From geometric quantization to Moyal quantization

José M. Gracia-Bondía,[1,2] and Joseph C. Várilly[3]

[2] Departamento de Física Teórica, Universidad Autónoma de Madrid, 28049 Madrid, Spain

[1] Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain

[3] Escuela de Matemática, Universidad de Costa Rica, 11501 San José, Costa Rica

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Abstract

We show how the Moyal product of phase-space functions, and the Weyl correspondence between symbols and operator kernels, may be obtained directly using the procedures of geometric quantization, applied to the symplectic groupoid constructed by "doubling" the phase space.

1 Introduction

Over the last two decades, several approaches to quantization of classical systems have been developed. The most mathematically thorough of these is the so-called method of geometric quantization [1–4], which seeks to manufacture the quantum-mechanical Hilbert space from the symplectic manifold of classical states. Other quantization procedures may be refined and extended by recasting them in the geometric quantization framework; thus, for example, the recent work of Tuynman [5] on BRST symmetry. The relations between different quantization schemes continue to merit attention [6, 7].

That the Moyal or phase-space approach to quantization [8–10] can be explicitly derived from the geometric quantization scheme was pointed out by Weinstein [12]. However, we are not aware of an explicit treatment in the literature; this note attempts to fill that gap. We spell out how these two approaches may be related, in the simplest case of a linear phase space. The idea needed to bridge the gap between both quantization schemes is the concept of symplectic groupoid, developed by Weinstein and co-workers [11–14].

The article is arranged as follows. In Sec. 2 we recall the definition of symplectic groupoid, and in Sec. 3 we briefly review the theory of pairings in geometric quantization, in order to establish the context. In Sec. 4 we show that the Weyl correspondence between Weyl symbols of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$ and their kernels, is given by a pairing of real polarizations of a particular symplectic groupoid, namely two copies of the flat phase-space $\mathbb{R}^{2n}$. We then show, in Sec 5, that the Moyal product of phase-space functions arises directly from the groupoid structure of the double phase space.

Two further applications of this viewpoint are given. In Sec. 6, we rederive the integral transformation introduced by Daubechies and Grossmann [15] to effect quantization in the coherent-state picture, from a pairing of a real and a complex polarization on the aforementioned groupoid. Finally, it is shown in Sec. 7 that the appearance of the ordinary Fourier transformation as a power of the Weyl correspondence map can be understood geometrically as a property of symplectic transformations on that groupoid.

2 Symplectic groupoids

If M is a manifold with symplectic form ω, we will denote by $\overline{M}$ the symplectic manifold $(M, -\omega)$. A groupoid is a set with a partially-defined associative multiplication. We recall the definition of a symplectic groupoid, as set forth in [12].

A *symplectic groupoid* consists of a pair of manifolds (G, G_0), where G has a symplectic form Ω and a partially defined multiplication with domain $G_2 \subset G \times G$, together with two submersions $\alpha\colon G \to G_0$, $\beta\colon G \to G_0$, and an involution $x \mapsto x^*$ of G, such that:

(1) the graph $\mathcal{M} = \{ (x, y, xy) : (x, y) \in G_2 \}$ of the multiplication is a Lagrangian submanifold of $\overline{G} \times \overline{G} \times G$;

(2) the set of "units" G_0 may be identified with a Lagrangian submanifold of G (also denoted by G_0);

(3) for any $x \in G$, we have $\alpha(x)x = x = x\beta(x)$; and $\alpha(x) = xx^*$, $\beta(x) = x^*x$; moreover, $(x, y) \in G_2$ if and only if $\beta(x) = \alpha(y)$;

(4) the graph $I = \{ (x, x^*) : x \in G \}$ of the involution is a Lagrangian submanifold of $G \times G$; and

(5) whenever (x, y) and $(y, z) \in G_2$, then (xy, z) and (x, yz) lie in G_2, and $(xy)z = x(yz)$.

As consequences of these postulates, we find that $\alpha(x^*) = \beta(x)$; that $\alpha(x)^* = \alpha(x) = \alpha(x)^2$ and $\beta(x)^* = \beta(x) = \beta(x)^2$; that $xx^*x = \alpha(x)x = x$; that $\alpha(\alpha(x)) = \alpha(x)$ and $\beta(\beta(x)) = \beta(x)$. Moreover, if $(x, y) \in G_2$, then

$$\alpha(xy) = xyy^*x^* = x\alpha(y)x^* = x\beta(x)x^* = xx^* = \alpha(x),$$

and also $\beta(xy) = \alpha(y^*x^*) = \alpha(y^*) = \beta(y)$.

As a notational convention, we write $G \rightrightarrows G_0$ to denote a symplectic groupoid, if α and β are understood.

Two general examples of symplectic groupoids deserve mention. One is the groupoid $T^*H \rightrightarrows \mathfrak{h}^*$, where H is a Lie group and $\mathfrak{h}^*$ is the dual of its Lie algebra. The maps α and β are given by right, resp. left, translation of a cotangent vector to the cotangent space at the identity of H.

Another example is that of the fundamental groupoid $\pi(M) \rightrightarrows M$ of a symplectic manifold (M, ω). Its elements are homotopy classes of smooth paths $\sigma\colon [0, 1] \to M$, with the usual concatenation product of paths whose endpoints match; reversing the path gives the involution. Here $\alpha([\sigma]) = \sigma(0)$, $\beta([\sigma]) = \sigma(1)$ are the endpoint assignment maps. The manifold M embeds in $\pi(M)$ as the submanifold of constant paths, which is Lagrangian with respect to the symplectic structure $\Omega = \alpha^*\omega - \beta^*\omega$ on $\pi(M)$.

When M is simply connected, $[\sigma]$ is determined by its endpoints, and the fundamental groupoid may be re-expressed as

$$M \times \overline{M} \rightrightarrows M.$$

We can then write $\alpha(q,p) = q$, $\beta(q,p) = p$, and identify M with the diagonal submanifold $\{ (q,q) : q \in M \}$. The multiplication and involution are given by:

$$(q,p) \cdot (p,r) = (q,r); \quad (q,p)^* = (p,q).$$

One can check that the graph of the product $\mathcal{M} = \{ (q,p;p,r;q,r) : q,p,r \in M \}$ is Lagrangian in $\overline{G} \times \overline{G} \times G$.

We now specialize further to the case $M = \mathbb{R}^{2n}$, with ω a nondegenerate alternating bilinear form on $\mathbb{R}^{2n}$. Writing $\hat{\omega}(u) \colon v \mapsto \omega(u,v)$ gives a skewsymmetric invertible map $\hat{\omega} \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n*}$. One obtains

$$\begin{aligned}\Omega((x,y),(z,w)) &= \omega(x,z) - \omega(y,w) \\ &= \hat{\omega}(x-y)\Big[\frac{z+w}{2}\Big] - \hat{\omega}(z-w)\Big[\frac{x+y}{2}\Big].\end{aligned}$$

On the other hand, $\mathbb{R}^{2n} \times \overline{\mathbb{R}}^{2n}$ can be identified with the cotangent bundle $T^*(\mathbb{R}^{2n})$. If (u,ξ), (v,η) are elements of $\mathbb{R}^{2n} \times \mathbb{R}^{2n*}$, regarded as local coordinates of covectors in $T^*(\mathbb{R}^{2n})$, the cotangent symplectic structure of $T^*(\mathbb{R}^{2n})$ reduces to the alternating bilinear form:

$$\Sigma((u,\psi),(v,\chi)) = \chi(u) - \psi(v).$$

Thus $\mathbb{R}^{2n} \times \overline{\mathbb{R}}^{2n}$ can be identified with $T^*(\mathbb{R}^{2n})$ as symplectic manifolds by the linear isomorphism

$$\Phi \colon (x,y) \mapsto \big(\tfrac{1}{2}(x+y), \hat{\omega}(x-y)\big) \tag{1}$$

for which $\Phi^*\Sigma = \Omega$.

3 Pairing in geometric quantization

We briefly recall here, in order to fix notation, those aspects of geometric quantization that we shall need to address.

Prequantization of an $2n$-dimensional symplectic manifold (M,ω) proceeds by finding a real-linear map $f \mapsto \hat{f}$ from the Poisson algebra of smooth functions on M to an algebra of operators on the Hilbert space of $L^2(M)$, for which $\hat{1} = 1$ and $\{f_1,f_2\}^\frown = (i/\hbar)[\hat{f}_1,\hat{f}_2]$. The right recipe is $\hat{f} = f - i\hbar\,\nabla_{X_f}$, where X_f is the Hamiltonian vector field of f and the covariant derivative ∇ is locally given by

$$\nabla_X = X - (i/\hbar)\,\theta(X). \tag{2}$$

Here θ is a symplectic potential, i.e., a one-form for which $d\theta = \omega$. When ω is not exact, local potentials must be patched together so that ∇ becomes a linear connection on a Hermitian complex line bundle $L \to M$, whose curvature form is $(-i/\hbar)\omega$, as is well-known. The elements of the prequantization Hilbert space are sections $s \in \Gamma L$ of this line bundle.

Geometric quantization then involves finding a positive polarization of (M,ω), i.e., a subbundle F of the complexified tangent bundle $T^*M^{\mathbb{C}}$, which is maximally isotropic for ω, with $F \cap \overline{F}$ of constant rank; which is integrable in the sense that both F and $F \cap \overline{F}$ are closed under the Lie bracket; and which is positive in that $-i\omega(\overline{Y}, Y) \geqslant 0$ whenever Y is a section of F.

A polarized section is any $s \in \Gamma L$ for which $\nabla_Y s = 0$ whenever $Y \in \Gamma F$. The quantizable observables are those $g \in C^\infty(M)$ for which $\mathrm{ad}(X_g)$ preserves ΓF. Then one checks that $\hat{g}$ preserves the space $\Gamma_F L$ of polarized sections. The remaining difficulty is to endow $\Gamma_F L$ – or some modification thereof – with a suitable inner product, in order that the quantizable observables be represented as operators on a Hilbert space. This is done by using the idea of a half-form *pairing* [16].

We follow the very precise treatment of pairings by Rawnsley [17, 18]. The canonical line bundle of F is $K^F = \Lambda^n F^0$, where $F^0 \subset T^*M^{\mathbb{C}}$ denotes covectors which vanish on F. For example, if M is a Kähler manifold with local holomorphic coordinates $(z_1, \dots, z_n)$, and F is spanned by $\partial/\partial\bar{z}_1, \dots, \partial/\partial\bar{z}_n$, then K^F is spanned by $dz_1 \wedge \cdots \wedge dz_n$; in this case we have $F \cap \overline{F} = 0$. A contrasting example, for which F is a real polarization, that is, $F = \overline{F}$, is obtained by taking local Darboux coordinates $(q_1, \dots, q_n; p_1, \dots, p_n)$ for M, with F spanned by $\partial/\partial p_1, \dots, \partial/\partial p_n$, whereupon K^F is spanned by $dq_1 \wedge \cdots \wedge dq_n$.

Suppose there are two positive polarizations F and P; it turns out that K^F and K^P are isomorphic as line bundles over M and that $\overline{K^F} \otimes K^P$ is a trivial bundle. There is an obvious map from this bundle to $\Lambda^{2n} T^*M^{\mathbb{C}}$ (replace tensor by exterior product), which is an isomorphism iff $\overline{F} \cap P = 0$. The Liouville volume $\lambda = (-1)^{n(n-1)/2} \omega^{\wedge n}/n!$ trivializes the latter bundle. Thus there is a pairing $\langle \alpha, \beta \rangle$ of $\alpha \in \Gamma K^F$ and $\beta \in \Gamma K^P$ defined by

$$i^n \langle \alpha, \beta \rangle \, \lambda = \bar{\alpha} \wedge \beta \tag{3}$$

provided $\overline{F} \cap P = 0$. In particular, if $\overline{F} \cap F = 0$, then $\langle \cdot, \cdot \rangle$ is an inner product on ΓK^F.

Matters are less straightforward if $\overline{F} \cap P \neq 0$. Here $\overline{F} \cap P = D^{\mathbb{C}}$ where D is an isotropic subbundle of TM. If $D^\perp$ is the symplectic orthogonal of D, then $D^\perp/D$ becomes a symplectic vector bundle (with an induced symplectic form ω_D), of which $\overline{F}/D$ and P/D are non-overlapping maximal-isotropic subbundles; thus we may apply the previous recipe to get a pairing of $K^{F/D}$ and $K^{P/D}$.

We can try to pull back to a pairing of K^F and K^P by suppressing the common real directions in D. Suppose that the foliation of M induced by D has a smooth space of leaves M/D, that D is spanned locally by $\partial/\partial y_1, \dots, \partial/\partial y_k$, and that $(x_1, \dots, x_k)$ are conjugate local coordinates to $(y_1, \dots, y_k)$; if

$$\begin{aligned}
\alpha &= a \, dx_1 \wedge \cdots \wedge dx_k \wedge dz_1 \wedge \cdots \wedge dz_{n-k} \in \Gamma K^F, \\
\beta &= b \, dx_1 \wedge \cdots \wedge dx_k \wedge dw_1 \wedge \cdots \wedge dw_{n-k} \in \Gamma K^P,
\end{aligned}$$

where the coefficient functions do not depend on the y_j, then we can define $\tilde{\alpha} = a \, d\tilde{z}_1 \wedge \cdots \wedge d\tilde{z}_{n-k}$ in $\Gamma K^{F/D}$, $\tilde{\beta} = b \, d\tilde{w}_1 \wedge \cdots \wedge d\tilde{w}_{n-k}$ in $\Gamma K^{P/D}$, where the tildes denote corresponding coordinates on M/D, and we can try to set $\langle \alpha, \beta \rangle = \langle \tilde{\alpha}, \tilde{\beta} \rangle$. It turns out, of course, that this recipe is coordinate-dependent, and in fact (after incorporating a correction factor of λ^2) the change of variables formula shows that the result is a 2-density on the leaf space M/D.

Since we could integrate a 1-density over M/D to get a scalar-valued inner product, we abandon K^F in favour of the vector bundle Q^F of "half-forms" on M which is defined by the requirement that $Q^F \otimes Q^F = K^F$; if $\alpha \in \Gamma K^F$, we write $\sqrt{\alpha} = \mu \in \Gamma Q^F$ if $\mu \otimes \mu = \alpha$. It can then be shown that

$\overline{Q^F} \otimes Q^P$ carries a pairing, whose values are 1-densities on M/D, determined (up to a sign) by the requirement that $\langle\sqrt{\alpha}, \sqrt{\beta}\rangle^2 = \langle\alpha, \beta\rangle$.

[We tiptoe past the crucial question of the existence of Q^F, for which there is a topological obstruction: (M, ω) must "admit metaplectic structures". This obstruction has been ingeniously overcome by Robinson and Rawnsley [3] by replacing metaplectic structures by Mp^c-structures, which always exist; the procedure is akin to passing from spin structures to spin^c structures on Riemannian manifolds.]

The final touch is to replace the prequantization bundle L by $L \otimes Q^F$, and let $\Gamma_F(L \otimes Q^F)$ denote its polarized sections (those killed by ∇_Y for $Y \in \Gamma F$). The pairing of two sections $s \otimes \sqrt{\alpha} \in \Gamma_F(L \otimes Q^F)$, $t \otimes \sqrt{\beta} \in \Gamma_P(L \otimes Q^P)$ is given by

$$\langle s \otimes \sqrt{\alpha}, t \otimes \sqrt{\beta}\rangle = \int_{M/D} (s, t)\, \langle\sqrt{\alpha}, \sqrt{\beta}\rangle, \tag{4}$$

where $(\cdot, \cdot)$ is the Hermitian metric on L. When $F = P$, the geometric quantization Hilbert space $\mathcal{H}_F$ is obtained by completing $\Gamma_F(L \otimes Q^F)$ with respect to this inner product.

4 Pairings and the Weyl correspondence

On the symplectic manifold $G_0 = \mathbb{R}^{2n}$, we take coordinates $(x', x'') \equiv (x'_1, \dots, x'_n, x''_1, \dots, x''_n)$, so that $\omega = dx' \wedge dx'' \equiv \sum_k dx'_k \wedge dx''_k$. (To avoid index clutter, we will henceforth just take $n = 1$.) We can regard ω as a bilinear symplectic form on $\mathbb{R}^2$, with $\omega(x, z) = x'z'' - x''z'$. Then $\hat{\omega}(x) = (-x'', x')$ in the dual space $\mathbb{R}^{2*}$.

The symplectic groupoid $G = \mathbb{R}^2 \times \overline{\mathbb{R}}^2$ has coordinates $(x', x''; y', y'')$, with which its symplectic form may be written as

$$\Omega = \pi_1^*\omega - \pi_2^*\omega = dx' \wedge dx'' - dy' \wedge dy''. \tag{5}$$

Thus $(x', y'; x'', -y'')$ are Darboux coordinates for G.

On the cotangent bundle $T^*\mathbb{R}^2$, we use Darboux coordinates (q_1, q_2, p_1, p_2); the symplectic form is $\Sigma = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. The symplectomorphism Φ of (1) is given explicitly by

$$q_1 = \frac{x' + y'}{2}, \quad q_2 = x' - y', \quad p_1 = x'' - y'', \quad p_2 = \frac{x'' + y''}{2}. \tag{6}$$

We consider the following two real polarizations of G. Set

$$F := \operatorname{span}\left\{\frac{\partial}{\partial x''}, \frac{\partial}{\partial y''}\right\}, \qquad P := \operatorname{span}\left\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_2}\right\}. \tag{7}$$

From (6), it follows that

$$\frac{\partial}{\partial p_1} = \frac{1}{2}\left(\frac{\partial}{\partial x''} - \frac{\partial}{\partial y''}\right), \quad \frac{\partial}{\partial p_2} = \frac{\partial}{\partial x''} + \frac{\partial}{\partial y''},$$

so we can rewrite $F = \operatorname{span}\{\partial/\partial p_1, \partial/\partial p_2\}$. Therefore $\overline{F} \cap P = D^{\mathbb{C}}$, where D is spanned by $\partial/\partial p_1$. By a slight abuse of notation, we can regard $\{q_1, q_2, p_2\}$ as local coordinates for the (affine) leaf space G/D, and the pairing $\Gamma Q^F \times \Gamma Q^F \to \mathcal{D}^1(G/D)$ is determined by

$$\langle\sqrt{dx' \wedge dy'}, \sqrt{dq_1 \wedge dp_2}\rangle = dq_1\, dq_2\, dp_2.$$

The polarized sections in $\Gamma_F L$ are of the form $f s_0$, where $f \in C^\infty(G)$ and s_0 is a nonvanishing section of L satisfying $\nabla_X s_0 = -(i/\hbar)\, \Theta_F(X) s_0$ and $(s_0, s_0) = 1$. The symplectic potential Θ_F for (G, Ω) may be taken to vanish on F; and so

$$\Theta_F = -x''\, dx' + y''\, dy' = -p_1\, dq_1 - p_2\, dq_2.$$

In this case $f s_0 \in \Gamma_F L$ if and only if $Xf = 0$ for $X \in F$, that is, $f = f(x', y')$. Likewise, if t_0 is a section of L satisfying $\nabla_X t_0 = -(i/\hbar)\, \Theta_P(X) t_0$ and $(t_0, t_0) = 1$, with

$$\Theta_P = -p_1\, dq_1 + q_2\, dp_2$$

being the symplectic potential which vanishes on P, then a typical element of $\Gamma_P L$ is of the form $g t_0$ with $g = g(q_1, p_2)$.

Clearly $t_0 = \phi_0 s_0$ for a nonvanishing $\phi_0 \in C^\infty(G)$; indeed, from $\nabla_X t_0 = (X\phi_0) s_0 + \phi_0\, \nabla_X s_0$ we obtain

$$\frac{d\phi_0}{\phi_0} = \frac{i}{\hbar}(\Theta_F - \Theta_P) = -\frac{i}{\hbar} d(q_2 p_2),$$

and so $\phi_0 = C\, \exp(-i q_2 p_2/\hbar)$ for some positive constant C. Since $(s_0, t_0) = \phi_0$, we can now compute the half-form pairing of $\alpha = f(x', y') s_0 \otimes \sqrt{dx' \wedge dy'}$ and $\beta = g(q_1, q_2) t_0 \otimes \sqrt{dq_1 \wedge dp_2}$ as

$$\begin{aligned}
\langle \alpha, \beta \rangle &= C \int \overline{f(x', y')}\, g(q_1, p_2)\, e^{-i q_2 p_2/\hbar}\, dq_1\, dq_2\, dp_2 \\
&= C \int \overline{f(x', y')}\, g\!\left(\frac{x' + y'}{2}, p_2\right) e^{i p_2 (y' - x')/\hbar}\, dp_2\, dx'\, dy' \\
&= \langle f, Tg \rangle_{L^2(\mathbb{R}^2)}\,,
\end{aligned}$$

where

$$Tg(x', y') := C \int g\!\left(\frac{x' + y'}{2}, \zeta\right) e^{i\zeta (y' - x')/\hbar}\, d\zeta \tag{8}$$

is the kernel of the operator – on $L^2(\mathbb{R})$ – whose Weyl symbol is g [19]. Unitarity of T is achieved by taking $C = (2\pi\hbar)^{-1}$.

In other words: the pairing of the non-transverse polarizations F and P of the symplectic groupoid $\mathbb{R}^2 \times \overline{\mathbb{R}}^2$ yields the well-known correspondence between kernels of Hilbert–Schmidt operators on $L^2(\mathbb{R})$ and the Weyl symbols of these operators. Thus the groupoid forms a bridge between conventional quantum mechanics and the phase-space formalism. It remains only to see how the symbol product may be obtained directly from this starting point.

5 The Moyal product from geometric quantization

The importance of symplectic groupoids in general is that the partial multiplication in G induces an associative product of polarized sections, so that *the geometric quantization Hilbert space becomes in fact a Hilbert algebra.* By suitably modifying its topology, one can obtain a C^*-algebra. This is in the spirit of noncommutative geometry [20–22]. Indeed, in [13,23], a symplectic groupoid structure on the torus $\mathbb{T}^2$, which depends on an irrational parameter, is shown to yield the “noncommutative torus” algebra considered by Rieffel and others [21,24].

On the other hand, the basic idea of Moyal quantization is that by working with functions on phase space, rather than wave functions, one may describe both states and observables of quantum-mechanical systems in classical terms; thus phase-space functions are to be equipped with a noncommutative product which give the quantum formalism directly without invoking a Hilbert space a priori. In Ref. [12] it is claimed that the Moyal product of phase-space functions is inherited from the groupoid structure of $\mathbb{R}^2 \times \overline{\mathbb{R}}^2 \rightrightarrows \mathbb{R}^2$, equipped with the polarization P of (7). We next verify this claim in detail.

For any groupoid G, we may define a convolution product of two functions f, g on G by

$$(f * g)(z) := \int_{\{xy=z\}} f(x)g(y)\, d\lambda_z(x, y),$$

where λ_z is some suitable measure on the set $\{\, (x, y) \in G_2 : xy = z \,\}$. For the symplectic groupoid $G = M \times \overline{M}$, this simplifies to:

$$(f * g)(x, y) := \int_M f(x, t)g(t, y)\, d\lambda(t),$$

where $\lambda = \lambda_{x,y}$ is (a multiple of) the Liouville volume on M.

When G has a real polarization with a regular leaf space, the polarized sections are represented (locally) by functions covariantly constant along the leaves; in general their convolution products will fail to be covariantly constant. To obtain a new polarized section, one must average over the leaves (by integration); by projection, one recovers a *twisted product* of functions on the leaf space.

In the case of $G = \mathbb{R}^2 \times \overline{\mathbb{R}}^2$, the diagonal $\Delta = \{\, (x', x''; x', x'') \in G : (x', x'') \in \mathbb{R}^2 \,\}$ is a Lagrangian submanifold of G which is transverse to the leaves $q_1 = \mathrm{const}_1$, $q_2 = \mathrm{const}_2$ of the polarization P; thus a polarized section is determined by its values on Δ, and we may identify Δ with the leaf space G/P.

Let us now regard Eq. (6) as a linear change of variables; we wish to rewrite the groupoid product

$$(x', x'', y', y'') = (x', x'', t', t'') \cdot (t', t'', y', y'') \tag{9}$$

in a more suitable form; we substitute

$$\begin{aligned}
&q = \tfrac{1}{2}(x' + y'), && q' = \tfrac{1}{2}(x' + t'), && q'' = \tfrac{1}{2}(t' + y'); \\
&p = \tfrac{1}{2}(x'' + y''), && p' = \tfrac{1}{2}(x'' + t''), && p'' = \tfrac{1}{2}(t'' + y''); \\
&\xi = x'' - y'', && \xi' = x'' - t'', && \xi'' = t'' - y''; \\
&\eta = y' - x', && \eta' = t' - x', && \eta'' = y' - t'.
\end{aligned} \tag{10}$$

Now Eq. (9) takes the form

$$(q, p, \xi, \eta) = (q', p', \xi', \eta') \cdot (q'', p'', \xi'', \eta''), \tag{11}$$

determined by the four relations

$$\begin{aligned}
q &= \tfrac{1}{2}(q' + q'') - \tfrac{1}{4}(\eta' - \eta''), \quad \xi = 2(p' - p''), \\
p &= \tfrac{1}{2}(p' + p'') + \tfrac{1}{4}(\xi' - \xi''), \quad \eta = 2(q'' - q').
\end{aligned} \tag{12}$$

Now $\alpha(q,p,\xi,\eta) = (q - \frac{1}{2}\eta, p + \frac{1}{2}\xi)$ and $\beta(q,p,\xi,\eta) = (q + \frac{1}{2}\eta, p - \frac{1}{2}\xi)$ in the new coordinates, so the partial product (11) is subject to the compatibility conditions:

$$q' + \tfrac{1}{2}\eta' = q'' - \tfrac{1}{2}\eta'', \qquad p' - \tfrac{1}{2}\xi' = p'' + \tfrac{1}{2}\xi''. \tag{13}$$

We may interpret the coordinate change (10) thus: the parameters (q,p) label points of the leaf space G/P (since Δ is the submanifold $\xi = \eta = 0$), while (ξ,η) are parameters along the leaves. Since $(x',x'',y',y'') = (q - \frac{1}{2}\eta, p + \frac{1}{2}\xi, q + \frac{1}{2}\eta, p - \frac{1}{2}\xi)$, each leaf carries a natural volume form $2^{-4}\, d\eta \wedge d\xi$.

The pointwise product of two functions on G representing sections in $\Gamma_P(L \otimes Q^P)$ is

$$(2\pi\hbar)^{-2} g(q',p') e^{-ip'\eta'/\hbar}\, h(q'',p'')\, e^{-ip''\eta''/\hbar},$$

which is of the form

$$f(q,p,q',p',q'',p'')\, e^{-ip\eta/\hbar}$$

with

$$\begin{aligned} f(q,p,q',p',q'',p'') &= (2\pi\hbar)^{-2} g(q',p')\, h(q'',p'')\, \exp\!\left(-\frac{i}{\hbar}(p'\eta' + +p''\eta'' - p\eta)\right) \\ &= (2\pi\hbar)^{-2} g(q',p')\, h(q'',p'') \\ &\qquad \times \exp\!\left(-\frac{2i}{\hbar}(pq' - qp' + p'q'' - q'p'' + p''q - q''p)\right), \end{aligned} \tag{14}$$

since the relations Eqs. (12) and (13) imply

$$\eta = 2(q'' - q'), \quad \eta' = 2(q'' - q), \quad \eta'' = 2(q - q').$$

The twisted product $(g \times h)(q,p)$ is thus an integral of the expression (14) over: (a) the parameter region $(t',t'') \in \mathbb{R}^2$ determined by (13) which underlies the (prequantized) convolution product, and (b) the leaf of P through the point $(q,p) \in \Delta$, which is parametrized by $(q - \frac{1}{2}\eta, p + \frac{1}{2}\xi)$. Since

$$\begin{aligned} dt' \wedge dt'' \wedge (2^{-4}\, d\eta \wedge d\xi) &= \tfrac{1}{4} d(q' + q'') \wedge d(p' + p'') \wedge d(q'' - q') \wedge d(p' - p'') \\ &= dq' \wedge dq'' \wedge dp' \wedge dp'', \end{aligned}$$

we finally arrive at

$$\begin{aligned} (g \times h)(q,p) = (2\pi\hbar)^{-2} \int_{\mathbb{R}^4} & g(q',p')\, h(q'',p'') \\ & \times \exp\!\left(-\frac{2i}{\hbar}(pq' - qp' + p'q'' - q'p'' + p''q - q''p)\right) dq'\, dq''\, dp'\, dp'', \end{aligned}$$

which is the Moyal product [8, 10] of the symbols g and h. Thus the geometric quantization data (G, Ω, P) indeed incorporate the essentials of Moyal quantization in the linear case.

6 The Daubechies–Grossmann transform

Some years ago, Daubechies and Grossmann [15] discovered an integral transformation similar to the well-known one of Bargmann and Segal [25], but more directly adapted to quantization in that it intertwined classical observables (i.e., functions on phase space) directly with the coherent-state transitions of the corresponding quantized operators. They noted that the new transformation differed from Bargmann's in two respects: the transformed operators acted on a space with double the usual number of variables, and that some mixing of the variables had occurred. We now show how these phenomena may be simply elucidated in terms of the symplectic groupoid $\mathbb{R}^{2n} \times \overline{\mathbb{R}}^{2n} \rightrightarrows \mathbb{R}^{2n}$.

The idea is to pair the "Moyal polarization" P of Eq. (7) with a certain complex polarization R. Specifically, write $z = x' + ix''$, $w = y' + iy''$, and take

$$R = \operatorname{span}\left\{ \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial w} \right\}.$$

Then $\overline{P} \cap R = 0$, and K^R is spanned by $dz \wedge d\bar{w}$. From Eq. (5), $\Omega = \frac{i}{2}(dz \wedge d\bar{z} - dw \wedge d\bar{w})$, and the symplectic potential vanishing on R is

$$\Theta_R = -\tfrac{i}{2}(\bar{z}\, dz + w\, d\bar{w}).$$

Elements of $\Gamma_R L$ are of the form $h(z, \bar{w})\, r_0$, where h is holomorphic in $(z, \bar{w})$ and $\nabla_X r_0 = -(i/\hbar)\, \Theta_R(X) r_0$. Thus $r_0 = \psi_0 t_0$ with $d\psi_0/\psi_0 = (i/\hbar)(\Theta_P - \Theta_R)$. It is convenient to use the complex notations on the symplectic groupoid $u = q_1 + ip_2$, $v = q_2 + ip_1$, and to write $d^2u = dq_1\, dp_2$, etc. We thus get

$$\psi_0 = C \exp\{-(z\bar{z} + w\bar{w} + \bar{u}v - u\bar{v})/4\hbar\}.$$

One finds that $\langle \sqrt{dq_1 \wedge dp_2}, \sqrt{dz \wedge d\bar{w}} \rangle = 1$, so if $\gamma = h(z, \bar{w}) r_0 \otimes \sqrt{dz \wedge d\bar{w}}$, then

$$\begin{aligned}
\langle \beta, \gamma \rangle &= C \int \overline{g(u)} h(z, \bar{w})\, e^{-(z\bar{z} + w\bar{w} + \bar{u}v - u\bar{v})/4\hbar}\, d^2u\, d^2v \\
&= C \int \overline{g(u)} h(u + \tfrac{1}{2}v, \bar{u} - \tfrac{1}{2}\bar{v})\, e^{-(2u\bar{u} + \bar{u}v - u\bar{v} + \frac{1}{2}v\bar{v})/4\hbar}\, d^2u\, d^2v \\
&= \langle g, Sh \rangle_{L^2(\mathbb{R}^2)},
\end{aligned}$$

with

$$\begin{aligned}
Sh(u) &= C \int h(u + \tfrac{1}{2}v, \bar{u} - \tfrac{1}{2}\bar{v})\, e^{-(2u\bar{u} + \bar{u}v - u\bar{v} + \frac{1}{2}v\bar{v})/4\hbar}\, d^2v \\
&= \int K(\bar{z}, w; u) h(z, \bar{w})\, e^{-(z\bar{z} + w\bar{w})/2\hbar}\, d^2z\, d^2w,
\end{aligned}$$

where K is computed from the reproducing kernel property of Gaussian integrals:

$$\begin{aligned}
K(\bar{z}, w; u) &= \frac{C}{(2\pi\hbar)^2} \int \exp\left(\frac{\bar{z}(u + \frac{1}{2}v) + w(\bar{u} - \frac{1}{2}\bar{v})}{2\hbar} - \frac{2u\bar{u} + \bar{u}v - u\bar{v} + \frac{1}{2}v\bar{v}}{4\hbar} \right) d^2v \\
&= \frac{2C}{\pi\hbar} \exp\left(\frac{-2u\bar{u} + 2\bar{z}u + 2w\bar{u} - \bar{z}w}{2\hbar} \right).
\end{aligned}$$

If $e_{\bar{a},b}(z,\bar{w}) = \exp\{(\bar{a}z+b\bar{w})/2\hbar\}$ denote coherent-state vectors in $(z,\bar{w})$-space, one checks that $\|Se_{\bar{a},b}\| = 2C(2\pi\hbar)^{3/2}\|e_{\bar{a},b}\|$, so the normalization $C = \frac{1}{2}(2\pi\hbar)^{-3/2}$ makes S unitary. Moreover, S^{-1} is given by the conjugate kernel:

$$Q(z,\bar{w};u) = \frac{2}{(2\pi\hbar)^{5/2}} \exp\left(\frac{-2u\bar{u} + 2z\bar{u} + 2\bar{w}u - z\bar{w}}{2\hbar}\right).$$

Apart from Gaussian-integral conventions, this is precisely the kernel of the Daubechies–Grossmann transformation which takes a Weyl symbol g to the coherent-state transition matrix:

$$\langle w \mid Q_g \mid z\rangle = \int Q(z,\bar{w};u)\, g(u)\, d^2u.$$

Thus the symplectic groupoid picture shows that this arises from the pairing of the polarizations P and R.

The comparison with the double Bargmann transformation, explored in [15], may now be clarified. The double Bargmann transformation is obtained from the pairing of the polarizations F and R; the "mixing" of variables noted in [15] comes from the combination of this pairing with that of Sec. 4.

7 Iteration of pairings

In [26] we proved, by a lengthy functional-analytic argument, that the Weyl transform is of finite order 24. We now show that this comes in fact from a simple identity among linear symplectomorphisms of the groupoid.

Let us write $q_1^{(0)} = x'$, $q_2^{(0)} = y'$, $p_1^{(0)} = x''$, $p_2^{(0)} = -y''$, and considering the symplectic linear map Ψ given by:

$$q_1^{(1)} = \frac{q_1^{(0)} + q_2^{(0)}}{\sqrt{2}}, \quad q_2^{(1)} = \frac{p_1^{(0)} - p_2^{(0)}}{\sqrt{2}}, \quad p_1^{(1)} = \frac{p_1^{(0)} + p_2^{(0)}}{\sqrt{2}}, \quad p_2^{(1)} = \frac{q_2^{(0)} - q_1^{(0)}}{\sqrt{2}}, \tag{15}$$

which is related to Eq. (6) by $p_2 \mapsto q_2$, $q_2 \mapsto -p_2$ and a rescaling by $\sqrt{2}$ factors. The pairing of the polarizations $F^{(j)} = \operatorname{span}\{\partial/\partial p_1^{(j)}, \partial/\partial p_2^{(j)}\}$ $(j = 0, 1)$ yields the unitary transformation of operator kernels:

$$Wg(q_1^{(0)}, q_2^{(0)}) = \frac{1}{2\pi\hbar} \int g\left(\frac{q_1^{(0)} + q_2^{(0)}}{\sqrt{2}}, t\right) e^{it(q_1^{(0)} - q_2^{(0)})/\sqrt{2}\hbar}\, dt,$$

which is essentially the Weyl transformation: compare Eq. (8).

It should be noted that W maps symbols g on position-momentum space to kernels Wg in a doubled position space, and by iterating W we implicitly identify these two interpretations of phase space. It is with respect to this identification that we establish the periodicity of the Weyl transform.

After three iterations of (15), the variables decouple in two pairs:

$$q_1^{(3)} = \frac{q_1^{(0)} + p_1^{(0)}}{\sqrt{2}}, \quad p_1^{(3)} = \frac{-q_1^{(0)} + p_1^{(0)}}{\sqrt{2}}, \quad q_2^{(3)} = \frac{q_2^{(0)} + p_2^{(0)}}{\sqrt{2}}, \quad p_2^{(3)} = \frac{-q_2^{(0)} + p_2^{(0)}}{\sqrt{2}},$$

and Ψ^6 becomes simply:

$$q_j^{(6)} = p_j^{(0)}, \quad p_j^{(6)} = -q_j^{(0)}, \qquad (j = 1, 2),$$

which is a complex structure on $\mathbb{R}^4$. The pairing of $F^{(0)}$ and $F^{(6)}$ yields the (inverse) Fourier transformation in the variables $(q_1^{(0)}, q_2^{(0)})$.

The Fourier transformation on $L^2(\mathbb{R}^n)$ is the image, under the metaplectic representation of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, of the complex structure $q \mapsto p$, $p \mapsto -q$ acting on Darboux coordinates on $\mathbb{R}^{2n}$. Now the symplectic group acts transitively on the set of real polarizations of $\mathbb{R}^{2n}$, and the unitary representation of the symplectic group given by pairing real polarizations is precisely the metaplectic representation. To be fully explicit, this is the projective representation of the symplectic group [3, 27–29], which is defined up to a phase factor; there ensues a $\mathrm{U}(1)$-valued group cocycle. The restriction of this cocycle to any given unitary subgroup can be made trivial. [The more customary choice of a (± 1)-valued cocycle [30] is "double-valued" even on unitary subgroups.] Now the commutant of the complex structure Ψ^6 is a unitary subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, containing Ψ. In selecting positive constants in formulas such as (8), we are following a consistent procedure. Thus the result of [26] is now seen to be the metaplectic image of the elementary geometric fact that Ψ^6 is a complex structure on the symplectic groupoid $\mathbb{R}^2 \times \overline{\mathbb{R}}^2$, and thus Ψ^{24} is the identity map.

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