Hopf algebras in noncommutative geometry

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Abstract

We give an introductory survey to the use of Hopf algebras in several problems of noncommutative geometry. The main example, the Hopf algebra of rooted trees, is a graded, connected Hopf algebra arising from a universal construction. We show its relation to the algebra of transverse differential operators introduced by Connes and Moscovici in order to compute a local index formula in cyclic cohomology, and to the several Hopf algebras defined by Connes and Kreimer to simplify the combinatorics of perturbative renormalization. We explain how characteristic classes for a Hopf module algebra can be obtained from the cyclic cohomology of the Hopf algebra which acts on it. Finally, we discuss the theory of noncommutative spherical manifolds and show how they arise as homogeneous spaces of certain compact quantum groups.

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Introduction

These are lecture notes for a course given at the Summer School on Geometric and Topological Methods for Quantum Field Theory, sponsored by the Centre International de Mathématiques Pures et Appliquées (CIMPA) and the Universidad de Los Andes, at Villa de Leyva, Colombia, from the 9th to the 27th of July, 2001.

These notes explore some recent developments which place Hopf algebras at the heart of the noncommutative approach to geometry and physics. Many examples of Hopf algebras are known from the literature on "quantum groups", some of which provide algebraic deformations of the classical transformation groups. The main emphasis here, however, is on certain other Hopf algebras which have recently appeared in two seemingly unrelated contexts: in the combinatorics of perturbative renormalization in quantum field theories, and in connection with local index formulas in noncommutative geometry.

These Hopf algebras act on "noncommutative spaces", and certain characteristic classes for these spaces can be obtained, by a canonical procedure, from corresponding invariants of the Hopf algebras. This comes about by pulling back the cyclic cohomology of the algebra representing the noncommutative space, which is the receptacle of Chern characters, to another cohomology of the Hopf algebra.

Recently, some interesting spaces have been discovered, the noncommutative spheres, which are completely specified by certain algebraic relations. They turn out to be homogeneous spaces under the action of certain Hopf algebras: in this way, these Hopf algebras appear as "quantum symmetry groups". We shall show how these symmetries arise from a class of quantum groups built from Moyal products on group manifolds.

Section 1 is introductory: it offers a snapshot of noncommutative geometry and the basic theory of Hopf algebras; as an example of how both theories interact, we exhibit the Connes–Moscovici Hopf algebra of differential operators in the one-dimensional case. Section 2 concerns the Hopf algebras which have been found useful in the perturbative approach to renormalization. We develop at length a universal construction, the Connes–Kreimer algebra of rooted trees, which is a graded, commutative, but highly noncocommutative Hopf algebra. Particular quantum field theories give rise to related Hopf algebras of Feynman graphs; we discuss briefly how these give a conceptual approach to the renormalization problem.

The third section gives an overview of cyclic cohomology for both associative and Hopf algebras, indicating how the latter provide characteristic classes for associative algebras on which they act. The final Section 4 explains how cyclic-homology Chern characters lead to new examples of noncommutative spin geometries, whose symmetry groups are compact quantum groups obtained from the Moyal approach to prequantization.

1   Noncommutative Geometry and Hopf Algebras

Noncommutative geometry, in the broadest sense, is the study of geometrical properties of singular spaces, by means of suitable "coordinate algebras" which need not be commutative. If the space in question is a differential manifold, its coordinates form a commutative algebra of smooth functions; but even in this case, adding a metric structure may involve operators which do not commute with the coordinates. One learns to replace the usual calculus of points, paths, integration domains, etc., by an alternative language involving the algebra of coordinates; by focusing only on those features

which do not require that the coordinates commute, one arrives at an algebraic (or operatorial) approach which is applicable to many singular spaces also.

1.1 The algebraic tools of noncommutative geometry

The first step is to replace a topological space X by its algebra of complex-valued continuous functions $C(X)$. If X is a compact (Hausdorff) space, then $C(X)$ is a commutative C^*-algebra with unit 1 and its norm $\|f\| := \sup_{x \in X} |f(x)|$ satisfies the C^*-property $\|f\|^2 = \|f^*f\|$. The first Gelfand–Naĭmark theorem [48] says that any commutative unital C^*-algebra A is of this form: $A = C(X)$ where $X = M(A)$ is the space of *characters* (nonzero homomorphisms) $\mu \colon A \to \mathbb{C}$, which is compact in the weak* topology determined by the maps $\mu \mapsto \mu(a)$, for $a \in \mathbb{C}$. Indeed, the characters of $C(X)$ are precisely the evaluation maps $\varepsilon_x \colon f \mapsto f(x)$ at each point $x \in X$.

We shall mainly deal with the compact case in what follows. A locally compact, but noncompact, space Y can be handled by passing to a compactification (that is, a compact space in which Y can be densely embedded). For instance, we can adjoin one "point at infinity": if $X = Y \uplus \{\infty\}$, then $\{ f \in C(X) : f(\infty) = 0 \}$ is isomorphic to $C_0(Y)$, the commutative C^*-algebra of continuous functions on Y "vanishing at infinity"; thus, by dropping the constant functions from $C(X)$, we get the commutative nonunital C^*-algebra $C_0(Y)$ as a stand-in for the locally compact space Y. There is also a maximal compactification $\beta Y := M(C_b(Y))$, called the Stone–Čech compactification, namely, the character space of the (unital) C^*-algebra of bounded continuous functions on Y.

The construction $X \mapsto C(X)$ yields a contravariant functor: to each continuous map $h \colon X_1 \to X_2$ between compact spaces there is a morphism[1] $\varphi_h \colon C(X_2) \to C(X_1)$ given by $\varphi_h(f) := f \circ h$.

By relaxing the commutativity requirement, we can regard *noncommutative C^*-algebras* (unital or not) as proxies for "noncommutative locally compact spaces". The characters, if any, of such an algebra may be said to label "classical points" of the corresponding noncommutative space. However, noncommutative C^*-algebras generally have few characters, so these putative spaces will have correspondingly few points. The recommended course of action, then, is to leave these pointless spaces behind and to adopt the language and techniques of algebras instead.

There is a second Gelfand–Naĭmark theorem [48], which states that any C^*-algebra, commutative or not, can be faithfully represented as a (norm-closed) algebra of bounded operators on a Hilbert space. The data for a "noncommutative topology" consist, then, of a pair $(A, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and A is a closed subalgebra of $\mathcal{L}(\mathcal{H})$.

► Vector bundles over a compact space also have algebraic counterparts. If X is compact and $E \xrightarrow{\pi} X$ is a complex vector bundle, the space $\Gamma(X, E)$ of continuous sections is naturally a module over $C(X)$, which is necessarily of the form $eC(X)^m$, where $e = e^2 \in M_m(C(X))$ is an idempotent matrix of elements of $C(X)$. More generally, if A is any algebra over $\mathbb{C}$, a right A-module of the form eA^m with $e = e^2 \in M_m(A)$ is called a *finitely generated projective module* over A. The Serre–Swan theorem [99] matches vector bundles over X with finitely generated projective modules over $C(X)$. The idempotent e may be constructed from the transition functions of the vector bundle by pulling back a standard idempotent from a Grassmannian bundle: see [45, §1.1] or [52, §2.1] for details.

A more concrete example is that of the tangent bundle over a compact Riemannian manifold M: by the Nash embedding theorem [101, Thm 14.5.1], one can embed M in some $\mathbb{R}^m$ so that the

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[1]By a morphism of unital C^*-algebras we mean a $*$-homomorphism preserving the units.

metric on TM is obtained from the ambient Euclidean metric; if $e(x)$ is the orthogonal projector on $\mathbb{R}^n$ with range $T_x M$, then $e = e^2 \in M_m(C(M))$ and the module $\Gamma(M, TM)$ of vector fields on M may be identified with the range of e.

In the noncompact case, one can use Rennie's nonunital version of the Serre–Swan theorem [84]: $C_0(Y)$-modules of the form $eC(X)^m$, where X is some compactification of Y and $e = e^2 \in M_m(C(X))$, consist of sections vanishing at infinity (i.e., outside of Y) of vector bundles $E \to X$. One can take X to be the *one-point* compactification of Y only if E is *trivial* at infinity; as a rule, the compactification to be used depends on the problem at hand.

If A is a C^*-algebra, we may replace e by an orthogonal projector (i.e., a selfadjoint idempotent) $p = p^* = p^2$ so that $eA^m \simeq pA^m$ as right A-modules. If A is faithfully represented by bounded operators on a Hilbert space $\mathcal{H}$, then $M_m(A)$ is an algebra of bounded operators on $\mathcal{H}^m = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (m times), so we can schematically write $e = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ as an operator on $e\mathcal{H}^m \oplus (1 - e)\mathcal{H}^m$; then $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the range projector on $e\mathcal{H}^m$.

The correspondence $E \mapsto \Gamma(X, E)$ is a covariant functor which carries topological invariants of X to algebraic invariants of $C(X)$. In particular, it identifies the K-theory group $K^0(X)$, formed by stable equivalence classes of vector bundles where $[E] + [F] := [E \oplus F]$ —here $\oplus$ denotes Whitney sum of vector bundles over X— with the group $K_0(C(X))$ formed by stable isomorphism classes of matrix projectors over $C(X)$ where $[p] + [q] := [p \oplus q]$ and $\oplus$ now denotes direct sum of projectors. The K-theory of C^*-algebras may be developed in an operator-theoretic way, see [8, 76, 108] and [52, Chap. 3], for instance; or purely algebraically, and the group $K_0(A)$ turns out to be the same in both approaches. (However, the group $K_1(A)$, formed by classes of unitaries in $M_m(A)$, does not coincide with the algebraic K_1-group in general: see, for instance, [95] or [52, p. 131].) The salient feature of both topological and C^*-algebraic K-theories is *Bott periodicity*, which says that two K-groups are enough: although one can define $K_j(A)$ is a systematic way for any $j \in \mathbb{N}$, it turns out that $K_{j+2}(A) \simeq K_j(A)$ by natural isomorphisms (in marked contrast to the case of purely algebraic K-theory).

▶ To deal with a (compact) differential manifold M (in these notes, we only treat differential manifolds without boundary), we replace the continuous functions in $C(M)$ by the dense subalgebra of *smooth* functions $\mathcal{A} = C^\infty(M)$. This is no longer a C^*-algebra, but it is complete in its natural topology (that of uniform convergence of functions, together with their derivatives of all orders), so it is a Fréchet algebra with a C^*-completion. Likewise, given a vector bundle $E \longrightarrow M$, we replace the continuous sections in $\Gamma(M, E)$ by the $\mathcal{A}$-module of smooth sections $\Gamma^\infty(M, E)$. The Serre–Swan theorem continues to hold, *mutatis mutandis*, in the smooth category.

In the noncommutative case, with no differential structure *a priori*, we need to replace the C^*-algebra A by a subalgebra $\mathcal{A}$ which should (a) be dense in $\mathcal{A}$; (b) be a Fréchet algebra, that is, it should be complete under some countable family of seminorms including the original C^*-norm of A; and (c) satisfy $K_0(\mathcal{A}) \simeq K_0(A)$. This last condition is not automatic: it is necessary that $\mathcal{A}$ be a *pre-C^*-algebra*, that is to say, it should be stable under the holomorphic functional calculus (which is defined in the larger algebra A). The proof of (c) for pre-C^*-algebras is given in [10]; see also [52, §3.8].

▶ The next step is to find an algebraic description of a Riemannian metric on a smooth manifold. This can be done in a principled way through a theory of "noncommutative metric spaces" at present

under construction by Rieffel [91–94]. But here we shall take a short cut, by defining metrics only over *spin* manifolds, using the Dirac operator as our instrument; this was, indeed, the original insight of Connes [23].

A metric $g = [g_{ij}]$ on the tangent bundle TM of a (compact) manifold M yields a contragredient metric $g^{-1} = [g^{rs}]$ on the cotangent bundle T^*M; so we can build a Clifford algebra bundle $\mathbb{C}\ell(M) \longrightarrow M$, whose fibre at x is $C\ell((T_x^*M)^\mathbb{C}, g_x^{-1})$, by imposing a suitable product structure on the complexified exterior bundle $(\Lambda^\bullet T^*M)^\mathbb{C}$. We assume that M supports a spinor bundle $S \longrightarrow M$, on which $\mathbb{C}\ell(M)$ acts fibrewise and irreducibly; on passing to smooth sections, we may write $c(\alpha)$ for the Clifford action of a 1-form α on spinors. The spinor bundle comes equipped with a Hermitian metric, so the squared norm $\|\psi\|^2 := \int_M |\psi(x)|^2 \sqrt{\det g}\, dx$ makes sense; the completion of $\Gamma^\infty(M, S)$ in this norm is the Hilbert space $\mathcal{H} = L^2(M, S)$ of square-integrable spinors. Locally, we may write the Clifford action of 1-forms as $c(dx^r) := h_\alpha^r \gamma^\alpha$, where the "gamma matrices" γ^α satisfy $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\, \delta^{\alpha\beta}$ and the coefficients h_α^r are real and obey $h_\alpha^r \delta^{\alpha\beta} h_\beta^s = g^{rs}$. The Dirac operator is locally defined as

$$\slashed{D} := -i\, c(dx^r) \left(\frac{\partial}{\partial x^r} - \omega_r \right), \tag{1.1}$$

where $\omega_r = \frac{1}{4} \widetilde{\Gamma}_{r\alpha}^\beta \gamma^\alpha \gamma^\beta$ are components of the *spin connection*, obtained from the Christoffel symbols $\widetilde{\Gamma}_{r\alpha}^\beta$ (in an orthogonal basis) of the Levi-Civita connection. The manifold M is spin whenever these local formulae patch together to give a well-defined spinor bundle. There is a well-known topological condition for this to happen (the second Stiefel–Whitney class $w_2(TM) \in H^2(M, \mathbb{Z}_2)$ must vanish [67]), and when it is fulfilled $\slashed{D}$ extends to a selfadjoint operator on $\mathcal{H}$ with compact resolvent [52, 67].

Apart from these local formulas, the Dirac operator has a fundamental algebraic property. If ψ is a spinor and $a \in C^\infty(M)$ is regarded as a multiplication operator on spinors, it can be checked that

$$\slashed{D}(a\psi) = -i\, c(da)\, \psi + a\, \slashed{D}\psi,$$

or, more simply,

$$[\slashed{D}, a] = -i\, c(da). \tag{1.2}$$

Following [6], we call a "generalized Dirac operator" any selfadjoint operator D on $\mathcal{H}$ satisfying $[D, a] = -i\, c(da)$ for $a \in C^\infty(M)$. Now $c(da)$ is a bounded operator on $L^2(M, S)$ whenever a is smooth, and its norm is that of the gradient of a, i.e., the vector field determined by $g(\operatorname{grad} a, X) := da(X) = X(a)$. A continuous function $a \in C(M)$ is called Lipschitz (with respect to the metric g) if its gradient is defined, almost everywhere, as an essentially bounded measurable vector field, i.e., $\|\operatorname{grad} a\|_\infty$ is finite. Now the Riemannian distance $d_g(p, q)$ between two points $p, q \in M$ is usually defined as the infimum of the lengths of (piecewise smooth) paths from p to q; but it is not hard to show (see [52, §9.3], for instance) that the distance can also be defined as a supremum:

$$d_g(p, q) = \sup\{ |a(p) - a(q)| : a \in C(M),\ \|\operatorname{grad} a\|_\infty \leqslant 1 \}. \tag{1.3}$$

The basic equation (1.2) allows to replace the gradient by a commutator with the Dirac operator:

$$d_g(p, q) = \sup\{ |a(p) - a(q)| : a \in C(M),\ \|[\slashed{D}, a]\| \leqslant 1 \}. \tag{1.4}$$

Thus, the Riemannian distance function d_g is entirely determined by $\slashed{D}$. Moreover, the metric g is in turn determined by d_g, according to the Myers–Steenrod theorem [77]. From the noncommutative

point of view, then, the Dirac operator assumes the role of the metric. This leads to the following basic concept.

Definition 1.1. A **spectral triple** is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a pre-C^*-algebra, $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators, and D is a selfadjoint operator on $\mathcal{A}$, with compact resolvent, such that the commutator $[D, a]$ is a bounded operator on $\mathcal{H}$, for each $a \in \mathcal{A}$.

Spectral triples comes in two parities, odd and even. In the odd case, there is nothing new; in the even case, there is a grading operator χ on $\mathcal{H}$ (a bounded selfadjoint operator satisfying $\chi^2 = 1$, making a splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$), such that the representation of $\mathcal{A}$ is even ($\chi a = a\chi$ for all $a \in \mathcal{A}$) and the operator D is odd, i.e., $\chi D = -D\chi$; thus each $[D, a]$ is a bounded odd operator on $\mathcal{H}$.

A **noncommutative spin geometry** is a spectral triple satisfying several extra conditions, which were first laid out by Connes in the seminal paper [25]. These conditions (or "axioms", as they are sometimes called) arise from a careful consideration of the algebraic properties of ordinary metric geometry. Seven such properties are put forward in [25]; here, we shall just outline the list. Some of the terminology will be clarified later on; a more complete account, with all prerequisites, is given in [52, §10.5].

1. *Classical dimension*: There is a unique nonnegative integer n, the "classical dimension" of the geometry, for which the eigenvalue sums $\sigma_N := \sum_{0 \leqslant k < N} \mu_k$ of the compact positive operator $|D|^{-n}$ satisfy $\sigma_N \sim C \log N$ as $N \to \infty$, with $0 < C < \infty$; the coefficient is written $C = \fint |D|^{-n}$, where $\fint$ denotes the "Dixmier trace" if $n \geqslant 1$. This n is even if and only if the spectral triple is even. (When $\mathcal{A} = C^\infty(M)$ and D is a Dirac operator, n equals the ordinary dimension of the spin manifold M).

2. *Regularity*: Not only are the operators a and $[D, a]$ bounded, but they lie in the smooth domain of the derivation $\delta(T) := [|D|, T]$. (When $\mathcal{A}$ is an algebra of functions and D is a Dirac operator, this smooth domain consists exactly of the C^∞ functions.)

3. *Finiteness*: The algebra $\mathcal{A}$ is a pre-C^*-algebra, and the space of smooth vectors $\mathcal{H}^\infty := \bigcap_k \mathrm{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$-module. (In the commutative case, this yields the smooth spinors.)

4. *Reality*: There is an antiunitary operator C on $\mathcal{H}$, such that $[a, Cb^*C^{-1}] = 0$ for all $a, b \in \mathcal{A}$ (thus $b \mapsto Cb^*C^{-1}$ is a commuting representation on $\mathcal{H}$ of the "opposite algebra" $\mathcal{A}^\circ$, with the product reversed); and moreover, $C^2 = \pm 1$, $CD = \pm DC$, and $C\chi = \pm\chi C$ in the even case, where the signs depend only on $n \bmod 8$. (In the commutative case, C is the charge conjugation operator on spinors.)

5. *First order*: The bounded operators $[D, a]$ commute with the opposite algebra representation: $[[D, a], Cb^*C^{-1}] = 0$ for all $a, b \in \mathcal{A}$.

6. *Orientation*: There is a *Hochschild n-cycle* $\mathbf{c}$ on $\mathcal{A}$ whose natural representative is $\pi_D(\mathbf{c}) = \chi$ (even case) or $\pi_D(\mathbf{c}) = 1$ (odd case). More on this later: such an n-cycle is usually a finite sum of terms like $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ which map to operators

$$\pi_D(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := a_0 [D, a_1] \ldots [D, a_n],$$

and $\mathbf{c}$ is the algebraic expression of the *volume form* for the metric determined by D.

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7. *Poincaré duality*: The index map of D determines a nondegenerate pairing on the K-theory of the algebra $\mathcal{A}$. (We shall not go into details, except to mention that in the commutative case, the Chern homomorphism matches this nondegeneracy with Poincaré duality in de Rham co/homology.)

It is very important to know that when $A = C^\infty(M)$ the usual apparatus of geometry on spin manifolds (spin structure, metric, Dirac operator) can be fully recovered from these seven conditions: for the full proof of this theorem, see [52, Chap. 11]. Another proof, assuming only that $\mathcal{A}$ is commutative, is developed by Rennie in [83].

1.2 Hopf algebras: introduction

The general scheme of replacing point spaces by function algebras and then moving on to noncommutative algebras also works for symmetry groups. Now, however, the interplay of algebra and topology is much more delicate. There are at least two ways of handling this issue. One is to leave topology aside and develop a purely algebraic theory of symmetry-bearing algebras: these are the Hopf algebras, sometimes called "quantum groups", about which there is already a vast literature. At the other extreme, one may insist on using C^*-algebras with special properties; in the unital case, there has emerged a useful theory of "compact quantum groups" [113], which only recently has been extended to the locally compact case also [66].

We begin with the more algebraic treatment, keeping to the compact case, i.e., all algebras will be unital unless indicated otherwise. The field of scalars may be taken as $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$, according to convenience; to cover all cases, we shall denote it by $\mathbb{F}$. In this section, $\otimes$ always means the algebraic tensor product.

Definition 1.2. A *bialgebra* is a vector space A over $\mathbb{F}$ which is both an algebra and a coalgebra in a compatible way. The *algebra* structure is given by $\mathbb{F}$-linear maps $m\colon A \otimes A \to A$ (the product) and $\eta\colon \mathbb{F} \to A$ (the unit map) where $xy := m(x, y)$ and $\eta(1) = 1_A$. The *coalgebra* structure is likewise given by linear maps $\Delta\colon A \to A \otimes A$ (the coproduct) and $\varepsilon\colon A \to \mathbb{F}$ (the counit map). We write $\iota\colon A \to A$, or sometimes ι_A, to denote the identity map on A. The required properties are:

1. Associativity: $m(m \otimes \iota) = m(\iota \otimes m) : A \otimes A \otimes A \to A$;

2. Unity: $m(\eta \otimes \iota) = m(\iota \otimes \eta) = \iota : A \to A$;

3. Coassociativity: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta : A \to A \otimes A \otimes A$;

4. Counity: $(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta = \iota : A \to A$;

5. Compatibility: Δ and ε are unital algebra homomorphisms.

The first two conditions, expressed in terms of elements x, y, z of A, say that $(xy)z = x(yz)$ and $1_A x = x 1_A = x$. The next two properties are obtained by "reversing the arrows". Commutativity may be formulated by using the "flip map" $\sigma\colon A \otimes A \to A \otimes A : x \otimes y \mapsto y \otimes x$: the bialgebra is *commutative* if $m\sigma = m : A \otimes A \to A$. Likewise, the bialgebra is called *cocommutative* if $\sigma\Delta = \Delta : A \to A \otimes A$.

The (co)associativity rules suggest the abbreviations

$$m^2 := m(m \otimes \iota) = m(\iota \otimes m), \qquad \Delta^2 := (\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

with obvious iterations $m^3 : A^{\otimes 4} \to A$, $\Delta^3 : A \to A^{\otimes 4}$; $m^r : A^{\otimes(r+1)} \to A$, $\Delta^r : A \to A^{\otimes(r+1)}$.

Exercise 1.1. If (C, Δ, ε) and $(C', \Delta', \varepsilon')$ are coalgebras, a counital coalgebra morphism between them is an $\mathbb{F}$-linear map $\ell : C \to C'$ such that $\Delta' \ell = (\ell \otimes \ell)\Delta$ and $\varepsilon' \ell = \varepsilon$. Show that the compatibility condition is equivalent to the condition that m and u are counital coalgebra morphisms. ◊

Definition 1.3. The vector space $\mathrm{Hom}(C, A)$ of $\mathbb{F}$-linear maps from a coalgebra (C, Δ, ε) to an algebra (A, m, η) has an operation of *convolution*: given two elements f, g of this space, the map $f * g \in \mathrm{Hom}(C, A)$ is defined as

$$f * g := m(f \otimes g)\Delta : C \to A.$$

Convolution is associative because

$$(f * g) * h = m((f * g) \otimes h)\Delta = m(m \otimes \iota)(f \otimes g \otimes h)(\Delta \otimes \iota)\Delta$$
$$= m(\iota \otimes m)(f \otimes g \otimes h)(\iota \otimes \Delta)\Delta = m(f \otimes (g * h))\Delta = f * (g * h).$$

This makes $\mathrm{Hom}(C, A)$ an algebra, whose unit is $\eta_A \varepsilon_C$:

$$f * \eta_A \varepsilon_C = m(f \otimes \eta_A \varepsilon_C)\Delta = m(\iota_A \otimes \eta_A)(f \otimes \iota_{\mathbb{F}})(\iota_C \otimes \varepsilon_C)\Delta = \iota_A f \iota_C = f,$$
$$\eta_A \varepsilon_C * f = m(\eta_A \varepsilon_C \otimes f)\Delta = m(\eta_A \otimes \iota_A)(\iota_{\mathbb{F}} \otimes f)(\varepsilon_C \otimes \iota_C)\Delta = \iota_A f \iota_C = f.$$

A *bialgebra morphism* is a linear map $\ell : H \to H'$ between two bialgebras, which is both a unital algebra homomorphism and a counital coalgebra homomorphism; that is, ℓ satisfies the four identities

$$\ell m = m'(\ell \otimes \ell), \qquad \ell \eta = \eta', \qquad \Delta' \ell = (\ell \otimes \ell)\Delta, \qquad \varepsilon' \ell = \varepsilon,$$

where the primes indicate coalgebra operations for H'.

A bialgebra morphism respects convolution, in the following ways; if $f, g \in \mathrm{Hom}(C, H)$ and $h, k \in \mathrm{Hom}(H', A)$ for some coalgebra C and some algebra A, then

$$\ell(f * g) = \ell m(f \otimes g)\Delta_C = m'(\ell \otimes \ell)(f \otimes g)\Delta_C = m'(\ell f \otimes \ell g)\Delta_C = \ell f * \ell g,$$
$$(h * k)\ell = m_A(h \otimes k)\Delta' \ell = m_A(h \otimes k)(\ell \otimes \ell)\Delta = m_A(h\ell \otimes k\ell)\Delta = h\ell * k\ell.$$

Definition 1.4. A **Hopf algebra** is a bialgebra H together with a (necessarily unique) convolution inverse S for the identity map $\iota = \iota_H$; the map S is called the **antipode** of H. Thus,

$$\iota * S = m(\iota \otimes S)\Delta = \eta\varepsilon, \qquad S * \iota = m(S \otimes \iota)\Delta = \eta\varepsilon.$$

A bialgebra morphism between Hopf algebras is automatically a Hopf algebra morphism, i.e., it exchanges the antipodes: $\ell S = S'\ell$. For that, it suffices to prove that these maps provide a left inverse and a right inverse for ℓ in $\mathrm{Hom}(H, H')$. Indeed, since the identity in $\mathrm{Hom}(H, H')$ is $\eta'\varepsilon$, it is enough to notice that

$$\ell S * \ell = \ell(S * \iota) = \ell\eta\varepsilon = \eta'\varepsilon = \eta'\varepsilon'\ell = (\iota' * S')\ell = \ell * S'\ell,$$

and associativity of convolution then yields

$$S'\ell = \eta'\varepsilon * S'\ell = \ell S * \ell * S'\ell = \ell S * \eta'\varepsilon = \ell S.$$

The antipode has an important pair of algebraic properties: it is an *antihomomorphism* for both the algebra and the coalgebra structures. Formally, this means

$$Sm = m\sigma(S \otimes S) \quad \text{and} \quad \Delta S = (S \otimes S)\sigma\Delta. \tag{1.5}$$

The first relation, evaluated on $a \otimes b$, becomes the familiar antihomomorphism property $S(ab) = S(b)S(a)$. We postpone its proof until a little later.

Example 1.1. The simplest example of a Hopf algebra is the "group algebra" $\mathbb{F}G$ of a finite group G. This is just the vector space over $\mathbb{F}$ with a basis labelled by the elements of G; the necessary linear maps are specified on this basis. The product is given by $m(x \otimes y) := xy$, linearly extending the group multiplication, and $\eta(1) := 1_G$ gives the unit map. The coproduct, counit and antipode satisfy $\Delta(x) := x \otimes x$, $\varepsilon(x) := 1$ and $S(x) := x^{-1}$, for each $x \in G$.

Exercise 1.2. In a general Hopf algebra H, a nonzero element g is called *grouplike* if $\Delta(g) := g \otimes g$. Show that this condition entails that g is invertible and that $\varepsilon(g) = 1$ and $S(g) = g^{-1}$. $\diamond$

There are two main "classical" examples of Hopf algebras: representative functions on a compact group and the enveloping algebra of a Lie algebra.

Example 1.2. Now let G be a compact topological group (most often, a Lie group), and let the scalar field $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. The Peter–Weyl theorem [13, III.3] shows that any unitary irreducible representation π of G is finite-dimensional, any matrix element $f(x) := \langle u \mid \pi(x)v \rangle$ is a continuous function on G, and the vector space $\mathcal{R}(G)$ generated by these matrix elements is a dense subalgebra (∗-subalgebra in the complex case) of $C(G)$. Elements of $\mathcal{R}(G)$ can be characterized as those continuous functions $f : G \to \mathbb{F}$ whose translates $f_t : x \mapsto f(t^{-1}x)$ generate a finite-dimensional subspace of $C(G)$; they are called *representative functions* on G.

The algebra $\mathcal{R}(G)$ is a G-bimodule in the sense of Wildberger [110] under left and right translation; indeed, it is the algebraic direct sum of the finite-dimensional irreducible G-subbimodules of $C(G)$.

The group structure of G makes $\mathcal{R}(G)$ a coalgebra. Indeed, we can identify the *algebraic* tensor product $\mathcal{R}(G) \otimes \mathcal{R}(G)$ with $\mathcal{R}(G \times G)$ in the obvious way —here is where the finite-dimensionality of the translates is used [52, Lemma 1.27]— by $(f \otimes g)(x, y) := f(x)g(y)$, and then

$$\Delta f(x, y) := f(xy) \tag{1.6}$$

defines a coproduct on $\mathcal{R}(G)$. The counit is $\varepsilon(f) := f(1)$, and the antipode is given by $Sf(x) := f(x^{-1})$.

Example 1.3. The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the quotient of the tensor algebra $\mathcal{T}(\mathfrak{g})$ by the two sided ideal I generated by the elements $XY - YX - [X, Y]$, for all $X, Y \in \mathfrak{g}$. (Here we write XY instead of $X \otimes Y$, to distinguish products within $\mathcal{T}(\mathfrak{g})$ from elements of $\mathcal{T}(\mathfrak{g}) \otimes \mathcal{T}(\mathfrak{g})$.) The coproduct and counit are defined on $\mathfrak{g}$ by

$$\Delta(X) := X \otimes 1 + 1 \otimes X, \tag{1.7}$$

and $\varepsilon(X) := 0$. These linear maps on $\mathfrak{g}$ extend to homomorphisms of the tensor algebra; for instance,

$$\Delta(XY) = \Delta(X)\Delta(Y) = XY \otimes 1 + X \otimes Y + Y \otimes X + 1 \otimes XY,$$

and thus

$$\Delta(XY - YX - [X, Y]) = (XY - YX - [X, Y]) \otimes 1 + 1 \otimes (XY - YX - [X, Y]),$$

so $\Delta(I) \subseteq I \otimes \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g}) \otimes I$. Clearly, $\varepsilon(I) = 0$, too. Therefore, I is both an ideal and a "coideal" in the full tensor algebra, so the quotient $\mathcal{U}(\mathfrak{g})$ is a bialgebra, in fact a Hopf algebra: the antipode is given by $S(X) := -X$.

From (1.7), the Hopf algebra $\mathcal{U}(\mathfrak{g})$ is clearly *cocommutative*. The word "universal" is appropriate because any Lie algebra homomorphism $\psi \colon \mathfrak{g} \to A$, where A is an unital associative algebra, extends uniquely (in the obvious way) to a unital algebra homomorphism $\Psi \colon \mathcal{U}(\mathfrak{g}) \to A$.

Example 1.4. Historically, an important example of a Hopf algebra is Woronowicz' q-deformation of SU(2). The compact group SU(2) consists of complex matrices $g = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}$, subject to the unimodularity condition $a^*a + c^*c = 1$. The matrix elements a and c, regarded as functions of g, generate the $*$-algebra $\mathcal{R}(\mathrm{SU}(2))$: that is, any matrix element of a unitary irreducible (hence finite-dimensional) representation of SU(2) is a polynomial in a, a^*, c, c^*.

Woronowicz found [111] a noncommutative $*$-algebra with two generators a and c, subject to the relations

$$ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c, \quad a^*a + c^*c = 1, \quad aa^* + q^2cc^* = 1,$$

where q is a *real* number, which can be taken in the range $0 < q \leqslant 1$. For the coalgebra structure, take Δ and ε be $*$-homomorphisms determined by

$$\Delta a := a \otimes a - qc^* \otimes c, \qquad \Delta c := c \otimes a + a^* \otimes c,$$

and $\varepsilon(a) := 1, \varepsilon(c) := 0$. One can check that, by applying Δ elementwise, the matrix $g := \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$ satisfies $\Delta(g) = g \otimes g$. The antipode S is the linear antihomomorphism determined by

$$S(a) := a^*, \quad S(a^*) := a, \quad S(c) := -qc, \quad S(c^*) := -q^{-1}c^*,$$

so that $x \mapsto S(x^*)$ is an antilinear homomorphism, indeed an involution: $S(S(x^*)^*) = x$ for all x. This last relation is a general property of Hopf algebras with an involution.

The initial interest of this example was that it could be represented by a $*$-algebra of bounded operators on a Hilbert space, whose closure was a C^*-algebra which could legitimately be called a *deformation* of $C(\mathrm{SU}(2))$; it has become known as $C(\mathrm{SU}_q(2))$. In this way, the "quantum group" $\mathrm{SU}_q(2)$ was born. Nowadays, many q-deformations of the classical groups are known, although q may not always be real: for example, to define $SL_q(2, \mathbb{R})$, one needs selfadjoint generators a and c satisfying $ac = qca$, which is only possible if q is a complex number of modulus 1.

▶ If $u_{ij}(x) := \langle e_i \mid \pi(x)e_j \rangle$, for $i, j = 1, \ldots, n$, are the matrix elements of an n-dimensional irreducible representation of a compact group G with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$, then (1.6) and $\pi(xy) = \pi(x)\pi(y)$ show that

$$\Delta u_{ij} = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \tag{1.8a}$$

and the coassociativity of Δ is manifested as

$$\Delta^2 u_{ij} = \sum_{k,l} u_{ik} \otimes u_{kl} \otimes u_{lj}, \tag{1.8b}$$

reflecting the associativity of matrix multiplication. This may be generalized by a notational trick due to Sweedler [100]: if a is an element of any Hopf algebra, we write

$$\Delta a =: \sum a_{:1} \otimes a_{:2} \quad \text{(finite sum)}.$$

(The prevalent custom is to write $\Delta a = \sum a_{(1)} \otimes a_{(2)}$, leading to a surfeit of parentheses.) The equality of $(\Delta \otimes \iota)(\Delta a) = \sum a_{:1:1} \otimes a_{:1:2} \otimes a_{:2}$ and $(\iota \otimes \Delta)(\Delta a) = \sum a_{:1} \otimes a_{:2:1} \otimes a_{:2:2}$ is expressed by rewriting both sums as

$$\Delta^2 a = \sum a_{:1} \otimes a_{:2} \otimes a_{:3}.$$

The matricial coproduct (1.8b) is a particular instance of this notation. The counit and antipode properties can now be rewritten as

$$\sum \varepsilon(a_{:1}) a_{:2} = \sum a_{:1} \varepsilon(a_{:2}) = a, \tag{1.9a}$$

$$\sum S(a_{:1}) a_{:2} = \sum a_{:1} S(a_{:2}) = \varepsilon(a) 1. \tag{1.9b}$$

The coalgebra antihomomorphism property of S is expressed as

$$\Delta(S(a)) = \sum S(a_{:2}) \otimes S(a_{:1}). \tag{1.10}$$

We can now prove the antipode properties (1.5). We show that $Sm \colon a \otimes b \mapsto S(ab)$ and $m\sigma(S \otimes S) \colon a \otimes b \mapsto S(b)S(a)$ are one-sided convolution inverses for m in $\mathrm{Hom}(H \otimes H, H)$, so they must coincide. The coproduct in $H \otimes H$ is $(\iota \otimes \sigma \otimes \iota)(\Delta \otimes \Delta) \colon a \otimes b \mapsto \sum a_{:1} \otimes b_{:1} \otimes a_{:2} \otimes b_{:2}$, and so

$$(Sm * m)(a \otimes b) = m(Sm \otimes m)\left(\sum a_{:1} \otimes b_{:1} \otimes a_{:2} \otimes b_{:2}\right) = \sum S(a_{:1} b_{:1}) a_{:2} b_{:2}$$
$$= (S * \iota)(ab) = \eta\varepsilon(ab) = \eta\varepsilon_{H\otimes H}(a \otimes b).$$

On the other hand, writing $\tau := m\sigma(S \otimes S)$,

$$(m * \tau)(a \otimes b) = m(m \otimes \tau)\left(\sum a_{:1} \otimes b_{:1} \otimes a_{:2} \otimes b_{:2}\right) = \sum a_{:1} b_{:1} S(b_{:2}) S(a_{:2})$$
$$= \varepsilon(b) \sum a_{:1} S(a_{:2}) = \varepsilon(a)\varepsilon(b) 1_H = \eta\varepsilon(ab) = \eta\varepsilon_{H\otimes H}(a \otimes b).$$

Thus, $Sm * m = \eta_H \varepsilon_{H\otimes H} = m * \tau$, as claimed. In like fashion, one can verify (1.10) by showing that $\Delta S * \Delta = \eta_{H\otimes H}\varepsilon = \Delta * ((S \otimes S)\sigma\Delta)$ in $\mathrm{Hom}(H, H \otimes H)$; we leave the details to the reader.

Exercise 1.3. Carry out the verification of $\Delta S = (S \otimes S)\sigma\Delta$. $\diamond$

Notice that in the examples $H = \mathcal{R}(G)$ and $H = \mathcal{U}(\mathfrak{g})$, the antipode satisfies $S^2 = \iota_H$, but this does not hold in the $\mathrm{SU}_q(2)$ case. We owe the following remark to Matthias Mertens [72, Satz 2.4.2]: $S^2 = \iota_H$ if and only if

$$\sum S(a_{:2}) a_{:1} = \sum a_{:2} S(a_{:1}) = \varepsilon(a) 1 \quad \text{for all} \quad a \in H. \tag{1.11}$$

Indeed, if $S^2 = \iota_H$, then

$$\sum S(a_{:2})\, a_{:1} = \sum S(a_{:2})\, S^2(a_{:1}) = S\big(\sum S(a_{:1})\, a_{:2}\big) = S(\varepsilon(a)\, 1) = \varepsilon(a)\, 1,$$

while the relation $\sum S(a_{:2})\, a_{:1} = \varepsilon(a)\, 1$ implies that

$$(S * S^2)(a) = \sum S(a_{:1})\, S^2(a_{:2}) = S\big(\sum S(a_{:2})\, a_{:1}\big) = S(\varepsilon(a)\, 1) = \varepsilon(a)\, 1,$$

so that (1.11) entails $S * S^2 = S^2 * S = \eta\varepsilon$, hence $S^2 = \iota_H$ is the (unique) convolution inverse for S. Now, the relations (1.11) clearly follow from (1.9b) if H is either commutative or cocommutative (in the latter case, $\Delta a = \sum a_{:1} \otimes a_{:2} = \sum a_{:2} \otimes a_{:1}$). It follows that $S^2 = \iota_H$ if H is either commutative or cocommutative.

▶ Just as locally compact but noncompact spaces are described by nonunital function algebras, one may expect that locally compact but noncompact groups correspond to some sort of "nonunital Hopf algebras". The lack of a unit requires substantial changes in the formalism. At the purely algebraic level, an attractive alternative is the concept of "multiplier Hopf algebra" due to van Daele [103,104].

If A is an algebra whose product is nondegenerate, that is, $ab = 0$ for all b only if $a = 0$, and $ab = 0$ for all a only if $b = 0$, then there is a unital algebra $M(A)$ such that $A \subseteq M(A)$, called the *multiplier algebra* of A, characterized by the property that $xa \in A$ and $ax \in A$ whenever $x \in M(A)$ and $a \in A$. Here, $M(A) = A$ if and only if A is unital. A coproduct on A is defined as a homomorphism $\Delta \colon A \to M(A \otimes A)$ such that, for all $a, b, c \in A$,

$$(\Delta a)(1 \otimes b) \in A \otimes A, \quad \text{and} \quad (a \otimes 1)(\Delta b) \in A \otimes A,$$

and the following coassociativity property holds:

$$(a \otimes 1 \otimes 1)\, (\Delta \otimes \iota)\, ((\Delta b)(1 \otimes c)) = (\iota \otimes \Delta)\, ((a \otimes 1)(\Delta b))\, (1 \otimes 1 \otimes c).$$

There are then two well-defined linear maps from $A \otimes A$ into itself:

$$T_1(a \otimes b) := (\Delta a)(1 \otimes b), \quad \text{and} \quad T_2(a \otimes b) := (a \otimes 1)(\Delta b).$$

We say that A is a *multiplier Hopf algebra* [103] if T_1 and T_2 are bijective.

When A is a (unital) Hopf algebra, one finds that $T_1^{-1}(a \otimes b) = ((\iota \otimes S)\Delta a)(1 \otimes b)$ and $T_2^{-1}(a \otimes b) = (a \otimes 1)((S \otimes \iota)\Delta b)$. In fact,

$$
\begin{aligned}
T_1\big(((\iota \otimes S)\Delta a)(1 \otimes b)\big) &= \sum T_1(a_{:1} \otimes S(a_{:2})b) = \sum a_{:1} \otimes a_{:2}S(a_{:3})b \\
&= \sum a_{:1} \otimes \varepsilon(a_{:2})b = a \otimes b,
\end{aligned}
$$

and $T_2\big((a \otimes 1)((S \otimes \iota)\Delta b)\big) = a \otimes b$ by a similar argument. The bijectivity of T_1 and T_2 is thus a proxy for the existence of an antipode. It is shown in [103] that from the stated properties of Δ, T_1 and T_2, one can construct both a counit $\varepsilon \colon A \to \mathbb{F}$ and an antipode S, though the latter need only be an antihomomorphism from A to $M(A)$.

The motivating example is the case where A is an algebra of functions on a locally compact group G (with finite support, say, to keep the context algebraic), and $\Delta f(x, y) := f(xy)$ as before. Then $T_1(f \otimes g) \colon (x, y) \mapsto f(xy)g(y)$ also has finite support and the formula $(T_1^{-1}F)(x, y) := F(xy^{-1}, y)$ shows that T_1 is bijective; similarly for T_2. A fully topological theory, generalizing Hopf algebras to

include $C_0(G)$ for any locally compact group G and satisfying Pontryagin duality, is now available: the basic paper on that is [66].

▶ *Duality* is an important aspect of Hopf algebras. If (C, Δ, ε) is a coalgebra, the linear dual space $C^* := \mathrm{Hom}(C, \mathbb{F})$ is an algebra, as we have already seen, where the product $f \otimes g \mapsto (f \otimes g)\Delta$ is just the restriction of Δ^t to $C^* \otimes C^*$; the unit is ε^t, where ^{t} denotes transpose. (By convention, we do not write the multiplication in $\mathbb{F}$, implicit in the identification $\mathbb{F} \otimes \mathbb{F} \simeq \mathbb{F}$.) However, if (A, m, u) is an algebra, then (A^*, m^t, u^t) need not be a coalgebra because m^t takes A^* to $(A \otimes A)^*$ which is generally much larger than $A^* \otimes A^*$. Given a Hopf algebra $(H, m, u, \Delta, \varepsilon, S)$, we can replace H^* by the subspace $H^\circ := \{\, f \in H^* : m^t(f) \in H^* \otimes H^* \,\}$; one can check that $(H^\circ, \Delta^t, \varepsilon^t, m^t, u^t, S^t)$ is again a Hopf algebra, called the *finite dual* (or "Sweedler dual") of H.

To see why H° is a coalgebra, we must check that $m^t(H^\circ) \subseteq H^\circ \otimes H^\circ$. So suppose that $f \in H^*$ satisfies $m^t(f) = \sum_{j=1}^m g_j \otimes h_j$, a *finite* sum with $g_j, h_j \in H^*$. We may suppose that the g_j are linearly independent, so we can find elements $a_1, \dots, a_m \in H$ such that $g_j(a_k) = \delta_{jk}$. Now

$$h_k(ab) = \sum_{j=1}^m g_j(a_k) h_j(ab) = f(a_k ab) = \sum_{j=1}^m g_j(a_k a) h_j(b),$$

so $m^t(h_k) = \sum_{j=1}^m f_{jk} \otimes h_j$, where $f_{jk}(a) := g_j(a_k a)$; thus $h_k \in H^\circ$. A similar argument shows that each $g_j \in H^\circ$, too.

However, H° is often too small to be useful: in practice, one works with two Hopf algebras H and H', where each may be regarded as included in the dual of the other. That is to say, we can write down a bilinear form $\langle a, f \rangle := f(a)$ for $a \in H$ and $f \in H'$ with an implicit inclusion $H' \hookrightarrow H^*$. The transposing of operations between the two Hopf algebras boils down to the following five relations, for $a, b \in H$ and $f, g \in H'$:

$$\langle ab, f \rangle = \langle a \otimes b, \Delta' f \rangle, \qquad \langle a, fg \rangle = \langle \Delta a, f \otimes g \rangle, \qquad \langle S(a), f \rangle = \langle a, S'(f) \rangle,$$
$$\varepsilon(a) = \langle a, 1_{H'} \rangle, \quad \text{and} \quad \varepsilon'(f) = \langle 1_H, f \rangle.$$

The nondegeneracy conditions which allow us to assume that $H' \subseteq H^*$ and $H \subseteq H'^*$ are: (i) $\langle a, f \rangle = 0$ for all $f \in H'$ implies $a = 0$, and (ii) $\langle a, f \rangle = 0$ for all $a \in H$ implies $f = 0$.

Let G be a compact connected Lie group whose Lie algebra is $\mathfrak{g}$. The function algebra $\mathcal{R}(G)$ is a commutative Hopf algebra, whereas $\mathcal{U}(\mathfrak{g})$ is a cocommutative Hopf algebra. On identifying $\mathfrak{g}$ with the space of left-invariant vector fields on the group manifold G, we can realize $\mathcal{U}(\mathfrak{g})$ as the algebra of left-invariant differential operators on G. If $X \in \mathfrak{g}$, and $f \in \mathcal{R}(G)$, we define

$$\langle X, f \rangle := Xf(1) = \frac{d}{dt}\Big|_{t=0} f(\exp tX),$$

and more generally, $\langle X_1 \dots X_n, f \rangle := X_1(\cdots (X_n f))(1)$; we also set $\langle 1, f \rangle := f(1)$. This yields a duality between $\mathcal{R}(G)$ and $\mathcal{U}(\mathfrak{g})$. Indeed, the Leibniz rule for vector fields, namely $X(fh) = (Xf)h + f(Xh)$, gives

$$\langle X, fh \rangle = Xf(1)h(1) + f(1)Xh(1) = (X \otimes 1 + 1 \otimes X)(f \otimes h)(1 \otimes 1)$$
$$= \Delta X(f \otimes h)(1 \otimes 1) = \langle \Delta X, f \otimes h \rangle, \tag{1.12}$$

while

$$\langle X \otimes Y, \Delta f \rangle = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} (\Delta f)(\exp tX \otimes \exp sY) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(\exp tX \exp sY)$$

$$= \frac{d}{dt}\Big|_{t=0} (Yf)(\exp tX) = X(Yf)(1) = \langle XY, f \rangle.$$

If $\langle D, f \rangle = 0$ for all $D \in \mathcal{U}(\mathfrak{g})$, then f has a vanishing Taylor series at the identity of G. Since representative functions are real-analytic [62], this forces $f = 0$. On the other hand, if $\langle D, f \rangle = 0$ for all f, the left-invariant differential operator determined by D is null, so $D = 0$ in $\mathcal{U}(\mathfrak{g})$. The remaining properties are easily checked.

Definition 1.5. The relation (1.12) shows that $\Delta X = X \otimes 1 + 1 \otimes X$ encodes the Leibniz rule for vector fields. In any Hopf algebra H, an element $h \in H$ for which $\Delta h = h \otimes 1 + 1 \otimes h$ is called **primitive**. It follows that $\varepsilon(h) = 0$ and that $S(h) = -h$. In the enveloping algebra $\mathcal{U}(\mathfrak{g})$, elements of $\mathfrak{g}$ are obviously primitive. If a and b are primitive, then so is $ab - ba$, so the vector space $\mathrm{Prim}(H)$ of primitive elements of H is actually a Lie algebra.

Indeed, since the field of scalars $\mathbb{F}$ has characteristic zero, the *only* primitive elements of $\mathcal{U}(\mathfrak{g})$ are those in $\mathfrak{g}$, i.e., $\mathrm{Prim}(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}$: see [11], [52, Lemma 1.21] or [74, Prop. 5.5.3]. (Over fields of prime characteristic, there are other primitive elements in $\mathcal{U}(\mathfrak{g})$ [74].)

▶ If H is a bialgebra and A is an algebra, and if $\phi, \psi \colon H \to A$ are algebra homomorphisms, their convolution $\phi * \psi \in \mathrm{Hom}(H, A)$ is a linear map, and will be also a homomorphism provided that A is *commutative*. Indeed, $\phi * \psi = m(\phi \otimes \psi)\Delta$ is a composition of three homomorphisms in this case; the commutativity of A is needed to ensure that $m \colon A \otimes A \to A$ is multiplicative. A particularly important case arises when $A = \mathbb{F}$.

Definition 1.6. A **character** of an algebra is a nonzero linear functional which is also multiplicative, that is,

$$\mu(ab) = \mu(a)\,\mu(b) \quad \text{for all} \quad a, b;$$

notice that $\mu(1) = 1$. The counit ε of a bialgebra is a character. Characters of a bialgebra can be convolved, since $\mu * \nu = (\mu \otimes \nu)\Delta$ is a composition of homomorphisms. The characters of a Hopf algebra H form a **group** $\mathcal{G}(H)$ under convolution, whose neutral element is ε; the inverse of μ is μS.

A **derivation** or "infinitesimal character" of a Hopf algebra H is a linear map $\delta \colon H \to \mathbb{F}$ satisfying

$$\delta(ab) = \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) \quad \text{for all} \quad a, b \in H.$$

This entails $\delta(1_H) = 0$. The previous relation can also be written as $m^t(\delta) = \delta \otimes \varepsilon + \varepsilon \otimes \delta$, which shows that δ belongs to H° and is primitive there; in particular, the bracket $[\delta, \partial] := \delta * \partial - \partial * \delta$ of two derivations is again a derivation. Thus the vector space $\mathrm{Der}_\varepsilon(H)$ of derivations is actually a Lie algebra.

In the commutative case, there is another kind of duality to consider: one that matches a Hopf algebra with its character group. A compact topological group G admits a normalized left-invariant integral (the Haar integral): this can be thought of as a functional $J \colon \mathcal{R}(G) \to \mathbb{R}$, where the left-invariance translates as $(\iota \otimes J)\Delta = \eta J$. (We leave it as an exercise to show that this corresponds to

the usual definition of an invariant integral.) The evaluations at points of G supply all the characters of this Hopf algebra: $\mathcal{G}(\mathcal{R}(G)) \simeq G$. Conversely, if H is a commutative Hopf algebra possessing such a left-invariant functional J, then its character group is compact, and $H \simeq \mathcal{R}(\mathcal{G}(H))$. These results make up the Tannaka–Kreĭn duality theorem —for the proofs, see [52] or [55]— and it is important either to use real scalars, or to consider only hermitian characters if complex scalars are used. The totality of all $\mathbb{C}$-valued characters of $\mathcal{R}(G)$, hermitian or not, is a complex group $G^{\mathbb{C}}$ called the complexification of G [13, III.8]; for instance, if $G = \mathrm{SU}(n)$, then $G^{\mathbb{C}} \simeq SL(n, \mathbb{C})$.

▶ The action of vector fields in $\mathfrak{g}$ and differential operators in $\mathcal{U}(\mathfrak{g})$ on the space of smooth functions on G, and more generally on any manifold carrying a transitive action of the group G, leads to the notion of a *Hopf action* of a Hopf algebra H on an algebra A.

Definition 1.7. Let H be a Hopf algebra. A (left) **Hopf H-module algebra** A is an algebra which is a (left) module for the algebra H such that $h \cdot 1_A = \varepsilon(h)\, 1_A$ and

$$h \cdot (ab) = \sum (h_{:1} \cdot a)(h_{:2} \cdot b) \tag{1.13}$$

whenever $a, b \in A$ and $h \in H$.

Grouplike elements act by endomorphisms of A, since $g \cdot (ab) = (g \cdot a)(g \cdot b)$ and $g \cdot 1 = 1$ if g is grouplike. On the other hand, primitive elements of H act by the usual Leibniz rule: $h \cdot (ab) = (h \cdot a)b + a(h \cdot b)$ and $h \cdot 1 = 0$ if $\Delta h = h \otimes 1 + 1 \otimes h$. Thus (1.13) is a sort of generalized Leibniz rule.

▶ Duality suggests that an *action* of $\mathcal{U}(\mathfrak{g})$ should manifest itself as a *coaction* of $\mathcal{R}(G)$.

Definition 1.8. A vector space V is called a right *comodule* for a Hopf algebra H if there is a linear map $\Phi\colon V \to V \otimes H$ (the right *coaction*) satisfying

$$(\Phi \otimes \iota)\Phi = (\iota \otimes \Delta)\Phi : V \to V \otimes H \otimes H, \qquad (\iota \otimes \varepsilon)\Phi = \iota : V \to V. \tag{1.14}$$

In Sweedler notation, we may write the coaction as $\Phi(v) =: \sum v_{:0} \otimes v_{:1}$, so $\sum v_{:0}\, \varepsilon(v_{:1}) = v$ and $\sum v_{:0:0} \otimes v_{:0:1} \otimes v_{:1} = \sum v_{:0} \otimes v_{:1:1} \otimes v_{:1:2}$; we can rewrite both sides of the last equality as $\sum v_{:0} \otimes v_{:1} \otimes v_{:2}$, where, by convention, $v_{:r} \in H$ for $r \neq 0$ while $v_{:0} \in V$.

Left H-comodules are similarly defined; a linear map $\Phi\colon V \to H \otimes V$ is a left coaction if

$$(\iota \otimes \Phi)\Phi = (\Delta \otimes \iota)\Phi \quad \text{and} \quad (\varepsilon \otimes \iota)\Phi = \iota;$$

it is convenient to write $\Phi(v) =: \sum v_{:-1} \otimes v_{:0}$ in this case.

If a H-comodule A is also an algebra and if the coaction $\Phi\colon A \to A \otimes H$ is an algebra homomorphism, we say that A is a (right) *H-comodule algebra*. In this case, $\sum (ab)_{:0} \otimes (ab)_{:1} = \sum a_{:0}b_{:0} \otimes a_{:1}b_{:1}$.

If H and U are two Hopf algebras in duality, then any right H-comodule algebra A becomes a left U-module algebra, under

$$X \cdot a := \sum a_{:0}\, \langle X, a_{:1} \rangle,$$

for $X \in U$ and $a \in A$. In symbols: X acts as the operator $(\iota \otimes \langle X |)\Phi$ on A. Indeed, it is enough to note that

$$\begin{aligned}
X \cdot (ab) &= \sum a_{:0}b_{:0}\, \langle X, a_{:1}b_{:1} \rangle = \sum a_{:0}b_{:0}\, \langle \Delta X, a_{:1} \otimes b_{:1} \rangle \\
&= \sum a_{:0}b_{:0}\, \langle X_{:1} \otimes X_{:2}, a_{:1} \otimes b_{:1} \rangle = \sum a_{:0}\, \langle X_{:1}, a_{:1} \rangle\, b_{:0}\, \langle X_{:2}, b_{:1} \rangle \\
&= \sum (X_{:1} \cdot a)\,(X_{:2} \cdot b).
\end{aligned}$$

The language of coactions is used to formulate what one obtains by applying the Gelfand cofunctor (loosely speaking) to the concept of a *homogeneous space under a group action*. If a compact group G acts transitively on a space M, one can write $M \approx G/K$, where K is the closed subgroup fixing a basepoint $z_0 \in M$ (i.e., K is the "isotropy subgroup" of z_0). Then any function on M is obtained from a function on G which is constant on right cosets of K. If $\mathcal{F}(G)$ and $\mathcal{F}(M)$ denote suitable algebras of functions on G and M (we shall be more precise about these algebras in a moment), then there is a corresponding algebra of right K-invariant functions

$$\mathcal{F}(G)^K := \{ \, f \in \mathcal{F}(G) : f(xw) = f(x) \text{ whenever } w \in K, \ x \in G \, \}.$$

If $\bar{x} \in M$ corresponds to the right coset xK in G/K, then

$$\zeta f(\bar{x}) := f(x)$$

defines an algebra isomorphism $\zeta \colon \mathcal{F}(G)^K \to \mathcal{F}(M)$. [For aesthetic reasons, one may prefer to work with left K-invariant functions; for that, one should instead identify M with the space $K \backslash G$ of left cosets of K.]

Suppose now that the chosen spaces of functions satisfy

$$\mathcal{F}(G) \otimes \mathcal{F}(M) \simeq \mathcal{F}(G \times M), \tag{1.15}$$

where $\otimes$ denotes, as before, the *algebraic* tensor product. Then we can define $\rho \colon \mathcal{F}(M) \to \mathcal{F}(G) \otimes \mathcal{F}(M)$ by $\rho f(x, \bar{y}) := f(\overline{xy})$. It follows that

$$[\rho \zeta f](x, \bar{y}) = \zeta f(\overline{xy}) = f(xy) = \Delta f(x, y) = [(\iota \otimes \zeta)\Delta f](x, \bar{y}), \tag{1.16}$$

so that $\rho \zeta = (\iota \otimes \zeta)\Delta : \mathcal{F}(G)^K \to \mathcal{F}(G) \otimes \mathcal{F}(M)$. Notice, in passing, that the coproduct Δ maps $\mathcal{F}(G)^K$ into $\mathcal{F}(G) \otimes \mathcal{F}(G)^K$, which consists of functions h on $G \times G$ such that $h(x, yw) = h(x, y)$ when $w \in K$. [Had we used left cosets and left-invariant functions, the corresponding relations would be $\Delta(\mathcal{F}(G)^K) \subseteq \mathcal{F}(G)^K \otimes \mathcal{F}(G)$, $\rho \colon \mathcal{F}(M) \to \mathcal{F}(M) \otimes \mathcal{F}(G)$, and $\rho \zeta = (\zeta \otimes \iota)\Delta$.] In Hopf algebra language, ρ defines a left [or right] coaction of $\mathcal{F}(G)$ on the algebra $\mathcal{F}(M)$, implementing the left [or right] action of the group G on M, and ζ intertwines this with left [or right] regular coaction on K-invariant functions induced by the coproduct Δ. We get an instance of the following definition.

Definition 1.9. In the lore of quantum groups —see, for instance, [61, §11.6]— a (left) **embedded homogeneous space** for a Hopf algebra H is a left H-comodule algebra A with coaction $\rho \colon A \to H \otimes A$, for which there exists a *subalgebra* $B \subseteq H$ and an algebra isomorphism $\zeta \colon B \to A$ such that $\rho \zeta = (\iota \otimes \zeta)\Delta : B \to H \otimes A$.

A right embedded homogeneous space is defined, *mutatis mutandis*, in the same way.

There are two ways to ensure that the relation (1.15) holds. One way is to choose $\mathcal{F}(G) := \mathcal{R}(G)$, which is a bona-fide Hopf algebra, and then to *define* $\mathcal{R}(M)$ as the image $\zeta(\mathcal{R}(G)^K)$ of the K-invariant representative functions. For instance, if $G = \mathrm{SU}(2)$ and $K = \mathrm{U}(1)$, so that $M \approx \mathbb{S}^2$ is the usual 2-sphere of spin directions, then $\mathcal{R}(G)$ is spanned by the matrix elements $\mathcal{D}^j_{mn}$ of the $(2j+1)$-dimensional unitary irreducible representations of $\mathrm{SU}(2)$: see [7], for example. Now $\mathcal{D}^j_{mn}$ is right $\mathrm{U}(1)$-invariant if and only if j is an integer (not a half-integer) and $n = 0$; moreover, the

functions $Y_{lm} := \sqrt{(2l+1)/4\pi}\, \mathcal{D}_{m0}^{l*}$ are the usual spherical harmonics on the 2-sphere. In other words: $\mathcal{R}(\mathbb{S}^2)$ is the algebra of spherical harmonics on $\mathbb{S}^2$.

▶ To move closer to noncommutative geometry, it would be better to use either *continuous* functions (at the C^*-algebra level) or *smooth* functions on G and M; that is, one should work with $\mathcal{F} = \mathbb{C}$ or with $\mathcal{F} = C^\infty$. Notice that formulas like (1.16) make perfect sense in those cases; but the tensor product relation (1.15) is false in the continuous or smooth categories, unless the algebraic $\otimes$ is replaced by a more suitable *completed* tensor product.

In the continuous case, for compact G and M, the relation

$$C(G) \otimes C(M) \simeq C(G \times M)$$

is valid, where $\otimes$ denotes the "minimal" tensor product of C^*-algebras. (There may be several compatible C^*-norms on a tensor product of two C^*-algebras; but they all coincide if the algebras are commutative.) In the smooth case, we may fall back on a theorem of Grothendieck [54], which says that

$$C^\infty(G) \, \widehat{\otimes} \, C^\infty(M) \simeq C^\infty(G \times M),$$

where $\widehat{\otimes}$ denotes the projective tensor product of Fréchet spaces. But then, it is necessary to go back and reexamine our definitions: for instance, the coproduct need only satisfy $\Delta(A) \subseteq A \otimes A$ for a completed tensor product, which is a much weaker statement than the original one — the formula $\Delta a = \sum a_{:1} \otimes a_{:2}$ need no longer be a finite sum, but only some kind of convergent series. The bad news is that, in the C^*-algebra case, the product map $m \colon A \otimes A \to A$ is usually not continuous; the counit ε and antipode S become unbounded linear maps and one must worry about their domains; and so on. We shall meet examples of these generalized Hopf algebras in subsection 4.2.

1.3 Hopf actions of differential operators: an example

The Hopf algebras which are currently of interest are typically neither commutative, like $\mathcal{R}(G)$, nor cocommutative, like $\mathcal{U}(\mathfrak{g})$. The enormous profusion of "quantum groups" which have emerged in the last twenty years provide many examples of such noncommutative, noncocommutative Hopf algebras: see [17,59,61,70] for catalogues of these. A class of Hopf algebras which are commutative but are not cocommutative were introduced a few years ago, first by Kreimer in a quantum field theory context [63], and independently by Connes and Moscovici [35] in connection with a local index formula for foliations; in both cases, the Hopf algebra becomes a device to organize complicated calculations. We shall discuss the QFT version at length in the next section; here we look at the geometric example first.

If one wishes to deal with gravity in a noncommutative geometric framework [26], one must be able to handle the geometrical invariants of spacetime under the action of local diffeomorphisms. We consider an oriented n-dimensional manifold M, without boundary. By local diffeomorphisms on M we mean diffeomorphisms $\psi \colon \operatorname{Dom} \psi \to \operatorname{Ran} \psi$, where both the domain $\operatorname{Dom} \psi$ and range $\operatorname{Ran} \psi$ are open subsets of M; and we shall always assume that ψ preserves the given orientation on M. Two such local diffeomorphisms can be composed if and only if the range of the first lies within the domain of the second, and any local diffeomorphism can be inverted: taken all together, they form what is called a pseudogroup. We let Γ be a subpseudogroup (with the discrete topology), and consider the pair (M, Γ).

The orbit space M/Γ has in most cases a very poor topology. The noncommutative geometry approach is to replace this singular space by an *algebra* which captures the action of Γ on M. The initial candidate, a "crossed product" algebra $C(M) \rtimes \Gamma$, still has a very complicated structure; but much progress can be made [22] by replacing M by the bundle $F \to M$ of oriented frames on M. This is a principal fibre bundle whose structure group is $\mathrm{GL}^+(n, \mathbb{R})$, the $n \times n$ matrices with positive determinant.

Any $\psi \in \Gamma$ admits a *prolongation* to the frame bundle described as follows. Let $x = (x_1, \ldots, x_n)$ be local coordinates on M and let $y = (y_1^1, y_1^2, \ldots, y_n^n)$ be local coordinates for the frame at x. To avoid a "debauch of indices", we mainly consider the 1-dimensional case, where $M \approx \mathbb{S}^1$ is a circle and F is a cylinder (but we use a matrix notation to indicate how to proceed for higher dimensions; the details for the general case are carefully laid out in [114]). Then ψ acts locally on F through $\tilde{\psi}$, given by

$$\tilde{\psi}(x, y) := (\psi(x), \psi'(x)y).$$

The point is that, while M need not carry any Γ-invariant measure, the top-degree differential form $\nu = y^{-2}\, dy \wedge dx$ on F is Γ-invariant:

$$\tilde{\psi}^* \nu = y^{-2} \psi'(x)^{-2}\, \psi'(x)\, dy \wedge \psi'(x)\, dx = \nu,$$

so we can build a Hilbert space $L^2(F, \nu)$ and represent the action of each $\psi \in \Gamma$ by the *unitary* operator U_ψ defined by $U_\psi \xi(x, y) := \xi(\tilde{\psi}^{-1}(x, y))$. It is slightly more convenient to work with the adjoint unitary operators $U_\psi^\dagger \xi(x, y) := \xi(\tilde{\psi}(x, y))$. These unitaries intertwine multiplication operators coming from functions on F (specifically, smooth functions with compact support) as follows:

$$U_\psi f U_\psi^\dagger = f^\psi, \quad \text{where} \quad f^\psi(x, y) := f(\tilde{\psi}^{-1}(x, y)). \tag{1.17}$$

The local action of Γ on F can be described in the language of smooth groupoids [38], or alternatively by introducing a "crossed product" algebra which incorporates the groupoid convolution. This is a pre-C^*-algebra $\mathcal{A}$ obtained by suitably completing the algebra

$$\mathrm{span}\{\, fU_\psi^\dagger : \psi \in \Gamma,\ f \in C_c^\infty(\mathrm{Dom}\,\tilde{\psi})\,\}.$$

The relation (1.17) gives the multiplication rule

$$(fU_\psi^\dagger)(gU_\phi^\dagger) = f(U_\psi^\dagger g U_\psi)U_\psi^\dagger U_\phi^\dagger = f(g \circ \tilde{\psi})U_{\phi\psi}^\dagger, \tag{1.18}$$

Any two such elements are composable, since the support of $f(g \circ \tilde{\psi})$ is a compact subset of $\mathrm{Dom}\,\tilde{\psi} \cap \tilde{\psi}^{-1}(\mathrm{Dom}\,\tilde{\phi}) \subseteq \mathrm{Dom}(\tilde{\phi}\tilde{\psi})$.

This construction is called the *smash product* in the Hopf algebra books: if H is a Hopf algebra and A is a left Hopf H-module algebra, the smash product is the algebra $A \# H$ which is defined as the vector space $A \otimes H$ with the product rule

$$(a \otimes h)(b \otimes k) := \sum a(h_{:1} \cdot b) \otimes h_{:2}k.$$

If h is a grouplike element of H, this reduces to $(a \otimes h)(b \otimes k) := a(h \cdot b) \otimes hk$, of which (1.18) is an instance.

A local basis $\{X, Y\}$ of vector fields on the bundle F is defined by the "vertical" vector field $Y := y\, \partial/\partial y$, generating translations along the fibres, and the "horizontal" vector field $X := y\, \partial/\partial x$,

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generating displacements transverse to the fibres. In higher dimensions, the basis contains n^2 vertical vector fields Y_j^i and n horizontal vector fields X_k [114]. Under the lifted action of Γ, Y is invariant:

$$\tilde{\psi}_* Y = \psi'(x) y \, \frac{\partial}{\partial \psi'(x) y} = y \, \frac{\partial}{\partial y} = Y,$$

but X is not. To see that, consider the 1-forms $\alpha := y^{-1} \, dx$ and $\omega := y^{-1} \, dy$. The form α is the so-called canonical 1-form on F, which is invariant since $\tilde{\psi}^* \alpha = y^{-1} \psi'(x)^{-1} \, d\psi(x) = y^{-1} \, dx = \alpha$, whereas ω is not invariant:

$$\tilde{\psi}^* \omega = y^{-1} \, dy + \psi'(x)^{-1} \, d\psi'(x) = y^{-1} \, dy + \frac{\psi''(x)}{\psi'(x)} \, dx.$$

This transformation rule shows that ω is a connection 1-form on the principal bundle $F \to M$; and the horizontality of X means, precisely, that $\omega(X) = 0$. Notice also that $\alpha(X) = 1$. Now the vector field $\tilde{\psi}_*^{-1} X$ can be computed from the two equations $\alpha(\tilde{\psi}_*^{-1} X) = \tilde{\psi}^* \alpha(\tilde{\psi}_*^{-1} X) = \alpha(X) = 1$ and $\tilde{\psi}^* \omega(\tilde{\psi}_*^{-1} X) = \omega(X) = 0$; we get

$$\tilde{\psi}_*^{-1} X = y \, \frac{\partial}{\partial x} - y^2 \frac{\psi''(x)}{\psi'(x)} \frac{\partial}{\partial y} = X - h_\psi Y, \tag{1.19a}$$

where

$$h_\psi(x, y) := y \, \frac{\psi''(x)}{\psi'(x)} = y \, \frac{\partial}{\partial x} \big(\log \psi'(x) \big). \tag{1.19b}$$

Any vector field Z on F determines a linear operator on $\mathcal{A}$, also denoted by Z, by

$$Z(f U_\psi^\dagger) := (Zf) \, U_\psi^\dagger, \tag{1.20}$$

which makes sense since $\operatorname{supp}(Zf) \subseteq \operatorname{supp} f \subset \operatorname{Dom} \tilde{\psi}$. When applied to products, this operator gives

$$\begin{aligned}
Z(f U_\psi^\dagger \, g U_\phi^\dagger) &= Z(f (g \circ \tilde{\psi})) U_{\phi\psi}^\dagger = (Zf)(g \circ \tilde{\psi}) U_{\phi\psi}^\dagger + f Z(g \circ \tilde{\psi}) U_{\phi\psi}^\dagger \\
&= (Zf) U_\psi^\dagger \, g U_\phi^\dagger + f U_\psi^\dagger \, (Z(g \circ \tilde{\psi}) \circ \tilde{\psi}^{-1}) U_\phi^\dagger \\
&= (Zf) U_\psi^\dagger \, g U_\phi^\dagger + f U_\psi^\dagger \, \tilde{\psi}_* Z(g) U_\phi^\dagger.
\end{aligned} \tag{1.21}$$

Since the vector field Y is invariant, $\tilde{\psi}_* Y = Y$, so the lifted operator Y is a derivation on the algebra $\mathcal{A}$:

$$Y(f U_\psi^\dagger \, g U_\phi^\dagger) = (Yf) U_\psi^\dagger \, g U_\phi^\dagger + f U_\psi^\dagger \, (Yg) U_\phi^\dagger,$$

Proposition 1.1. *The operator X on $\mathcal{A}$ is* not *a derivation; however, there is a derivation λ_1 on $\mathcal{A}$ such that X obeys the generalized Leibniz rule*

$$X(ab) = X(a)b + aX(b) + \lambda_1(a)Y(b) \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{1.22}$$

Proof. Using the invariance of Y and (1.19a), we get

$$\tilde{\psi}_* X - X = \tilde{\psi}_* (X - \tilde{\psi}_*^{-1} X) = \tilde{\psi}_* (h_\psi Y) = (h_\psi \circ \tilde{\psi}^{-1}) Y,$$

and it follows that

$$f U_\psi^\dagger (\tilde{\psi}_* X(g) - Xg) U_\phi^\dagger = f U_\psi^\dagger (h_\psi \circ \tilde{\psi}^{-1})(Yg) U_\phi^\dagger = f\, h_\psi U_\psi^\dagger (Yg) U_\phi^\dagger.$$

If we define

$$\lambda_1(f U_\psi^\dagger) := h_\psi f U_\psi^\dagger, \tag{1.23}$$

then (1.21) for $Z = X$ now reads

$$X(f U_\psi^\dagger\, g U_\phi^\dagger) = X(f U_\psi^\dagger)\, g U_\phi^\dagger + f U_\psi^\dagger\, X(g U_\phi^\dagger) + \lambda_1(f U_\psi^\dagger)\, Y(g U_\phi^\dagger).$$

Thus, (1.22) holds on generators. We leave the reader to check that the formula extends to finite products of generators, provided that λ_1 is indeed a derivation. Now (1.19b) implies

$$h_{\phi\psi}(x, y) = y\, \frac{\partial}{\partial x} \big(\log \phi'(\psi(x)) + \log \psi'(x)\big) = h_\phi(\tilde{\psi}(x, y)) + h_\psi(x, y),$$

so that $h_{\phi\psi} = \tilde{\psi}^* h_\phi + h_\psi$, and the derivation property of λ_1 follows:

$$
\begin{aligned}
\lambda_1(f U_\psi^\dagger\, g U_\phi^\dagger) &= (\tilde{\psi}^* h_\phi + h_\psi)\, f(g \circ \tilde{\psi}) U_{\phi\psi}^\dagger \\
&= f\left((h_\phi g) \circ \tilde{\psi}\right) U_{\phi\psi}^\dagger + h_\psi f U_\psi^\dagger\, g U_\phi^\dagger \\
&= (f U_\psi^\dagger)(h_\phi g U_\phi^\dagger) + (h_\psi f U_\psi^\dagger)(g U_\phi^\dagger). \qquad \square
\end{aligned}
$$

Consider now the Lie algebra obtained from the operators X, Y and λ_1. The vector fields X, Y have the commutator $[y\, \partial/\partial y, y\, \partial/\partial x] = y\, \partial/\partial x$ and the corresponding operators on $\mathcal{A}$ satisfy $[Y, X] = X$. Next, $[Y, \lambda_1](f U_\psi^\dagger) = f(Y h_\psi) U_\psi^\dagger$, and from $Y h_\psi = h_\psi$ we get $[Y, \lambda_1] = \lambda_1$. Similarly, $[X, \lambda_1](f U_\psi^\dagger) = f(X h_\psi) U_\psi^\dagger$, where $X h_\psi = y\, \partial/\partial x\big(y\, \psi''(x)/\psi'(x)\big) = y^2\, \partial^2/\partial x^2\big(\log \psi'(x)\big)$. Introduce

$$h_\psi^n = y^n\, \frac{d^n}{dx^n} \log \psi'(x),$$

for $n = 1, 2, \ldots$, and define $\lambda_n(f U_\psi^\dagger) := f h_\psi^n U_\psi^\dagger$, then $\lambda_2 = [X, \lambda_1]$ and by induction we obtain $\lambda_{n+1} = [X, \lambda_n]$ for all n. Clearly $Y h_\psi^n = n h_\psi^n$, which implies $[Y, \lambda_n] = n\lambda_n$. The operators λ_n commute among themselves. We have constructed a Lie algebra, linearly generated by X, Y, and all the λ_n.

We can make the associative algebra with these same generators into a Hopf algebra [35] by defining their coproducts as follows. Since Y and λ_1 act as derivations, they must be primitive:

$$\Delta Y := Y \otimes 1 + 1 \otimes Y, \tag{1.24a}$$

$$\Delta \lambda_1 := \lambda_1 \otimes 1 + 1 \otimes \lambda_1. \tag{1.24b}$$

The coproduct of X can be read off from (1.22):

$$\Delta X := X \otimes 1 + 1 \otimes X + \lambda_1 \otimes Y. \tag{1.24c}$$

Moreover, $\varepsilon(Y) = \varepsilon(\lambda_1) = 0$ since Y and λ_1 are primitive, and $\varepsilon(X) = 0$ since $X = [Y, X]$ is a commutator; moreover, $\varepsilon(\lambda_n) = 0$ for all $n \geqslant 2$ for the same reason. The commutation relations yield the remaining coproducts; for instance,

$$\Delta \lambda_2 := [\Delta X, \Delta \lambda_1] = \lambda_2 \otimes 1 + 1 \otimes \lambda_2 + \lambda_1 \otimes \lambda_1.$$

The antipode is likewise determined: $S(Y) = -Y$ and $S(\lambda_1) = -\lambda_1$ since Y and λ_1 are primitive, and $(\iota * S)(X) = \varepsilon(X)1 = 0$ gives $X + S(X) + \lambda_1 Y = 0$, so $S(X) = -X + \lambda_1 Y$. The relation $S(\lambda_{n+1}) = [S(\lambda_n), S(X)]$ yields all $S(\lambda_n)$ by induction.

Definition 1.10. The Hopf algebra H_{CM} generated as an algebra by X, Y and λ_1, with the coproduct determined by (1.24) and the indicated counit and antipode, will be called the *Connes–Moscovici Hopf algebra*.

Exercise 1.4. Show that the commutative subalgebra generated by $\{\lambda_n : n = 1, 2, 3, \dots\}$ is indeed a Hopf subalgebra which is not cocommutative. ◊

The example H_{CM} arose in connection with a local index formula computation, which is already very involved when the base space M has dimension 1 (the case treated above). In higher dimensions, one may start [114] with the vertical vector fields $Y^i_j = y^\mu_j \, \partial/\partial y^\mu_i$ and a matrix-valued connection 1-form $\omega^i_j = (y^{-1})^i_\mu (dy^\mu_j + \Gamma^\mu_{\alpha\beta} y^\alpha_j \, dx^\beta)$, which may be chosen torsion-free, with Christoffel symbols $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$. With respect to this connection form, there are horizontal vector fields $X_k = y^\mu_k (\partial/\partial x^\mu - \Gamma^\nu_{\alpha\mu} y^\alpha_j \, \partial/\partial y^j_\nu)$. One obtains the Lie algebra relations $[Y^j_i, Y^l_k] = \delta^j_k Y^l_i - \delta^l_i Y^j_k$ and $[Y^j_i, X_k] = \delta^j_k X_i$, involving "structure constants"; however, $[X_k, X_l] = R^i_{jkl} Y^j_i$ where R^i_{jkl} are the components of the curvature of the connection ω, and these coefficients are in general not constant, for $n > 1$.

At first, Connes and Moscovici decided to use flat connections only [35], which entails $[X_k, X_l] = 0$; then, on lifting the Y^i_j and the X_k using (1.20), a higher-dimensional analogue of H_{CM} is obtained. For instance, one gets [114]:

$$\Delta X_k = X_k \otimes 1 + 1 \otimes X_k + \lambda^i_{kj} \otimes Y^j_i,$$

where the λ^i_{kj} are derivations of the form (1.23).

A better solution was later found [38]: one can allow commutation relations like $[X_k, X_l] = R^i_{jkl} Y^j_i$ if one modifies the original setup to allow for "transverse differential operators with non-constant coefficients". The algebra $\mathcal{A}$ remains the same as before, but the base field $\mathbb{C}$ is replaced by the algebra $\mathcal{R} = C^\infty(F)$ of smooth functions on F. Now $\mathcal{A}$ is an $\mathcal{R}$-bimodule under the commuting left and right actions

$$\alpha(b) : fU^\dagger_\psi \mapsto b \cdot (fU^\dagger_\psi) := (bf)U^\dagger_\psi, \tag{1.25a}$$

$$\beta(b) : fU^\dagger_\psi \mapsto (fU^\dagger_\psi) \cdot b := (b \circ \tilde\psi) \cdot (fU^\dagger_\psi) = (f(b \circ \tilde\psi))U^\dagger_\psi. \tag{1.25b}$$

Letting H now denote the algebra of operators on $\mathcal{A}$ generated by these operators (1.25) and the previous ones (1.20), then we no longer have a Hopf algebra over $\mathbb{C}$, but $(H, \mathcal{R}, \alpha, \beta)$ gives an instance of a more general structure called a *Hopf algebroid* over $\mathcal{R}$ [69]. For instance, the coproduct is an $\mathcal{R}$-bimodule map from H into $H \otimes_\mathcal{R} H$, where elements of this range space satisfy $(h \cdot b) \otimes_\mathcal{R} k = h \otimes_\mathcal{R} (b \cdot k)$ by construction, for any $b \in \mathcal{R}$. Just as Hopf algebras are the noncommutative counterparts of groups, Hopf algebroids are the noncommutative counterparts of groupoids: see [69, 115] for instance. For the details of these recent developments, we refer to [38].

2 The Hopf Algebras of Connes and Kreimer

2.1 The Connes–Kreimer algebra of rooted trees

A very important Hopf algebra structure is the one found by Kreimer [63] to underlie the combinatorics of subdivergences in the computation of perturbative expansions in quantum field theory. Such calculations involve several layers of complication, and it is no small feat to remove one such layer by organizing them in terms of a certain coproduct: indeed, the corresponding antipode provides a method to obtain suitable counterterms. Instead of addressing this matter from the physical side, the approach taken here is algebraic, in order first to understand why the Hopf algebras which emerge are in the nature of things.

A given Feynman graph represents a multiple integral (say, over momentum space) where the integrand is assembled from a definite collection of Rules, and before renormalization will often be superficially divergent, as determined by power counting. Even if not itself divergent, it may well contain one or several subgraphs which yield divergent partial integrations: the first order of business is to catalogue and organize the various graphs according to this nesting of subdivergences. Kreimer's coproduct separates out the divergences of subgraphs from those of the overall graph. In consequence, when expressed in terms of suitable generators of a Hopf algebra, the coproduct turns out to be polynomial in its first tensor factor, but merely linear in the second factor, and is therefore highly noncocommutative. Our starting point is to find a source of Hopf algebras with this kind of noncocommutativity.

▶ We start with an apparently unrelated digression into the homological classification of (associative) algebras.

There is a natural homology theory for associative algebras, linked with the name of Hochschild. Given an algebra $\mathcal{A}$ over any field $\mathbb{F}$ of scalars, one forms a *complex* by setting $C_n(\mathcal{A}) := \mathcal{A}^{\otimes(n+1)}$, and defining the boundary operator $b\colon C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A})$ by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},$$

where the last term "turns the corner". By convention, $b = 0$ on $C_0(\mathcal{A}) = \mathcal{A}$. One checks that $b^2 = 0$ by cancellation. For instance, $b(a_0 \otimes a_1) := [a_0, a_1]$, while

$$b(a_0 \otimes a_1 \otimes a_2) := a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1.$$

There are two important variants of this definition. One comes from the presence of a "degenerate subcomplex" $D_\bullet(\mathcal{A})$ where, for each $n = 0, 1, 2, \ldots$, the elements of $D_n(\mathcal{A})$ are finite sums of terms of the form $a_0 \otimes \cdots \otimes a_j \otimes \cdots \otimes a_n$, with $a_j = 1$ for some $j = 1, 2, \ldots, n$; elements of the quotient $\Omega^n \mathcal{A} := C_n(\mathcal{A})/D_n(\mathcal{A}) = \mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n}$, where $\overline{\mathcal{A}} = \mathcal{A}/\mathbb{F}$, are sums of expressions $a_0\, da_1 \cdots da_n$ where $d(ab) = da\, b + a\, db$. The direct sum $\Omega^\bullet \mathcal{A} = \bigoplus_{n \geqslant 0} \Omega^n \mathcal{A}$ is the universal graded differential algebra generated by $\mathcal{A}$ in degree zero; using it, b can be rewritten as

$$b(a_0\, da_1 \cdots da_n) := a_0 a_1\, da_2 \cdots da_n + \sum_{j=1}^{n-1} (-1)^j a_0\, da_1 \cdots d(a_j a_{j+1}) \cdots da_n$$

$$+ (-1)^n a_n a_0\, da_1 \cdots da_{n-1}. \tag{2.1}$$

The second variant involves replacing the algebra $\mathcal{A}$ in degree 0 by any $\mathcal{A}$-bimodule $\mathcal{E}$, and taking $C_n(\mathcal{A}, \mathcal{E}) := \mathcal{E} \otimes \mathcal{A}^{\otimes n}$; in the formulas, the products $a_n a_0$ and $a_0 a_1$ make sense even when $a_0 \in \mathcal{E}$. We write its homology as $H_\bullet(\mathcal{A}, \mathcal{E})$ and abbreviate $HH_n(\mathcal{A}) := H_n(\mathcal{A}, \mathcal{A})$.

Hochschild *cohomology*, with values in an $\mathcal{A}$-bimodule $\mathcal{E}$, is defined using cochains in $C^n = C^n(\mathcal{A}, \mathcal{E})$, the vector space of n-linear maps $\psi \colon \mathcal{A}^n \to \mathcal{E}$; this itself becomes an $\mathcal{A}$-bimodule by writing $(a' \cdot \psi \cdot a'')(a_1, \ldots, a_n) := a' \cdot \psi(a_1, \ldots, a_n) \cdot a''$. The coboundary map $b \colon C^n \to C^{n+1}$ is given by

$$b\psi(a_1, \ldots, a_{n+1}) := a_1 \cdot \psi(a_2, \ldots, a_{n+1}) + \sum_{j=1}^{n} (-1)^j \psi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$
$$+ (-1)^{n+1} \psi(a_1, \ldots, a_n) \cdot a_{n+1}. \tag{2.2}$$

The standard case is $\mathcal{E} = \mathcal{A}^*$ as an $\mathcal{A}$-bimodule, where for $\psi \in \mathcal{A}^*$ we put $(a' \cdot \psi \cdot a'')(c) := \psi(a'' c a')$. Here, we identify $\psi \in C^n(\mathcal{A}, \mathcal{E})$ with the $(n+1)$-linear map $\varphi \colon \mathcal{A}^{n+1} \to \mathbb{C}$ given by $\varphi(a_0, a_1, \ldots, a_n) := \psi(a_1, \ldots, a_n)(a_0)$; then, from the first summand in (2.2) we get $a_1 \cdot \psi(a_2, \ldots, a_{n+1})(a_0) = \psi(a_2, \ldots, a_{n+1})(a_0 a_1) = \varphi(a_0 a_1, \ldots, a_{n+1})$, while the last summand gives $\psi(a_1, \ldots, a_n) \cdot a_{n+1}(a_0) = \psi(a_1, \ldots, a_n)(a_{n+1} a_0) = \varphi(a_{n+1} a_0, \ldots, a_n)$. In this case, (2.2) reduces to

$$b\varphi(a_0, \ldots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \varphi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, \ldots, a_n). \tag{2.3}$$

The n-th Hochschild cohomology group is denoted $H^n(\mathcal{A}, \mathcal{E})$ in the general case, and we also write $HH^n(\mathcal{A}) := H^n(\mathcal{A}, \mathcal{A}^*)$.

Suppose that $\mu \colon \mathcal{A} \to \mathbb{F}$ is a character of $\mathcal{A}$. We denote by $\mathcal{A}_\mu$ the bimodule obtained by letting $\mathcal{A}$ act on itself on the left by the usual multiplication, but on the right through μ:

$$a' \cdot c \cdot a'' := a' c \, \mu(a'') \quad \text{for all} \quad a', a'', c \in \mathcal{A}.$$

In (2.2), the last term on the right must be replaced by $(-1)^{n+1} \varphi(a_1, \ldots, a_n) \mu(a_{n+1})$.

▶ We return now to the Hopf algebra setting, by considering a dual kind of Hochschild cohomology for coalgebras. Actually, we now consider a *bialgebra* B; the dual of the coalgebra (B, Δ, ε) is an algebra B^*, and the unit map η for B transposes to a character η^t of B^*. Thus we may define the Hochschild cohomology groups $H^n(B^*, B^*_{\eta^t})$. An "n-cochain" now means a linear map $\ell \colon B \to B^{\otimes n}$ which transposes to an n-linear map $\varphi = (B^*)^n \to B^*$ by writing $\varphi(a_1, \ldots, a_n) := \ell^t(a_1 \otimes \cdots \otimes a_n)$. Its coboundary is defined by

$$\langle a_1 \otimes \cdots \otimes a_{n+1}, b\ell(x) \rangle := \langle b\varphi(a_1, \ldots, a_{n+1}), x \rangle, \qquad x \in B.$$

We compute $b\ell$ using (2.2). First,

$$\langle a_1 \cdot \varphi(a_2, \ldots, a_{n+1}), x \rangle = \langle a_1 \otimes \varphi(a_2, \ldots, a_{n+1}), \Delta x \rangle = \langle a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}, (\iota \otimes \ell)\Delta x \rangle.$$

Next, if $\Delta_j \colon B^{\otimes n} \to B^{\otimes(n+1)}$ is the homomorphism which applies the coproduct on the jth factor only, then $\langle \varphi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}), x \rangle = \langle a_1 \otimes \cdots \otimes a_{n+1}, \Delta_j(\ell(x)) \rangle$. Finally, notice that

$\langle\varphi(a_1,\ldots,a_n)\eta^t(a_{n+1}),x\rangle = \langle a_1\otimes\cdots\otimes a_{n+1},\ell(x)\otimes 1\rangle$. Thus the Hochschild coboundary operator simplifies to

$$b\ell(x) := (\iota\otimes\ell)\Delta(x) + \sum_{j=1}^{n}(-1)^j\Delta_j(\ell(x)) + (-1)^{n+1}\ell(x)\otimes 1. \tag{2.4}$$

In particular, a linear form $\lambda\colon B\to\mathbb{F}$ is a 0-cochain, and $b\lambda = (\iota\otimes\lambda)\Delta - \lambda\otimes 1$ is its coboundary; and a 1-*cocycle* is a linear map $\ell\colon B\to B$ satisfying

$$\Delta\ell = \ell\otimes 1 + (\iota\otimes\ell)\Delta. \tag{2.5}$$

The simplest example of a nontrivial 1-cocycle obeying (2.5) come from integration of polynomials in the algebra $B = \mathbb{F}[X]$; we make $\mathbb{F}[X]$ a cocommutative coalgebra by declaring the indeterminate X to be *primitive*, so that $\Delta(X) = X\otimes 1 + 1\otimes X$ and $\varepsilon(X) = 0$. We immediately get the binomial expansion $\Delta(X^k) = (\Delta X)^k = \sum_{j=0}^{k}\binom{k}{j}X^{k-j}\otimes X^j$. If λ is any linear form on $\mathbb{F}[X]$, then

$$b\lambda(X^k) = (\iota\otimes\lambda)\Delta(X^k) - \lambda(X^k)\otimes 1 = \sum_{j=1}^{k}\binom{k}{j}\lambda(X^{k-j})X^j,$$

so $b\lambda$ is a linear transformation of polynomials which does not raise the degree. Therefore, the integration map $\ell(X^k) := X^{k+1}/(k+1)$ is *not* a 1-coboundary, but it is a 1-cocycle:

$$\Delta(\ell(X^k)) = \frac{1}{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}X^{k+1-j}\otimes X^j = \frac{X^{k+1}}{k+1}\otimes 1 + \sum_{j=1}^{k+1}\frac{1}{j}\binom{k}{j-1}X^{k+1-j}\otimes X^j$$

$$= \ell(X^k)\otimes 1 + \sum_{r=0}^{k}\frac{1}{r+1}\binom{k}{r}X^{k-r}\otimes X^{r+1} = \ell(X^k)\otimes 1 + (\iota\otimes\ell)(\Delta(X^k)).$$

This simple example already shows what the "Hochschild equation" (2.5) is good for: it allows *a recursive definition of the coproduct* Δ, with the assistance of a degree-raising operation ℓ. Indeed, $\mathbb{F}[X]$ is a simple example of a *connected, graded bialgebra*.

Definition 2.1. A bialgebra $H = \bigoplus_{n=0}^{\infty}H^{(n)}$ is a **graded bialgebra** if it is graded both as an algebra and as a coalgebra:

$$H^{(m)}H^{(n)}\subseteq H^{(m+n)} \quad\text{and}\quad \Delta(H^{(n)})\subseteq\bigoplus_{p+q=n}H^{(p)}\otimes H^{(q)}. \tag{2.6}$$

It is called **connected** if the degree-zero piece consists of scalars only: $H^{(0)} = \mathbb{F}1 = \operatorname{im}\eta$.

In a connected graded bialgebra, we can write the coproduct with a modified Sweedler notation: if $a\in H^{(n)}$, then

$$\Delta a = a\otimes 1 + 1\otimes a + \sum a'_{:1}\otimes a'_{:2}, \tag{2.7}$$

where the terms $a'_{:1}$ and $a'_{:2}$ all have degrees between 1 and $n-1$. Indeed, for the counit equations (1.9a) to be satisfied, Δa must contain the terms $a\otimes 1$ in $H^{(n)}\otimes H^{(0)}$ and $1\otimes a$ in $H^{(0)}\otimes H^{(n)}$; the remaining terms have intermediate bidegrees. On applying $\varepsilon\otimes\iota$, we get $a = (\varepsilon\otimes\iota)(\Delta a) = \varepsilon(a)1 + a + \sum\varepsilon(a'_{:1})a'_{:2}$, so that $\varepsilon(a) = 0$ when $n\geqslant 1$: in a connected graded bialgebra, the "augmentation ideal" $\ker\varepsilon$ is $\bigoplus_{n=1}^{\infty}H^{(n)}$, so that $H = \mathbb{F}1\oplus\ker\varepsilon$.

In fact, H is a *Hopf* algebra, since the grading allows us to define the antipode recursively [73, §8]. Indeed, the equation $m(S \otimes \iota)\Delta = \eta\varepsilon$ may be solved thus: if $a \in H^{(n)}$, we can obtain $0 = \varepsilon(a)\, 1 = S(a) + a + \sum S(a'_{:1})\, a'_{:2}$, where each term $a'_{:1}$ has degree less than n, just by setting

$$S(a) := -a - \sum S(a'_{:1})\, a'_{:2}. \tag{2.8}$$

Likewise, $m(\iota \otimes T)\Delta = \eta\varepsilon$ is solved by setting $T(1) := 1$ and recursively defining $T(a) := -a - \sum T(a'_{:2})\, a'_{:1}$. It follows that $T = S * \iota * T = S$, so we have indeed constructed a convolution inverse for ι.

In the same way, if there is a 1-cocycle ℓ which raises the degree, then (2.5) gives a recursive recipe for the coproduct: start with $\Delta(1) := 1 \otimes 1$ in degree zero (since H is connected, that will suffice), and use

$$\Delta(\ell(a)) := \ell(a) \otimes 1 + (\iota \otimes \ell)\Delta(a)$$

as often as necessary. The point is that, at each level, coassociativity is maintained:

$$(\iota \otimes \Delta)\Delta(\ell(a)) = (\iota \otimes \Delta)(\ell(a) \otimes 1 + (\iota \otimes \ell)(\Delta a)) = \ell(a) \otimes 1 \otimes 1 + (\iota \otimes \Delta\ell)(\Delta a)$$
$$= \ell(a) \otimes 1 \otimes 1 + (\iota \otimes \ell)(\Delta a) \otimes 1 + (\iota \otimes \iota \otimes \ell)(\iota \otimes \Delta)(\Delta a),$$

whereas

$$(\Delta \otimes \iota)\Delta(\ell(a)) = (\Delta \otimes \iota)(\ell(a) \otimes 1 + (\iota \otimes \ell)(\Delta a))$$
$$= \ell(a) \otimes 1 \otimes 1 + (\iota \otimes \ell)(\Delta a) \otimes 1 + (\iota \otimes \iota \otimes \ell)(\Delta \otimes \iota)(\Delta a),$$

where we have used the trivial relation $(\Delta \otimes \iota)(\iota \otimes \ell) = (\iota \otimes \iota \otimes \ell)(\Delta \otimes \iota)$. The only remaining issues are (i) whether such a 1-cocycle ℓ exists; and (ii) whether any $c \in H^{(n+1)}$ is a sum of products of elements of the form $\ell(a)$ with a of degree at most n.

▶ Both questions are answered by producing a *universal* example of a pair (H, ℓ) consisting of a connected graded Hopf algebra and a 1-cocycle ℓ. It was pointed out by Connes and Kreimer [30] that their Hopf algebra of *rooted trees* gives precisely this universal example. (Kreimer had first introduced a Hopf algebra of "parenthesized words" [63], where the nesting of subdivergences was indicated by parentheses, but rooted trees are nicer, and both Hopf algebras are isomorphic by the same universality.)

Definition 2.2. A **rooted tree** is a tree (a finite, connected graph without loops) with oriented edges, in which all the vertices but one have exactly one incoming edge, and the remaining vertex, the *root*, has only outgoing edges.

Here are the rooted trees with at most four vertices (up to isomorphism). To draw them, we place the root at the top with a ⋄ symbol, and denote the other vertices with • symbols:

$$t_1 \quad t_2 \quad t_{31} \quad t_{32} \quad t_{41} \quad t_{42} \quad t_{43} \quad t_{44}$$
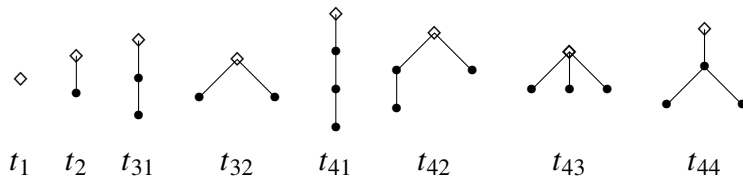
The **algebra of rooted trees** H_R is the commutative algebra generated by symbols T, one for each isomorphism class of rooted trees, plus a unit 1 corresponding to the empty tree. We shall write

the product of trees as the juxtaposition of their symbols. There is an obvious grading making H_R a graded algebra, by assigning to each tree T the number of its vertices $\#T$. The counit $\varepsilon : H_R \to \mathbb{F}$ is the linear map defined by $\varepsilon(1) := 1$ and $\varepsilon(T_1 T_2 \ldots T_n) = 0$ if $T_1, \ldots, T_n$ are trees; this ensures that $H_R = \mathbb{F}\,1 \oplus \ker \varepsilon$. To get a coproduct satisfying (2.7), we must give a rule which shows how a tree may be cut into subtrees with complementary sets of vertices. A *simple cut c* of a tree T is the removal of some of its edges, in such a way that along the path from the root to any vertex, at most one edge is removed. Here, for instance, are the possible simple cuts of t_{44}:

Among the subtrees of T produced by a simple cut, exactly one, the "trunk" $R_c(T)$, contains the root of T. The remaining "pruned" branches also form one or more rooted trees, whose product is denoted by $P_c(T)$. The formula for the coproduct can now be given, on the algebra generators, as

$$\Delta T := T \otimes 1 + 1 \otimes T + \sum_c P_c(T) \otimes R_c(T), \tag{2.9}$$

where the sum extends over all simple cuts of the tree T; as well as $\Delta 1 := 1 \otimes 1$, of course. Here are the coproducts of the trees listed above:

$$\Delta t_1 = t_1 \otimes 1 + 1 \otimes t_1,$$
$$\Delta t_2 = t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1,$$
$$\Delta t_{31} = t_{31} \otimes 1 + 1 \otimes t_{31} + t_2 \otimes t_1 + t_1 \otimes t_2,$$
$$\Delta t_{32} = t_{32} \otimes 1 + 1 \otimes t_{32} + 2 t_1 \otimes t_2 + t_1^2 \otimes t_1,$$
$$\Delta t_{41} = t_{41} \otimes 1 + 1 \otimes t_{41} + t_{31} \otimes t_1 + t_2 \otimes t_2 + t_1 \otimes t_{31},$$
$$\Delta t_{42} = t_{42} \otimes 1 + 1 \otimes t_{42} + t_1 \otimes t_{32} + t_2 \otimes t_2 + t_1 \otimes t_{31} + t_2 t_1 \otimes t_1 + t_1^2 \otimes t_2.$$
$$\Delta t_{43} = t_{43} \otimes 1 + 1 \otimes t_{43} + 3 t_1 \otimes t_{32} + 3 t_1^2 \otimes t_2 + t_1^3 \otimes t_1,$$
$$\Delta t_{44} = t_{44} \otimes 1 + 1 \otimes t_{44} + t_{32} \otimes t_1 + 2 t_1 \otimes t_{31} + t_1^2 \otimes t_2. \tag{2.10}$$

In this way, H_R becomes a connected graded commutative Hopf algebra; clearly, it is not cocommutative. In order to prove that this Δ is coassociative, we need only produce the appropriate 1-cocycle L which raises the degree by 1. The linear operator L – also known as B^+ [30] – is defined, on each product of trees, by sprouting a new common root.

Definition 2.3. Let $L \colon H_R \to H_R$ be the linear map given by $L(1) := t_1$ and

$$L(T_1 \ldots T_k) := T, \tag{2.11}$$

where T is the rooted tree obtained by conjuring up a new vertex as its root and extending edges from this vertex to each root of $T_1, \ldots, T_k$. Notice, in passing, that *any* tree T with $n + 1$ vertices equals $L(T_1 \cdots T_k)$, where $T_1, \ldots, T_k$ are the rooted trees, with n vertices in all, formed by removing every edge outgoing from the root of T.

For instance,

$$L\left(\vcenter{\hbox{$\wedge$ tree}} \right) = \vcenter{\hbox{tree}} \qquad \text{and} \qquad L\left(\vcenter{\hbox{tree tree}} \right) = \vcenter{\hbox{tree}}.$$

Checking the Hochschild equation (2.5) is a matter of bookkeeping: see [30, p. 229] or [52, p. 603], for instance. Here, an illustration will suffice:

$$\Delta\!\left(L\!\left(\vcenter{\hbox{tree}}\right)\right) = \Delta\!\left(\vcenter{\hbox{tree}}\right) = \vcenter{\hbox{tree}} \otimes 1 + 1 \otimes \vcenter{\hbox{tree}} + \vcenter{\hbox{tree}} \otimes \diamond + 2\,\diamond \otimes \vcenter{\hbox{tree}} + \diamond\diamond \otimes \vcenter{\hbox{tree}}$$

$$= L\!\left(\vcenter{\hbox{tree}}\right) \otimes 1 + (\iota \otimes L)\!\left(1 \otimes \vcenter{\hbox{tree}} + \vcenter{\hbox{tree}} \otimes 1 + 2\,\diamond \otimes \vcenter{\hbox{tree}} + \diamond\diamond \otimes \diamond\right)$$

$$= L\!\left(\vcenter{\hbox{tree}}\right) \otimes 1 + (\iota \otimes L)\Delta\!\left(\vcenter{\hbox{tree}}\right).$$

Finally, suppose that a pair (H, ℓ) is given; we want to define a Hopf algebra morphism $\rho \colon H_R \to H$ such that

$$\rho(L(a)) = \ell(\rho(a)), \tag{2.12}$$

where a is a product of trees. Since $L(a)$ may be any tree of degree $\#a + 1$, we may regard this as a recursive definition (on generators) of an algebra homomorphism, starting from $\rho(1) := 1_H$. The only thing to check is that it also yields a coalgebra homomorphism, which again reduces to an induction on the degree of a:

$$\begin{aligned}
\Delta(\rho(L(a))) = \Delta(\ell(\rho(a))) &= \ell(\rho(a)) \otimes 1 + (\iota \otimes \ell)\Delta(\rho(a)) \\
&= \ell(\rho(a)) \otimes 1 + (\iota \otimes \ell)(\rho \otimes \rho)(\Delta a) \\
&= \rho(L(a)) \otimes 1 + (\rho \otimes \rho)(\iota \otimes L)(\Delta a) \\
&= (\rho \otimes \rho)\big(L(a) \otimes 1 + (\iota \otimes L)(\Delta a)\big) = (\rho \otimes \rho)\Delta(L(a)),
\end{aligned}$$

where in the third line, by using $\ell(\rho(a'_{.2})) = \rho(L(a'_{.2}))$, we have implicitly relied on the property (2.7) that the nontrivial components of the coproduct Δa have lower degree than a.

▶ Since the Hopf algebra H_R is commutative, we may look for a cocommutative Hopf algebra in duality with it. Now, there is a structure theorem for connected graded cocommutative Hopf algebras, arising from contributions of Hopf, Samelson, Leray, Borel, Cartier, Milnor, Moore and Quillen,[2] commonly known as the Milnor–Moore theorem, which states that such a Hopf algebra H is necessarily isomorphic to $\mathcal{U}(\mathfrak{g})$, with $\mathfrak{g}$ being the Lie algebra of primitive elements of H.

This dual Hopf algebra is constructed as follows. Each rooted tree T gives not only an algebra generator for H_R, but also a derivation $Z_T \colon H_R \to \mathbb{F}$ defined by

$$\langle Z_T, T_1 \ldots T_k \rangle := 0 \quad \text{unless } k = 1 \text{ and } T_1 = T; \qquad \langle Z_T, T \rangle := 1.$$

Also, $\langle Z_T, 1 \rangle = 0$ since $Z_T \in \mathrm{Der}_\varepsilon(H)$ (Definition 1.6). Notice that the ideal generated by products of two or more trees is $(\ker \varepsilon)^2$, and any derivation δ vanishes there, since $\delta(ab) =$

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[2]The historical record is murky; this list of contributors is due to P. Cartier.

$\delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) = 0$ whenever $a, b \in \ker \varepsilon$. Therefore, derivations are determined by their values on the subspace $H_R^{(1)}$ spanned by single trees – which equals $L(H_R)$, by the way – and reduce to linear forms on this subspace; thus $\mathrm{Der}_\varepsilon(H)$ can be identified with the (algebraic) dual space $H_R^{(1)*}$. We denote by $\mathfrak{h}$ the linear subspace spanned by all the Z_T.

Let us compute the Lie bracket $[Z_R, Z_S] := (Z_R \otimes Z_S - Z_S \otimes Z_R)\Delta$ of two such derivations. Using (2.9) and $\langle Z_R, 1 \rangle = \langle Z_S, 1 \rangle = 0$, we get

$$\langle Z_R \otimes Z_S, \Delta T \rangle = \sum_c \langle Z_R, P_c(T) \rangle \langle Z_S, R_c(T) \rangle,$$

where $\langle Z_R, P_c(T) \rangle = 0$ unless $P_c(T) = R$ and $\langle Z_S, R_c(T) \rangle = 0$ unless $R_c(T) = S$; in particular, the sum ranges only over simple cuts which remove just *one* edge of T. Let $n(R, S; T)$ be the number of one-edge cuts c of T such that $P_c(T) = R$ and $R_c(T) = S$; then

$$\langle [Z_R, Z_S], T \rangle = \langle Z_R \otimes Z_S - Z_S \otimes Z_R, \Delta T \rangle = n(R, S; T) - n(S, R; T),$$

and this expression vanishes altogether except for the finite number of trees T which can be produced either by grafting R on S or by grafting S on R. Evaluation of the derivation $[Z_R, Z_S]$ on a product $T_1 \dots T_k$ of two or more trees gives zero, since each $T_j \in \ker \varepsilon$. Therefore,

$$[Z_R, Z_S] = \sum_T \big(n(R, S; T) - n(S, R; T)\big) Z_T,$$

which is a finite sum. In particular, $[Z_R, Z_S] \in \mathfrak{h}$, and so $\mathfrak{h}$ is a Lie subalgebra of $\mathrm{Der}_\varepsilon(H)$. The linear duality of $H_R^{(1)}$ with $\mathfrak{h}$ then extends to a duality between the graded Hopf algebras H_R and $\mathcal{U}(\mathfrak{h})$.

It is possible to give a more concrete description of the Hopf algebra $\mathcal{U}(\mathfrak{h})$ in terms of *another* Hopf algebra of rooted trees H_{GL}, which is cocommutative rather than commutative. This structure was introduced by Grossman and Larson [53] and is described in [52, §14.2]; here we mention only that the multiplicative identity is the tree t_1 and that the primitive elements are spanned by those trees which have only one edge outgoing from the root. Panaite [79] has shown that $\mathfrak{h}$ is isomorphic to the Lie algebra of these primitive trees – by matching each Z_T to the tree $L(T)$ – so that $\mathcal{U}(\mathfrak{h}) \simeq H_{GL}$.

In [30], another binary operation among the Z_T was introduced by setting

$$Z_R \star Z_S := \sum_T n(R, S; T) Z_T.$$

This is *not* the convolution $(Z_R \otimes Z_S)\Delta$, nor is it even associative, although it is obviously true that $Z_R \star Z_S - Z_S \star Z_R = [Z_R, Z_S]$. This nonassociative bilinear operation satisfies the defining property of a *pre-Lie algebra* [15]:

$$(Z_R \star Z_S) \star Z_T - Z_R \star (Z_S \star Z_T) = (Z_R \star Z_T) \star Z_S - Z_R \star (Z_T \star Z_S).$$

Indeed, both sides of this equation express the formation of new trees by grafting both S and T onto the tree R. The combinatorics of this operation are discussed in [16], and several computations with it are developed in [18] and [60].

▶ The characters of H_R form a group $\mathcal{G}(H_R)$ (under convolution): see Definition 1.6. This group is infinite-dimensional, and can be thought of as the set of grouplike elements in a suitable completion of the Hopf algebra $U = \mathcal{U}(\mathfrak{h})$. To see that, recall that U is a graded connected Hopf algebra; denote by e its counit. Then the sets $(\ker e)^m = \sum_{k \geqslant m} \mathfrak{h}^k$, for $m = 1, 2, \ldots$, form a basis of neighbourhoods of 0 for a vector space topology on U, and the grading properties (2.6) entail that all the Hopf operations are continuous for this topology. (The basic neighbourhoods of 0 in $U \otimes U$ are the powers of the ideal $1 \otimes \ker e + \ker e \otimes 1$.) We can form the *completion* $\widehat{U}$ of this topological vector space, which is again a Hopf algebra since all the Hopf operations extend by continuity; an element of $\widehat{U}$ is a series $\sum_{k \geqslant 0} z_k$ with $z_k \in \mathfrak{h}^k$ for each $k \in \mathbb{N}$, since the partial sums form a Cauchy sequence in U. The closure of $\mathfrak{h}$ within $\widehat{U}$ is $\mathrm{Der}_\varepsilon(H)$.

For example, consider the exponential given by $\varphi_T := \exp Z_T = \sum_{n \geqslant 0} (1/n!) \, Z_T^n$; in any evaluation

$$\varphi_T(T_1 \ldots T_k) = \sum_{n \geqslant 0} \frac{1}{n!} \langle Z_T^{\otimes n}, \Delta^{n-1}(T_1 \ldots T_k) \rangle,$$

the series has only finitely many nonzero terms. More generally, $\varphi := \exp \delta \in \widehat{U}$ makes sense for each $\delta \in \mathrm{Der}_\varepsilon(H)$; and $\varphi \in \mathcal{G}(H_R)$ since $\Delta \varphi = \exp(\Delta \delta) = \exp(\varepsilon \otimes \delta + \delta \otimes \varepsilon) = \varphi \otimes \varphi$ by continuity of Δ. In fact, the exponential map is a bijection between $\mathrm{Der}_\varepsilon(H)$ and $\mathcal{G}(H_R)$, whose inverse is provided by the logarithmic series $\log(1 - x) := - \sum_{k \geqslant 1} x^k / k$; for if μ is a character, the equation $\mu = \exp(\log \mu)$ holds in $\widehat{U}$, and

$$\Delta(\log \mu) = \Delta(\log(\varepsilon - (\varepsilon - \mu))) = \log(\varepsilon \otimes \varepsilon - \Delta(\varepsilon - \mu)) = \log(\mu \otimes \mu)$$
$$= \log(\varepsilon \otimes \mu) + \log(\mu \otimes \varepsilon) = \varepsilon \otimes \log \mu + \log \mu \otimes \varepsilon,$$

so that $\log \mu \in \mathrm{Der}_\varepsilon(H)$. See [55, Chap. X] or [56, Chap. XVI] for a careful discussion of the exponential map. In view of this bijection, we can regard the commutative Hopf algebra H_R as an algebra of affine coordinates on the group $\mathcal{G}(H_R)$, in the spirit of Tannaka–Kreĭn duality.

▶ In any Hopf algebra, whether cocommutative or not, the determination of the *primitive elements* plays an important part. If in any tree T, the longest path from the root to a leaf contains k edges, then the coproduct ΔT is a sum of at least $k + 1$ terms. In the applications to renormalization, T represents a possibly divergent integration with k nested subdivergences, while the primitive tree t_1 corresponds to an integration without subdivergences. A primitive algebraic *combination* of trees represents a collection of integrations where some of these divergences may cancel. For that reason alone, it would be desirable to describe all the primitive elements of H_R and then, as far as possible, to rebuild H_R from its primitives. This is a work in progress [12, 18, 46], which deserves a few comments here.

To begin with, since t_1 is primitive and $\Delta t_2 = t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1$, the combination $p_2 := t_2 - \frac{1}{2} t_1^2$ is also primitive. One can check that $p_3 := t_{31} - t_1 t_2 + \frac{1}{3} t_1^3$ is primitive, too.

For each $k = 1, 2, \ldots$, let t_k denote the "stick" tree with $k - 1$ edges and k vertices in a vertical progression. (In particular, t_3 and t_4 are the trees previously referred to as t_{31} and t_{41}, respectively.) A simple cut severs t_k into two shorter sticks, and so

$$\Delta t_k = \sum_{0 \leqslant r \leqslant k} t_r \otimes t_{k-r}, \tag{2.13}$$

with $t_0 := 1$ by convention. Thus the sticks generate a cocommutative graded Hopf subalgebra H_l of H_R.

To find the primitives in H_l, we follow the approach of [18]. Consider the formal power series $g(x) := \sum_{k \geqslant 0} t_k x^k$ whose coefficients are sticks. Then the equation (2.13) can be read as saying that $g(x)$ is grouplike in $H_l[\![x]\!]$, that is, $\Delta g(x) = g(x) \otimes g(x)$. If we can find a power series $p(x) = \sum_{r \geqslant 1} p_r x^r$, where each p_r is homogeneous of degree r in the grading of H_l, such that $\exp(p(x)) = g(x)$, the corresponding equation will be $\Delta p(x) = p(x) \otimes 1 + 1 \otimes p(x)$; on comparing coefficients of each x^r, we see that each p_r is primitive. The equation $\exp(p(x)) = g(x)$ is solved as

$$\sum_{r \geqslant 1} p_r x^r = \log\left(1 + \sum_{k \geqslant 1} t_k x^k\right),$$

by developing the Taylor series of $\log(1 + x)$. Since a monomial $t_1^{m_1} t_2^{m_2} \dots t_r^{m_r}$ has degree $m_1 + 2m_2 + \dots + rm_r$, the general formula [46, Prop. 9.3] is quickly found to be

$$p_r = \sum_{m_1 + 2m_2 + \dots + rm_r = r} (-1)^{m_1 + \dots + m_r + 1} \frac{(m_1 + \dots + m_r - 1)!}{m_1! \dots m_r!} t_1^{m_1} \dots t_r^{m_r},$$

where the sum ranges over the partitions of the positive integer r.

▶ Nonstick primitives are more difficult to come by, but an algorithm which provides many of them is found in [18], based on formal differential calculus. Indeed, this "differential" approach can be extended, in principle, to deal efficiently with the more elaborate Hopf algebras of Feynman diagrams discussed in the next subsection.

For each $a \in H_R$, the expression

$$\Pi_a := \sum S(a_{:1}) \, da_{:2} \tag{2.14}$$

where d denotes an ordinary exterior derivative, may be regarded as a 1-form on G; it is a straightforward generalization of the familiar (matrix-valued) 1-form $g^{-1} \, dg$ on a group manifold, whose matrix elements are $\sum_j (g^{-1})_{ij} \, dg_{jk}$. We can treat such expressions algebraically, as a "first-order differential calculus" on a Hopf algebra, in the sense of Woronowicz [112]. The commutativity of H_R shows that these 1-forms have the following derivation property:

$$\Pi_{ab} = \sum S(a_{:1} b_{:1}) \, d(a_{:2} b_{:2}) = \sum S(b_{:1}) S(a_{:1}) a_{:2} \, db_{:2} + S(b_{:1}) b_{:2} S(a_{:1}) \, da_{:2} = \varepsilon(a) \, \Pi_b + \Pi_a \, \varepsilon(b).$$

In particular, $\Pi_a = 0$ for $a \in (\ker \varepsilon)^2$, so we need only consider Π_a for $a \in H_R^{(1)}$. Each Π_a can be thought of as a "left-invariant" 1-form, as follows.

Exercise 2.1. Let G be a compact Lie group and let $\mathcal{R}(G)$ be its Hopf algebra of representative functions. If L_t denotes left translation by $t \in G$, then $L_t^* f(x) = f(t^{-1} x) = \Delta f(t^{-1}, x) = \sum f_{:1}(t^{-1}) \, f_{:2}(x)$, so that $L_t^* f = \sum f_{:1}(t^{-1}) \, f_{:2}$ for $f \in \mathcal{R}(G)$. Let Π_f be the smooth 1-form on G defined by (2.14); prove that $L_t^* \Pi_f = \Pi_f$ for all $t \in G$. ◊

Each left-invariant 1-form (2.14) satisfies a "Maurer–Cartan equation":

$$d\Pi_a = -\sum \Pi_{a_{:1}} \wedge \Pi_{a_{:2}}.$$

Indeed, since $0 = d(\varepsilon(a) \, 1) = \sum d(S(a_{:1}) \, a_{:2}) = \sum d(S(a_{:1})) \, a_{:2} + S(a_{:1}) \, da_{:2}$, we find that

$$d(S(a)) = \sum d(S(a_{:1}) \, \varepsilon(a_{:2})) = \sum d(S(a_{:1})) \, \varepsilon(a_{:2}) = \sum d(S(a_{:1})) \, a_{:2} \, S(a_{:3})$$
$$= -\sum S(a_{:1}) \, da_{:2} \, S(a_{:3}),$$

in analogy with $d(g^{-1}) = -g^{-1} \, dg \, g^{-1}$. Therefore,

$$d\Pi_a = \sum d(S(a_{:1})) \wedge da_{:2} = -\sum S(a_{:1}) \, da_{:2} \wedge S(a_{:3}) \, da_{:4} = -\sum \Pi_{a_{:1}} \wedge \Pi_{a_{:2}}.$$

Suppose now that we are given some element $a \in H_R^{(1)}$ for which $d\Pi_a = 0$. The bijectivity of the exponential map for $\mathcal{G}(H_R)$ suggests that this closed 1-form should be exact: $\Pi_a = db$ for some $b \in H_R$. It is clear from (2.14) that the equation $\Pi_a = db$ can hold only if b is primitive. Theorem 2 of [18] uses the Poincaré lemma technique to provide a formula for b, namely,

$$b := -\Phi^{-1}(S(a)),$$

where Φ is the operator which grades H_R by the number of trees in a product: $\Phi(T_1 \ldots T_k) := k \, T_1 \ldots T_k$. Notice that $b = a + c$, where $c \in (\ker \varepsilon)^2$ is a sum of higher-degree terms.

Exercise 2.2. Show that $a = $ $+$ $- 2$ satisfies $d\Pi_a = 0$, and compute that

$$b = \text{} + \text{} - 2 \text{} - \diamond \text{} + \text{}.$$

Verify directly that b is indeed primitive. ◇

It is still not a trivial matter to find linear combinations of trees satisfying $d\Pi_a = 0$, but it clearly is much easier to verify this property than to check primitivity directly on a case-by-case basis.

▶ Finally, we comment on the link between H_R and the Hopf algebra H_{CM} of differential operators, developed in [30]. This is found by extending H_R to a larger (but no longer commutative) Hopf algebra $\widetilde{H}_R$. Since H_R is graded by the number of vertices per tree, we regard the subspace $H_R^{(1)}$ of single trees as an abelian Lie algebra, and introduce an extra generator Y with the commutation rule

$$[Y, T] := (\#T) \, T.$$

For each simple cut c of T, it is clear that $\#P_c(T) + \#R_c(T) = \#T$; a glance at (2.9) then shows that $\Delta[Y, T] = (\#T) \, \Delta T = [Y \otimes 1 + 1 \otimes Y, \Delta T]$. This forces Y to be primitive:

$$\Delta Y := Y \otimes 1 + 1 \otimes Y, \tag{2.15}$$

in order to get $\Delta[Y, T] = [\Delta Y, \Delta T]$ for consistency.

Another important operator on $\mathcal{H}_R$ is the so-called *natural growth* of trees. We define $N(T)$, for each tree T with vertices $v_1, \ldots, v_n$, by setting $N(T) := T_1 + T_2 + \cdots + T_n$, where each T_j is obtained from T by adding a leaf to v_j. For example,

$$N(\diamond) := \text{}, \qquad N\left(\text{}\right) := \text{} + \text{},$$

$$N\left(\text{} + \text{}\right) := \text{} + 3 \text{} + \text{} + \text{}.$$

In symbols, we write these relations as

$$N(t_1) = t_2, \qquad N^2(t_1) = N(t_2) = t_{31} + t_{32},$$
$$N^3(t_1) = N(t_{31} + t_{32}) = t_{41} + 3t_{42} + t_{43} + t_{44}.$$

We rename these $\delta_1 := t_1$, $\delta_2 := N(\delta_1)$, $\delta_3 := N^2(\delta_1)$, $\delta_4 := N^3(\delta_1)$, and in general $\delta_{n+1} := N^n(\delta_1)$ for any n. Notice that δ_{n+1} is a sum of $n!$ trees.

N, defined on the algebra generators, extends uniquely to a *derivation* $N \colon H_R \to H_R$. Now, we can add one more generator X with the commutation rule

$$[X, T] := N(T).$$

The Jacobi identity forces $[Y, X] = X$, as follows:

$$[[Y, X], T] = [[Y, T], X] + [Y, [X, T]] = (\#T)\, [T, X] + [Y, N(T)]$$
$$= -(\#T)\, N(T) + (\#T + 1)\, N(T) = N(T) = [X, T].$$

What must the coproduct ΔX be? Proposition 3.6 of [30] – see also Proposition 14.6 of [52] – proves that

$$\Delta N(T) = (N \otimes \iota)\Delta T + (\iota \otimes N)\Delta T + [\delta_1 \otimes Y, \Delta T] \qquad (2.16)$$

for each rooted tree T. The argument is as follows: to get $\Delta N(T)$, we grow an extra leaf on T and then cut the resulting trees in every allowable way. If the new edge is not cut, then it belongs either to a pruned branch or to the trunk which remains after a cut has been made on the original tree T; this amounts to $(N \otimes \iota)\Delta T + (\iota \otimes N)\Delta T$. On the other hand, if the new edge is cut, the new leaf contributes a solitary vertex δ_1 to P_c; the new leaf must have been attached to the trunk $R_c(T)$ at any one of the latter's vertices. Since $(\#R_c)R_c = [Y, R_c]$, the terms wherein the new leaf is cut amount to $[\delta_1 \otimes Y, \Delta T]$. The equation (2.16) accounts for both possibilities. Then, since $\Delta[X, T] = [\Delta X, \Delta T]$ must hold, we get

$$\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y. \qquad (2.17)$$

Let $\widetilde{H}_R$ be the algebra generated by X, Y and H_R. We can extend the counit and antipode to it as follows. Since Y is primitive, we must take $\varepsilon(Y) := 0$ and $S(Y) := -Y$. Then, on applying $(\iota \otimes \varepsilon)$ to (2.17), $\varepsilon(X) := 0$ follows; and by applying $m(\iota \otimes S)$ or $m(S \otimes \iota)$ to it, we also get $0 = X + S(X) - \delta_1 Y$, which forces $S(X) := -X + \delta_1 Y$.

Now (2.15) and (2.17) reproduce exactly the coproducts (1.24) for the differential operators Y and X of the Hopf algebra H_{CM}. Indeed, since δ_1, like $\lambda_1 \in H_{CM}$, is primitive and since $\delta_{n+1} = N(\delta_n) = [X, \delta_n]$, the correspondence $X \mapsto X$, $Y \mapsto Y$, $\lambda_n \mapsto \delta_n$ maps H_{CM} isomorphically into $\widetilde{H}_R$.

2.2 Hopf algebras of Feynman graphs and renormalization

In this subsection, we shall describe briefly some other Hopf algebras which underlie the structure of a renormalizable quantum field theory. Rather than going into the details of perturbative renormalization, we shall merely indicate how such Hopf algebras are involved.

In a given QFT, one is faced with the problem of computing correlations (Green functions) from a perturbative expansion whose terms are labelled by Feynman graphs Γ, and consist of multiple integrals where the integrand is completely specified by the combinatorial structure of Γ (its vertices, external and internal lines, and loops) according to a small number of Feynman rules. Typically, one works in momentum space of D dimensions, and a preliminary count of the powers of the momenta in the integrand indicates, in many cases, a superficially divergent integral; even if the graph Γ itself passes this test, it may contain subgraphs corresponding to superficially divergent integrals. The main idea of renormalization theory is to associate a "counterterm" to each superficially divergent subgraph, in order to obtain a finite result by subtraction.

The first step in approaching such calculations is to realize that all superficially divergent subgraphs must be dealt with, in a recursive fashion, before finally assigning a finite value to the full graph Γ. Thus, each graph Γ determines a nesting of divergent subgraphs: this nesting is codified by a *rooted tree*, where the root represents the full graph, provided that the Γ does not contain overlapping divergences. (Even if overlapping divergences do occur, one can replace the single rooted tree by a sum over rooted trees after disentangling the overlaps: see [64] for a detailed analysis.) A "leaf" is a divergent subgraph which itself contains no further subdivergences.

The combinatorial algebra is worked out in considerable detail in a recent article of Connes and Kreimer [31]: the following remarks can be taken as an incentive for a closer look at that paper. See also the survey of Kreimer [65] for a detailed discussion of the conceptual framework. The authors of [31] consider ϕ^3 theory in $D = 6$ dimensions; but one could equally well start with ϕ^4 theory for $D = 4$ [49], or QED, or any other well-known theory.

Definition 2.4. Let Φ stand for any particular QFT. The Hopf algebra H_Φ is a commutative algebra generated by one-particle irreducible (1PI) graphs: that is, connected graphs with at least two vertices which cannot be disconnected by removing a single line. The product is given by disjoint union of graphs: $\Gamma_1 \Gamma_2$ means $\Gamma_1 \uplus \Gamma_2$. The counit is given by $\varepsilon(\Gamma) := 0$ on any generator, with $\varepsilon(\emptyset) := 1$ (we assign the empty graph to the identity element). The *coproduct* Δ is given, on any 1PI graph Γ, by

$$\Delta\Gamma := \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma, \tag{2.18}$$

where the sum ranges over all subgraphs which are divergent and proper (in the sense that removing one internal line cannot increase the number of its connected components); γ may be either connected or a disjoint union of several connected pieces. The notation Γ/γ denotes the (connected, 1PI) graph obtained from Γ by replacing each component of γ by a single vertex.

To see that Δ is coassociative, we may reason as follows. We may replace the right hand side of (2.18) by a single sum over $\emptyset \subseteq \gamma \subseteq \Gamma$, allowing $\gamma = \emptyset$ or $\gamma = \Gamma$ and setting $\Gamma/\Gamma := 1$. We observe that if $\gamma \subseteq \gamma' \subseteq \Gamma$, then γ'/γ can be regarded as a subgraph of Γ/γ; moreover, it is obvious that

$$(\Gamma/\gamma)/(\gamma'/\gamma) \simeq \Gamma/\gamma'. \tag{2.19}$$

The desired relation $(\Delta \otimes \iota)(\Delta\Gamma) = (\iota \otimes \Delta)(\Delta\Gamma)$ can now be expressed as

$$\sum_{\emptyset \subseteq \gamma \subseteq \gamma' \subseteq \Gamma} \gamma \otimes \gamma'/\gamma \otimes \Gamma/\gamma' = \sum_{\emptyset \subseteq \gamma \subseteq \Gamma,\, \emptyset \subseteq \gamma'' \subseteq \Gamma/\gamma} \gamma \otimes \gamma'' \otimes (\Gamma/\gamma)/\gamma'',$$

so coassociativity reduces to proving, for each subgraph γ of Γ, that

$$\sum_{\gamma \subseteq \gamma' \subseteq \Gamma} \gamma'/\gamma \otimes \Gamma/\gamma' = \sum_{\emptyset \subseteq \gamma'' \subseteq \Gamma/\gamma} \gamma'' \otimes (\Gamma/\gamma)/\gamma''.$$

Choose γ' so that $\gamma \subseteq \gamma' \subseteq \Gamma$; then $\emptyset \subseteq \gamma'/\gamma \subseteq \Gamma/\gamma$. Reciprocally, to every $\gamma'' \subseteq \Gamma/\gamma$ there corresponds a unique γ' such that $\gamma \subseteq \gamma' \subseteq \Gamma$ and $\gamma'/\gamma = \gamma''$; the previous equality now follows from the identification (2.19).

We have now defined H_Φ as a bialgebra. To make sure that it is a *Hopf* algebra, it suffices to show that it is graded and connected, whereby the antipode comes for free. Several grading operators Υ are available, which satisfy the two conditions (2.6):

$$\Upsilon(\Gamma_1\Gamma_2) = \Upsilon(\Gamma_1) + \Upsilon(\Gamma_2) \quad \text{and} \quad \Upsilon(\gamma) + \Upsilon(\Gamma/\gamma) = \Upsilon(\Gamma)$$

whenever γ is a divergent proper subgraph of Γ. One such grading is the loop number $\ell(\Gamma) := I(\Gamma) - V(\Gamma) + 1$, if Γ has $I(\Gamma)$ internal lines and $V(\Gamma)$ vertices. If $\ell(\Gamma) = 0$, then Γ would be a tree graph, which is never 1PI; thus $\ker \ell$ consists of scalars only, so H_Φ is connected. The antipode is now given recursively by (2.8):

$$S(\Gamma) = -\Gamma + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} S(\gamma)\, \Gamma/\gamma. \tag{2.20}$$

As it stands, the Hopf algebra H_Φ corresponds to a formal manipulation of graphs. It remains to understand how to match these formulas to expressions for numerical values, whereby the antipode S delivers the counterterms. This is done in two steps. First of all, the Feynman rules for the unrenormalized theory can be thought of as prescribing a linear map

$$f \colon H_\Phi \to \mathcal{A},$$

into some commutative algebra $\mathcal{A}$, that is multiplicative on disjoint unions: $f(\Gamma_1\Gamma_2) = f(\Gamma_1)\,f(\Gamma_2)$. In other words, f is actually a homomorphism of algebras. For instance, $\mathcal{A}$ is often an algebra of Laurent series in some (complex) regularization parameter ε: in dimensional regularization, after adjustment by a mass unit μ so that each $f(\Gamma)$ is dimensionless, one computes the corresponding integral in dimension $d = D + \varepsilon$, for $\varepsilon \neq 0$. We shall also suppose that $\mathcal{A}$ is the direct sum of two *subalgebras*:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-.$$

Let $T \colon \mathcal{A} \to \mathcal{A}_-$ be the projection on the second subalgebra, with $\ker T = \mathcal{A}_+$. When $\mathcal{A}$ is a Laurent-series algebra, one takes $\mathcal{A}_+$ to be the holomorphic subalgebra of Taylor series and $\mathcal{A}_-$ to be the subalgebra of polynomials in $1/\varepsilon$ without constant term; the projection T picks out the pole part, as in a minimal subtraction scheme. Now T is not a homomorphism, but the property that both its kernel and image are subalgebras is reflected in a "multiplicativity constraint":

$$T(ab) + T(a)\,T(b) = T(T(a)\,b) + T(a\,T(b)) \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{2.21}$$

Exercise 2.3. Check (2.21) by examining the four cases $a \in \mathcal{A}_\pm$, $b \in \mathcal{A}_\pm$ separately. ◊

The second step is to invoke the renormalization scheme. It can now be summarized as follows. If Γ is 1PI and is *primitive* (i.e., it has no subdivergences), we set

$$C(\Gamma) := -T(f(\Gamma)), \quad \text{and then} \quad R(\Gamma) := f(\Gamma) + C(\Gamma),$$

where $C(\Gamma)$ is the *counterterm* and $R(\Gamma)$ is the desired finite value: in other words, for primitive graphs one simply removes the pole part. Next, we may recursively define Bogoliubov's $\overline{R}$-operation by setting

$$\overline{R}(\Gamma) := f(\Gamma) + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} C(\gamma)\, f(\Gamma/\gamma),$$

with the proviso that

$$C(\gamma_1 \ldots \gamma_r) := C(\gamma_1) \ldots C(\gamma_r), \tag{2.22}$$

whenever $\gamma = \gamma_1 \ldots \gamma_r$ is a disjoint union of several components. The final result is obtained by removing the pole part of the previous expression: $C(\Gamma) := -T(\overline{R}(\Gamma))$ and $R(\Gamma) := \overline{R}(\Gamma) + C(\Gamma)$. In summary,

$$C(\Gamma) := -T\left[f(\Gamma) + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} C(\gamma)\, f(\Gamma/\gamma) \right], \tag{2.23a}$$

$$R(\Gamma) := f(\Gamma) + C(\Gamma) + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} C(\gamma)\, f(\Gamma/\gamma). \tag{2.23b}$$

The equation (2.23a) is what is meant by saying that "the antipode delivers the counterterm": one replaces S in the calculation (2.20) by C to obtain the right hand side, before projection with T. From the definition of the coproduct in H_Φ, (2.23b) is a *convolution* in $\mathrm{Hom}(H_\Phi, \mathcal{A})$, namely, $R = C * f$. To show that R is multiplicative, it is enough to verify that the counterterm map C is multiplicative, since the convolution of homomorphisms is a homomorphism because $\mathcal{A}$ is commutative. In other words, we must check that (2.22) and (2.23a) are compatible.

This is easy to do by induction on the degree of the grading of H_Φ. We shall use the modified Sweedler notation of (2.7), to simplify the calculation. Starting from $C(1) := 1_\mathcal{A}$, we define, for $a \in \ker \varepsilon$,

$$C(a) := -T\left[f(a) + \sum C(a'_{:1})\, f(a'_{:2}) \right], \tag{2.24}$$

assuming $C(b)$ to be already defined, and multiplicative, whenever b has smaller degree than a. By comparing the expansions of $\Delta(ab)$ and $(\Delta a)(\Delta b)$, we see that

$$\sum (ab)'_{:1} \otimes (ab)'_{:2} = a \otimes b + b \otimes a + \sum ab'_{:1} \otimes b'_{:2} + b'_{:1} \otimes ab'_{:2}$$
$$+ a'_{:1} b \otimes a'_{:2} + a'_{:1} \otimes a'_{:2} b + a'_{:1} b'_{:1} \otimes a'_{:2} b'_{:2}.$$

Using the multiplicativity constraint (2.21) and the definition $C(a) := -T(\overline{R}(a))$, we get

$$\begin{aligned}
C(a)C(b) &= T\left[\overline{R}(a)\right] T\left[\overline{R}(b)\right] = -T\left[\overline{R}(a)\,\overline{R}(b) + C(a)\,\overline{R}(b) + \overline{R}(a)\,C(b)\right]\\
&= -T\big[f(a)f(b) + C(a)f(b) + f(a)C(b) + \sum C(a)C(b'_{:1})f(b'_{:2}) + f(a)C(b'_{:1})f(b'_{:2})\\
&\quad + \sum C(a'_{:1})f(a'_{:2})C(b) + C(a'_{:1})f(a'_{:2})f(b) + C(a'_{:1})f(a'_{:2})C(b'_{:1})f(b'_{:2})\big]\\
&= -T\big[f(a)f(b) + C(a)f(b) + C(b)f(a) + \sum C(ab'_{:1})f(b'_{:2}) + C(b'_{:1})f(ab'_{:2})\\
&\quad + \sum C(a'_{:1}b)f(a'_{:2}) + C(a'_{:1})f(a'_{:2}b) + C(a'_{:1}b'_{:1})f(a'_{:2}b'_{:2})\big]\\
&= -T\big[f(ab) + \sum C((ab)'_{:1})f((ab)'_{:2})\big] = C(ab),
\end{aligned}$$

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where, in the penultimate line, we have used the assumed multiplicativity of C in lower degrees.

▶ The decomposition $R = C * f$ has a further consequence. Assume that the unrenormalized integrals, although divergent at $\varepsilon = 0$, make sense on the circle S in the complex plane where $|\varepsilon| = |d - D| = r_0$, say. Evaluation at any $d = z$ defines a character $\chi_z \colon \mathcal{A} \to \mathbb{C}$ of the Laurent-series algebra. Composing this character with $f \colon H_\Phi \to \mathcal{A}$ gives a loop of characters of H_Φ:

$$\gamma(z) := \chi_z \circ f, \quad \text{for any} \quad z \in S.$$

Likewise, $\gamma_-(z) := \chi_z \circ C$ and $\gamma_+(z) := \chi_z \circ R$ define characters of H_Φ – here is where we use the multiplicativity of C and R – and $R = C * f$ entails $\gamma_+(z) = \gamma_-(z)\gamma(z)$, or equivalently,

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z), \quad \text{for all} \quad z \in S. \tag{2.25}$$

The properties of the subalgebras $\mathcal{A}_+$ and $\mathcal{A}_-$ show that $\gamma_+(z)$ extends holomorphically to the disc $|z - D| < r_0$, while $\gamma_-(z)$ extends holomorphically to the outer region $|z - D| > r_0$ with $\gamma_-(\infty)$ being finite. Since a function holomorphic on both regions must be constant (Liouville's theorem), we can normalize the factorization (2.25) just by setting $\gamma_-(\infty) := 1$. The renormalization procedure thus corresponds to replacing the loop $\{\gamma(z) : z \in S\}$ by the finite evaluation $\gamma_+(D)$.

The decomposition (2.25) of a group-valued loop is known as the *Birkhoff factorization*, and arises in the study of linear systems of differential equations

$$y'(z) = A(z)\,y(z),$$

where $A(z)$ is a meromorphic $n \times n$ matrix-valued function with simple poles. The solution involves constructing a loop around one of these poles z_0 with values in the Lie group $\mathrm{GL}(n, \mathbb{C})$. We refer to [82, Chap. 8] for an instructive discussion of this problem. Any such loop factorizes as follows:

$$\gamma(z) = \gamma_-(z)^{-1}\,\lambda(z)\,\gamma_+(z),$$

where $\gamma_+(z)$ is holomorphic for $|z - z_0| < r_0$, $\gamma_-(z)$ is holomorphic for $|z - z_0| > r_0$ with $\gamma_-(\infty) = 1$, and $\{\lambda(z) : |z - z_0| = r_0\}$ is a loop with values in the n-torus of diagonal matrices. The loop λ provides clutching functions for n line bundles over the Riemann sphere, and these are obstructions to the solvability of the differential system. However, in our context, the Lie group $\mathrm{GL}(n, \mathbb{C})$ is replaced by the topologically trivial group $\mathcal{G}(H_\Phi)$, so that the loop λ becomes trivial and the decomposition (2.25) goes through as stated, thereby providing a general recipe for computing finite values in renormalizable theories.

3   Cyclic Cohomology

3.1   Hochschild and cyclic cohomology of algebras

We have already discussed briefly, in subsection 2.1, the Hochschild cohomology of associative algebras. Recall that a Hochschild n-cochain, for an algebra over the complex field, is a multilinear map $\varphi \colon \mathcal{A}^{n+1} \to \mathbb{C}$, with the coboundary map given by (2.3). These n-cochains make up an $\mathcal{A}$-bimodule $C^n = C^n(\mathcal{A}, \mathcal{A}^*)$; the n-cocycles $Z^n = \{\varphi \in C^n : b\varphi = 0\}$ and the n-coboundaries $B^n = \{b\psi : \psi \in C^{n-1}\}$ conspire to form the Hochschild cohomology module $HH^n(\mathcal{A}) := Z^n/B^n$. A 0-cocycle τ is a trace on $\mathcal{A}$, since $\tau(a_0 a_1) - \tau(a_1 a_0) = b\tau(a_0, a_1) = 0$.

In the commutative case, when $\mathcal{A} = C^\infty(M)$ is an algebra of smooth functions on a manifold M (we take $\mathcal{A}$ unital and M compact, as before), there is a theorem of Connes [21], which dualizes an older result in algebraic geometry due to Hochschild, Kostant and Rosenberg [57], to the effect that Hochschild classes for $C^\infty(M)$ correspond exactly to de Rham currents on M. (Currents are the objects which are dual to differential forms, and can be thought of as formal linear combinations of domains for line and surface integrals within M.) The correspondence $[\varphi] \mapsto C_\varphi$ is given by skewsymmetrization of φ in all arguments but the first:

$$\int_{C_\varphi} a_0 \, da_1 \wedge \cdots \wedge da_k := \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi \, \varphi(a_0, a_{\pi(1)}, \ldots, a_{\pi(k)}).$$

Dually, Hochschild *homology* classes on $C^\infty(M)$ correspond to differential *forms* on M; that is, $HH_k(C^\infty(M)) \simeq \mathcal{A}^k(M)$ for $k = 0, 1, \ldots, \dim M$.

On the de Rham side, the vector spaces $\mathcal{D}_k(M)$ of currents of dimension k form a complex, but with zero maps between them, so that each Hochschild class $[\varphi]$ matches with a single current C_φ rather than with its homology class. To deal with the homology *classes*, we must bring in an algebraic expression for the de Rham boundary. This turns out to be a degree-lowering operation on Hochschild cochains: if $\psi \in C^k$, then $B\psi \in C^{k-1}$, given by

$$B\psi(a_0, \ldots, a_{k-1}) := \sum_{j=0}^{k-1} (-1)^{j(k-1)} \psi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1})$$
$$+ (-1)^{(j-1)(k-1)} \psi(a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}, 1), \qquad (3.1)$$

does the job. Indeed, if C is a k-current and φ_C is the (already skewsymmetric) cochain

$$\varphi_C(a_0, a_1, \ldots, a_k) := \int_C a_0 \, da_1 \wedge \cdots \wedge da_k,$$

then $\varphi_C(a_0, \ldots, a_{k-1}, 1) = 0$, and therefore

$$B\varphi_C(a_0, \ldots, a_{k-1}) = \sum_{j=0}^{k-1} (-1)^{j(k-1)} \int_C da_j \wedge \cdots \wedge da_{k-1} \wedge da_0 \wedge \cdots \wedge da_{j-1}$$
$$= \sum_{j=0}^{k-1} \int_C da_0 \wedge \cdots \wedge da_{k-1} = k \int_{\partial C} a_0 \, da_1 \wedge \cdots \wedge da_{k-1},$$

by using Stokes' theorem; thus $B\varphi_C = k \, \varphi_{\partial C}$. Up to the normalization factor $k = \deg C$, the algebraic operator B delivers the de Rham boundary. Thus, the algebraic picture for de Rham homology involves a cohomology of algebras which uses both b and B.

▶ Dually, the Hochschild homology of algebras supports a degree-raising operator, also called B, which is closely related related to the de Rham coboundary (that is, the exterior derivative). Indeed, if we use the version of Hochschild homology where the chains belong to the universal graded differential algebra $\Omega^\bullet \mathcal{A}$, with b given by (2.1), then $B \colon \Omega^k \mathcal{A} \to \Omega^{k+1} \mathcal{A}$ is given simply by

$$B(a_0 \, da_1 \cdots da_k) := \sum_{j=0}^{k} (-1)^{kj} \, da_j \cdots da_k \, da_0 \cdots da_{j-1}. \qquad (3.2)$$

which mimics the operation $\omega \mapsto k \, d\omega$ on differential k-forms. In the manifold case, the various da_j anticommute, but for more general algebras they do not, so the cyclic summation in (3.2) is unavoidable. From the formula, it is obvious that $B^2 = 0$. One checks easily that $bB + Bb = 0$, too.

Exercise 3.1. If $e \in \mathcal{A}$ is an idempotent element, that is, $e^2 = e$, and k is even, check that

$$b(e \, (de)^k) = e \, (de)^{k-1}, \qquad\qquad b((de)^k) = (2e - 1) \, (de)^{k-1},$$
$$B(e \, (de)^k) = (k + 1) \, (de)^{k+1}, \qquad\qquad B((de)^k) = 0.$$

If k is odd, show that instead,

$$b(e(de)^k) = b((de)^k) = 0 \quad \text{and} \quad B(e(de)^k) = B((de)^k) = 0. \qquad\qquad \diamond$$

Moving back to cohomology, one can check that $b^2 = 0$, $B^2 = 0$, and $bB + Bb = 0$ hold there, too. This gives rise to a *bicomplex*:

Folding this up along the diagonals, we get a "total complex" whose coboundary operator is $b + B$, and whose module in degree n is

$$C^n \oplus C^{n-2} \oplus C^{n-4} \oplus \cdots \oplus C^{\#n},$$

where $\#n = 0$ or 1 according as n is even or odd. The cohomology of this total complex is, by definition, the **cyclic cohomology** $HC^\bullet(\mathcal{A})$ of the algebra $\mathcal{A}$. (The letters HC stand for "homologie cyclique": on replacing C^k by $\Omega^k(\mathcal{A})$ and running all the arrows backwards, we get a dual bicomplex; the homology $HC_\bullet(\mathcal{A})$ of its total complex is the *cyclic homology* of $\mathcal{A}$.)

▶ There is an alternative description of cyclic cohomology, which in some ways is simpler. Let τ be the operation of cyclic permutation of the arguments of a Hochschild cochain:

$$\tau\varphi(a_0, \ldots, a_n) := \varphi(a_n, a_0, \ldots, a_{n-1}). \qquad\qquad (3.3)$$

We say that φ is *cyclic* if $\tau\varphi = (-1)^n \varphi$ – notice that $(-1)^n$ is the sign of this cyclic permutation – and denote the subspace of cyclic n-cochains by $C_\lambda^n = C_\lambda^n(\mathcal{A})$ (the notation $\lambda = (-1)^n\tau$ is often used).

If $Z^n_\lambda(\mathcal{A})$ and $B^n_\lambda(\mathcal{A})$ are the respective cyclic n-cocycles and cyclic n-coboundaries, an exercise in homological algebra shows that $HC^n(\mathcal{A}) \simeq Z^n_\lambda(\mathcal{A})/B^n_\lambda(\mathcal{A})$.

Let us compute $HC^\bullet(\mathcal{A})$ for a simple example: the algebra $\mathcal{A} = \mathbb{C}$, which is the coordinate algebra of a single point. The module $\mathbb{C}^n$ is one-dimensional, since $\varphi(a_0, \ldots, a_n) = a_0 \cdots a_n \varphi(1, 1, \ldots, 1)$; it has a basis element φ^n determined by $\varphi^n(1, 1, \ldots, 1) := 1$. Clearly, $b\varphi^n = \sum_{j=0}^{n+1}(-1)^j \varphi^{n+1} = 0$ or φ^{n+1}, according as n is even or odd. We also find that $B\varphi^n = 0$ or $2n\varphi^{n-1}$, according as n is even or odd. The total complex is of the form

$$\mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{d_1} \mathbb{C}^2 \xrightarrow{0} \mathbb{C}^2 \xrightarrow{d_2} \mathbb{C}^3 \xrightarrow{0} \mathbb{C}^3 \xrightarrow{d_3} \cdots$$

each d_j being injective with range of codimension 1; for instance, $d_2(\varphi^3, \varphi^1) = (\varphi^4, 7\varphi^2, 2\varphi^0)$. The alternative approach, using cyclic n-cocycles, argues more simply that $\tau\varphi^n = \varphi^n$, so that $Z^n_\lambda(\mathbb{C}) = \mathbb{C}$ or 0 according as n is even or odd, while $B^n_\lambda(\mathbb{C}) = 0$ for all n. Either way, $HC^n(\mathbb{C}) = \mathbb{C}$ if n is even, and $HC^n(\mathbb{C}) = 0$ if n is odd.

This periodicity might seem surprising: the de Rham cohomology of a one-point space is $\mathbb{C}$ in degree zero, and 0 in all higher degrees. Now we may notice that there is an obvious "shifting operation" S on the bicomplex, moving all modules right and up by one step (and pushing the total complex along by two steps); it leaves behind the first column, which is just the Hochschild complex of $\mathcal{A}$. At the level of cohomology, we get a pair of maps

$$HC^{n-2}(\mathcal{A}) \xrightarrow{S} HC^n(\mathcal{A}) \xrightarrow{I} HH^n(\mathcal{A}),$$

which actually splice together into a long exact sequence:

$$\cdots \longrightarrow HC^n(\mathcal{A}) \xrightarrow{I} HH^n(\mathcal{A}) \xrightarrow{B} HC^{n-1}(\mathcal{A}) \xrightarrow{S} HC^{n+1}(\mathcal{A}) \xrightarrow{I} HH^{n+1}(\mathcal{A}) \longrightarrow \cdots$$

whose connecting homomorphism comes from the aforementioned B at the level of cochains. The detailed calculations which back up these plausible statements are long and tedious; they are given in [68, Chap. 2] for cyclic homology, and in [52, §10.1] is the cohomological setting. The upshot is that, by iterating the periodicity operator S, one can compute two direct limits, which capture the main algebraic invariants of $\mathcal{A}$.

Definition 3.1. The periodicity maps $S\colon HC^n \to HC^{n+2}$ define two directed systems of abelian groups; their inductive limits

$$HP^0(\mathcal{A}) := \varinjlim HC^{2k}(\mathcal{A}), \qquad HP^1(\mathcal{A}) := \varinjlim HC^{2k+1}(\mathcal{A}),$$

are called the even and odd *periodic cyclic cohomology* groups of the algebra $\mathcal{A}$. In particular, $HP^0(\mathbb{C}) = \mathbb{C}$ and $HP^1(\mathbb{C}) = 0$.

In the commutative case $\mathcal{A} = C^\infty(M)$, it turns out that $HC^\bullet(\mathcal{A})$ does not quite capture the de Rham homology of M. The exact result – see [24, Thm. III.2.2] or [52, Thm. 10.5] – is

$$HC^k(C^\infty(M)) \simeq Z^{\mathrm{dR}}_k(M) \oplus H^{\mathrm{dR}}_{k-2}(M) \oplus H^{\mathrm{dR}}_{k-4}(M) \oplus \cdots \oplus H^{\mathrm{dR}}_{\#k}(M),$$

where $Z^{\mathrm{dR}}_k(M)$ is the vector space of closed k-currents on M, $H^{\mathrm{dR}}_r(M)$ is the rth de Rham homology group of M, and $\#k = 0$ or 1 according as k is even or odd. However, one may use S to promote

the closed k-currents, two degrees at a time, until the full de Rham homology is obtained, since $Z_k^{dR}(M) = 0$ for $k > \dim M$; then we get de Rham homology exactly, albeit rolled up into even and odd degrees:

$$HP^0(C^\infty(M)) \simeq H_{\text{even}}^{dR}(M), \qquad HP^1(C^\infty(M)) \simeq H_{\text{odd}}^{dR}(M).$$

There is also a dual result, which matches a periodic variant of the cyclic homology of $C^\infty(M)$ with the even/odd de Rham cohomology of M.

▶ The importance of this algebraic scheme for de Rham co/homology is that it provides many *Chern characters*, even for highly noncommutative algebras. Generally speaking, Chern characters are tools to compute algebraic invariants from the more formidable K-theory and K-homology of algebras. The idea is to associate, to any pair of classes $[x] \in K_\bullet(\mathcal{A})$ and $[D] \in K^\bullet(\mathcal{A})$ another pair of classes $\mathrm{ch}_\bullet x \in HC_\bullet(\mathcal{A})$ and $\mathrm{ch}^\bullet D \in HC^\bullet(\mathcal{A})$, given by explicit and manageable formulas, so that the *index pairing* $\langle [x], [D] \rangle$ can be computed from a cyclic co/homology pairing $\langle \mathrm{ch}_\bullet x, \mathrm{ch}^\bullet D \rangle$, which is usually more tractable. We look at the K-theory version first, and distinguish the even and odd cases.

Suppose first that $e = e^2$ is an idempotent in $\mathcal{A}$, representing a class $[e] \in K_0(\mathcal{A})$; we define $\mathrm{ch}\, e := \sum_{k=0}^\infty \mathrm{ch}_k\, e \in \Omega^{\text{even}}\mathcal{A}$, where the component chains are

$$\mathrm{ch}_k\, e := (-1)^k \frac{(2k)!}{k!} (e - \tfrac{1}{2}) (de)^{2k} \in \Omega^{2k}\mathcal{A},$$

It follows from Exercise 3.1 that $(b + B)(\mathrm{ch}\, e) = 0$. Next, if $u \in \mathcal{A}$ is invertible, representing a class $[u] \in K_1(\mathcal{A})$; we define $\mathrm{ch}\, u := \sum_{k=0}^\infty \mathrm{ch}_{k+\frac{1}{2}}\, u \in \Omega^{\text{odd}}\mathcal{A}$, with components

$$\mathrm{ch}_{k+\frac{1}{2}}\, u := (-1)^k k!\, u^{-1}\, du\, (d(u^{-1})\, du)^k = k!\, (u^{-1}\, du)^{2k+1} \in \Omega^{2k+1}\mathcal{A}.$$

Again, one checks that $(b + B)(\mathrm{ch}\, u) = 0$. Actually, it is fairly rare that K-theory classes arise from idempotents or invertibles in the original algebra $\mathcal{A}$; more often, e and u belong to $M_r(\mathcal{A})$, the algebra of $r \times r$ matrices with entries in $\mathcal{A}$, for some $r = 1, 2, 3, \dots$; so in the definitions we must insert a trace over these matrix elements; the previous equations must be modified to

$$\mathrm{ch}_k\, e := (-1)^k \frac{(2k)!}{k!} \mathrm{tr}\big((e - \tfrac{1}{2}) (de)^{2k}\big) \in \Omega^{2k}\mathcal{A}, \tag{3.4a}$$

$$\mathrm{ch}_{k+\frac{1}{2}}\, u := k!\, \mathrm{tr}(u^{-1}\, du)^{2k+1} \in \Omega^{2k+1}\mathcal{A}. \tag{3.4b}$$

For instance, $\mathrm{tr}(e\, de\, de) = \sum e_{ij}\, de_{jk}\, de_{ki}$. The pairing of, say, the 2-chain $\mathrm{ch}_1\, e$ and a 2-cochain φ is given by

$$\langle \varphi, \mathrm{ch}_1\, e \rangle = -2 \sum \varphi(e_{ij} - \tfrac{1}{2}\delta_{ij}, e_{jk}, e_{ki}).$$

▶ The Chern character from K-homology to cyclic cohomology is trickier to define. First of all, what is a K-cycle over the algebra $\mathcal{A}$? It turns out that it is just a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, of Definition 1.1: an even spectral triple is a K^0-cycle, an odd spectral triple is a K^1-cycle. The unboundedness of the selfadjoint operator D may cause trouble, but one can always replace D (using the homotopy $D \mapsto D\, |D|^{-t}$ for $0 \leqslant t \leqslant 1$) with its *sign operator* $F := D\, |D|^{-1}$, which is a symmetry, that is, a bounded selfadjoint operator such that $F^2 = 1$. The compactness of $|D|^{-1}$ translates to the condition that $[F, a]$ be compact for each $a \in \mathcal{A}$; in the even case, F anticommutes

with the grading operator χ, just like D does. The triple $(\mathcal{A}, \mathcal{H}, F)$, satisfying these conditions, is called a *Fredholm module*; it represents the same K-homology class as the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Although F is bounded, it is analytically a much more singular object than D, as a general rule. For instance, if $D = (2\pi i)^{-1} d/d\theta$ is the Dirac operator on the unit circle $\mathbb{S}^1$, one finds that F is given by a principal-value integral:

$$Fh(\alpha) = \mathrm{P}\!\int_0^1 i\, h(\alpha - \theta)\, \cot \pi\theta\, d\theta,$$

which is a trigonometric version of the *Hilbert transform* on $L^2(\mathbb{R})$,

$$Fh(x) = \frac{i}{\pi}\, \mathrm{P}\!\int \frac{h(x-t)}{t}\, dt := \frac{i}{\pi}\, \lim_{\varepsilon \downarrow 0} \int_{|t| > \varepsilon} \frac{h(x-t)}{t}\, dt.$$

This can be seen by writing both operators in a Fourier basis for $\mathcal{H} = L^2(\mathbb{S}^1)$:

$$D(e^{2\pi i k\theta}) = k\, e^{2\pi i k\theta}, \qquad F(e^{2\pi i k\theta}) = (\mathrm{sign}\, k)\, e^{2\pi i k\theta},$$

with the convention that $\mathrm{sign}\, 0 = 1$. This analytic intricacy of F must be borne in mind when regarding the formula for the Chern character of its K-homology class, which is given by the cyclic n-cocycle

$$\tau_F^n(a_0, \dots, a_n) := \frac{\Gamma(\frac{n}{2} + 1)}{2\, n!}\, \mathrm{Tr}\big(\chi F[F, a_0] \dots [F, a_n]\big), \tag{3.5}$$

provided n is large enough that the operator in parentheses is trace-class. (The Fredholm module is said to be "finitely summable" if this is true for a large enough n.) One can always replace n by $n+2$, because it turns out that $S\tau_F^n$ and τ_F^{n+2} are cohomologous, so that the Chern character is well-defined as a periodic class. Much effort has gone into finding more tractable "local index formulas" for this Chern character, in terms of more easily computable cocycles: see [34] or [5].

▶ An important example of a cyclic 1-cocycle – historically one of the first to appear in the literature [1, 2] – is the *Schwinger term* of a $1 + 1$-dimensional QFT. In that context, there is a fairly straightforward "second quantization" in Fock space: we recall here only a few aspects of the formalism. In "first quantization", one starts with a real vector space V of solutions of a Dirac-type equation $(i\, \partial/\partial t - D)\psi = 0$, together with a symmetric bilinear form g making it a *real* Hilbert space. If E_+ and E_- denote the orthogonal projectors on the subspaces of positive- and negative-frequency solutions, respectively, the sign operator is $F := E_+ - E_-$; moreover, $J := iF = iE_+ - iE_-$ is an orthogonal complex structure on V (in other words, $J^2 = -1$), which can be used to make V into a *complex* Hilbert space V_J with the scalar product

$$\langle u \mid v \rangle_J := g(u, v) + i g(Ju, v).$$

(In examples representing charged fields, V is already a complex Hilbert space with an "original" complex structure $Q = i$; the construction of the new Hilbert space with complex structure J is equivalent to "filling up the Dirac sea", and Q is the charge, a generator of global gauge transformations.)

The fermion Fock space $\mathcal{F}_J(V)$ is simply the exterior algebra over V_J; the scalars in $\Lambda^0 V$ are the multiples of the vacuum vector $|0\rangle$. If $\{u_j\}$ is an orthonormal basis for V_J, there are corresponding

creation and annihiliation operators on $\mathcal{F}_J(V)$:

$$a_i^\dagger(u_1 \wedge \cdots \wedge u_k) := u_i \wedge u_1 \wedge \cdots \wedge u_k,$$

$$a_i(u_1 \wedge \cdots \wedge u_k) := \sum_{j=1}^{k} (-1)^{j-1} \langle u_i \mid u_j \rangle_J \, u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_k.$$

Any real-linear operator B on V can be written as $B = B_+ + B_-$ where $B_+ := \frac{1}{2}(B - JBJ)$ gives a complex-linear operator on V_J because it commutes with J, but $B_- := \frac{1}{2}(B + JBJ)$ is antilinear: $JB_- = -B_-J$. A skewsymmetric operator B is quantizable, by a result of Shale and Stinespring [96], if and only if $[J, B] = 2JB_-$ is Hilbert–Schmidt operator, and the second-quantization rule is $B \mapsto \dot\mu(B)$, where $\dot\mu(B)$ is the following operator on Fock space:

$$\dot\mu(B) := \frac{1}{2} \sum_{k,l} \langle u_k \mid B_- u_l \rangle_J \, a_k^\dagger a_l^\dagger + 2 \langle u_k \mid B_+ u_l \rangle_J \, a_k^\dagger a_l - \langle B_- u_l \mid u_k \rangle_J \, a_l a_k. \tag{3.6}$$

The rule complies with normal ordering, because $\langle 0 \mid \dot\mu(B) \mid 0 \rangle = 0$, i.e., the vacuum expectation value is zero. However, this implies that (3.6) is not quite a representation of the Lie algebra $\{ B = -B^t : B_- \text{ is Hilbert–Schmidt} \}$. The anomalous commutator, or Schwinger term, is given by

$$[\dot\mu(A), \dot\mu(B)] - \dot\mu([A, B]) = -\frac{1}{2} \mathrm{Tr}[A_-, B_-].$$

This is a well-known result: see [51] or [52, Thm. 6.7] for a proof. The trace here is taken on the Hilbert space V_J; notice that, although $[A_-, B_-]$ is a traceclass commutator, its trace need not vanish, because it is the commutator of *antilinear* operators.

The claim is that $\alpha(A, B) := -\frac{1}{2} \mathrm{Tr}[A_-, B_-]$ defines a cyclic 1-cocycle on the algebra generated by such A and B. For that, we rewrite it in terms of a trace of operators on the complexified space $V^\mathbb{C} := V \oplus iV$; any real-linear operator B on V extends to a $\mathbb{C}$-linear operator on $V^\mathbb{C}$ in the obvious way: $B(u + iv) := B(u) + iB(v)$. For instance, $F := E_+ - E_-$ where E_+ and E_- now denote complementary orthogonal projectors on $V^\mathbb{C}$. Taking now the trace over $V^\mathbb{C}$, too, we find that

$$\alpha(A, B) = \frac{1}{8} \mathrm{Tr}(F[F, A][F, B]). \tag{3.7}$$

To see that, first notice that $F[F, B] = B - FBF = -[F, B]F$, and so $\mathrm{Tr}(F[F, A][F, B]) = \mathrm{Tr}([F, B]F[F, A]) = -\mathrm{Tr}(F[F, B][F, A])$. The right hand side of (3.7) is unchanged under skewsymmetrization: $\frac{1}{8} \mathrm{Tr}(F[F, A][F, B]) = \frac{1}{2} \mathrm{Tr}(A_- F B_-) = -\frac{1}{4} \mathrm{Tr}(F[A_-, B_-])$. Thus, in turn, equals

$$-\tfrac{1}{4} \mathrm{Tr}(F[A_-, B_-]) = -\tfrac{1}{4} \mathrm{Tr}(E_+ [A_-, B_-] E_+) + \tfrac{1}{4} \mathrm{Tr}(E_- [A_-, B_-] E_-)$$

$$= -\tfrac{1}{2} \mathrm{Tr}(E_+ A_- E_- B_- E_+ - E_+ B_- E_- A_- E_+) = \alpha(A, B).$$

This is a cyclic cochain, since $\alpha(A, B) = -\alpha(B, A)$; and it is a cocycle because

$$b\alpha(A, B, C) = \tfrac{1}{8} \mathrm{Tr}(F[F, AB][F, C] - F[F, A][F, BC] + F[F, CA][F, B])$$

$$= \tfrac{1}{8} \mathrm{Tr}(FA[F, B][F, C] - F[F, A][F, B]C + FC[F, A][F, B] + F[F, C]A[F, B])$$

$$= \tfrac{1}{8} \mathrm{Tr}(FA[F, B][F, C] - [F, A][F, B]FC + FC[F, A][F, B] - [F, C]FA[F, B])$$

$$= 0.$$

The Schwinger term is actually just a multiple of the Chern character τ_F^1, as specified by (3.5), of the Fredholm module defined by F. The Shale–Stinespring condition shows that $F[F, A][F, B]$ is trace-class, so that, in this case, the character formula makes sense already for $n = 1$.

3.2 Cyclic cohomology of Hopf algebras

We now take a closer look at the algebraic operators b and B, in the cohomological setting. They can be built up from simpler constituents. First of all, the coboundary $b: C^{n-1} \to C^n$ may be written as $b = \sum_{i=0}^{n}(-1)^i \delta_i$, where

$$\delta_i \varphi(a_0, \ldots, a_n) := \varphi(a_0, \ldots, a_i, a_{i+1}, \ldots, a_n), \qquad i = 0, 1, \ldots, n-1,$$
$$\delta_n \varphi(a_0, \ldots, a_n) := \varphi(a_n a_0, \ldots, a_{n-1}).$$

We also introduce maps $\sigma_j: C^{n+1} \to C^n$, for $j = 0, 1, \ldots, n$, given by

$$\sigma_j \varphi(a_0, \ldots, a_n) := \varphi(a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_n),$$

and recall the "cyclic permuter" $\tau: C^n \to C^n$ of (3.3):

$$\tau \varphi(a_0, \ldots, a_n) := \varphi(a_n, a_0, \ldots, a_{n-1}).$$

Notice that $\tau^{n+1} = 1$ on C^n. The operator B is built from the σ_j and τ, as follows. The "cyclic skewsymmetrizer" $N := \sum_{k=0}^{n}(-1)^{nk}\tau^k$ acts on C^n as

$$N\varphi(a_0, \ldots, a_n) = \varphi(a_0, \ldots, a_n) + \sum_{k=1}^{n}(-1)^{nk}\varphi(a_{n-k+1}, \ldots, a_n, a_0, \ldots, a_{n-k}).$$

The formula (3.1) now reduces to

$$B = (-1)^n N(\sigma_0 \tau^{-1} + \sigma_n) : C^{n+1} \to C^n.$$

The algebraic structure of cyclic cohomology is essentially determined by the relations between the elementary maps δ_i, σ_j and τ. For instance, the associativity of the algebra $\mathcal{A}$ is captured by the rule $\delta_{i+1}\delta_i = \delta_i^2$ as maps from C^{n-1} to C^{n+1}. Here is the full catalogue of these composition rules:

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if} \quad i < j;$$
$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad \text{if} \quad i \leqslant j;$$
$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j, \\ \iota & \text{if } i = j \text{ or } j+1, \\ \delta_{i-1}\sigma_j & \text{if } i > j+1; \end{cases}$$
$$\tau \delta_i = \delta_{i-1}\tau : C^{n-1} \to C^n \text{ for } i = 1, \ldots, n, \quad \text{and} \quad \tau\delta_0 = \delta_n,$$
$$\tau \sigma_j = \sigma_{j-1}\tau : C^{n+1} \to C^n \text{ for } j = 1, \ldots, n, \quad \text{and} \quad \tau\sigma_0 = \sigma_n \tau^2,$$
$$\tau^{n+1} = \iota \text{ on } C^n. \tag{3.8}$$

The first three rules, not involving τ, arise when working with simplexes of different dimensions, where the "face maps" δ_i identify an $(n-1)$-simplex with the ith face of an n-simplex, while the "degeneracy maps" σ_j reduce an $(n+1)$-simplex to an n-simplex by collapsing the edge from the jth to the $(j+1)$st vertex into a point. A set of simplexes, one in each dimension, together with maps δ_i and σ_j complying with the above rules, forms the so-called "simplicial category" Δ – see [68],

for instance – and any other instance of those rules defines a functor from Δ to another category: in other words, Δ is a universal model for those rules.

By bringing in the next three rules involving τ also, Connes defined a "cyclic category" Λ which serves as a universal model for cyclic cohomology [20]. Essentially, one supplements Δ with the maps which cyclically permute the vertices of each simplex (an ordering of the vertices is given). The point of this exercise is its universality, so that any system of maps complying with (3.8) gives a bona-fide cyclic cohomology theory, complete with periodicity properties and so on. Indeed, one can show [52, Lemma 10.4] that if $\gamma \colon C^{n-1} \to C^n$ is defined by $\gamma := \sum_{k=1}^{n} (-1)^k k\, \delta_k$, then $S := (n^2 + n)^{-1} b\gamma$ defines the periodicity operator on cyclic $(n-1)$-cocycles.

▶ Important cyclic cocycles, such as the characteristic classes for the algebras which typically arise in noncommutative geometry, can be quite difficult to compute. This is especially true for crossed product algebras, such as those of subsection 1.3. It is time to discuss how this problem may be addressed by transfer from cyclic cocycles of an associated Hopf algebra which acts on the algebra in question.

We recall from subsection 1.3 that such a crossed product algebra $\mathcal{A}$, obtained from the action of local diffeomorphisms on the frame bundle over a manifold, carries an action of a certain Hopf algebra H of differential (and multiplication) operators, where the Hopf action itself codifies the generalized Leibniz rules for these operators. To define characteristic classes in $HC^\bullet(\mathcal{A})$, we introduce a new cyclic cohomology for H and then show how to map H-classes to $\mathcal{A}$-classes.

This cyclic cohomology for H was introduced in [35] and developed further in [36–38] and also in [39, 40]. Its definition will make full use of the Hopf algebra structure, so we proceed in a "categorical" fashion. We shall first assume that the antipode S is *involutive*, that is, $S^2 = \iota_H$. As indicated earlier, this holds true for commutative or cocommutative Hopf algebras, although not for the Hopf algebra H_{CM} of subsection 1.3; but that case can be handled by making a suitable adjustment later on.

To set up the cyclic cohomology of H, we start with the algebras $C^n(H) := H^{\otimes n}$ for $n = 1, 2, 3, \ldots$ and $C^0(H) := \mathbb{C}$ (or $\mathbb{F}$, if one prefers other kinds of scalars). This looks superficially like the *chain* complex for associative algebras, but we shall make it a cochain complex by (once again) taking advantage of duality to replace products by coproducts, and so on. The "simplicial" operations are defined by

$$
\begin{aligned}
\delta_0(h^1 \otimes \cdots \otimes h^{n-1}) &:= 1 \otimes h^1 \otimes \cdots \otimes h^{n-1}, \\
\delta_i(h^1 \otimes \cdots \otimes h^{n-1}) &:= h^1 \otimes \cdots \otimes \Delta(h^i) \otimes \cdots \otimes h^{n-1}, \quad i = 1, \ldots, n-1, \\
\delta_n(h^1 \otimes \cdots \otimes h^{n-1}) &:= h^1 \otimes \cdots \otimes h^{n-1} \otimes 1, \\
\sigma_j(h^1 \otimes \cdots \otimes h^{n+1}) &:= \varepsilon(h^{j+1})\, h^1 \otimes \cdots \otimes h^j \otimes h^{j+2} \otimes \cdots \otimes h^{n+1}.
\end{aligned}
\tag{3.9}
$$

For $n = 0$, these reduce to $\delta_0(1) := 1$, $\delta_1(1) := 1$, and $\sigma_0(h) := \varepsilon(h)$. The relation $\delta_{i+1}\delta_i = \delta_i^2$ of (3.8) expresses the coassociativity of Δ and the equation $\Delta(1) = 1 \otimes 1$; the relations $\sigma_j \delta_j = \sigma_j \delta_{j+1} = \iota$ are equivalent to $(\iota \otimes \varepsilon)\Delta = (\varepsilon \otimes \iota)\Delta = \iota$; and the remaining relations involving the δ_i and the σ_j only are trivial.

To define the cyclic permuter τ, we first note that $H^{\otimes n}$ is itself an H-module algebra under the "diagonal" action of H:

$$
h \cdot (k^1 \otimes \cdots \otimes k^n) := (\Delta^{n-1} h)(k^1 \otimes \cdots \otimes k^n) = \sum h_{:1} k^1 \otimes h_{:2} k^2 \otimes \cdots \otimes h_{:n} k^n.
$$

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We then define

$$\tau(h^1 \otimes \cdots \otimes h^n) := S(h^1) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1) = \Delta^{n-1}(S(h^1))\,(h^2 \otimes \cdots \otimes h^n \otimes 1)$$
$$= \sum S(h^1_{:n})h^2 \otimes S(h^1_{:n-1})h^3 \otimes \cdots \otimes S(h^1_{:2})h^n \otimes S(h^1_{:1}). \qquad (3.10)$$

The cyclicity property of τ is a consequence of the following calculation.

Proposition 3.1. *The map* $\tau\colon H^{\otimes n} \to H^{\otimes n}$ *satisfies*

$$\tau^{n+1}(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = S^2(h^1) \otimes S^2(h^2) \otimes \cdots \otimes S^2(h^n). \qquad (3.11)$$

Proof. First we compute $\tau^2(h^1 \otimes h^2 \otimes \cdots \otimes h^n)$. The diagonal action of $S(S(h^1_{:n})\, h^2) = S(h^2)\, S^2(h^1_{:n})$ gives

$$\tau^2(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = \sum S(h^2_{:n})\, S^2(h^1_{:n})\, S(h^1_{:n-1})\, h^3 \otimes S(h^2_{:n-1})\, S^2(h^1_{:n+1})\, S(h^1_{:n-2})\, h^4$$
$$\otimes \cdots \otimes S(h^2_{:2})\, S^2(h^1_{:2n-2})\, S(h^1_{:1}) \otimes S(h^2_{:1})\, S^2(h^1_{:2n-1}).$$

Observe that $\sum S^2(h_{:2})\, S(h_{:1}) = S\big(\sum h_{:1}\, S(h_{:2})\big) = S(\varepsilon(h)\, 1) = \varepsilon(h)\, 1$. A further simplification is $\sum \varepsilon(h_{:2})S^2(h_{:3})\, S(h_{:1}) = \sum S^2(h_{:2})\, S(h_{:1}) = \varepsilon(h)\, 1$, so the terms $S^2(h^1_{:n+k})\, S(h^1_{:n-k-1})$ telescope from left to right, leaving

$$\tau^2(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = \sum S(h^2_{:n})\, h^3 \otimes S(h^2_{:n-1})\, h^4 \otimes \cdots \otimes S(h^2_{:2}) \otimes S(h^2_{:1})\, S^2(h^1),$$

where the sum runs over the terms in $\Delta^{n-1}S(h^2)$. After $n-1$ iterations of this process, we obtain

$$\tau^n(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = \sum S(h^n_{:n}) \otimes S(h^n_{:n-1})\, S^2(h^1) \otimes \cdots \otimes S(h^n_{:2})\, S^2(h^{n-2}) \otimes S(h^n_{:1})\, S^2(h^{n-1}),$$

and, since $\Delta^{n-1}(S(1)) = 1 \otimes \cdots \otimes 1$, the final iteration gives (3.11). $\qquad\square$

This shows that the condition $S^2 = \iota_H$ is necessary and sufficient to give $\tau^{n+1} = \iota$ on $C^n(H)$. We leave the remaining relations in (3.8) to the reader.

▶ However, it turns out that S^2 is *not* the identity in the Hopf algebra H_{CM}. For instance,

$$S^2(X) = S(-X + \lambda_1 Y) = (X - \lambda_1 Y) + S(Y)\, S(\lambda_1) = X + [Y, \lambda_1] = X + \lambda_1.$$

The day is saved by the existence of a character δ of H_{CM} such that the "twisted antipode" $S_\delta := \eta\delta * S$ is involutive. Indeed, since X and λ_1 are commutators, any character satisfies $\delta(X) = \delta(\lambda_1) = 0$, so any character is determined by its value on the other algebra generator, Y. We set $\delta(Y) := 1$. (Recall that $\varepsilon(Y) = 0$.) Now

$$S_\delta(h) := (\eta\delta * S)(h) = \sum \delta(h_{:1})\, S(h_{:2}),$$

so the twisted antipode does satisfy $S_\delta^2 = \iota_H$.

Exercise 3.2. Show this by verifying $S_\delta^2(X) = X$, $S_\delta^2(Y) = Y$, and $S_\delta^2(\lambda_1) = \lambda_1$ directly. $\qquad\Diamond$

The relation with the coproduct is given by

$$\Delta(S_\delta(h)) = \sum S(h_{:2}) \otimes S_\delta(h_{:1}), \qquad \Delta^2(S_\delta(h)) = \sum S(h_{:3}) \otimes S(h_{:2}) \otimes S_\delta(h_{:1}),$$

and more generally, $\Delta^{n-1}(S_\delta(h)) = \sum S(h_{:n}) \otimes \cdots \otimes S(h_{:2}) \otimes S_\delta(h_{:1})$. It is also worth noting that

$$\sum S_\delta(h_{:1})\, h_{:2} = \sum \delta(h_{:1})\, S(h_{:2})\, h_{:3} = \sum \delta(h_{:1})\, \varepsilon(h_{:2})\, 1 = \delta(h)\, 1.$$

The crossed product algebra $\mathcal{A}$ on which H_{CM} acts carries a distinguished faithful *trace*, given by integration over the frame bundle F with the Γ-invariant volume form ν:

$$\varphi(fU_\psi^\dagger) := 0 \text{ if } \psi \neq \iota, \qquad \varphi(f) := \int_F f\, d\nu. \tag{3.12}$$

It follows from (1.18) that, for $a = fU_\psi^\dagger$ and $b = gU_\psi$, the equality $\varphi(ab) = \varphi(ba)$ reduces to $\int_F f(g \circ \tilde{\psi})\, d\nu = \int_F (f \circ \tilde{\psi}^{-1})g\, d\nu$, so that the Γ-invariance of ν yields the tracial property of φ.

If $f \in C_c^\infty(F)$, it is easily checked that $\int_F (Xf)\, d\nu = 0$ and that $\int_F (Yf)\, d\nu = \int_F f\, d\nu$, using integration by parts. Moreover, since $\lambda_1(f) := h_\iota f$ from (1.23) and $h_\iota = 0$, we also get $\int_F (\lambda_1 f)\, d\nu = 0$. These identities are enough to confirm that

$$\varphi(h \cdot a) = \delta(h)\, \varphi(a), \quad \text{for all} \quad h \in H_{CM},\ a \in \mathcal{A}.$$

It is standard to call a functional μ on $\mathcal{A}$ "invariant" under a Hopf action if the relation $\mu(h \cdot a) = \varepsilon(h)\, \mu(a)$ holds. Since the character δ takes the place of the counit here, we may say that the trace φ is a δ-*invariant* functional.

This δ-invariance may be reformulated as a rule for *integration by parts*, as pointed out in [37]:

$$\varphi((h \cdot a)\, b) = \varphi(a\, (S_\delta(h) \cdot b)). \tag{3.13}$$

Indeed, one only needs to observe that

$$\varphi((h \cdot a)\, b) = \sum \varphi((h_{:1} \cdot a)\, \varepsilon(h_{:2})\, b) = \sum \varphi((h_{:1} \cdot a)\, (h_{:2}\, S(h_{:3}) \cdot b))$$
$$= \sum \varphi(h_{:1} \cdot (a\, (S(h_{:2}) \cdot b))) = \sum \delta(h_{:1})\, \varphi(a\, (S(h_{:2}) \cdot b)) = \varphi(a\, (S_\delta(h) \cdot b)).$$

The cyclic permuter τ must be redefined to take account of the twisted antipode S_δ, as follows:

$$\tau(h^1 \otimes \cdots \otimes h^n) := S_\delta(h^1) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1) = \Delta^{n-1}(S_\delta(h^1))\, (h^2 \otimes \cdots \otimes h^n \otimes 1)$$
$$= \sum S(h_{:n}^1)h^2 \otimes S(h_{:n-1}^1)h^3 \otimes \cdots \otimes S(h_{:2}^1)h^n \otimes S_\delta(h_{:1}^1).$$

A straightforward modification of the proof of Proposition 3.1 yields the following identity [40, Prop. 4.4]:

$$\tau^{n+1}(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = S_\delta^2(h^1) \otimes S_\delta^2(h^2) \otimes \cdots \otimes S_\delta^2(h^n).$$

Thus, $S_\delta^2 = \iota_H$ entails $\tau^{n+1} = \iota$ on $C^n(H)$.

▶ The cyclic cohomology $HC_\delta^\bullet(H)$ is now easily defined. The maps $b \colon C^{n-1}(H) \to C^n(H)$ and $B \colon C^{n+1}(H) \to C^n(H)$ are given by the very same formulae as before:

$$b := \sum_{i=0}^n (-1)^i \delta_i, \qquad B := (-1)^n N(\sigma_0 \tau^{-1} + \sigma_n),$$

where $N := \sum_{k=0}^n (-1)^{nk} \tau^k$ on $C^n(H)$.

Exercise 3.3. Show that $h \in H$ is a cyclic 1-cocycle if and only if h is primitive and $\delta(h) = 0$. ◊

It remains to show how $HC^\bullet_\delta(H)$ and $HC^\bullet(\mathcal{A})$ are related; the trace φ provides the link. For each $n = 0, 1, 2, \ldots$, we define a linear map $\gamma_\varphi \colon C^n(H) \to C^n(\mathcal{A}, \mathcal{A}^*)$ by setting $\gamma_\varphi(1) := \varphi$ and

$$\gamma_\varphi(h^1 \otimes \cdots \otimes h^n) : (a_0, \ldots, a_n) \mapsto \varphi(a_0 \, (h^1 \cdot a_1) \ldots (h^n \cdot a_n)).$$

Following [37], we call γ_φ the *characteristic map* associated to φ.

It is easy to check that γ_φ intertwines the maps δ_i, σ_j and τ defined on the two cochain complexes. For instance, if $i = 1, 2, \ldots, n-1$, then

$$
\begin{aligned}
\gamma_\varphi \delta_i(h^1 \otimes \cdots \otimes h^n) : (a_0, \ldots, a_{n+1}) &\mapsto \gamma_\varphi(h^1 \otimes \cdots \otimes \Delta(h^i) \otimes \cdots \otimes h^{n-1})\,(a_0, \ldots, a_{n+1}) \\
&= \varphi(a_0\,(h^1 \cdot a_1) \cdots (h^i_{:1} \cdot a_i)\,((h^i_{:2} \cdot a_{i+1}) \cdots (h^n \cdot a_{n+1})) \\
&= \varphi(a_0\,(h^1 \cdot a_1) \cdots (h^i \cdot (a_i a_{i+1})) \cdots (h^n \cdot a_{n+1})) \\
&= \delta_i \gamma_\varphi(h^1 \otimes \cdots \otimes h^n)\,(a_0, \ldots, a_{n+1}).
\end{aligned}
$$

To match the cyclic actions, we first recall that

$$\tau(h^1 \otimes h^2 \otimes \cdots \otimes h^n) = S_\delta(h^1) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1) = \sum S(h^1_{:2}) \cdot (h^2 \otimes \cdots \otimes h^n) \otimes S_\delta(h^1_{:1}).$$

Write $b := (h^2 \cdot a_1) \cdots (h^n \cdot a_{n-1})$; then, using the "integration by parts" formula, we get

$$
\begin{aligned}
\gamma_\varphi \tau(h^1 \otimes h^2 \otimes \cdots \otimes h^n)\,(a_0, \ldots, a_n) &= \sum \varphi\big(a_0\,(S(h^1_{:2}) \cdot b)\,S_\delta(h^1_{:1}) \cdot a_n\big) \\
&= \sum \varphi\big(h^1_{:1} \cdot (a_0\,S(h^1_{:2}) \cdot b)\,a_n\big) = \sum \varphi\big(a_n\,(h^1_{:1} \cdot a_0)\,(h^1_{:2}\,S(h^1_{:3}) \cdot b)\big) \\
&= \sum \varphi\big(a_n\,(h^1_{:1} \cdot a_0)\,\varepsilon(h^1_{:2})b\big) = \varphi(a_n\,(h^1 \cdot a_0)\,b) \\
&= \varphi\big(a_n\,(h^1 \cdot a_0)\,(h^2 \cdot a_1) \ldots (h^n \cdot a_{n-1})\big) = \tau\gamma_\varphi(h^1 \otimes h^2 \otimes \cdots \otimes h^n)\,(a_0, \ldots, a_n).
\end{aligned}
$$

In retrospect, we can see what lies behind the definition of τ on $C^n(H)$: on reading the last calculation backwards, we see that the formula for τ is predetermined in order to fulfil $\gamma_\varphi \tau = \tau \gamma_\varphi$ for any δ-invariant trace φ.

► We conclude with two variations on this algorithm for characteristic classes. The first concerns algebras which support a Hopf action but have no natural δ-invariant trace. In the theory of locally compact quantum groups [66], another possibility arises, namely that instead of a trace the algebra supports a linear functional φ such that $\varphi(ab) = \varphi(b(\sigma \cdot a))$ where σ is a grouplike "modular element" of the Hopf algebra. If φ is also δ-invariant for a character δ such that $\delta(\sigma) = 1$, only two further modifications of the elementary maps (3.9) and (3.10) are needed:

$$
\begin{aligned}
\delta_n(h^1 \otimes \cdots \otimes h^{n-1}) &:= h^1 \otimes \cdots \otimes h^{n-1} \otimes \sigma, \\
\tau(h^1 \otimes \cdots \otimes h^n) &:= S_\delta(h^1) \cdot (h^2 \otimes \cdots \otimes h^n \otimes \sigma).
\end{aligned}
$$

This time, the computation in Proposition 3.1 leads to

$$\tau^{n+1}(h^1 \otimes \cdots \otimes h^n) = \sigma^{-1} S^2_\delta(h^1)\, \sigma \otimes \cdots \otimes \sigma^{-1} S^2_\delta(h^n)\, \sigma.$$

Thus, the necessary and sufficient condition for $\tau^{n+1} = \iota$ is $S^2_\delta(h) = \sigma h \sigma^{-1}$ for all h. See [36] and [52, §14.7] for the detailed construction of the characteristic map in this "modular" case.

The other variant concerns the application to the original problem of finding characteristic classes for foliations, in the higher-dimensional cases, as discussed at the end of subsection 1.3. What is needed is a cohomology theory which takes account of the Hopf algebroid structure, when the coefficient is $\mathcal{R} = C^\infty(F)$ instead of $\mathbb{C}$. The formula (3.12) continues to define a Γ-invariant faithful trace on the algebra $\mathcal{A}$. Now, however, instead of seeking a special character δ, the main role is taken by the integration-by-parts formula (3.13). The twisted antipode in that formula is replaced by a map $\widetilde{S}\colon H \to H$, subject to four requirements: (a) that it be an algebra antihomomorphism; (b) which is involutive, that is, $\widetilde{S}^2 = \iota_H$; (c) that it exchange the algebroid actions of (1.25), namely, $\widetilde{S}\beta = \alpha$; and (d) that $m(\widetilde{S} \otimes_{\mathcal{R}} \iota)\Delta = \beta\varepsilon\widetilde{S}$. Connes and Moscovici show in [38] that a unique map $\widetilde{S}$ satisfying these properties exists, and with its help one can again build a cyclic cohomology theory for the Hopf algebroid of transverse differential operators, which provides the needed invariants of $\mathcal{A}$.

4 Noncommutative Homogeneous Spaces

4.1 Chern characters and noncommutative spheres

A fundamental theme of noncommutative geometry is the determination of geometric quantities from the spectra of certain operators on Hilbert space. An early precursor is Weyl's theorem on the dimension and volume of a compact Riemannian manifold: these are determined by the growth of the eigenvalues of the Laplacian. For spin manifolds, one can obtain the same data from the asymptotics of the spectra of the Dirac operator $\slashed{D}$. This phenomenon forms the background for the study of spectral triples. We know, for instance, that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ over the algebra $\mathcal{A} = C^\infty(M)$, complying with the seven requirements listed in subsection 1.1, provides a spin structure and a Riemannian metric on M for which D equals $\slashed{D}$ plus a torsion term.

A question raised in the paper which introduced these seven conditions [25] is whether the manifold itself – or its algebra of smooth coordinates – may be extracted from spectral data. The key property here is the orientation or volume-form condition:

$$\pi_D(\mathbf{c}) = \chi, \quad \text{with} \quad \mathbf{c} \in C_n(\mathcal{A}) \quad \text{such that} \quad b\,\mathbf{c} = 0, \tag{4.1}$$

where n is the classical dimension of the spin geometry. In view of the isomorphism between $HH_n(C^\infty(M)) \simeq \mathcal{A}^n(M)$, there is a unique n-form v matched to the class $[\mathbf{c}]$ of the Hochschild n-cycle. It turns out that (4.1) entails that v is nonvanishing on M, so that, suitably normalized, it is a volume form; in fact, it is the Riemannian volume for the metric associated to the Dirac-type operator D.

To see how this works, recall that the standard volume form on the 2-sphere $\mathbb{S}^2$ is

$$v = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \in \mathcal{A}^2(\mathbb{S}^2). \tag{4.2}$$

The corresponding Hochschild 2-cycle is

$$\mathbf{c} := \tfrac{i}{2}\big(x\,(dy\,dz - dz\,dy) + y\,(dz\,dx - dx\,dz) + z\,(dx\,dy - dy\,dx)\big) \in \Omega^2(C^\infty(\mathbb{S}^2)), \tag{4.3}$$

and (4.1) becomes

$$\tfrac{i}{2}\big(x\,[[D,y],[D,z]] + y\,[[D,z],[D,x]] + z\,[[D,x],[D,y]]\big) = \chi. \tag{4.4}$$

The algebra $\mathcal{A} = C^\infty(\mathbb{S}^2)$ is generated by the three commuting coordinates x, y, z, subject to the constraint $x^2 + y^2 + z^2 = 1$. It is important to note that one can vary the metric on $\mathbb{S}^2$ while keeping the volume form ν fixed; one usually thinks of the round metric $g = dx^2 + dy^2 + dz^2$ which is SO(3)-invariant, but one can compose g with any volume-preserving diffeomorphism of $\mathbb{S}^2$ to get many another metric g' whose volume form is also ν. Therefore, the D in the equation (4.1) is not uniquely determined; it may be a Dirac operator $D = \slashed{D}_{g'}$ obtained from any such metric g' (the Hilbert space $\mathcal{H}$ is the vector space of square-integrable spinors on $\mathbb{S}^2$).

On the other hand, one may think of (4.4) as a (highly nonlinear) equation for the coordinates x, y, z. To see how this comes about, we collect the three coordinates for the 2-sphere into a single orthogonal projector (selfadjoint idempotent),

$$e := \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \tag{4.5}$$

in the algebra of 2×2 matrices, $M_2(C^\infty(\mathbb{S}^2))$. This is actually the celebrated *Bott projector*, whose class $[e] \in K_0(C^\infty(\mathbb{S}^2)) = K^0(\mathbb{S}^2)$ is nontrivial. It is easy to check the following identity in exterior algebra:

$$\text{tr}((e - \tfrac{1}{2}) \, de \wedge de) = \tfrac{i}{2} \nu \in \mathcal{A}^2(\mathbb{S}^2).$$

Now, up to normalization and replacement of the exterior derivative by the differential of the universal graded differential algebra $\Omega^\bullet(C^\infty(\mathbb{S}^2))$, the left hand side is just the term $\text{ch}_1 \, e$ of the cyclic-homology Chern character of $[e]$. Notice that $\text{ch}_0 \, e = \text{tr}(e - \tfrac{1}{2})$ vanishes also. The cyclic homology computations preceding (3.4) show that, in full generality,

$$b(\text{ch}_1 \, e) = -B(\text{ch}_0 \, e),$$

so that the vanishing $\text{ch}_0 \, e = 0$ is enough to guarantee that $\text{ch}_1 \, e$ is a *Hochschild* cycle: $b(\text{ch}_1 \, e) = 0$.

▶ We now switch to a different point of view. Suppose we wish to produce examples of spectral triples $(\mathcal{A}, \mathcal{H}, D, C, \chi)$ satisfying the seven conditions for a noncommutative spin geometry. We first fix the classical dimension, which for convenience we shall suppose to be even: $n = 2m$. Then we start from the orientation condition:

$$\pi_D(\text{ch}_m \, e) = \chi, \tag{4.6a}$$

subject to the constraints

$$\text{ch}_0 \, e = 0, \quad \text{ch}_1 \, e = 0, \quad \dots, \quad \text{ch}_{m-1} \, e = 0, \tag{4.6b}$$

which guarantee that $\text{ch}_m \, e$ will be a Hochschild $2m$-cycle.

Consider (4.6) as a system of equations for an "unknown" projector $e \in M_r(\mathcal{A})$, r being a suitable matrix size. What does this system tell us about the coordinate algebra $\mathcal{A}$?

In Connes' survey paper [27], the answer is given in detail for the case $n = 2, r = 2$: it turns out that (4.6b) forces $\mathcal{A}$ to be commutative, and (4.6a) ensures that its character space is the 2-sphere. We summarize the argument, following our [52, §11.A]. First of all, the selfadjointness $e^* = e$ and the equation $\text{ch}_0 \, e = \text{tr}(e - \tfrac{1}{2}) = 0$ allow us to write e in the form (4.5), where x, y, z are selfadjoint elements of $\mathcal{A}$. The positivity of the projector e implies $-1 \leq z \leq 1$ (here we are implicitly

assuming that $\mathcal{A}$ is a dense subalgebra of a C^*-algebra). The idempotence $e^2 = e$ boils down to a pair of equations

$$(1 \pm z)^2 + x^2 + y^2 \pm i[x, y] = 2(1 \pm z),$$
$$(1 \mp z)(x \pm iy) + (x \pm iy)(1 \pm z) = 2(x \pm iy),$$

which simplify to $[x, y] = [y, z] = [z, x] = 0$ and $x^2 + y^2 + z^2 = 1$. Thus, x, y, z generate a *commutative* algebra $\mathcal{A}$. Moreover, by regarding them as commuting selfadjoint operators in a faithful representation of $\mathcal{A}$, the equation $x^2 + y^2 + z^2 = 1$ tells us that their joint spectrum in $\mathbb{R}^3$ is a closed subset V of the sphere $\mathbb{S}^2$: the C^*-completion of $\mathcal{A}$ is $C(V)$.

This partial description of $\mathcal{A}$ has not yet used the main equation (4.6a), whose role is to confirm that V is all of $\mathbb{S}^2$. For convenience, we abbreviate $\mathrm{d}a := [D, a]$ (at this stage, d is just an unspecified derivation on $\mathcal{A}$). Since

$$\mathrm{d}e = \frac{1}{2} \begin{pmatrix} \mathrm{d}z & \mathrm{d}x - i\,\mathrm{d}y \\ \mathrm{d}x + i\,\mathrm{d}y & -\mathrm{d}z \end{pmatrix},$$

a short calculation gives

$$\chi = \mathrm{tr}((e - \tfrac{1}{2})\,\mathrm{d}e\,\mathrm{d}e) = \tfrac{i}{2}(x\,[\mathrm{d}y, \mathrm{d}z] + y\,[\mathrm{d}z, \mathrm{d}x] + z\,[\mathrm{d}x, \mathrm{d}y]).$$

This is of the form $\pi_D(\mathbf{c}) = \chi$, where $\mathbf{c}$ is just the Hochschild 2-cycle of the formula (4.3). The corresponding volume form on V is precisely (4.2): but this volume is nonvanishing on all of $\mathbb{S}^2$, so we conclude that $V = \mathbb{S}^2$. The pre-C^*-algebra $\mathcal{A}$, generated by x, y, z, is none other than $C^\infty(\mathbb{S}^2)$!

▶ The odd-dimensional case $n = 2m + 1$ uses the odd Chern character (3.4b), and its orientation condition is $\pi_D(\mathrm{ch}_{m+\frac{1}{2}} u) = 1$, with constraints $\mathrm{ch}_{k+\frac{1}{2}} u = 0$ for $k = 0, 1, \ldots, m - 1$. The unitarity condition $u^* u = u u^* = 1$ may be assumed. For instance, in dimension three, Connes and Dubois-Violette [29] have shown that, under the sole constraint $\mathrm{ch}_{1/2} u = \mathrm{tr}(u^{-1}\,du) = 0$, all solutions of the equation $\pi_D(\mathrm{ch}_{3/2} u) = 1$ form a 3-parameter family of algebras; one of these is the commutative algebra $C^\infty(\mathbb{S}^3)$, but the others are noncommutative.

▶ Moving on now to dimension 4, we take $e = e^* = e^2$ in $M_4(\mathcal{A})$, and look for solutions of (4.6) with $2m = 4$. In [27], a commutative solution is again found, by using a "quaternionic" prescription reminiscent of the Connes–Lott approach to the Standard Model (see [24, VI.5] or [71] for the story of how quaternions enter in that approach). One writes e in 2×2 blocks:

$$e := \frac{1}{2} \begin{pmatrix} (1 + z)1_2 & q \\ q^* & (1 - z)1_2 \end{pmatrix}, \quad \text{where} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}. \tag{4.7}$$

Here again, z is a selfadjoint element of $\mathcal{A}$ such that $-1 \leqslant z \leqslant 1$, and $e^2 = e$ yields the equalities $qq^* = (1 - z^2) = q^*q$ and $[z\,1_2, q] = 0$. Since $qq^* = q^*q$ is diagonal, we find that $z, \alpha, \alpha^*, \beta, \beta^*$ are commuting elements of $\mathcal{A}$, subject to the constraint $\alpha\alpha^* + \beta\beta^* = 1 - z^2$: these are coordinate relations for a closed subset of $\mathbb{S}^4$. Once more, the equation (4.6a) produces the standard volume form supported on the full sphere, and the conclusion is that $\mathcal{A} = C^\infty(\mathbb{S}^4)$: the ordinary 4-sphere emerges as a solution to the cohomological equation (4.6) in dimension four.

Now, the particular quaternionic form of q in (4.7) is merely an Ansatz, and Landi soon pointed out that one could equally well try

$$q = \begin{pmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{pmatrix}, \quad \text{with} \quad \lambda \in \mathbb{C}.$$

The consequences are worked out in a recent paper by Connes and Landi [33] —see also [28]. One finds that $qq^* = (1 - z^2) = q^*q$ and $[z\,1_2, q] = 0$ still hold, but these relations now lead to

$$\alpha\beta = \bar{\lambda}^{-1}\beta\alpha, \qquad \alpha^*\beta = \bar{\lambda}\,\beta\alpha^*,$$
$$\beta\beta^* = \beta^*\beta, \qquad \alpha\alpha^* + \beta\beta^* = 1 - z^2 = \alpha^*\alpha + \lambda\bar{\lambda}\,\beta^*\beta. \qquad (4.8a)$$

The computation of $\mathrm{ch}_1(e)$, carried out in [41], yields

$$\mathrm{ch}_1(e) = \tfrac{1}{8}(1 - \lambda\bar{\lambda})\big(z\,[d\beta, d\beta^*] + \beta^*\,[dz, d\beta] + \beta\,[d\beta^*, dz]\big),$$

which vanishes if and only if λ is a complex number of modulus 1.

In particular, this scheme parts company with the ever-popular deformations where $\lambda = q$ would be a real number other than ± 1. Foremost among these are the well-known Podleś spheres $\mathbb{S}^2_{qc}$, which were originally constructed [80] as homogeneous spaces of the quantum group $\mathrm{SU}_q(2)$. Other higher-dimensional q-spheres currently on the market are described in [9, 14, 41, 58, 98]; the C^*-algebra construction of $\mathbb{S}^n_q$ by Hong and Szymański [58], in particular, is quite far-reaching. However, none of these arises from a Hochschild cycle in the manner described above. On the other hand, Aschieri and Bonechi [3] have constructed, with R-matrix techniques, a multiparameter family of quantum spaces which yields the spheres described here as limiting cases; see also [4].

By assuming $|\lambda| = 1$, $\lambda = e^{2\pi i\theta}$ from now on, the relations (4.8a) simplify to

$$\alpha\beta = \lambda\,\beta\alpha, \qquad \alpha^*\beta = \bar{\lambda}\,\beta\alpha^*,$$
$$\alpha\alpha^* = \alpha^*\alpha, \qquad \beta\beta^* = \beta^*\beta, \qquad \alpha\alpha^* + \beta\beta^* = 1 - z^2, \qquad (4.8b)$$

which determines a noncommutative algebra $\mathcal{A}$, baptized $C^\infty(\mathbb{S}^4_\theta)$ by Connes and Landi.

4.2 How Moyal products yield compact quantum groups

To construct a spin geometry over $\mathcal{A} = C^\infty(\mathbb{S}^4_\theta)$, we need a representation of this algebra on a suitable Hilbert space. The key is to notice that the relation $\alpha\beta = e^{2\pi i\theta}\beta\alpha$ of (4.8b), for normal operators α and β (that is, $\alpha\alpha^* = \alpha^*\alpha$ and $\beta\beta^* = \beta^*\beta$), is closely related to the definition of the noncommutative torus [19, 85]. This is a pre-C^*-algebra $C^\infty(\mathbb{T}^2_\theta)$ with two generators u and v which are unitary: $uu^* = u^*u = 1$, $vv^* = v^*v = 1$, subject only to the commutation relation

$$uv = e^{2\pi i\theta}\,vu. \qquad (4.9)$$

One can then define "spherical coordinates" (u, v, ϕ, ψ) for the noncommutative space $\mathbb{S}^4_\theta$ by setting

$$\alpha =: u\sin\psi\cos\phi, \qquad \beta =: v\sin\psi\sin\phi, \qquad z =: \cos\psi, \qquad (4.10)$$

where ϕ, ψ are ordinary angular coordinates. It is clear that this is equivalent to (4.8), for $\lambda = e^{2\pi i\theta}$.

There is a canonical action of the ordinary 2-torus $\mathbb{T}^2$ on the algebra $C^\infty(\mathbb{T}^2_\theta)$, obtained from the independent rotations $u \mapsto e^{2\pi i\phi_1} u$, $v \mapsto e^{2\pi i\phi_2} v$ that respect (4.9). By substituting these rotations in (4.10), we also obtain an action of $\mathbb{T}^2$ on $C^\infty(\mathbb{S}^4_\theta)$.

In the commutative case $\theta = 0$, this becomes an action of the abelian Lie group $\mathbb{T}^2$ by rotations on the compact manifold $\mathbb{S}^4$, and these rotations are isometries for the round metric on $\mathbb{S}^4$. Any

smooth function on $\mathbb{S}^4$ can be decomposed as a generalized Fourier series $f = \sum_r f_r$, indexed by $r = (r_1, r_2) \in \mathbb{Z}^2$, where f_r satisfies

$$(e^{2\pi i \phi_1}, e^{2\pi i \phi_2}) \cdot f_r = e^{2\pi i (r_1 \phi_1 + r_2 \phi_2)} f_r.$$

Indeed, each f_r is of the form $u^{r_1} v^{r_2} h(\phi, \psi)$, in terms of the coordinates (4.10); all such functions form the spectral subspace E_r of $C^\infty(\mathbb{S}^4)$. The same is true of $C^\infty(\mathbb{S}^4_\theta)$ when $\theta \neq 0$.

If $g_s = u^{s_1} v^{s_2} k(\phi, \psi)$, then $g_s \in E_s$ and $e^{2\pi i \theta r_2 s_1} f_r g_s = u^{r_1 + s_1} v^{r_2 + s_2} hk$ lies in E_{r+s}, so we may identify the algebra $C^\infty(\mathbb{S}^4_\theta)$ with the vector space $C^\infty(\mathbb{S}^4)$ of smooth functions on the ordinary 4-sphere, gifted with the new product:

$$f_r * g_s := e^{2\pi i \theta r_2 s_1} f_r g_s, \tag{4.11a}$$

defined on homogeneous elements $f_r \in E_r$, $g_s \in E_s$. Since the Fourier series $f = \sum_r f_r$ converges rapidly in the Fréchet topology of $C^\infty(\mathbb{S}^4)$, one can show that this recipe defines a continuous bilinear operation on that space. A more symmetric-looking operation, which yields an isomorphic algebra, is given by

$$f_r \times g_s := e^{\pi i \theta (r_2 s_1 - r_1 s_2)} f_r g_s. \tag{4.11b}$$

This deserves to be called a *Moyal product* of functions on $\mathbb{S}^4$. Indeed, suppressing the coordinates ϕ, ψ yields exactly the Moyal product on $C^\infty(\mathbb{T}^2)$, which has long been recognized to give the smooth algebras $C^\infty(\mathbb{T}^2_\theta)$ of the noncommutative 2-tori [109].

The only nonobvious feature of the products (4.11) is their associativity. To check it, we generalize a little. Suppose that M is a compact Riemannian manifold on which an l-dimensional torus acts by isometries (there is no shortage of examples of that). Then one can decompose $C^\infty(M)$ into spectral subspaces indexed by $\mathbb{Z}^l$. A "twisted" product of two homogeneous functions f_r and g_s may be defined by

$$f_r * g_s := \rho(r, s) f_r g_s, \tag{4.12}$$

where the phase factors $\{ \rho(r, s) \in \mathrm{U}(1) : r, s \in \mathbb{Z}^l \}$ make up a 2-cocycle on the additive group $\mathbb{Z}^l$. The cocycle relation

$$\rho(r, s + t)\rho(s, t) = \rho(r, s)\rho(r + s, t) \tag{4.13}$$

ensures that the new product is associative. To define such a cocycle, one could take [106]:

$$\rho(r, s) := \exp\{-2\pi i \sum_{j<k} r_j \theta_{jk} s_k\},$$

where $\theta = [\theta_{jk}]$ is a real $l \times l$ matrix. Complex conjugation of functions remains an involution for the new product provided that the matrix θ is *skewsymmetric*. (When $l = 2$, it is customary to replace the matrix θ by the real number $\theta_{12} = -\theta_{21}$ and, rather sloppily, call this number θ, too; but in higher dimensions one is forced to deal with a matrix of parameters.) The product (4.12) defines a C^*-algebra which, when $M = \mathbb{T}^l$, is isomorphic to that of the noncommutative torus $C(\mathbb{T}^l_\theta)$ with parameter matrix θ, as we shall soon see.

Moreover, we may define a "Moyal product":

$$f_r \times g_s := \sigma(r, s) f_r g_s, \tag{4.14}$$

by replacing ρ by its skewsymmetrized version,

$$\sigma(r, s) := \exp\{-\pi i \sum_{j,k=1}^l r_j \theta_{jk} s_k\}, \tag{4.15}$$

which is again a group 2-cocycle; in fact, ρ and σ are cohomologous as group cocycles [86], therefore they define isomorphic C^*-algebras.

▶ To see why (4.14) should be called a Moyal product, let us briefly recall the real thing. The quantum product of two functions on the phase space $\mathbb{R}^{2m}$ was introduced by Moyal [75] using a series development in powers of $\hbar$ whose first nontrivial term gives the Poisson bracket; later, it was noticed [81] that it could be rewritten in an integral form [50]:

$$(f \times_J g)(x) := (\pi\hbar)^{-2m} \iint f(x+s)g(x+t)\, e^{2is \cdot Jt/\hbar}\, ds\, dt,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the skewsymmetric matrix giving the standard symplectic structure on $\mathbb{R}^{2m}$ (and the dot is the usual scalar product). This is in fact the Fourier transform of the "twisted convolution" of phase-space functions which goes back to von Neumann's work on the Schrödinger representation [78]. For suitable classes of functions and distributions on $\mathbb{R}^{2m}$, it is an oscillatory integral, which yields Moyal's series development as an *asymptotic* expansion in powers of $\hbar$ [44, 107].

This integral form of the Moyal product is the starting point for a general deformation theory of C^*-algebras, which was undertaken by Rieffel [87]. He gave it a mildly improved presentation by rewriting it as

$$(f \times_J g)(x) := \iint f(x+Js)g(x+t)\, e^{2\pi is \cdot t}\, ds\, dt,$$

taking $\hbar = 2$ and rescaling the measure on $\mathbb{R}^{2m}$. He then replaced the functions f, g by elements a, b of any C^*-algebra A, and the translations $f(x) \mapsto f(x+t)$ by a strongly continuous action α of $\mathbb{R}^l$ on A by automorphisms; and he replaced the original matrix J by any skewsymmetric real $l \times l$ matrix, still called J, ending up with

$$a \times_J b := \iint_{\mathbb{R}^l \times \mathbb{R}^l} \alpha_{Js}(a)\alpha_t(b)\, e^{2\pi is \cdot t}\, ds\, dt. \tag{4.16}$$

This formula makes sense, as an oscillatory integral, for elements a, b in the subalgebra $A^\infty := \{ a \in A : t \mapsto \alpha_t(a) \text{ is smooth} \}$, which is a Fréchet pre-C^*-algebra (as a subalgebra of the original C^*-algebra A).

We wish to complete the algebra $(A^\infty, \times_J)$ to a C^*-algebra A_J, which in general is not isomorphic to A (for instance, A may be commutative while the new product is not). The task is to find a new norm $\| \cdot \|_J$ on A^∞ with the C^*-property $\|a^* \times_J a\|_J = \|a\|_J^2$; then A_J is just the completion of A^∞ in this norm. Rieffel achieved this by considering the left multiplication operators $L_a^J = L^J(a)$ given by

$$L_a^J f(x) := \iint \alpha_{x+Js}(a)f(x+t)\, e^{2\pi is \cdot t}\, ds\, dt,$$

where f is a smooth A-valued function which is rapidly decreasing at infinity. A particular "Schwartz space" of such functions f is identified in [87], on which the obvious A-valued pairing $(f \mid g) := \int_{\mathbb{R}^l} f(x)^* g(x)\, dx$ yields a Hilbert-space norm by setting $\interleave f \interleave^2 := \|(f \mid f)\|_A$. It can then be shown that if $a \in A^\infty$, L_a^J is a bounded operator on this Hilbert space; $\|a\|_J$ is defined to be the

operator norm of L_a^J. Importantly, L^J is a homomorphism:

$$
\begin{aligned}
L^J(a \times_J b) f(x) &= \iint \alpha_{x+Js}(a \times_J b) f(x+t) \, e^{2\pi i s \cdot t} \, ds \, dt \\
&= \iiiint \alpha_{x+Js+Ju}(a) \alpha_{x+Js+v}(b) f(x+t) \, e^{2\pi i (s \cdot t + u \cdot v)} \, du \, dv \, ds \, dt \\
&= \iiiint \alpha_{x+Ju'}(a) \alpha_{x+v+Js}(b) f(x+v+t') \, e^{2\pi i (s \cdot t' + u' \cdot v)} \, ds \, dt' \, du' \, dv \\
&= \iint \alpha_{x+Ju'}(a) L_b^J f(x+v) \, e^{2\pi i u' \cdot v} \, du' \, dv \\
&= L_a^J L_b^J f(x),
\end{aligned}
\tag{4.17}
$$

so that $L^J(a \times_J b) = L^J(a) L^J(b)$. The calculation uses only the change of variable $t' := t - v$, $u' := s + u$, for which $s \cdot t + u \cdot v = s \cdot t' + u' \cdot v$.

Rieffel's construction provides a *deformation* $A \mapsto A_J$ of C^*-algebras which is explicit only on the smooth subalgebra A^∞. This construction has several useful functorial properties which we now list, referring to the monograph [87] for the proofs.

- If A and B are two C^*-algebras carrying the respective actions α and β of $\mathbb{R}^l$, and if $\phi \colon A \to B$ is a $*$-homomorphism intertwining them: $\phi \alpha_t = \beta_t \phi$ for all t, then $\phi(A^\infty) \subseteq B^\infty$ and the restriction of ϕ to A^∞ extends uniquely to a $*$-homomorphism $\phi_J \colon A_J \to B_J$.

- The map ϕ_J is injective if and only if ϕ is injective, and ϕ_J is surjective if and only if ϕ is surjective.

- When $A = B$ and $\alpha = \beta$, we may take $\phi = \alpha_s$ for any s, because $\alpha_s \alpha_t = \alpha_{s+t} = \alpha_t \alpha_s$ for all t; thus $\alpha_J \colon s \mapsto (\alpha_s)_J$ is an action of $\mathbb{R}^l$ on A_J by automorphisms, whose restriction to A^∞ coincides with the original action α.

- Deforming (A_J, α_J) with another skewsymmetric matrix K gives a C^*-algebra isomorphic to A_{J+K}. In particular, if $K = -J$, the second deformation recovers the original algebra A.

- The smooth subalgebra $(A_J)^\infty$ of A_J under the action α_J coincides exactly with the original smooth subalgebra A^∞ (although their products are different).

When the action α of $\mathbb{R}^l$ is periodic, so that $\alpha_t = \iota_A$ for each t in a subgroup L, then α is effectively an action of the abelian group $H = \mathbb{R}^l / L$, and $H \simeq \mathbb{T}^k \times \mathbb{R}^{l-k}$ for some k. Suppose that H is compact, i.e., $k = l$ and $H \simeq \mathbb{T}^l$. Then A^∞ decomposes into spectral subspaces $\{ E_p : p \in L \}$ where $\alpha_s(a_p) = e^{2\pi i p \cdot s} a_p$ for $a_p \in E_p$. If $b_q \in E_q$ also, one can check [87, Prop. 2.21] that

$$
a_p \times_J b_q = e^{-2\pi i p \cdot Jq} a_p b_q.
$$

On comparing this with (4.14), we see that if $A = C(\mathbb{T}^l)$ and $J := \frac{1}{2}\theta$, then A_J is none other than the noncommutative l-torus $C(\mathbb{T}_\theta^l)$. Moreover, if $A = C(\mathbb{S}^4)$ and θ is a real number, then the rotation action of $\mathbb{T}^2$ on $\mathbb{S}^4$ and the parameter matrix

$$
Q := \frac{1}{2} \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}
$$

define a deformation such that $C(\mathbb{S}^4)_Q \simeq C(\mathbb{S}_\theta^4)$.

▶ We now apply this machinery to the case of the C^*-algebra $C(G)$, where G is a compact connected Lie group. The dense subalgebra $\mathcal{R}(G)$ is a Hopf algebra: we may ask how its coalgebra structure is modified by this kind of deformation. The answer is: not at all! It turns out that, for suitable parameter matrices J, the coproduct remains an algebra homomorphism for the new product $\times_J$. This was seen early on by Dubois-Violette [42] in the context of Woronowicz' compact quantum groups: he noticed that the matrix corepresentations of $C(\mathrm{SU}_q(N))$ and similar bialgebras could be seen as different products on the same coalgebra.

There are many ways in which a torus can act on G. Indeed, any connected abelian closed subgroup H of G is a torus; by the standard theory of compact Lie groups [13, 97], any such H is included in a maximal torus, and all maximal tori are conjugate. Thus H can act on G by left translation, right translation, or conjugation. In what follows, we shall focus on the action of the doubled torus $H \times H$ on G, given by

$$(h, k) \cdot x := hxk^{-1}. \tag{4.18}$$

The corresponding action on $C(G)$ is $[(h, k) \cdot f](x) := f(h^{-1}xk)$. If $\mathfrak{h}$ is the Lie algebra of H, we may pull this back to a periodic action of the $\mathfrak{h} \oplus \mathfrak{h}$ on $C(G)$. For notational convenience, we choose and fix a basis for the vector space $\mathfrak{h} \simeq \mathbb{R}^l$, which allows to write the exponential mapping as a homomorphism $e \colon \mathbb{R}^l \to H$ whose kernel is the integer lattice $\mathbb{Z}^l$. If $\lambda := e(1, 1, \ldots, 1)$, we may write $\lambda^s := e(s)$ for $s \in \mathbb{R}^l$; and the action of $\mathfrak{h} \oplus \mathfrak{h}$ on $C(G)$ becomes

$$[\alpha(s, t)f](x) := f(\lambda^{-s}x\lambda^t). \tag{4.19}$$

The coefficient matrix J for the Moyal product (4.16) is now a skewsymmetric matrix in $M_{2l}(\mathbb{R})$. It is argued in [90] – see also [105, §4] – that compatibility with the coalgebra structure is to be expected only if J splits as the direct sum of two opposing $l \times l$ matrices:

$$J := \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} \tag{4.20}$$

where $Q \in M_l(\mathbb{R})$ is evidently skewsymmetric. Here, we accept this as an Ansatz and explore where it leads.

The Moyal product on the group manifold G can now be written as

$$(f \times_J g)(x) := \int_{\mathfrak{h}^4} f(\lambda^{-Qs}x\lambda^{-Qt})g(\lambda^{-u}x\lambda^v)\, e^{2\pi i(s \cdot u + t \cdot v)}\, ds\, dt\, du\, dv. \tag{4.21}$$

We remind ourselves that this makes sense as an oscillatory integral provided $f, g \in C^\infty(G)$, since the smooth subalgebra of $C(G)$ for the action (4.19) certainly includes $C^\infty(G)$; it could, however, be larger, for instance if the torus H is not maximal.

In subsection 1.2, the coproduct, counit and antipode for the Hopf algebra $\mathcal{R}(G)$ are defined by

$$\Delta f(x, y) := f(xy), \qquad \varepsilon(f) := f(1), \qquad Sf(x) := f(x^{-1}). \tag{4.22}$$

These formulas make sense in $C^\infty(G)$, which includes $\mathcal{R}(G)$ since representative functions are real-analytic; or even in $C(G)$. In accordance with the remarks at the end of subsection 1.2, we

shall now discard the algebraic tensor product and work in the smooth category. The coproduct may now be regarded as a homomorphism

$$\Delta : C^\infty(G) \to C^\infty(G \times G),$$

the counit is a homomorphism $\varepsilon \colon C^\infty(G) \to \mathbb{C}$, and the coalgebra relations $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and $(\varepsilon \otimes \iota)\Delta = (\iota \otimes \varepsilon)\Delta = \iota$ continue to hold. Moreover, the antipode S is an algebra antiautomorphism of $C^\infty(G)$.

Let us check that all of those statements continue to hold when the pointwise product of functions in $C^\infty(G)$ is replaced by a Moyal product. The following calculations are taken from [88]; they all make use of changes of variable similar to that of (4.17). First of all,

$$
\begin{aligned}
(\Delta f \times_J \Delta g)(x, y) &= \int_{\mathfrak{h}^8} f(\lambda^{-Qs} x \lambda^{-Qt-Qs'} y \lambda^{-Qt'}) g(\lambda^{-u} x \lambda^{v-u'} y \lambda^{v'})\, e^{2\pi i(s \cdot u + t \cdot v + s' \cdot u' + t' \cdot v')}\, ds \cdots dv' \\
&= \int_{\mathfrak{h}^8} f(\lambda^{-Qs} x \lambda^{-Qt''} y \lambda^{-Qt'}) g(\lambda^{-u} x \lambda^{-u''} y \lambda^{v'})\, e^{2\pi i(s \cdot u + t'' \cdot v + s' \cdot u'' + t' \cdot v')}\, ds \cdots dv' \\
&= \int_{\mathfrak{h}^6} f(\lambda^{-Qs} x \lambda^{-Qt''} y \lambda^{-Qt'}) g(\lambda^{-u} x \lambda^{-u''} y \lambda^{v'})\, e^{2\pi i(s \cdot u + t' \cdot v')}\, \delta(t'')\, \delta(u'')\, ds \cdots dv' \\
&= \int_{\mathfrak{h}^4} f(\lambda^{-Qs} xy \lambda^{-Qt'}) g(\lambda^{-u} xy \lambda^{v'})\, e^{2\pi i(s \cdot u + t' \cdot v')}\, ds\, dt'\, du\, dv' \\
&= (f \times_J g)(xy) = \Delta(f \times_J g)(x, y).
\end{aligned}
$$

Integrations like $\int_{\mathfrak{h}} e^{2\pi i t'' \cdot v}\, dv = \delta(t'')$ are a convenient shorthand for the Fourier inversion theorem. Next,

$$(f \times_J g)(1) = \int_{\mathfrak{h}^4} f(\lambda^{-Q(s+t)}) g(\lambda^{v-u})\, e^{2\pi i(s \cdot u + t \cdot v)}\, ds\, dt\, du\, dv,$$

which simplifies to

$$
\begin{aligned}
\int_{\mathfrak{h}^4} f(\lambda^{-Qs'}) g(\lambda^{v'})\, & e^{2\pi i(s' \cdot u + t \cdot v')}\, ds'\, dt\, du\, dv' \\
&= \int_{\mathfrak{h}^2} f(\lambda^{-Qs'}) g(\lambda^{v'})\, \delta(s')\, \delta(v')\, ds'\, dv' = f(1)\, g(1),
\end{aligned}
$$

so $\varepsilon(f \times_J g) = \varepsilon(f)\varepsilon(g)$. Finally, if Q is invertible, then

$$
\begin{aligned}
(Sf \times_J Sg)(x) &= \int_{\mathfrak{h}^4} f(\lambda^{Qt} x^{-1} \lambda^{Qs}) g(\lambda^{-v} x^{-1} \lambda^{u})\, e^{2\pi i(s \cdot u + t \cdot v)}\, ds\, dt\, du\, dv \\
&= (\det Q)^{-2} \int_{\mathfrak{h}^4} f(\lambda^{-t'} x^{-1} \lambda^{s'}) g(\lambda^{-v} x^{-1} \lambda^{-u})\, e^{-2\pi i(Q^{-1} t' \cdot v + Q^{-1} s' \cdot u)}\, ds'\, dt'\, du\, dv \\
&= \int_{\mathfrak{h}^4} f(\lambda^{-t'} x^{-1} \lambda^{s'}) g(\lambda^{-Qv'} x^{-1} \lambda^{-Qu'})\, e^{2\pi i(t' \cdot v' + s' \cdot u')}\, ds'\, dt'\, du'\, dv' \\
&= (g \times_J f)(x^{-1}) = S(g \times_J f)(x),
\end{aligned}
$$

where the skewsymmetry of Q has been used. On the other hand, if $Q = 0$, then $f \times_J g = fg$ and the calculation reduces to $(Sf \times_J Sg)(x) = f(x^{-1})g(x^{-1}) = S(g \times_J f)(x)$; since we may integrate separately over the nullspace of Q and its orthogonal complement, the relation $Sf \times_J Sg = S(g \times_J f)$ holds in general.

Exercise 4.1. Show, by similar calculations, that

$$m(\iota \otimes S)(\Delta f) = m(S \otimes \iota)(\Delta f) = \varepsilon(f)\,1$$

whenever $f \in C^{\infty}(G)$. ◊

The functoriality of Rieffel's construction then lifts these maps to the C^*-level, without further calculation. That is: the maps Δ, ε and S, defined as above on smooth functions only, extend respectively to a $*$-homomorphism $\Delta_J\colon C(G)_J \to C(G)_J \bar{\otimes} C(G)_J$ (using the minimal tensor product of C^*-algebras), a character $\varepsilon_J\colon C(G)_J \to \mathbb{C}$, and a $*$-antiautomorphism $S_J\colon C(G)_J \to C(G)_J$.

However, the Moyal product itself on $C^{\infty}(G)$ generally need not extend to a continuous linear map from $C(G)_J \otimes C(G)_J$ to $C(G)_J$. This may happen because the product map m is generally *not* continuous for the minimal tensor product. (There is an interesting category of "Hopf C^*-algebras", introduced by Vaes and van Daele [102], which does have continuous products, but the link with Moyal deformations remains to be worked out.)

The C^*-algebras $C(G)_J$, arising from Moyal products whose coefficient matrices are of the form (4.20), are fully deserving of the name *compact quantum groups*. Indeed, they are thus baptized in [88]. They differ from the compact quantum groups of Woronowicz [113] in that they explicitly define the algebraic operations on smooth subalgebras, and are thus well-adapted to the needs of noncommutative geometry.

4.3 Isospectral deformations of homogeneous spin geometries

The Connes–Landi spheres $\mathbb{S}^4_\theta$ can now be seen as homogeneous spaces for compact quantum groups. The ordinary 4-sphere is certainly a homogeneous space; in fact, it is – almost by definition – an orbit of the 5-dimensional rotation group: thus, $\mathbb{S}^4 \approx \mathrm{SO}(5)/\mathrm{SO}(4)$. Now, $\mathrm{SO}(5)$ is a compact simple Lie group of rank two; that is to say, its maximal torus is $\mathbb{T}^2$. We can exhibit this maximal torus as the group of block-diagonal matrices

$$h = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 & & & \\ -\sin\phi_1 & \cos\phi_1 & & & \\ & & \cos\phi_2 & \sin\phi_2 & \\ & & -\sin\phi_2 & \cos\phi_2 & \\ & & & & 1 \end{pmatrix}.$$

By regarding $\mathbb{S}^4$ as the orbit of $(0,0,0,0,1)$ in $\mathbb{R}^5$, whose isotropy subgroup is $\mathrm{SO}(4)$, we see that the maximal torus of $\mathrm{SO}(4)$ is also $\mathbb{T}^2$. When the 4-sphere is identified as the right-coset space $\mathrm{SO}(5)/\mathrm{SO}(4)$, and the doubled torus $\mathbb{T}^2 \times \mathbb{T}^2$ is made to act on $\mathrm{SO}(5)$ by left-right multiplication as in (4.18), then the right action of the second $\mathbb{T}^2$ is absorbed in the cosets, but the left action of the first $\mathbb{T}^2$ passes to the quotient. This is a group-theoretical description of how the 2-torus acts by rotations on the 4-sphere. The action is isometric since the left translations preserve the invariant metric on the group, and also preserve the induced $\mathrm{SO}(5)$-invariant metric on the coset space.

There is an immediate generalization, proposed in [105], which highlights the nature of this torus action. Consider a tower of subgroups

$$H \leqslant K \leqslant G,$$

where G is a compact connected Lie group, K is a closed subgroup of G, and H is a closed connected *abelian* subgroup of K, i.e., a torus. The example we have just seen reappears in higher dimensions as

$$\mathbb{T}^l \leqslant SO(2l) \leqslant SO(2l+1), \quad \text{with} \quad \mathbb{S}^{2l} \approx SO(2l+1)/SO(2l).$$

Odd-dimensional spheres yield a slightly different case:

$$\mathbb{T}^l \leqslant SO(2l+1) \leqslant SO(2l+2), \quad \text{with} \quad \mathbb{S}^{2l+1} \approx SO(2l+2)/SO(2l+1).$$

This time, H is a maximal torus in K but not in G.

Since $H \leqslant K$, the left-right action (4.18) of $H \times H$ on both G and K induces a left action of H on the quotient space $M := G/K$, since the right action of H is absorbed in the right K-cosets. If we deform $C(G)$, under the action of $H \times H$, by means of a Moyal product with parameter matrix $J = Q \oplus (-Q)$, the natural thing to expect is that the C^*-algebra $C(G/K)$ undergoes a deformation governed by Q only. We now prove this, following [105].

It helps to recall the discussion of homogeneous spaces at the end of subsection 1.2. We are now in a position to replace the generic function space $\mathcal{F}(G)$ used there by either $C^\infty(G)$ or $C(G)$, according to need. In particular, the algebra isomorphism $\zeta \colon C^\infty(G)^K \to C^\infty(G/K)$ given by $\zeta f(\bar{x}) := f(x)$ intertwines the coproduct Δ on $C^\infty(G)$ with the coaction $\rho \colon C^\infty(M) \to C^\infty(G) \otimes C^\infty(G/K)$ defined by $\rho f(x, \bar{y}) := f(\overline{xy})$.

We can distinguish three abelian group actions here. First there is action α of $\mathfrak{h} \oplus \mathfrak{h}$ on $C(G)$, already given by (4.19). Next, the formula $(\beta_t h)(\bar{x}) := h(\overline{\lambda^{-t}x})$ determines an action β of $\mathfrak{h}$ on $C(G/K)$. Then there is action γ of $\mathfrak{h}$ on $C(G)^K$ where $(\gamma_t f)(x) := f(\lambda^{-t}x)$; it can be regarded as an action of $\mathfrak{h} \oplus \mathfrak{h}$ where the second factor acts trivially, so that γ is just the restriction of α to the subspace $C(G)^K$ of $C(G)$.

Let $f, g \in C^\infty(G)^K$ be smooth right K-invariant functions. Then

$$
\begin{aligned}
(f \times_J g)(x) &= \int_{\mathfrak{h}^4} f(\lambda^{-Qs}x\lambda^{-Qt})g(\lambda^{-u}x\lambda^v)\, e^{2\pi i(s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv \\
&= \int_{\mathfrak{h}^4} f(\lambda^{-Qs}x)g(\lambda^{-u}x)\, e^{2\pi i(s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv \\
&= \int_{\mathfrak{h}^2} f(\lambda^{-Qs}x)g(\lambda^{-u}x)\, e^{2\pi i s\cdot u}\, ds\, du = (f \times_Q g)(x),
\end{aligned}
$$

where the Q-product comes from the action γ on $C(G)^K$. On passing to $C^\infty(G/K)$ with the isomorphism ζ, which obviously intertwines the actions γ and β, this calculation shows that

$$\zeta(f \times_J g) = \zeta f \times_Q \zeta g \quad \text{for all} \quad f, g \in C^\infty(G)^K.$$

In other words, the J-product on $C(G)$ induces the Q-product, as claimed.

The reason for this bookkeeping with actions and isomorphisms is to be able to lift everything to the C^*-level, using Rieffel's functoriality theorems. First, since $\zeta\gamma_t = \beta_t\zeta$ for each $t \in \mathfrak{h}$, the isomorphism $\zeta^{-1} \colon C^\infty(G/K) \to C^\infty(G)^K$ extends to a $*$-isomorphism of $C(G/K)_Q$ onto $C(G)^K_Q$. Since γ is the restriction of α to $C(G)^K$, the inclusion $C^\infty(G)^K \hookrightarrow C^\infty(G)$ is equivariant for the actions γ and α, so it extends to an injective $*$-homomorphism from $C(G)^K_Q$ to $C(G)_J$. We may summarize by saying that the isomorphism and inclusion

$$C(G/K) \simeq C(G)^K \hookrightarrow C(G)$$

restricts to the smooth subalgebras

$$C^\infty(G/K) \simeq C^\infty(G)^K \hookrightarrow C^\infty(G),$$

and from there extends to an isomorphism and inclusion

$$C(G/K)_Q \simeq C(G)_Q^K \hookrightarrow C(G)_J.$$

This shows that the deformed C^*-algebra $C(G/K)_Q$ is an *embedded homogeneous space* for the compact quantum group $C(G)_J$.

Example 4.1. To get the noncommutative spheres of Connes and Landi, just take $G = \mathrm{SO}(2l + 1)$, $K = \mathrm{SO}(2l)$ and let $H = \mathbb{T}^l$ be the maximal torus for both. Then let $Q = \frac{1}{2}\theta$, where θ is any real skewsymmetric $l \times l$ matrix. The resulting deformation of $C(\mathbb{S}^{2l})$ is the C^*-algebra $C(\mathbb{S}_\theta^{2l})$, and its smooth subalgebra (for the $\mathbb{T}^l$-action) is just $C^\infty(\mathbb{S}_\theta^{2l}) := C^\infty(\mathbb{S}^{2l})$ with the Moyal product $\times_Q$.

The odd-dimensional spheres $\mathbb{S}^{2l+1} = \mathrm{SO}(2l + 2)/\mathrm{SO}(2l + 1)$ may be deformed in like manner, using, say, the maximal torus $\mathbb{T}^l$ of $\mathrm{SO}(2l+1)$. However, in this case, since this torus is not maximal in the full group $\mathrm{SO}(2l+2)$, one can regard the $\mathbb{T}^l$-rotations as an action of the torus $\mathbb{T}^{l+1}$ that is trivial in one direction. The algebras $C(\mathbb{S}_\theta^{2l+1})$, with their $\mathbb{T}^l$-actions and the corresponding deformations of $C(\mathrm{SO}(2l + 2))$, have recently been discussed extensively by Connes and Dubois-Violette [29] from the cohomological standpoint.

In fact, in [29], the noncommutative spheres are constructed in another way, by directly obtaining generators and relations for the corresponding algebras from twistings of Clifford algebras, as already outlined in [33], before checking that those algebras also come from θ-deformations. The advantage of this procedure is that what is obtained is manifestly spherical, in the sense that the homology-sphere condition (4.6), or its odd-dimensional counterpart, is built-in [43].

The simplest *nonspherical* examples in even dimensions are $\mathbb{CP}^2 \simeq \mathrm{SU}(3)/\mathrm{U}(2)$ and the 6-dimensional flag manifold $\mathbb{F}^6 = \mathrm{SU}(3)/\mathbb{T}^2$. With $G = \mathrm{SU}(3)$ and $H = K = \mathbb{T}^2$ and any irrational $\theta = 2Q_{12}$, one obtains a family of 6-dimensional quantized flag manifolds.

▶ We have outlined a general construction of noncommutative algebras, including all the Connes–Landi spheres, which come equipped with dense pre-C^*-algebras. The final step is to build noncommutative spin geometries based on these algebras. This was done by Connes and Landi for their spheres [33] by means of an *isospectral deformation*. It was observed in [105], and likewise in [29], that their algorithm extends directly to any of the aforementioned quantum homogeneous spaces, with only notational changes.

The compact homogeneous manifold G/K can be regarded as a Riemannian manifold, since it has a G-invariant metric. We shall assume that G/K also has a homogeneous spin structure (this is not always the case; for instance, $\mathbb{CP}^2$ is only spin^c, while $\mathrm{SU}(3)/\mathrm{SO}(3)$ does not even admit a spin^c structure [47, §2.4]), and we let $\slashed{D}$ be the corresponding Dirac operator; it is a selfadjoint operator on the Hilbert space $\mathcal{H}$ of square-integrable spinors. As we shall see, in the end we only need that the metric, and the Dirac operator, be invariant under the action of the torus H rather than the full group G.

It is important to remark that the action of H by isometries on G/K does not lift directly to the spinor space $\mathcal{H}$ (or, if one prefers, to the spinor bundle S). Rather, in view of the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ where $n = \dim G/K$, there is a double covering $\widetilde{H} \xrightarrow{\pi} H$ and a homomorphism $\widetilde{H} \to \mathrm{Aut}(S)$ which covers the homomorphism $H \to \mathrm{Isom}(G/K)$ [29, §13]. This yields a group of

unitaries $\{\, V_{\tilde{x}} : \tilde{x} \in \widetilde{H} \,\}$ on $\mathcal{H}$ which preserve the subspace $\Gamma^\infty(G/K, S)$ of smooth spinors and cover the isometries $\{\, I_x : x \in H \,\}$ of G/K. More precisely: if $\phi, \psi \in \Gamma^\infty(G/K, S)$ and $f \in C^\infty(G/K)$, then

$$V_{\tilde{x}}(f\psi) = I_x(f)\, V_{\tilde{x}}\psi, \quad \text{and} \quad (V_{\tilde{x}}\phi)^\dagger V_{\tilde{x}}\psi = I_x(\phi^\dagger\psi),$$

where $x = \pi(\tilde{x})$. Consequently, the Dirac operator $D\!\!\!\!/$ on $\mathcal{H}$ commutes with each $V_{\tilde{x}}$.

Now choose a basis $X_1, \ldots, X_l$ of the Lie algebra $\mathfrak{h}$, and for $j = 1, \ldots, l$, let p_j be the selfadjoint operator representing X_j on $\mathcal{H}$; if $\exp\colon \mathfrak{h} \to H$ and $\mathrm{Exp}\colon \mathfrak{h} \to \widetilde{H}$ denote the exponential maps, then $\pi(\mathrm{Exp}(tX_j)) = \exp(tX_j/2)$ and $p_j = -i\frac{d}{dt}\big|_{t=0} V_{\mathrm{Exp}(tX_j)}$. Therefore, the spectrum of each operator p_j lies either in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$. For each $r \in \mathbb{R}^l$, we may define a unitary operator

$$\sigma(p, r) := \exp\{-2\pi i \sum_{j,k} p_j Q_{jk} r_k\}, \tag{4.23}$$

by formally replacing half of the arguments of the group cocycle (4.15) with the operators p_j; its inverse is the similarly defined operator $\sigma(r, p)$. These operators commute with each other and also with $D\!\!\!\!/$, but they do not commute with the representation of $C^\infty(G/K)$ on $\mathcal{H}$ (multiplication of spinors by functions).

The unitary conjugations $T \mapsto \sigma(p, t)T\sigma(t, p)$ define an action of $\mathbb{R}^l$ on the algebra of bounded operators on $\mathcal{H}$, which is periodic on account of the half-integer spectra of the p_j, and this action gives a grading of operators into spectral subspaces, indexed by $\mathbb{Z}^l$. Therefore, any bounded operator T in the common smooth domain of these transformations has a decomposition $T = \sum_{r \in \mathbb{Z}^l} T_r$, where the components satisfy the commutation rules

$$\sigma(p, r)\, T_s = T_s\, \sigma(p + s, r) \quad \text{for} \quad r, s \in \mathbb{Z}^l.$$

For any multiplication operator f obtained from the representation of the algebra $C^\infty(G/K)$ on spinors, this grading coincides with the previous decomposition $f = \sum_{r \in \mathbb{Z}^l} f_r$.

The operator $\mathbb{Z}^l$-grading allows us to define a "left twist" of T by

$$L(T) := \sum_{r \in \mathbb{Z}^l} T_r\, \sigma(p, r).$$

If $f, g \in C^\infty(G/K)$, the group cocycle property (4.13) of σ shows that

$$L(f)L(g) = \sum_{r,s} f_r\, \sigma(p, r)\, g_s\, \sigma(p, s) = \sum_{r,s} f_r\, g_s\, \sigma(p + s, r)\, \sigma(p, s)$$
$$= \sum_{r,s} f_r\, g_s\, \sigma(r, s)\, \sigma(p, r + s) = L(f \times_Q g),$$

on account of (4.14). Therefore, L yields a representation of $C^\infty(G/K)_Q := (C^\infty(G/K), \times_Q)$ on $\mathcal{H}$. In other words, the Moyal product gives not only an abstract deformation of the algebra $C^\infty(G/K)$, but also – more importantly – it yields a deformation of the spinor *representation* of $C^\infty(G/K)$, without disturbing the underlying Hilbert space.

The recipe for creating new spin geometries should now be clear: one deforms the algebra (and its representation), while keeping unchanged all the other terms of the spectral triple: the Hilbert space $\mathcal{H}$ together with its grading χ if $\dim(G/K)$ is even, the operator $D\!\!\!\!/$, and the charge conjugation C. This deformation is *isospectral* [33] in the tautological sense that the spectrum in question is that of the operator $D\!\!\!\!/$, which remains the same.

It remains to check that the new spectral triple satisfies the conditions governing a spin geometry. First of all, each $[\slashed{D}, L(f)]$, for $f \in C^\infty(G/K)$, must be a bounded operator; this is ensured by noting that

$$[\slashed{D}, L(f)] = \sum_r [\slashed{D}, f_r]\, \sigma(p, r) = L([\slashed{D}, f]),$$

since each $[\slashed{D}, f]$ is bounded. The grading operator χ is unaffected by the torus action on G/K since the metric is taken to be H-invariant: this implies $L(\chi) = \chi$. In view of the previous equation, the orientation equation $\pi_{\slashed{D}}(\mathbf{c}) = \chi$ survives after application of L to both sides.

The reality condition is more interesting. The charge conjugation operator C on spinors [52, Chap. 9] commutes with all $\sigma(p, r)$ (again, due to H-invariance of the metric). It follows from (4.23) and the antilinearity of C that $C p_j C^{-1} = -p_j$ for each j. We can now define a "right twist"

$$R(T) := C L(T)^* C^{-1} = \sum_{r \in \mathbb{Z}^l} \sigma(r, p)\, C T_r^* C^{-1} = \sum_{r \in \mathbb{Z}^l} C T_r^* C^{-1}\, \sigma(r, p).$$

Now, C intertwines multiplication operators from $C^\infty(G/K)$ with their complex conjugates: $C f^* C^{-1} = f$ for f. Therefore, $R(f) = \sum_{r \in \mathbb{Z}^l} f_r\, \sigma(r, p)$, from which one can check that $R(f) R(g) = R(f \times_{-Q} g)$; in other words, R gives an *anti*representation of $C^\infty(G/K)_Q$ on $\mathcal{H}$. This commutes with the representation L:

$$L(f) R(g) = \sum_{r,s} f_r\, \sigma(p, r)\, g_s\, \sigma(s, p) = \sum_{r,s} f_r g_s\, \sigma(p + s, r)\, \sigma(s, p)$$

$$= \sum_{r,s} g_s f_r\, \sigma(s, p + r)\, \sigma(p, r) = \sum_{r,s} g_s\, \sigma(s, p)\, f_r\, \sigma(p, r) = R(g) L(f).$$

The first-order property of the spin geometry is now immediate

$$[[\slashed{D}, L(f)], R(g)] = \sum_{r,s} \sigma(p, r)\, [[\slashed{D}, f_r], g_s]\, \sigma(s, p) = 0,$$

since $[[\slashed{D}, f_r], g_s] = 0$ in the commutative case (the commutator $[\slashed{D}, f_r]$ is an operator of order zero which commutes with multiplication operators). Regularity and finiteness are straightforward, since the smooth subalgebra $C^\infty(G/K)$ does not grow or shrink under deformations. Poincaré duality also goes through, on account of another theorem of Rieffel, to the effect that the K-theory of the pre-C^*-algebras remains unaffected by deformations [89].

The construction is now complete. We sum up with the following Proposition.

Proposition 4.1. *Let $H \leqslant K \leqslant G$ be a tower of compact connected Lie groups where H is a torus, such that G/K admits a spin structure. Let $(C^\infty(G/K), \mathcal{H}, \slashed{D}, C, \chi)$ denote any commutative spin geometry over $C^\infty(G/K)$ where $\mathcal{H}$ is the spinor space and $\slashed{D}$ is the Dirac operator for an H-invariant metric on G/K. Then there is a noncommutative spin geometry obtained from it by isospectral deformation, whose algebra $C^\infty(G/K)_Q$ is that of any quantum homogeneous space obtained from a Moyal product $\times_Q$ on $C^\infty(G/K)$.* $\qquad\square$

Acknowledgements

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