)|^r \, dx \right)^{1/r}$; For $r, s > 1$, a bilinear (r,s)-sparse form is defined by

$$\Lambda_{\mathcal{S},r,s}(f,g) = \sum_{S \in \mathcal{S}} \langle f \rangle_{S,r} \langle g \rangle_{S,s} |S|.$$

We say that an operator T is in Sparse(r,s) if there exists a sparse collection $\mathcal{S}$ such that

$$\langle Tf, g \rangle \lesssim \Lambda_{\mathcal{S},r,s}(f,g),$$

for all compactly supported functions f, g.

We can make this more precise in terms of a "sparse norm", considering the best implied constant in the previous inequality.

Boundedness:

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- The sparse domination also gives us vector valued inequalities.

Universal domination:

There is one sparse form *"to rule them all"*...

Lemma (Lacey, M.)

Given finitely supported functions f, g, there is a sparse form Λ^* and a constant $C > 0$ such that for any other sparse operator Λ we have

$$\Lambda(f, g) \leq C\Lambda^*(f, g).$$

This can be easily proven to be true with several kinds of sparse forms and in different contexts.

Sparse bounds for discrete spherical averages

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(joint work with Robert Kesler and Michael Lacey)

Let $\mathcal{A}_\lambda f = d\sigma_\lambda * f$ with $d\sigma_\lambda$ normalized spherical measure on a sphere of radius λ in $\mathbb{R}^d$, for $d \geq 3$. The Stein spherical maximal operator is

$$\mathcal{A}f(x) = \sup_{\lambda > 0} \mathcal{A}_\lambda f, \tag{1}$$

with f non-negative, compactly supported and bounded.

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In the continuous case, this estimate holds, and is sharp, up to the boundary:

Theorem (Lacey)

Let $d \geq 3$ *and set* $\mathbf{R}_d$ *to be the polygon with vertices* $R_0 = (\frac{d-1}{d}, \frac{1}{d})$, $R_1 = (\frac{d-1}{d}, \frac{d-1}{d})$, $R_2 = (\frac{d^2-d}{d^2+1}, \frac{d^2-d+2}{d^2+1})$, *and* $R_3 = (0,1)$. *(See Figure 1.) Then, for all* $(\frac{1}{p}, \frac{1}{q})$ *in the interior* $\mathbf{R}_d$, *we have the sparse bound* $\|\mathcal{A}\|_{p,q} < \infty$.

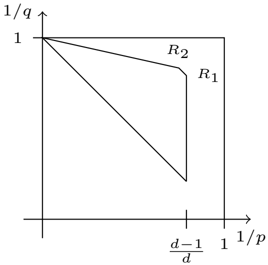

Figure: Sparse bounds hold for points $(1/p, 1/q)$ in the interior of the four sided region $\mathbf{R}_d$. The point R_1 is $(\frac{d-1}{d}, \frac{d-1}{d})$ and R_2 is $(\frac{d^2-d}{d^2+1}, \frac{d^2-d+2}{d^2+1})$.

In the discrete setting: If $\lambda^2 \in \mathbb{Z}$, and dimension $d \geq 5$, define

$$A_\lambda f(x) = \lambda^{2-d} \sum_{n \in \mathbb{Z}^d \, : \, |n| = \lambda} f(x-n) \tag{2}$$

for functions $f \in \ell^2(\mathbb{Z}^d)$.

We restrict attention to the case of $d \geq 5$ as in that case for all $\lambda^2 \in \mathbb{N}$, the cardinality of $\{n \in \mathbb{Z}^d \ : \ |n| = \lambda\} \simeq \lambda^{d-2}$.

Let $Af = \sup_{\lambda^2 \in \mathbb{N}} A_\lambda f$. This is the maximal function of Magyar, Stein and Wainger.

Theorem (Kesler, Lacey, M.)

Let $\mathbf{Z}_d$ be the polygon with vertices

$$Z_j = \tfrac{d-4}{d-2} R_j + \tfrac{2}{d-2}(\tfrac{1}{2}, \tfrac{1}{2}), \qquad j = 0, 1, 2, \tag{3}$$

and $Z_3 = (0, 1)$. (See Figure 2.) There holds:

1. *For all $(\frac{1}{p}, \frac{1}{q})$ in the interior $\mathbf{Z}_d$, we have the sparse bound $\|Af\|_{p,q} < \infty$.*
2. *With $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$, there holds*

$$\langle A\mathbf{1}_F, \mathbf{1}_G \rangle \lesssim \sup_{\mathcal{S}} \Lambda_{\mathcal{S}, \frac{d}{d-2}, \frac{d}{d-2}}(\mathbf{1}_F, \mathbf{1}_G). \tag{4}$$

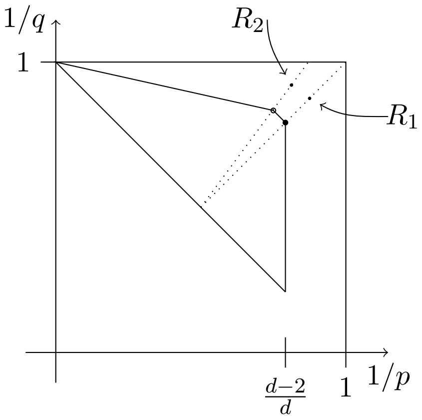

Figure: Sparse bounds for the discrete spherical maximal function hold for points $(1/p, 1/q)$ in the interior of the four sided figure above. The dotted lines pass through the points $(1/2, 1/2)$ and the points R_1 and R_2 of Figure 1. Circles along these lines are the points Z_1 and Z_2. The restricted weak-type sparse bound (4) holds at the filled in circle, $Z_1 = (\frac{d}{d-2}, \frac{d}{d-2})$.

$Z_0 = (\frac{d-2}{d}, \frac{2}{d})$, $Z_1 = (\frac{d-2}{d}, \frac{d-2}{d})$ and

$$Z_2 = \Big(\frac{d^3-4d^2+4d+1}{d^3-2d^2+d-2}, \frac{d^3-4d^2+6d-7}{d^3-2d^2+d-2}\Big).$$

The sparse bound near the point $(\frac{d-2}{d}, \frac{2}{d})$ implies the maximal inequality of Magyar, Stein and Wainger, namely, $A : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$, for $p > \frac{d}{d-2}$.

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The sparse bound (4) requires that both functions be indicator sets, and so is of restricted weak type. It implies the restricted weak type inequality of Ionescu.

These inequalities imply a wide range of weighted and vector valued inequalities, all of which are new.

For the proof, we decompose the maximal function into a series of terms, guided by the Hardy-Littlewood circle method decomposition developed by Magyar, Stein and Wainger.

For each part, we need only one estimate, either an ℓ^2 ('high frequency') estimate, or an endpoint ('low frequency') estimate.

Proof of the sparse bound

Fix a large dyadic cube E, functions $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$ supported on E.

We say that $\tau : E \to \{1, \ldots, \ell E\}$ is an *admissible stopping time* if for any subcube $Q \subset E$ with $\langle f \rangle_Q > C \langle f \rangle_E$, for some large constant C (chosen later), we have $\min_{x \in Q} \tau(x) > \ell Q$.

Lemma

Let $(\frac{1}{p}, \frac{1}{q})$ be in the interior of $\mathbf{Z}_d$. For any dyadic cube, functions $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$ supported on E, and any admissible stopping time τ, there holds

$$|E|^{-1} \langle A_\tau f, g \rangle \lesssim \langle f \rangle_E^{1/p} \langle g \rangle_E^{1/q}.$$

Proof of First Part of Theorem: Let $\mathcal{Q}_E$ be the maximal dyadic subcubes of E for which $\langle f \rangle_{3Q} > C \langle f \rangle_{3E}$, for a large constant C. Observe that we have, for an appropriate choice of admissible $\tau(x)$,

$$\Big\langle \sup_{\lambda \le \ell(E)} A_\lambda f, g \Big\rangle \le \langle A_\tau f, g \rangle + \sum_{Q \in \mathcal{Q}_E} \Big\langle \sup_{\lambda \le \ell(Q)} A_\lambda (f \mathbf{1}_{3Q}), g \mathbf{1}_Q \Big\rangle.$$

The first term is controlled by the lemma. For appropriate constant $C \simeq 3^d$, we have a sparse condition

$$\sum_{Q \in \mathcal{Q}_E} |Q| \le \tfrac{1}{4} |E|.$$

And we recurse to prove the sparse bounds for all indicator functions in the interior of $\mathbf{Z}_d$.

Proof of lemma:

We use indicator functions to be able to interpolate.

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Fix $(1/\bar{p}, 1/\bar{q}) \in \mathbf{R}_d$. For all sufficiently small $0 < \epsilon < 1$ so that $(1/(\bar{p}+\epsilon), 1/(\bar{q}+\epsilon)) \in \mathbf{R}_d$, and integers $N \in \mathbb{N}$, we can write $A_\tau f \leq M_1 + M_2$ where

$$|E|^{-1}\langle M_1, g\rangle \lesssim N^{1+\epsilon}\langle f\rangle_E^{\frac{1}{\bar{p}+\epsilon}}\langle g\rangle_E^{\frac{1}{\bar{q}+\epsilon}}, \tag{5}$$

$$|E|^{-1}\langle M_2, g\rangle \lesssim N^{d\epsilon+\frac{4-d}{2}}\langle f\rangle_E^{1/2}\langle g\rangle_E^{1/2}. \tag{6}$$

Implied constants depend upon $\bar{p}, \bar{q}$ and ϵ, but we do not track the dependence.

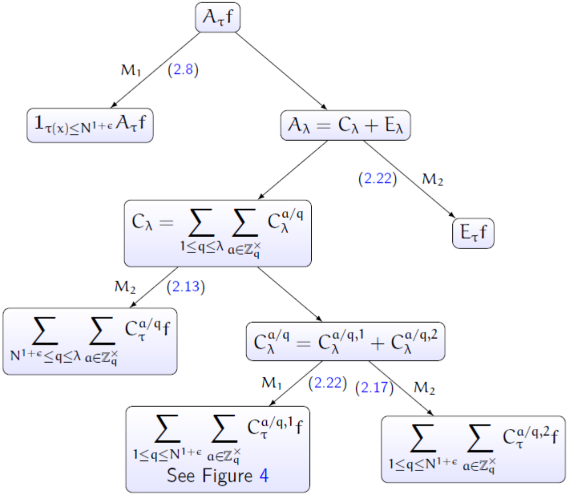
Then we have

$$|E|^{-1}\langle A_\tau f, g\rangle \lesssim N^{1+\epsilon}\langle f\rangle_E^{\frac{1}{\bar{p}+\epsilon}}\langle g\rangle_E^{\frac{1}{\bar{q}+\epsilon}} + N^{d\epsilon+\frac{4-d}{2}}\langle f\rangle_E^{1/2}\langle g\rangle_E^{1/2}.$$

Optimizing over N to minimize the right hand side, and letting $(1/\bar{p}, 1/\bar{q})$ and $0 < \epsilon < 1$ vary completes the proof.

We see that the value of p is given by

$$\frac{1}{p} = \frac{1}{\bar{p}} + \frac{2}{d-2}\left(\frac{1}{2} - \frac{1}{\bar{p}}\right) = \frac{2}{d-2}\cdot\frac{1}{2} + \frac{d-4}{d-2}\cdot\frac{1}{\bar{p}},$$


$A_\tau f$
M_1
(2.8)
$1_{\tau(x)\leq N^{1+\epsilon}} A_\tau f$
$A_\lambda = C_\lambda + E_\lambda$
(2.22)
M_2
$C_\lambda = \sum_{1\leq q\leq\lambda} \sum_{a\in\mathbb{Z}_q^\times} C_\lambda^{a/q}$
$E_\tau f$
M_2
(2.13)
$\sum_{N^{1+\epsilon}\leq q\leq\lambda} \sum_{a\in\mathbb{Z}_q^\times} C_\tau^{a/q} f$
$C_\lambda^{a/q} = C_\lambda^{a/q,1} + C_\lambda^{a/q,2}$
M_1
(2.22)
(2.17)
M_2
$\sum_{1\leq q\leq N^{1+\epsilon}} \sum_{a\in\mathbb{Z}_q^\times} C_\tau^{a/q,1} f$
See Figure 4
$\sum_{1\leq q\leq N^{1+\epsilon}} \sum_{a\in\mathbb{Z}_q^\times} C_\tau^{a/q,2} f$

Small values of τ:

The first contribution to M_1 is the term $M_{1,1} = \mathbf{1}_{\tau \leq N^{1+\epsilon}} A_\tau f$. We apply the continuous inequalities.

Take $\phi(x) = \sum_{n \in \mathbb{Z}^d} \mathbf{1}_F(n) \mathbf{1}_{n+[-1,1)^d}(x)$. By some reductions, these are indicator functions.

We can compare the discrete and continuous spherical averages as follows.

$$A_\tau f(x) \lesssim \tau \mathcal{A}_\tau \phi(x). \tag{7}$$

If $\tau \leq N^{1+\epsilon}$, we see that $M_{1,1}$ satisfies (5).

CASE $\tau > N^{1+\epsilon}$:

We decompose $A_\lambda f$ using the circle method: Let $e(x) = e^{2\pi i x}$ and for integers q, $e_q(x) = e(x/q)$.

$$A_\lambda f = C_\lambda f + E_\lambda f,$$

$$C_\lambda f = \sum_{1 \le q \le \lambda} \sum_{a \in \mathbb{Z}_q^\times} e_q(-\lambda^2 a) C_\lambda^{a/q} f,$$

$$c_\lambda^{a/q}(\xi) = \widehat{C_\lambda^{a/q}}(\xi) = \sum_{\ell \in \mathbb{Z}_q^d} G(a/q, \ell) \widetilde{\psi}_q(\xi - \ell/q) \widetilde{d\sigma_\lambda}(\xi - \ell/q),$$

$$G(a/q, \ell) = q^{-d} \sum_{n \in \mathbb{Z}_q^d} e_q(|n|^2 a + n \cdot \ell).$$

THE ERROR TERM E_λ: The first contribution to M_2 is $M_{2,1} = |E_\tau f|$. The inequality below is from M-S-W, and it implies that $M_{2,1}$ satisfies (6).

$$\Big\| \sup_{\Lambda \le \lambda \le 2\Lambda} |E_\lambda \cdot| \Big\|_{2\to 2} \lesssim \Lambda^{\frac{4-d}{2}}, \qquad \Lambda \ge 1.$$

LARGE DENOMINATORS: The second contribution to M_2 is

$$M_{2,2} = \Big| \sum_{N^{1+\epsilon} \le q \le \tau} e_q(-\lambda^2 a) C_\tau^{a/q} f \Big|.$$

For this term, we have.

$$\| \sup_{\lambda > q} |C_\lambda^{a/q} f| \|_2 \lesssim q^{-\frac{d}{2}} \|f\|_2. \tag{8}$$

SMALL DENOMINATORS: This is the case

$$\sum_{1 \le q \le N^{1+\epsilon}} \sum_{a \in \mathbb{Z}_q^\times} e_q(-\lambda^2 a) C_\tau^{a/q} f$$

Write $C_\tau^{a/q} = C_\tau^{a/q,1} + C_\tau^{a/q,2}$, where for an integer $1 \le Q \le N/2$, and $Q \le q < 2Q$, define

$$\widehat{C_\lambda^{a/q,1}}(\xi) = \sum_{\ell \in \mathbb{Z}^d} G(a,\ell,q) \widetilde{\psi}_q(\xi - \ell/q) \widetilde{\psi}_{\lambda Q/N}(\xi - \ell/q) \widetilde{d\sigma_\lambda}(\xi - \ell/q).$$

SMALL DENOMINATORS: THE ℓ^2 PART : We then show that

$$\Big\| \sup_{N^{1+\epsilon} \leq \lambda \leq \ell(E)} |C_\lambda^{a/q,2} f| \Big\|_2 \lesssim q^{-1} N^{-\frac{d-2}{2}} \|f\|_2.$$

It follows that

$$\Big\| \sum_{1 \leq q \leq N^{1+\epsilon}} \sum_{a \in \mathbb{Z}_q^\times} e_q(-\lambda^2 a) C_\tau^{a/q,2} f \Big\|_2 \lesssim N^{-\frac{d-4}{2}+\epsilon} \|f\|_2.$$

This is the third and final contribution to M_2.

To prove this part, we make use of:

To prove this part, we make use of:

- Factorization and transference argument (M-S-W).
- A Lemma of Bourgain for smooth multipliers.
- Stationary phase estimates over the continuous spherical measure.

Small Denominators: The Sparse Part

$$M_{1,2} f = \sum_{1 \le q \le N} \sum_{a \in \mathbb{Z}_q^\times} e_q(-\lambda^2 a) C_\tau^{a/q,1} f$$

The Gauss sum map $\ell \mapsto G(a/q, \ell)$ is the Fourier transform of $e_q(|x|^2 a)$.

We control by a product of maximal averages over annuli and Ramanujan sums.

After further decomposition, we obtain the desired bound (with an absorbing term).

The endpoint estimate

The relevant lemma is:

Lemma

For $f = \mathbf{1}_F$ supported on cube $3E$, there is a pre-sparse collection $\mathcal{Q}_E$ so that for all $\mathcal{Q}_E$ admissible $\tau = \tau(x)$, and all $g = \mathbf{1}_G$ supported on E, we have

$$\langle A_\tau f, g \rangle \lesssim \langle f \rangle_{3E, \frac{d}{d-2}} \langle g \rangle_{E, \frac{d}{d-2}} |E|.$$

For integers $N > 1$, there is a decomposition

$$A_\tau f \leq M_1 + M_2,$$
$$\langle M_1, g\rangle \lesssim N^2 \langle f\rangle_{3E,1} \langle g\rangle_{E,1} |E|,$$
and
$$\langle M_2, g\rangle \lesssim N^{-\frac{d-4}{2}} \langle f\rangle_{3E,2} \langle g\rangle_{E,2} |E|.$$

Recalling that $f = \mathbf{1}_F$ and $g = \mathbf{1}_G$, the right sides above are comparable for $N \simeq \left[\langle f\rangle_{3E,1}\langle g\rangle_{E,1}\right]^{-\frac{1}{d}}$, and this proves the main lemma.

Interpolation of sparse bounds

Lemma

Suppose that a sub-linear operator T satisfies the bound below for $1 < p, q < \infty$. For a fixed function f and all $|g| \le \mathbf{1}_G$, there is a sparse collection $\mathcal{S}$ so that

$$|\langle Tf, g\rangle| \lesssim \Lambda_{\mathcal{S},p,q}(f, \mathbf{1}_G)$$

Then,

$$|\langle Tf, g\rangle| \lesssim \sup_{\mathcal{S}} \Lambda_{\mathcal{S},p,q}(f, g), \qquad q < r < \infty.$$

About sharpness

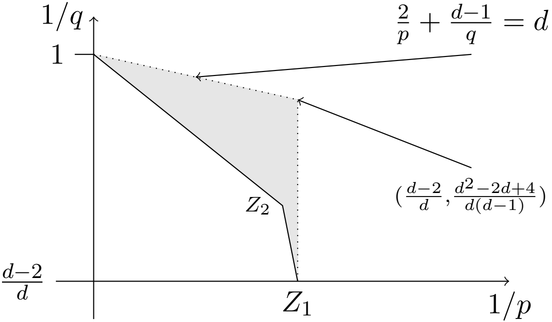


Figure: The horizontal line is set at $\frac{1}{q} = \frac{d-2}{d}$, for reasons of clarity. Sparse bounds hold below the solid line from $(0, 1)$, to Z_2 to $Z_1 = (\frac{d-2}{d}, \frac{d-2}{d})$. (Recall that Z_2 is defined in (18).) They cannot hold to the right of Z_1, nor above the dotted line. The gray area is unresolved.

Thank you!