The Moyal representation of quantum mechanics and special function theory

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Abstract

It is shown that the phase-space formulation of quantum mechanics is a rich source of special function identities. The Moyal formalism is reviewed for two phase spaces: the real plane and the sphere; and this is used to derive identities for Airy, Laguerre, Kummer and theta functions and for $\mathrm{SU}(2)$ rotation matrix elements, several of which are new.

1 Introduction

Some forty years ago, Moyal [1] proposed an attractive description of quantum mechanical systems, taking place in phase space, i.e., the same arena as classical mechanics. Moyal noticed that Wigner's recipe [1] associating a function on phase space (the so-called Wigner distribution) to a density operator on Hilbert space was essentially the inverse of the celebrated Weyl correspondence rule [3]. Thus the way was opened to represent Quantum Mechanics of spinless particles as a statistical theory in phase space.

After a long period of neglect, Moyal's formulation nowadays finds a wide range of application [4]. Also, mathematically very similar formalisms sprang up in apparently remote areas, such as signal theory [5], photometry, and (via the use of Gabor functions) in neurophysiology [6]. Recently, two of the present authors extended Moyal's approach to cover spinning particles as well [7].

The purpose of this article is to outline a 'mathematical application' of Moyal's theory, to wit, the derivation of properties of special functions. Such an endeavour has little precedent: indeed, we know only of Peetre's paper [8] as a forerunner. More recently, there has been some work about symbolic calculus on the Heisenberg group, which bears a close relationship with our work [9]. Nevertheless, there the emphasis was put on the functional-analytic side and few explicit formulae were given. We write here, rather, for those who are fond of beautiful and surprising equalities. They will not be new for the most part, as Peetre's formulae for Laguerre polynomials of course were not. What is new is the manner of their derivation. We hope to convince the reader that Moyal's theory is indeed a rich mine for special function theory.

The approach sketched in this paper would seem to go against the grain, as the tide has run for many years now to putting order in the bewildering chaos of special functions, reinterpreting them as matrix elements of unitary irreducible representations of Lie groups [10]. But in fact, at a deeper level, this is not so. If we substitute another suitable group for the Heisenberg group in the 'quantization' procedure, we get again special function formulae after our fashion: and we do this for SU(2) in Section 4 of the paper. In fact, Moyal's formulation of Quantum Mechanics is, mathematically speaking, a branch of Harmonic Analysis under a somewhat novel form, in much the same way as conventional Quantum Mechanics mirrors the theory of unitary representations of Lie groups.

In order to avoid cumbersome arguments, we shall remain on a formal level and shall neglect most questions of convergence. We can put it in no better way than Peetre's [8]: "In most instances it is obvious how to put things an a more rigorous status, but certainly then most of the present simplicity will be hidden. . . "

Section 2 of the paper contains a resumé of the Moyal representation of Quantum Mechanics in the simplest ordinary phase space $\mathbb{R}^2$. Our presentation is based on the Grossmann–Royer reflection operators [11]. They are but the simplest instance of a *Stratonovich–Weyl quantizer*, which is the bridge between the Moyal representation and the usual brand of harmonic analysis (in this case, Fourier analysis on the Heisenberg group).

In Section 3, two basic tricks are introduced: Wigner's 'transfer formula' and the 'functional calculus' method. They are employed to (re)derive properties of Airy functions, Hermite and Laguerre polynomials and Kummer's hypergeometric function. When one seeks to invert the functional calculus, three interesting families of polynomials appear. Then we exploit the connection between Moyal Quantum Mechanics and the holograph transform of signal theory. Using the Poisson formula, a discretization of the holograph transform identities, known as the radial pixel identity, is obtained. In this way, we derive several new identities for Jacobian theta-null values.

Section 4 introduces the Moyal representation for spin and exploits it to give a new expression for the SU(2) representation matrix elements, in terms of canonical coordinates of the first kind on the group manifold.

2 Moyal quantum mechanics on $\mathbb{R}^2$

The Weyl correspondence between symbols (i.e., complex functions or distributions on phase space) and standard quantum-mechanical operators in modernly stated by means of the Grossmann–Royer reflection operators [11]. Namely, to each point $u := (q, p)$ of phase space we associate an operator $\Pi(u)$ by:

$$\Pi(u)\Psi(\zeta) := \exp\Big[\frac{2i}{\hbar} p(\zeta - q)\Big]\Psi(2q - \zeta) \tag{1a}$$

in configuration space, or

$$\Pi'(u)\Phi(\xi) := \exp\Big[\frac{2i}{\hbar} q(p - \xi)\Big]\Phi(2p - \xi) \tag{1b}$$

in momentum space. In (1) $\hbar$ denotes Planck's constant and Ψ, Φ are wavefunctions belonging respectively to the Hilbert spaces $L^2(\mathbb{R}, d\zeta) =: \mathcal{H}_\zeta$ and $L^2(\mathbb{R}, d\xi) =: \mathcal{H}_\xi$. The operators Π and Π'

are intertwined by the Fourier transform $\mathcal{F}$: $\Pi' = \mathcal{F}\,\Pi\,\mathcal{F}^{-1}$, where:

$$\mathcal{F}\Psi(\xi) := \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} \exp\Bigl[-\frac{i}{\hbar}\,\xi\zeta\Bigr]\Psi(\zeta)\,d\zeta$$

connects position and momentum wavefunctions.

The operators Π, Π' are unitary and self-adjoint. An easy formal calculation gives the important formulas:

$$\begin{aligned} \operatorname{tr}\Pi(u) &= \tfrac{1}{2}, \\ \operatorname{tr}\Pi(u)\Pi(v) &= \frac{\pi\hbar}{2}\delta(u-v), \\ \operatorname{tr}\Pi(u)\Pi(v)\Pi(w) &=: \frac{1}{8}L(u,v,w) := \frac{1}{2}\exp\Bigl[\frac{2i}{\hbar}\bigl(\sigma(u,v)+\sigma(v,w)+\sigma(w,u)\bigr)\Bigr], \end{aligned} \tag{2}$$

where σ is the standard symplectic form: namely, if $u = (q,p)$ and $v = (q',p')$, then $\sigma(u,v) := qp' - q'p$.

A symbol f is mapped in a one-to-one linear way to an operator W_f:

$$W_f := \frac{1}{\pi\hbar}\int_{\mathbb{R}^2} f(u)\Pi(u)\,du \tag{3}$$

on the Hilbert space $\mathcal{H}_\zeta$, say.

From (2) we infer that:

(i) By using the same family of operators Π, formula (3) may be inverted:

$$f(u) = 2\operatorname{tr} W_f\Pi(u). \tag{4}$$

If A is an operator on $\mathcal{H}_\zeta$, we shall write $W_A(u) := 2\operatorname{tr} A\Pi(u)$, so $W_{W_A} = A$ and $W_{W_f} = f$.

(ii) The constant function 1 corresponds to the identity operator.

(iii) The product of two operators $W_f W_g$ is equal to $W_{f\times g}$ where the *twisted (quantum, Moyal) product* of two symbols f, g is given by

$$f \times g(u) = \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(v)\,g(w)\,L(u,v,w)\,dv\,dw \tag{5}$$

with the property $\int f \times g(u)\,du = \int f(u)\,g(u)\,du$.

Clearly, the whole procedure works the same using Π' on $\mathcal{H}_\xi$ instead.

Once one has (5), one can forget about the Hilbert spaces $\mathcal{H}_\zeta$, $\mathcal{H}_\xi$ and their operatorial theory, and work exclusively with the symbols. This is, in short, Moyal Quantum Mechanics.

We shall find it convenient to use Dirac's notation $|\Psi\rangle$ for wavefunctions $\Psi(\zeta)$ seen as vectors of $\mathcal{H}_\zeta$. The inner product is written $\langle\Psi_1 \mid \Psi_2\rangle_{\mathcal{H}_\zeta} = \int_{\mathbb{R}} \overline{\Psi_1(\zeta)}\,\Psi_2(\zeta)\,d\zeta$. We suppress the subscript $\mathcal{H}_\zeta$ when there is no risk of confusion. Consider the rank one operator $|\Psi_1\rangle\langle\Psi_2|$ associated to a couple Ψ_1, Ψ_2 of wavefunctions:

$$\bigl(|\Psi_1\rangle\langle\Psi_2|\bigr)\,\Psi(\zeta) = \langle\Psi_2 \mid \Psi\rangle\,\Psi_1(\zeta).$$

Then:

$$W_{|\Psi_1\rangle\langle\Psi_2|}(u) = 2\langle\Psi_2 \mid \Pi(u)\Psi_1\rangle = 2\int_{\mathbb{R}} \overline{\Psi_2(q+y)}\,\Psi_1(q-y)\exp\Big[\frac{2i}{\hbar}py\Big]\,dy. \tag{6a}$$

An analogous formula can be worked out in momentum space. One finds

$$W_{|\Phi_1\rangle\langle\Phi_2|}(u) = 2\int_{\mathbb{R}} \overline{\Phi_2(p+y)}\,\Phi_1(p-y)\exp\Big[-\frac{2i}{\hbar}qy\Big]\,dy. \tag{6b}$$

When $\Psi_1 = \Psi_2$, the previous expressions are, but for a constant factor, the famous Wigner distributions [1]. It is however advisable, as argued by Dahl [12] on physical grounds, to treat *states* (i.e., the symbols corresponding to projectors $|\Psi\rangle\langle\Psi|$) and *transitions* (i.e., the symbols of operators of the form $|\Psi_1\rangle\langle\Psi_2|$ with $\Psi_1 \neq \Psi_2$) on the same footing. Note that $W_{|\Psi_2\rangle\langle\Psi_1|} = \overline{W}_{|\Psi_1\rangle\langle\Psi_2|}$. Because of (2) there holds in all generality:

$$\langle\Psi_2 \mid A\Psi_1\rangle = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} W_A(u)\,W_{|\Psi_1\rangle\langle\Psi_2|}(u)\,du.$$

This is the formula which gives the aspect of a statistical theory in phase space to Moyal Quantum Mechanics. We denote by $\overline{\mathcal{H}_\zeta}$ the same space $\mathcal{H}_\zeta$, but with the multiplication law $(\alpha, \Psi) \mapsto \bar{\alpha}\Psi$ for $\alpha \in \mathbb{C}$ and the inner product $\langle\Psi_1 \mid \Psi_2\rangle_{\overline{\mathcal{H}_\zeta}} := \overline{\langle\Psi_1 \mid \Psi_2\rangle}_{\mathcal{H}_\zeta}$. Then $\Psi_1 \oplus \Psi_2 \mapsto W_{|\Psi_1\rangle\langle\Psi_2|}$ extends to a unitary isomorphism of the Hilbert space $\mathcal{H}_\zeta \oplus \overline{\mathcal{H}_\zeta}$ onto $L^2(\mathbb{R}^2, (2\pi\hbar)^{-1}du)$.

In practice, states are obtained, both in conventional and Moyal Quantum Mechanics, as eigenstates of given observables. Let us then look at dynamics in Moyal's formulation. Suppose that H is a real symbol such that W_H is selfadjoint. The basic equation:

$$i\hbar\frac{\partial\Xi_H}{\partial t}(u;t) = H \times \Xi_H(u;t) \tag{7}$$

simply translates into the Moyal language the semigroup differential equation for the unitary operator $U_H(t) := \exp[-(i/\hbar)W_H t]$. Here $U_H = W_{\Xi_H}$. We shall call Ξ_H the *Moyal propagator* or *evolution function* corresponding to the Hamiltonian H.

The formula (see Appendix B in the book by Taylor in [9]):

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} U_H(t)e^{itE/\hbar}\,dt = P_H(dE),$$

for the projection-valued measure P_H ('resolution of the identity') corresponding to the selfadjoint operator W_H by the spectral theorem, translates into:

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \Xi_H(u;t)e^{itE/\hbar}\,dt = W_{P_H(dE)} =: \Gamma_H(u;dE). \tag{8}$$

Here Γ_H is a measure on the real line with distributional values in general. In all cases we shall meet, this measure is either discrete or absolutely continuous with respect to Lebesgue measure; so we simply write $\Gamma_H(u;dE) = \Gamma_H(u;E)\,dE$.

The corresponding 'eigenvalue equation' is clearly:

$$H \times \Gamma_H(u;E) = \Gamma_H(u;E) \times H = E\,\Gamma_H(u;E) \tag{9}$$

and the *spectrum* $\operatorname{sp} H$ of W_H is the support of $P_H(dE)$, or equivalently, the support of $\Gamma_H(u;E)$ in the E-variable. (For details of rigour on this matter, see Appendix A of [13]).

The projection-valued measure P_H is the cornerstone of the functional calculus for the selfadjoint operator W_H. We can now analogously build a *twisted functional calculus* with the symbols, with (for our purposes) an all-important difference: its elements are *concrete functions* (or distributions) in phase space. Some important elements of a functional calculus are:

(1) The aforesaid *evolution function*:

$$\Xi_H(u;t) = \int_{\operatorname{sp} H} \Gamma_H(u;E)\, e^{-itE/\hbar}\, dE. \tag{10}$$

(2) The *resolvent function*:

$$R_H(u;\lambda) = \int_{\operatorname{sp} H} \frac{\Gamma_H(u;E)}{E-\lambda}\, dE, \tag{11}$$

for $\lambda \in \mathbb{C}$, $\lambda \notin \operatorname{sp} H$, which verifies $R_H(u;\lambda) \times (H-\lambda) = 1$.

(3) The *twisted powers*:

$$H^{\times n}(u) := H \times \cdots \times H(u) = \int_{\operatorname{sp} H} E^n\, \Gamma_H(u;E)\, dE. \tag{12}$$

Now we reveal our strategy: one can try to solve equations (7) or (9) directly, or one can obtain Ξ_H, Γ_H as the transforms (4), through use of Green functions and Schrödinger wavefunctions of standard Quantum Mechanics. Equating the results of both procedures, many interesting relationships emerge. A second device is direct use of the twisted functional calculus.

Equations (7) and (9) are obviously very difficult to solve in general (this has deterred potential users of Moyal formalism for quantum physics). There is however an exception, to wit, when the symbol H is some real quadratic polynomial in q, p. In that case the corresponding operators are selfadjoint (see Chapter 1 of the book by Taylor in [9], and [13]), and moreover (7), (9) degenerate into linear second-order differential equations which can, in fact, be easier to solve than the corresponding equations in the conventional formalism.

This quantization scheme can be generalized to phase spaces other than $\mathbb{R}^2$. The starting point is to realize that $\mathbb{R}^2$, with its natural symplectic structure, may be considered as a coadjoint orbit of the 3-dimensional Heisenberg group $\mathbb{H}_3$; and that the Grossmann–Royer operators $\Pi(u)$ transform covariantly under the coadjoint action of $\mathbb{H}_3$ on $\mathbb{R}^2$ and conjugation by the associated unitary irreducible representation U of $\mathbb{H}_3$ on the Hilbert space $\mathcal{H}_\zeta$. Let f be a function defined on the underlying manifold of $\mathbb{H}_3$ and set, for $\lambda \in \mathbb{R}^*$ indexing the set of generic coadjoint orbits or the nontrivial unitary irreducible representations of $\mathbb{H}_3$:

$$\hat{f}(\lambda) := \int_{\mathbb{H}_3} f(g)\, U^\lambda(g)\, dg,$$

where dg is the Haar measure on $\mathbb{H}_3$. This is the (operatorial) Fourier transform of f.

The abstract Plancherel formula:

$$f(0) = c \int_{-\infty}^{\infty} |\lambda|\ \operatorname{tr}[\hat{f}(\lambda)]\, d\lambda \quad \text{(for a suitable constant } c\text{)}$$

is well known. As indicated above, one has

$$U^\lambda(g)\,\Pi^\lambda(u)\,U^\lambda(g^{-1}) = \Pi^\lambda(g\cdot u)$$

where $g\cdot u$ denotes the coadjoint action and Π^λ is always given in the Schrödinger representation by (1a) with the formal identification $\lambda = \hbar$. Now we apply the inverse Weyl correspondence to the representation operators themselves:

$$E(g,u;\lambda) := 2\operatorname{tr}\bigl[U^\lambda(g)\,\Pi^\lambda(u)\bigr],$$

and think of E as a (scalar) Fourier kernel. Then it is clear from the foregoing that:

$$\tilde f(u;\lambda) := \int_{\mathbb{H}_3} E(g,u;\lambda)\,f(g)\,dg = W_{\hat f(\lambda)}(u).$$

By direct computation, E is seen to be simply the ordinary Fourier kernel in three dimensions: that is to say, the ordinary Plancherel theorem and the Plancherel theorem for the Heisenberg group are equivalent, as has been pointed out by Howe [14] and by Taylor [9] from slightly different points of view. The Weyl correspondence ferries back and forth between $\tilde f$ and $\hat f$.

The important fact is that one can duplicate this scheme for several other Lie groups. The seminal paper by Stratonovich [15] pointed this out for the group $\mathrm{SU}(2)$, and this approach to spin quantization has been explored in detail by two of us in [7]. In that general context the equivalent of E may be highly nontrivial. We shall then speak of a *Stratonovich–Weyl correspondence*.

Before we get down to business, we shall be mindful of Dahl's contention and take advantage of the remark following (6): we state the spectral theorem for Moyal Quantum Mechanics in a fairly general form.

(a) The solutions of the system:

$$\begin{aligned} H\times\Gamma_H(u;E,E') &= E\,\Gamma_H(u;E,E'),\\ \Gamma_H\times H(u;E,E') &= E'\,\Gamma_H(u;E,E'), \end{aligned} \tag{13}$$

span the space $L^2(\mathbb{R}^2,(2\pi\hbar)^{-1}\,du)$. The permissible values of E or E' give the spectrum. Omitting the u-variable for simplicity of notation, we have $\Gamma_H(E,E') = \overline{\Gamma}_H(E',E)$ and $\Gamma_H(E,E) = \Gamma_H(E)$ as defined in (8).

(b) The Γ_H can be appropriately normalized so that the following relations hold:

$$\Gamma_H(E,E')\times\Gamma_H(E'',E''') = \delta_{E',E''}\,\Gamma_H(E,E'''). \tag{14a}$$

(c) If we denote by $(\cdot\,|\,\cdot)$ the inner product on $L^2(\mathbb{R}^2,(2\pi\hbar)^{-1}du)$, then we have the orthogonality condition:

$$(\Gamma_H(E,E')\mid\Gamma_H(E'',E''')) = \delta_{E,E''}\,\delta_{E',E'''}\,. \tag{15}$$

Note that this implies:

$$\int_{\mathbb{R}^2}\Gamma_H(E,E')\,\frac{du}{2\pi\hbar} = \delta_{EE'}\,.$$

(d) The functional calculus gives in general:

$$g^{\times}(H) = \sum_{E \in \operatorname{sp} H} g(E)\, \Gamma_H(E) \tag{16}$$

with an obvious notation for the 'twisted' function g of the Hamiltonian H. The equations (10)–(12) are already instances of (16).

In particular:

$$\sum_{\operatorname{sp} H} \Gamma_H(E) = 1; \qquad \sum_{\operatorname{sp} H} E\, \Gamma_H(E) = H.$$

(e) In view of (a), (b) and (c), the following closure relations hold:

$$\sum_{E,E' \in \operatorname{sp} H} \Gamma_H(u; E, E')\, \Gamma_H(v; E', E) = 2\pi\hbar\, \delta(u - v),$$

$$\sum_{E,E',E'' \in \operatorname{sp} H} \Gamma_H(u; E, E')\, \Gamma_H(v; E', E'')\, \Gamma_H(w; E'', E) = L(u, v, w). \tag{17}$$

Of course, these are but (2) under another guise.

We have assumed in the notation that H has a simple pure point spectrum. Examples of continuous spectra will shortly appear. Then the transitions $\Gamma_H(E, E')$ are generalized functions outside the Hilbert space; formula (13) applies in a weak sense, and (14a) then reads:

$$\Gamma_H(E, E') \times \Gamma_H(E'', E''') = \delta(E' - E'')\, \Gamma(E, E'''). \tag{14b}$$

The new rendition of the other formulas (15), (16) and (17), wherein integrals substitute for the Fourier–Dirichlet series, is also straightforward.

3 Special functions

To reach our conclusions, we shall apply the foregoing theory to three illustrative examples, which are also of great physical significance: the free-fall Hamiltonian $H = (p^2/2m) + Fq$; the harmonic oscillator $H = (p^2/2m) + kq^2$ with $k > 0$; and the harmonic barrier (same expression with $k < 0$).

It is known [13, 16] that for any quadratic Hamiltonian $H = \frac{1}{2}u^t B u + c^t u$, where $u = \binom{q}{p}$, $u^t = (q, p)$, B is a nonsingular 2×2 symmetric matrix, and c is a constant vector, the solution of (7) is

$$\Xi_H(u; t) = \frac{2^n e^{-i\beta(t)/\hbar}}{\sqrt{\det(1 + M(t))}} \exp\Big[-\frac{i}{\hbar} g_H(u; t)\Big], \tag{18}$$

where $M(t)$ is the symplectic matrix $\exp(-JBt)$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

$$g_H(u; t) = (u + \tfrac{1}{2}a(t))^t J (M(t) - 1)(M(t) + 1)^{-1}(u + \tfrac{1}{2}a(t)) - u^t J a(t);$$

$$a(t) = \int_0^t M(s) Jc\, ds; \qquad \beta(t) = \frac{1}{2}\int_0^t \int_0^s c^t M(r - s) Jc\, dr\, ds.$$

By employing this result, one can get at the spectrum of H and, through (8), the 'resolution of the identity' $\Gamma_H(E)$, in a single stroke. The 'eigenvalue equations' (13) remain useful, however, to find the nondiagonal Γ_H.

3.1 Free-fall Hamiltonian and Airy functions

We take $F = m = 1$ to avoid encumbering our notation unnecessarily. Writing $H = \frac{1}{2}p^2 + q$, from (18) we obtain

$$\Xi_H(u;t) = \exp\left[-\frac{i}{\hbar}\left(Ht + \frac{t^3}{24}\right)\right]. \tag{19}$$

Recalling

$$\mathrm{Ai}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ivx + \tfrac{1}{3}iv^3)\,dv,$$

we get:

$$\Gamma_H(u;E) = \sqrt[3]{\frac{8}{\hbar^2}}\ \mathrm{Ai}\left(\sqrt[3]{\frac{8}{\hbar^2}}(H - E)\right). \tag{20}$$

The twisted product (5) has the following asymptotic expansion [17]:

$$f \times g \sim \sum_{|\alpha|=0}^{\infty}\left(\frac{i\hbar}{2}\right)^{|\alpha|}\frac{1}{\alpha!}\,\partial^\alpha f\,\hat{\partial}^\alpha g, \tag{21}$$

where

$$\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2; \quad |\alpha| := \alpha_1 + \alpha_2; \quad \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial q^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial p^{\alpha_2}}; \quad \hat{\partial}^\alpha = (-1)^{\alpha_2}\frac{\partial^{\alpha_1}}{\partial p^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial q^{\alpha_2}}.$$

This expansion is exact whenever one of the factors is a polynomial. Using this, and separating real and imaginary parts in (13), we get:

$$\begin{aligned}\left(\frac{p^2}{2} + q - \frac{\hbar^2}{8}\frac{\partial^2}{\partial q^2}\right)\Gamma_H(q,p;E,E') &= \frac{E+E'}{2}\,\Gamma_H(q,p;E,E'),\\ \left(-p\,\frac{\partial}{\partial q} + \frac{\partial}{\partial p}\right)\Gamma_H(q,p;E,E') &= -\frac{i}{\hbar}(E - E')\,\Gamma_H(q,p;E,E').\end{aligned}$$

In the new canonical variables (H, p), this reads:

$$\begin{aligned}\left[\frac{\hbar^2}{8}\frac{\partial^2}{\partial H^2} - \left(H - \frac{E+E'}{2}\right)\right]\Gamma_H(E,E') &= 0,\\ \frac{\partial\Gamma_H(E,E')}{\partial p} &= -\frac{i}{\hbar}(E - E')\,\Gamma_H(E,E'),\end{aligned}$$

whose (suitably normalized) regular solution is:

$$\Gamma_H(E,E') = \sqrt[3]{\frac{8}{\hbar^2}}\ \mathrm{Ai}\left(\sqrt[3]{\frac{8}{\hbar^2}}\left(H - \frac{E+E'}{2}\right)\right)e^{ip(E'-E)/\hbar}.$$

In particular, we recover (20).

Now the solution of the standard Schrödinger equation for the free-fall is again an Airy function! We have, in fact, the equation:

$$-\frac{\hbar^2}{2}\frac{\partial^2\Psi_{H,E}}{\partial q^2} + q\,\Psi_{H,E} = E\,\Psi_{H,E}$$

whose regular solution [normalized by $\langle\Psi_{H,E} \mid \Psi_{H,E'}\rangle = \delta(E - E')$] is

$$\Psi_{H,E} = \sqrt[3]{\frac{2}{\hbar^2}}\ \mathrm{Ai}\biggl(\sqrt[3]{\frac{2}{\hbar^2}}(q - E)\biggr).$$

Now we write down equation (6a) for this case and see what happens. After eliminating constants irrelevant to the matter at hand, we get

$$\mathrm{Ai}\bigl[\sqrt[3]{4}(a^2 + \tfrac{1}{2}(b + b'))\bigr] = \sqrt[3]{2}\, e^{i(b'-b)a} \int_{-\infty}^{\infty} \mathrm{Ai}(b' + c)\ \mathrm{Ai}(b - c)\, e^{2iac}\, dc. \tag{22a}$$

Moreover, from the general relation:

$$\Psi_{H,E}(q)\, \overline{\Psi}_{H,E'}(q) = \frac{1}{2\pi\hbar} \int_{\mathbb{R}} \Gamma_H(q, p; E, E')\, dp,$$

which follows from (6a), or working directly on (22a), we obtain

$$\mathrm{Ai}(b)\, \mathrm{Ai}(b') = \frac{1}{\pi\sqrt[3]{2}} \int_{-\infty}^{\infty} \mathrm{Ai}\bigl(\sqrt[3]{4}(a^2 + \tfrac{1}{2}(b + b'))\bigr) e^{ia(b-b')}\, da. \tag{22b}$$

These curious nonlinear relations among Airy functions seem to be new. Special cases have appeared in [18] and [19].

Many properties of Airy functions can be derived through (20) from the twisted functional calculus. We shall not go into that; instead we note only that the family of polynomials $G^1_n(x)$ defined by $G^1_n(H) := H^{\times n}$ (with $\hbar = 1$) obeys, by (21), the following recurrence relation:

$$G^1_n(x) = x\, G^1_{n-1}(x) - \frac{1}{8}\, \frac{dG^1_{n-1}}{dx^2}(x) \quad \text{for} \quad n \geqslant 1; \qquad G^1_0(x) = 1.$$

Then, from (12) and (20):

$$G^1_n(x) = \int (x - \tfrac{1}{2}y)^n\, \mathrm{Ai}(y)\, dy.$$

In particular, the moments of the Airy function are calculated to be

$$\mu_n(\mathrm{Ai}) = (-1)^n 2^n G^1_n(0).$$

The possibility of inverting the functional calculus is interesting. Note that $\Xi_H(u; t)$ is formally an exponential:

$$\Xi_H(u; t) = \sum_{n=0}^{\infty} \frac{(-i\hbar t)^n}{n!}\, H^{\times n}.$$

We write $\mathrm{Exp}(-iHt/\hbar) := \Xi_H(u; t)$. From (19), with $\hbar = 1$, we get:

$$\mathrm{Exp}\biggl[-it\Bigl(H - \frac{t^2}{24}\Bigr)\biggr] = e^{-itH}.$$

Formally,

$$e^{ax} = \mathrm{Exp}\biggl[a\Bigl(x + \frac{a^2}{24}\Bigr)\biggr] =: g_1(x; a) =: \sum_{n=0}^{\infty} T^{\times}_n(x)\, a^n.$$

We are treating g_1 as a generating function; from the differential equation for g_1:

$$\frac{\partial g_1}{\partial a} = (x + \tfrac{1}{8}a^2) \times g_1,$$

one gets the recurrence relation for the polynomials T_n:

$$nT_n(x) - x\,T_{n-1}(x) - \tfrac{1}{8}T_{n-3}(x) = 0 \quad \text{for} \quad n \geqslant 3;$$
$$T_0(x) = 1; \quad T_1(x) = x; \quad T_2(x) = \tfrac{1}{2}x^2.$$

If f denotes an ordinary function of H such that $f(H) = \sum_{n=0}^{\infty} a_n H^n$, it will follow that $f(H) = g^{\times}(H)$ with $g(H) = \sum_{n=0}^{\infty} a_n\, n!\, T_n(H)$.

Proceeding in the same way, we shall get two more families of polynomials respectively associated to the harmonic oscillator and the harmonic barrier Hamiltonians. The last one turns out to be quite interesting (see Subsection 3.3 below).

From (14b) and the analogues of (15, 17), there issue several integral formulas. We shall not bother to write them explicitly, except for:

$$\begin{aligned} \iiint \operatorname{Ai}(p^2 + 2q - a)\ \operatorname{Ai}(p'^2 + 2q' - a')\ \operatorname{Ai}(p''^2 + 2q'' - a'') \\ e^{i[p(a'-a'')+p'(a''-a)+p''(a-a')]}\, da\, da'\, da'' \\ = \frac{1}{8}\exp\bigl[2i(qp' - q'p + q'p'' - q''p' + q''p - qp'')\bigr]. \end{aligned}$$

3.2 Harmonic oscillator Hamiltonian and Laguerre functions

Writing $H = \frac{1}{2}(p^2 + q^2)$, we obtain from (18):

$$\Xi_H(u;t) = \sec\frac{t}{2}\exp\biggl(-\frac{2i}{\hbar}H\tan\frac{t}{2}\biggr). \tag{23}$$

The eigenvalue equations (13), on the other hand, turn out to be

$$\begin{aligned} \frac{p^2+q^2}{2} - \frac{\hbar^2}{8}\biggl(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\biggr)\Gamma_H(q,p;E,E') &= \frac{E+E'}{2}\,\Gamma_H(q,p;E,E'), \\ \biggl(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\biggr)\Gamma_H(q,p;E,E') &= -\frac{i}{\hbar}(E-E')\,\Gamma_H(q,p;E,E'). \end{aligned} \tag{24}$$

If we introduce the new canonical variables (H,τ), where $\tau = \arctan(p/q)$, the (appropriately normalized) regular solutions of (24) are:

$$f_{mn}(H,\tau) = 2(-1)^n\sqrt{\frac{n!}{m!}}\biggl(\frac{4H}{\hbar}\biggr)^{(m-n)/2} e^{-i\tau(m-n)} L_n^{m-n}\biggl(\frac{4H}{\hbar}\biggr)e^{-2H/\hbar} \tag{25a}$$

if $m \geqslant n$; $f_{mn} = \bar{f}_{nm}$ if $n < m$. Here L_n^{m-n} denotes the associated Laguerre polynomial of type $m-n$ and degree n, with $m,n \in \mathbb{N}$; the allowed values of E, E' are $(2m+1)\hbar/2$, $(2n+1)\hbar/2$.

In what follows we take $\hbar = 2$ for convenience. In terms of the original variables, (25a) then reads:

$$f_{mn}(q,p) = 2(-1)^n\sqrt{\frac{n!}{m!}}\,(q^2+p^2)^{(m-n)/2}e^{-i(m-n)\arctan(p/q)}L_n^{m-n}(q^2+p^2)\,e^{-(q^2+p^2)/2} \tag{25b}$$

for $m \geqslant n$, say.

From (16) one sees that the following equation holds:

$$\sec\frac{t}{2}\,e^{-iH\tan(t/2)} = \sum_{n=0}^{\infty} f_{nn}(H)e^{-i(2n+1)t}. \tag{26}$$

As observed by Peetre [8], this equation gives the generating function formula for the Laguerre polynomials.

A number of interesting consequences can be extracted from (26). If δ_0 denotes the Dirac measure concentrated at the origin of phase space, by taking limits in (26) as $t \longrightarrow \pi/2$, we get

$$\delta_0 = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n f_{nn}. \tag{27}$$

The functions f_{mn}, with $m \neq n$ in general, were introduced first by Groenewold [20] and by Bartlett and Moyal [21]. They have been rediscovered many times [16, 22], which surely is a measure of usefulness. All their basic properties, such as:

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} f_{mn}(q,p)\,dq\,dp = \delta_{mn}, \tag{28a}$$

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} f_{mn}(q,p)\,f_{rs}(q,p)\,dq\,dp = \delta_{nr}\,\delta_{ms}, \tag{28b}$$

$$\mathcal{F}(f_{mn}) = (-i)^{m+n}f_{mn}, \tag{28c}$$

follow easily from our apparatus. For the last one, for instance, note that $\mathcal{F}f_{mn}(q,p) = Ff_{mn}(-p,q)$, where $Ff := f\times\delta_0$ is the 'symplectic Fourier transform'. Using (27) we obtain $F(f_{mn}) = (-1)^n f_{mn}$ and $\mathcal{F}(f_{mn}) = (-i)^{m+n} f_{mn}$.

The orthonormal basis $\{f_{mn}\}_{m,n\in\mathbb{N}}$ has theoretical importance. It has been employed by two of us [23] to give a rather complete account of duality and functional-analytic properties of twisted product theory.

The f_{mn} could be employed to derive orthogonality and other properties of the associated Laguerre polynomials, in much the same way as Peetre exploited the 'diagonal' f_{nn} to reobtain properties of the ordinary Laguerre polynomials. We shall not go into that straightforward matter. Instead, we note that they are related to Hermite polynomials. It is well known that the solutions of the harmonic oscillator problem in the conventional formulation are:

$$\Psi_{H,E_n}(q) = (2\pi)^{-1/4}(2^n n!)^{-1/2}H_n\Big(\frac{q}{\sqrt{2}}\Big)e^{-q^2/4}$$

with $E_n = 2n+1$ (recall that $\hbar = 2$).

Use of (6a) gives then:

$$\int_{\mathbb{R}} e^{-x^2/2} H_n(x-\bar{z})\, H_m(x+z)\, dx = 2^m \sqrt{\pi}\, n!\, z^{m-n} L_n^{m-n}(2|z|^2) \quad (m \geqslant n),$$

which is a well-known integral [24, 7.377].

Now, the f_{mn} themselves are eigenfunctions of an harmonic-oscillator-type Schrödinger equation in two variables (see (24) and also (28c)). Accordingly,

$$f_{mn}(q,p) = \sum_{k+l=m+n} c_{mn}^{kl} h_k(q)\, h_l(p),$$

where $h_k(q) := (2^{k-1}k!)^{-1/2} H_k(q) e^{-q^2/2}$, and we have computed that

$$c_{mn}^{kl} = 2^{(m-n)/2} i^{2m+l} \binom{m+n}{l}^{1/2} \binom{m+n}{m}^{-1/2} P_m^{l-m,k-m}(0),$$

where $P_m^{l-m,k-m}$ denotes the usual Jacobi polynomial.

Finally, from (17), we obtain

$$\sum_{m,n,k=0}^{\infty} f_{mn}(q_1,p_1)\, f_{nk}(q_2,p_2)\, f_{km}(q_3,p_3) = 4\exp\bigl[i(q_1p_2 - q_2p_1 + q_2p_3 - q_3p_2 + q_3p_1 - q_1p_3)\bigr]$$

which no doubt holds in quite a strong sense. Surprisingly, no one seems to have remarked this before.

Now we turn to the functional calculus, with makes use only of the diagonal elements. Our primary interest is to find an explicit formula for the twisted resolvent function for the harmonic oscillator. Here we cannot follow Peetre, as he uses $(-1)^n f_{nn}$ for his diagonal elements, so in fact he essentially calculates the *Hankel transform* of our resolvent and other twisted functions. (This discrepancy can be traced back to the use of the twisted convolution in [8] instead of the twisted product.)

For calculating the resolvent we have at our disposal at least two different methods. Besides

$$R_H(q,p;\lambda) = e^{-H} \sum_{n=0}^{\infty} \frac{(-1)^n L_n(2H)}{n + \frac{1}{2}(1-\lambda)}, \tag{29}$$

which is (11), we note that the following Laplace-transform formula is available:

$$R_H(q,p;\lambda) = \int_0^{\infty} \frac{\exp(-H\tanh\beta + \lambda\beta)}{\cosh\beta}\, d\beta, \tag{30}$$

the integral being valid for $\operatorname{Re}\lambda < 1$. From the last formula, by a change of variables, we get

$$\begin{aligned} R_H(q,p;\lambda) &= \int_0^1 e^{-Hx}(1+x)^{(\lambda-1)/2}(1-x)^{(-\lambda-1)/2}\, dx \\ &= \int_0^{\pi/2} e^{-H\cos\theta}\Bigl(\tan\frac{\theta}{2}\Bigr)^{-\lambda} d\theta = \frac{2}{1-\lambda}\Phi_1\Bigl(1, \frac{1-\lambda}{2}, \frac{3-\lambda}{2}; -1, -H\Bigr), \end{aligned}$$

where Φ_1 denotes a confluent hypergeometric function in two variables [25]. From (29) and (30) we extract

$$\sum_{n=0}^{\infty}(-1)^n \frac{L_n(x)}{n+a} = \frac{e^{x/2}}{a}\,\Phi_1\Big(1, a, 1+a; -1, -\frac{x}{2}\Big).$$

This summation formula appears to be new.

In particular, the 'twisted inverse' $H^{\times -1}$ of the harmonic oscillator Hamiltonian is

$$H^{\times -1}(q,p) = R_H(q,p;0) = \int_0^{\pi/2} e^{-H\cos\theta}\,d\theta = \frac{\pi}{2}(I_0(H) - L_0(H)). \tag{31}$$

Here I_0 is the modified Bessel function of order 0 and L_0 is the modified Struve function of order 0. It is amusing to see how, as $I_0(0) = 1$ and $L_0(0) = 0$, the series (29) for $H = 0$ reduces to Gregory's series:

$$\frac{\pi}{2} = 2\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big).$$

Note that

$$H \times f(H) = H\,f(H) - f'(H) - H\,f''(H)$$

so $H^{\times -1}$ could also be sought as a series solution of the differential equation:

$$H\,f''(H) + f'(H) - H\,f(H) = -1.$$

One obtains:*

$$f(H) = \sum_{n=0}^{\infty} a_n H^n, \quad \text{with} \quad a_{2n} = \frac{\pi}{2\cdot 4^n (n!)^2}\,, \quad a_{2n+1} = -\frac{1}{(2n+1)!!}\,.$$

This coincides with (31) – consult, for instance, [26]. In contrast, the 'parametrix' method for twisted inversion suggested in [27] leads right away to very clumsy expressions.

We can associate two families of polynomials to the harmonic oscillator problem, in the same way as before. The family $H^{\times n} =: G_n^2(H)$ does not seem very interesting. On the other hand, if Exp denotes the quantum oscillator exponential, we get by obvious manipulations of (23):

$$e^{xa} = \frac{\operatorname{Exp}(x \operatorname{arctanh} a)}{\sqrt{1-a^2}} =: g_2(x;a).$$

We again treat $g_2(x;a)$ as a generating function:

$$g_2(x;a) = \sum_{n=0}^{\infty} Q_n^{\times}(x) a^n.$$

From the differential equation for g_2:

$$(1-a^2)\,\frac{\partial g_2}{\partial a} = (x+a) \times g_2,$$

---

[*] We are indebted to J. F. Ávila for performing this calculation.

one quickly gets the recurrence relation for the Q polynomials:

$$nQ_n(x) - x\,Q_{n-1}(x) - (n-1)Q_{n-2}(x) = 0 \quad \text{if} \quad n \geqslant 2, \qquad Q_0(x) = 1, \quad Q_1(x) = x.$$

If $f(H) = \sum_{n=0}^{\infty} a_n H^n$ denotes an ordinary function of the harmonic oscillator Hamiltonian, then $f(H) = g^{\times}(H)$, where $g(H) = \sum_{n=0}^{\infty} a_n n!\, Q_n(H)$. The Q_n are apparently new; nevertheless, they are very similar on the surface to the so-called 'continuous Hahn polynomials' that pop out by the same trick in Subsection 3.3.

3.3 Harmonic barrier Hamiltonian and Kummer functions

We take $H = \frac{1}{2}(p^2 - q^2)$ in this subsection. Formula (18) yields at once:

$$\Xi_H(u;t) = \operatorname{sech}\frac{t}{2}\exp\Big(-\frac{2i}{\hbar}\,H\tanh\frac{t}{2}\Big).$$

On the other hand, (13) can be solved using the same techniques as before and gives (with the appropriate normalization):

$$\Gamma_H(u;E,E') = \frac{1}{\hbar}\operatorname{sech}\frac{\pi(E+E')}{2\hbar}\,e^{-2iH/\hbar}{}_1F_1\bigg(\frac{1}{2} - \frac{i(E+E')}{2\hbar}, 1; \frac{4iH}{\hbar}\bigg)\exp\Big[\frac{i}{\hbar}(E-E')\operatorname{arctanh}\frac{p}{q}\Big].$$

It is amusing that the reality of the Γ_H for $E = E'$ entails a particular case of Kummer's transformation:

$$e^{-z/2}{}_1F_1(a,1,z) = e^{z/2}{}_1F_1(1-a,1,-z).$$

Another very elementary fact in our context will give the integral representation of Kummer's function: we must have (with $\hbar = 2$):

$$\begin{aligned}\Gamma_H(u;E) &= \int_{-\infty}^{\infty}\operatorname{sech}\frac{t}{2}\,e^{-i(H\tanh(t/2)-Et/2)}\,dt\\ &= 2e^{-iH}\int_0^1 e^{2iHu}u^{-(1+iE)/2}(1-u)^{-(1-iE)/2}\,du\end{aligned}$$

using the change of variables $\tanh t/2 = 1 - 2u$. If we put $2iH = z$, $1 - iE = 2a$, and take into account $\Gamma(\alpha)\Gamma(1-\alpha) = \pi\csc\pi\alpha$, this gives

$${}_1F_1(a,1,z) = \frac{1}{\Gamma(a)\Gamma(1-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{-a}\,dt.$$

Now we investigate again the two associated families of polynomials. We obtain, from (21) once more $H^{\times n} = G_n^3(H)$, where

$$G_n^3(H) = H\,G_{n-1}^3(H) + \frac{dG_{n-1}^3}{dH}(H) + H\,\frac{d^2G_{n-1}^3(H)}{dH^2} \quad \text{for} \quad n \geqslant 1; \quad G_0^3(H) = 1.$$

The functional calculus then gives

$$2e^{ix}G_n^3(x) = \int_{-\infty}^{\infty}\frac{y^n}{\cosh\frac{1}{2}\pi y}\,{}_1F_1(\tfrac{1}{2}(1-iy),1,2ix)\,dy.$$

In particular:

$$G^3_n(0) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ |E_n| & \text{if } n \text{ is even,} \end{cases} \tag{32}$$

where E_n are the Euler numbers.

Proceeding as before for inverting the functional calculus, we arrive at a new generating function:

$$e^{ax} = \frac{\operatorname{Exp}(x \arctan a)}{\sqrt{1+a^2}} =: g_3(x;a) =: \sum_{n=0}^{\infty} S^\times_n(x)\, a^n.$$

From the differential equation for g_3:

$$(1+a^2)\,\frac{\partial g_3}{\partial a} = (x-a) \times g_3,$$

we derive the recurrence relation for the S_n:

$$nS_n(x) - x\,S_{n-1}(x) + (n-1)S_{n-2}(x) = 0 \quad \text{for} \quad n \geqslant 2; \qquad S_0(x) = 1, \quad S_1(x) = x.$$

It is clear that the S_n exhibit parity. Suppose that they satisfy an orthonormality relation, with weight function $w(x)$. We note that we must then have $\int_{-\infty}^{\infty} x^{2n} w(x)\,dx = G^3_{2n}(0)$. From (32), it is thus obvious that these polynomials are indeed orthogonal on the real line, with positive weight function $w(x) = \frac{1}{2}\operatorname{sech}(\pi x/2)$.

The S_n are none other than the classical Meixner–Pollaczek polynomials, which have been recently discussed under the name 'continuous Hahn polynomials' [28]. These appeared in an investigation about a discrete version of quantum mechanics; but we find our approach to be simpler and more direct.

Again, several integral relations can be derived from the spectral theorem. We write down only:

$$\begin{aligned}
&\iiint da\, da'\, da''\ \operatorname{sech}\frac{\pi a}{2}\ \operatorname{sech}\frac{\pi a'}{2}\ \operatorname{sech}\frac{\pi a''}{2}\ {}_1F_1\big(\tfrac{1}{2}(1-ia), 1; i(p^2-q^2)\big) \\
&\qquad {}_1F_1\big(\tfrac{1}{2}(1-ia'), 1; i(p'^2-q'^2)\big)\ {}_1F_1\big(\tfrac{1}{2}(1-ia''), 1; i(p''^2-q''^2)\big) \\
&\qquad \exp\big[-i((a'-a'')\operatorname{arctanh}(p/q) + (a''-a)\operatorname{arctanh}(p'/q') + (a-a')\operatorname{arctanh}(p''/q''))\big] \\
&= 8\exp\Big[\frac{i}{2}(p^2+p'^2+p''^2-q^2-q'^2-q''^2+2qp'-2q'p+2q'p''-2q''p'+2q''p-2qp'')\Big].
\end{aligned}$$

3.4 Steps towards identities for theta-null values

As Howe has well said [14], the "rather Hinduish multiplicity-in-one is characteristic of the theory of [the Heisenberg group] and adds greatly to its richness." Substituting Weil's form of the representations for Schrödinger's (see, for instance, [29]) leads directly to the use of the Poisson formula and the theory of theta functions.

It was C. G. J. Jacobi who invented the theta functions in the 1820's. The classical first-order theta function is defined by the Fourier series

$$\theta(z,\tau) := \sum_{\mu\in\mathbb{Z}} e^{-\pi\mu^2\tau} e^{2\pi i\mu z}$$

which is normally convergent within the domain $\{\, (z,\tau) \in \mathbb{C}^2 : \operatorname{Re}\tau > 0 \,\}$. In the usual reduced characteristics notation, we have

$$\theta(z,\tau) = \theta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau);$$

see, for instance, the monograph by Rauch and Lebowitz [30]. The function

$$\tau \mapsto \theta(0,\tau) = \theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau) = \sum_{\mu\in\mathbb{Z}} e^{-\pi\mu^2\tau}$$

is holomorphic in the open right half-plane $\{\, \tau \in \mathbb{C} : \operatorname{Re}\tau > 0 \,\}$ and is called the theta-null value (of characteristic $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$).

Now let Planck's constant be normalized by setting $h = 1$. Then $\hbar = 1/2\pi$ and the real line $\mathbb{R}$ carries Lebesgue measure dt. For wavefunctions $\Psi, \Phi \in L^2(\mathbb{R})$, the Wigner function (6a) in position space takes the form

$$W_{|\Psi\rangle\langle\Phi|}(q,p) = \int_{\mathbb{R}} \Psi(q + \tfrac{1}{2}t)\, \overline{\Phi}(q - \tfrac{1}{2}t)\, \exp[-2\pi i p t]\, dt.$$

The holographic transformation $\mathcal{H}$ is defined [31, 31] by the identity

$$\mathcal{H}(\Psi,\Phi;x,y) := \int_{\mathbb{R}} \Psi(t + \tfrac{1}{2}x)\, \overline{\Phi}(t - \tfrac{1}{2}x)\, \exp[2\pi i y t]\, dt \tag{33}$$

for all pairs $(x,y) \in \mathbb{R}^2$. If $\check{\Phi}$ denotes the reflection $t \mapsto \Phi(-t)$ of Φ, we have the identity

$$\mathcal{H}(\Psi,\Phi;x,y) = \tfrac{1}{2} W_{|\Psi\rangle\langle\check{\Phi}|}(\tfrac{1}{2}x, -\tfrac{1}{2}y).$$

In signal theory, the holographic transform provides a mixing of coherent waves with amplitudes Ψ and Φ; this relates to the interpretation of Wigner functions as transitions.

The symmetry between the position and momentum representations (6a) and (6b) of the Wigner functions translates into a corresponding symmetry of the holographic transforms. It is readily seen from (6), or more directly from (33), that

$$\mathcal{H}(\widehat{\Psi},\widehat{\Phi};x,y) = \mathcal{H}(\Psi,\Phi;-y,x) \tag{34}$$

where $\widehat{\Psi} = \mathcal{F}_{\mathbb{R}}\Psi$ is the Fourier transform of Ψ. In fact, this is a simple special case of a more general identity

$$\mathcal{H}(\Psi_\sigma,\Phi_\sigma;\sigma(a,b)) = \mathcal{H}(\Psi,\Phi;a,b)$$

where σ is a symplectic transformation of $\mathbb{R}^2$ and Ψ_σ is the image of Ψ under the corresponding element of the metaplectic representation of the symplectic group $\mathrm{Sp}(2,\mathbb{R})$: see [31], for example.

From (33) one can verify directly that

$$\mathcal{F}_{\mathbb{R}^2}\big(\mathcal{H}(\Psi,\Psi;\cdot,\cdot)\,\overline{\mathcal{H}(\Phi,\Phi;\cdot,\cdot)}\big)(x,y) = \big|\mathcal{H}(\widehat{\Psi},\widehat{\Phi};x,y)\big|^2$$

which, in view of (34), yields

$$\mathcal{F}_{\mathbb{R}^2}\big(\mathcal{H}(\Psi,\Psi;\cdot,\cdot)\,\overline{\mathcal{H}(\Phi,\Phi;\cdot,\cdot)}\big)(x,y) = \big|\mathcal{H}(\Psi,\Phi;-y,x)\big|^2.$$

Now an application of the Poisson summation formula yields the general pixel identity

$$\sum_{(\mu,\nu)\in\mathbb{Z}^2} \mathcal{H}(\Psi,\Psi;\mu,\nu)\,\overline{\mathcal{H}(\Phi,\Phi;\mu,\nu)} = \sum_{(\mu,\nu)\in\mathbb{Z}^2} \left|\mathcal{H}(\Psi,\Phi;\mu,\nu)\right|^2$$

for all wavefunctions $\Psi, \Phi \in L^2(\mathbb{R})$.

Due to the radial isotropy of the harmonic oscillator wavefunctions and (25a), we get from the last identity the 'radial pixel identity' for $m \geqslant n \geqslant 0$:

$$\sum_{(\mu,\nu)\in\mathbb{Z}^2} l_m^0(\pi(\mu^2+\nu^2))\, l_n^0(\pi(\mu^2+\nu^2)) = \frac{n!}{m!}\,\pi^{m-n} \sum_{(\mu,\nu)\in\mathbb{Z}^2} (\mu^2+\nu^2)^{m-n}\big[l_n^{m-n}(\pi(\mu^2+\nu^2))\big]^2, \quad (35)$$

where $l_n^\alpha(x) := e^{-x/2} L_n^\alpha(x)$.

As special cases of the identity (35) we may relate the odd powers of π to the theta-null value and its derivatives at the point $\tau = 1$:

- Case $m = 1$, $n = 0$:

$$\pi = \sum_{\mu\in\mathbb{Z}} e^{-\pi\mu^2} \Bigg/ 4 \sum_{\mu\in\mathbb{Z}} \mu^2 e^{-\pi\mu^2}. \quad (36)$$

  To see this, notice that (35) reduces to

$$\sum_{(\mu,\nu)\in\mathbb{Z}^2} (1-\pi(\mu^2+\nu^2))e^{-\pi(\mu^2+\nu^2)} = \sum_{(\mu,\nu)\in\mathbb{Z}^2} \pi(\mu^2+\nu^2)e^{-\pi(\mu^2+\nu^2)}$$

  which becomes

$$\sum_{\mu\in\mathbb{Z}} e^{-\pi\mu^2} - 2\pi \sum_{\mu\in\mathbb{Z}} \mu^2 e^{-\pi\mu^2} = 2\pi \sum_{\mu\in\mathbb{Z}} \mu^2 e^{-\pi\mu^2},$$

  yielding (36).

Similar elementary manipulations yield the formulas:

- Case $m = 2$, $n = 1$:

$$\pi^3 = 15 \sum_{\mu\in\mathbb{Z}} (8\pi^2\mu^4 - 1)e^{-\pi\mu^2} \Bigg/ 32 \sum_{\mu\in\mathbb{Z}} \mu^6 e^{-\pi\mu^2}.$$

- Case $m = 3$, $n = 2$:

$$\pi^5 = 45 \sum_{\mu\in\mathbb{Z}} (16\pi^4\mu^8 - 140\pi^2\mu^4 + 21)e^{-\pi\mu^2} \Bigg/ 64 \sum_{\mu\in\mathbb{Z}} \mu^{10} e^{-\pi\mu^2}.$$

- Case $m = 4$, $n = 3$:

$$\pi^7 = \frac{91 \sum_{\mu\in\mathbb{Z}} (256\pi^6\mu^{12} - 15840\pi^4\mu^8 + 166320\pi^2\mu^4 - 25245)e^{-\pi\mu^2}}{1024 \sum_{\mu\in\mathbb{Z}} \mu^{14} e^{-\pi\mu^2}}.$$

These special cases have also been proved in [33] and numerically checked by Martin Schmidt.

Remark 1. The three cases 3.1–3.3 give all the interesting mathematics that can be extracted using the 'transfer formula' and 'functional calculus' methods from Moyal theory in $\mathbb{R}^2$, because all the nontrivial Hamiltonians are canonically equivalent to one of these [13]. One can go to higher-dimensional spaces, but the spectrum will be in general degenerate, which entails additional complications. An important exception, which warrants further study, is the generic Hamiltonian $H = \frac{1}{2}u^t Bu$, where $u \in \mathbb{R}^4$ and B is a 4×4 symmetric matrix such that the eigenvalues $\pm\alpha \pm i\beta$ of JB are all distinct. Then (with $\hbar = 2$) one can compute:

$$\Xi_H(u;t) = \frac{2}{\cosh\alpha t + \cos\beta t} \exp\left[-i\,\frac{\alpha^{-1}H_1 \sinh\alpha t + \beta^{-1}H_2 \sin\beta t}{\cosh\alpha t + \cos\beta t}\right]$$

with $H_1 + H_2 = H$. The associated functional calculus will give interesting properties for new special functions in two variables. It is no longer true that a twisted function of H is an (ordinary) function of H alone, as we consistently found in $\mathbb{R}^2$.

Remark 2. Another useful device is the use of limiting processes. For instance, a Hamiltonian of the form $\frac{1}{2}(p^2 + kq^2) + q$ is of the harmonic oscillator type [respectively, of the harmonic barrier type] as long as $k > 0$ [respectively, $k < 0$]. Reworking our formulas to include k explicitly and taking the limit $k \to 0$ should give interesting limit formulas for Laguerre and Kummer functions in terms of Airy functions.

Remark 3. The Laguerre polynomials L_n^α can be expressed in terms of the Poisson–Charlier polynomials [34]:

$$c_n(x;a) := {}_2F_0(-n,-x;-a^{-1}) = n!(-a)^{-n}L_n^{x-n}(a). \tag{37}$$

Notice that the argument x of the Poisson–Charlier polynomial appears as a *parameter* to the Laguerre polynomial and, vice versa, the parameter a of c_n is the argument of L_n^{x-n}. The importance of the Poisson–Charlier polynomials stems from their *discrete* orthogonality relation [34]:

$$\sum_{x=0}^{\infty} c_n(x;a)\, c_m(x;a)\, \frac{a^x}{x!} = e^a a^n n!\, \delta_{mn}. \tag{38}$$

It is also possible to express the Laguerre functions in terms of the generalized Bateman functions k_n^m (see [35]).

On using (35) and (37), we obtain a type of trace formula for the integer values of the Poisson–Charlier polynomials:

$$\begin{aligned}\sum_{(\mu,\nu)\in\mathbb{Z}^2} (-1)^{m+n} \frac{(\mu^2+\nu^2)^{m+n}}{(m+n)!}\, e^{-\pi(\mu^2+\nu^2)} c_m(m;\pi(\mu^2+\nu^2))\, c_n(n;\pi(\mu^2+\nu^2)) \\ = \sum_{(\mu,\nu)\in\mathbb{Z}^2} \frac{(\mu^2+\nu^2)^{m+n}}{(m+n)!}\, e^{-\pi(\mu^2+\nu^2)} \big[c_n(m;\pi(\mu^2+\nu^2))\big]^2\end{aligned}$$

whenever $m \geqslant n \geqslant 0$. This can be thought of as a two-dimensional discrete overcompleteness relation corresponding to the one-dimensional discrete orthogonality relation (38).

Remark 4. A new insight into the deep links between Moyal theory and harmonic analysis has been revealed in the discovery of the mysterious role played by *ordinary* (not twisted) convolution in the theory of Wigner functions. By this means, two of us have shown in [36] that Hudson's theorem [37] fails for mixed states (for a thorough exposition of Hudson's theorem, see [38]; also, [39] is relevant in this context).

4 Moyal representation for spin and special functions

We now show how the Moyal approach to spin quantization yields more information about special functions. The appropriate phase space is the sphere $\mathbb{S}^2$, which is a coadjoint orbit for $\mathrm{SU}(2)$. One chooses a unitary irreducible representation π_j of the compact Lie group $\mathrm{SU}(2)$ on $\mathbb{C}^{2j+1}$, for some positive half-integer j. The role of the operators $\Pi(u)$ is played by the $\mathrm{SU}(2)$ *Stratonovich–Weyl operator kernel* $\Delta^j(\boldsymbol{n})$ which associates to each $\boldsymbol{n}$ on $\mathbb{S}^2$ a $(2j+1)\times(2j+1)$ selfadjoint matrix, satisfying appropriate conditions.

If $\widetilde{R} \in \mathrm{SU}(2)$ and if $R \in \mathrm{SO}(3)$ denotes its image under the covering homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, then $\mathrm{SU}(2)$-covariance of Δ^j means that

$$\pi_j(\widetilde{R})\,\Delta^j(\boldsymbol{n})\,\pi_j(\widetilde{R})^{-1} = \Delta^j(R\boldsymbol{n}) \tag{39}$$

for $\widetilde{R} \in \mathrm{SU}(2)$, $\boldsymbol{n} \in \mathbb{S}^2$. The matrix elements of $\pi_j(\widetilde{R})$ in the standard presentation are customarily written $\mathcal{D}^j_{mn}(\widetilde{R})$.

Let $Z^j_{sr}(\boldsymbol{n})$ denote the (r,s)-element of the matrix $\Delta^j(\boldsymbol{n})$, with $r,s = -j,\dots,j-1,j$. It is shown in [7] that these matrix elements are:

$$Z^j_{sr}(\boldsymbol{n}) := \frac{\sqrt{4\pi}}{2j+1}\sum_{l=0}^{2j}\sqrt{2l+1}\,\langle j\,l\,r\,(s-r)\mid j\,s\rangle\,Y_{l,s-r}(\boldsymbol{n}), \tag{40}$$

where Y_{lm} denotes the usual spherical harmonics and $\langle j\,l\,r\,(s-r)|j\,s\rangle$ is a Clebsch–Gordan coefficient. (In [7], the indices r, s are erroneously switched.)

Using the well-known formula [40] for transforming spherical harmonics:

$$Y_{lm}(R\boldsymbol{n}) = \sum_{n=-l}^{l}\mathcal{D}^{l*}_{mn}(\widetilde{R})\,Y_{ln}(\boldsymbol{n}), \tag{41}$$

from (40) and (41) one derives [7]:

$$Z^j_{sr}(R\boldsymbol{n}) = \sum_{p,q=-j}^{j}\mathcal{D}^j_{rp}(\widetilde{R})\,\mathcal{D}^{j*}_{sq}(\widetilde{R})\,Z^j_{qp}(\boldsymbol{n}), \tag{42}$$

which is just (39) in explicit form.

A property directly analogous to (2) is valid for the $\mathrm{SU}(2)$ case:

$$\begin{aligned}\operatorname{tr}\Delta^j(\boldsymbol{n}) &= 1,\\ \operatorname{tr}\bigl(\Delta^j(\boldsymbol{m})\,\Delta^j(\boldsymbol{n})\bigr) &= \frac{4\pi}{2j+1}\,K^j(\boldsymbol{m},\boldsymbol{n}),\\ \operatorname{tr}\bigl(\Delta^j(\boldsymbol{m})\,\Delta^j(\boldsymbol{n})\,\Delta^j(\boldsymbol{k})\bigr) &=: L^j(\boldsymbol{m},\boldsymbol{n},\boldsymbol{k}),\end{aligned} \tag{43}$$

where

$$K^j(\boldsymbol{m}, \boldsymbol{n}) = \sum_{l=0}^{2j} \sum_{s=-l}^{l} Y_{ls}(\boldsymbol{m})\, Y_{ls}^*(\boldsymbol{n})$$

is the *reproducing kernel* of the space $\mathcal{H}_{2j}$ of spherical harmonics of degree $\leqslant 2j$.

Now one can define the *Stratonovich–Weyl correspondence* between 'symbols' – functions on the phase space $\mathbb{S}^2$ belonging to the function space $\mathcal{H}_{2j}$ – and operators on $\mathbb{C}^{2j+1}$ by the formulas

$$W_f = \frac{2j+1}{4\pi} \int_{\mathbb{S}^2} f(\boldsymbol{n})\, \Delta^j(\boldsymbol{n})\, d\boldsymbol{n}; \qquad W_A(\boldsymbol{n}) = \operatorname{tr}(A\, \Delta^j(\boldsymbol{n}))$$

(with $d\boldsymbol{n} = \sin\theta\, d\theta\, d\phi$ if $\boldsymbol{n} = (\theta, \phi)$ in spherical coordinates). Once more, we have $W_{W_A} = A$ and $W_{W_f} = f$, and the constant function 1 corresponds to the identity operator.

From (43), one verifies that the product of two operators W_f, W_g is equal to $W_{f\times g}$ where the $\mathrm{SU}(2)$ *twisted product* in $\mathcal{H}_{2j}$ is given by

$$f \times g(\boldsymbol{n}) = \Big(\frac{2j+1}{4\pi}\Big)^2 \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} f(\boldsymbol{m})\, g(\boldsymbol{k})\, L^j(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{k})\, d\boldsymbol{m}\, d\boldsymbol{k}$$

with the property that $\int_{\mathbb{S}^2} f \times g(\boldsymbol{n})\, d\boldsymbol{n} = \int_{\mathbb{S}^2} f(\boldsymbol{n})\, g(\boldsymbol{n})\, d\boldsymbol{n}$.

With these ingredients, one can carry out the quantization scheme outlined in Section 2 for the $\mathrm{SU}(2)$ case, using the formal analogues of equations (7) and (9): this is developed more fully in [7]. The special functions $Z^j_{rs}(\boldsymbol{n})$ are the primary computational tools, since they have the orthogonality and product properties:

$$\frac{2j+1}{4\pi} \int_{\mathbb{S}^2} Z^j_{rs}(\boldsymbol{n})\, Z^j_{tu}(\boldsymbol{n})\, d\boldsymbol{n} = \delta_{ru}\, \delta_{st}; \qquad Z^j_{rs} \times Z^j_{tu} = \delta_{st}\, Z^j_{ru}; \tag{44}$$

as is evident from the Stratonovich–Weyl correspondence, since $Z^j_{rs} = W_{|j,r\rangle\langle j,s|}$ where $|j, r\rangle$ denotes a spin eigenvector. They constitute an interesting subject in themselves.

From (40), we get in particular the *spin eigenstates*:

$$Z^j_{mm}(\boldsymbol{n}) = \sum_{l=0}^{2j} \frac{2l+1}{2j+1} \langle jl\, m0 \mid jm\rangle\, P_l(\cos\theta), \tag{45}$$

where the P_l are the Legendre polynomials. If $W^j_z := \sum_{m=-j}^{j} m Z^j_{mm}$ is the symbol associated to the J_z spin operator, then

$$W^j_z(\boldsymbol{n}) = \sum_{m=-j}^{j} \sum_{l=0}^{2j} m\, \frac{2l+1}{2j+1} \langle jl\, m0 \mid jm\rangle\, P_l(\cos\theta) = \sqrt{j(j+1)}\, \cos\theta.$$

Let us proceed as indicated at the end of Section 2. The *Fourier kernel E* defined by

$$E(g; j, \boldsymbol{n}) := \operatorname{tr}\big(\pi_j(g)\, \Delta^j(\boldsymbol{n})\big)$$

for $g \in \mathrm{SU}(2)$, $\boldsymbol{n} \in \mathbb{S}^2$, is of central importance for the harmonic analysis of $\mathrm{SU}(2)$. Note that $E(g; j, \boldsymbol{n})$ is the Stratonovich–Weyl symbol of $\pi_j(g)$. Explicitly,

$$E(g; j, \boldsymbol{n}) = \sum_{r,s=-j}^{j} Z^j_{sr}(\boldsymbol{n})\, \mathcal{D}^j_{sr}(g). \tag{46}$$

From the orthogonality properties (44) of the Z^j_{rs} functions, and the orthogonality properties of the matrix elements $\mathcal{D}^j_{mn}$ with respect to the normalized Haar measure dg on $\mathrm{SU}(2)$, from (46) we derive at once:

$$\begin{aligned} \mathcal{D}^j_{mn}(g) &= \frac{2j+1}{4\pi}\int_{\mathbb{S}^2} E(g;j,\boldsymbol{n})\,Z^j_{nm}(\boldsymbol{n})\,d\boldsymbol{n}, \\ Z^j_{rs}(\boldsymbol{n}) &= (2j+1)\int_{\mathrm{SU}(2)} \overline{E(g;j,\boldsymbol{n})}\,\mathcal{D}^j_{sr}(g)\,dg. \end{aligned} \tag{47}$$

We now show, by a specific example, how this machinery may be used to generate identities for special functions associated to the representation theory of $\mathrm{SU}(2)$. The representative functions $\mathcal{D}^j_{mn}$ are most often written explicity in terms of the Eulerian angle parametrization of $\mathrm{SU}(2)$, although some other parametrizations are occasionally employed [40]. However, an $\mathrm{SU}(2)$-element is most naturally written in the angle-axis parametrization:

$$g = g(\psi,\boldsymbol{m}) := \exp(-\tfrac{i}{2}\psi\boldsymbol{m}\cdot\boldsymbol{\sigma}) \quad \text{with} \quad |\boldsymbol{m}|^2 = 1.$$

We thus require a formula which expresses $\mathcal{D}^j_{mn}(g(\psi,\boldsymbol{m}))$ directly in terms of ψ and $\boldsymbol{m}$.

Let us write $\boldsymbol{n}_0$ for the north-pointing vector in $\mathbb{S}^2$, so that the one-parameter subgroup of $\mathrm{SU}(2)$ generated by $-\frac{i}{2}\sigma_z$ is $t \mapsto g(t,\boldsymbol{n}_0)$; applying the homomorphism and the Stratonovich–Weyl correspondence, this yields the evolution equation:

$$i\,\frac{\partial E}{\partial t}(g(t,\boldsymbol{n}_0);j,\boldsymbol{n}) = W^j_z \times E(g(t,\boldsymbol{n}_0);j,\boldsymbol{n})$$

whose solution is clearly

$$E(g(t,\boldsymbol{n}_0);j,\boldsymbol{n}) = \sum_{k=-j}^{j} e^{-ikt}\,Z^j_{kk}(\boldsymbol{n}). \tag{48}$$

A general expression for the Fourier kernel may now be found from its covariance properties. Let us, by an *abus de notation*, write $Z^j_{kk}(\cos\theta)$ instead of $Z^j_{kk}(\boldsymbol{n})$ for the spin states, in view of their cylindrical symmetry (45). Then since $g(\psi,\boldsymbol{m})$ is conjugate within $\mathrm{SU}(2)$ to $g(\psi,\boldsymbol{n}_0)$ by an element whose associated rotation takes $\boldsymbol{n}_0$ to $\boldsymbol{m}$, we obtain from (42), (46) and (48):

$$E(g(\psi,\boldsymbol{m});j,\boldsymbol{n}) = \sum_{k=-j}^{j} e^{-ik\psi} Z^j_{kk}(\boldsymbol{m}\cdot\boldsymbol{n}). \tag{49}$$

From (47) and (49) we now derive

$$\begin{aligned} \mathcal{D}^j_{mn}(g(\psi,\boldsymbol{m})) &= \sum_{k=-j}^{j} e^{-ik\psi}\,\frac{2j+1}{4\pi}\int_{\mathbb{S}^2} Z^j_{kk}(\boldsymbol{m}\cdot\boldsymbol{n})\,Z^j_{nm}(\boldsymbol{n})\,d\boldsymbol{n} \\ &= \sum_{l,l'=0}^{2j}\sum_{k=-j}^{j} \frac{(2l+1)\sqrt{2l'+1}}{(2j+1)\sqrt{4\pi}}\,e^{-ik\psi}\,\langle jl\,k0 \mid jk\rangle\,\langle jl'\,m(n-m) \mid jn\rangle \\ &\qquad \int_{\mathbb{S}^2} P_l(\boldsymbol{m}\cdot\boldsymbol{n})\,Y_{l',n-m}(\boldsymbol{n})\,d\boldsymbol{n} \\ &= \sum_{l=0}^{2j} \frac{\sqrt{2l+1}}{2j+1}\,\langle jl\,m(n-m) \mid jn\rangle \sum_{k=-j}^{j} \langle jl\,k0 \mid jk\rangle\,e^{-ik\psi}\sqrt{4\pi}\,Y_{l',n-m}(\boldsymbol{m}). \end{aligned} \tag{50}$$

(The integration relies on the known expansion

$$P_l(\boldsymbol{m}\cdot\boldsymbol{n}) = \sum_{s=-l}^{l} Y_{ls}(\boldsymbol{m})\,\overline{Y}_{ls}(\boldsymbol{n})$$

and the orthonormality of the spherical harmonics over $\mathbb{S}^2$.) This explicit development of the representative functions in terms of Clebsch–Gordan coefficients and spherical harmonics only, in the angle-axis parametrization, appears to be new.

Writing $\boldsymbol{m} = (\theta, \phi)$ in spherical polar coordinates, we can, for instance, write down the following special cases of (50) for spin one:

$$\begin{aligned}
\mathcal{D}^1_{00}(g(\psi,\boldsymbol{m})) &= \cos\psi + (1-\cos\psi)\cos^2\theta,\\
\mathcal{D}^1_{11}(g(\psi,\boldsymbol{m})) &= \tfrac{1}{2}(1+\cos\psi) - i\sin\psi\cos\theta - \tfrac{1}{2}(1-\cos\psi)\cos^2\theta,\\
\mathcal{D}^1_{01}(g(\psi,\boldsymbol{m})) &= -\frac{1}{\sqrt{2}}\,e^{i\phi}\sin\theta\big[i\sin\psi + (1-\cos\psi)\cos\theta\big],\\
\mathcal{D}^1_{-1,1}(g(\psi,\boldsymbol{m})) &= -\tfrac{1}{2}(1-\cos\psi)e^{2i\phi}\sin^2\theta.
\end{aligned}$$

5 Conclusion

The general scheme outlined in this paper can in principle be applied to other Lie groups having no evident connection with Quantum Mechanics; the point of contact is the theory of Kirillov, Kostant and Souriau which allows one to define a natural symplectic structure on the coadjoint orbits of the group: see [41,42], for example. This is merely Classical Mechanics, in principle; but whenever a ‘Stratonovich–Weyl correspondence’ can be found, a bridge is built to operator theory on the representation spaces and standard noncommutative harmonic analysis, a twisted product appears, and a systematic comparison of the twisted product formalism with the conventional operator theory yields many identities between special functions. Some of these identities will be old, and are merely derived in a fresh and illuminating manner; and several will be new. Thus the apparatus of Moyal Quantum Mechanics provides a means of demonstrating unexpected connections among many parts of mathematics.

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