Sparse bounds for Bochner-Riesz and Maximal Bochner-Riesz

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Some background on sparse bounds

Introduction

A collection of cubes $\mathcal{S}$ is c-sparse if for each $S \in \mathcal{S}$ there is $E_S \subseteq S$ such that

1. $|E_S| > c|S|$,
2. $\| \sum_{S \in \mathcal{S}} \mathbb{1}_{E_S} \|_\infty \leq c^{-1}$.

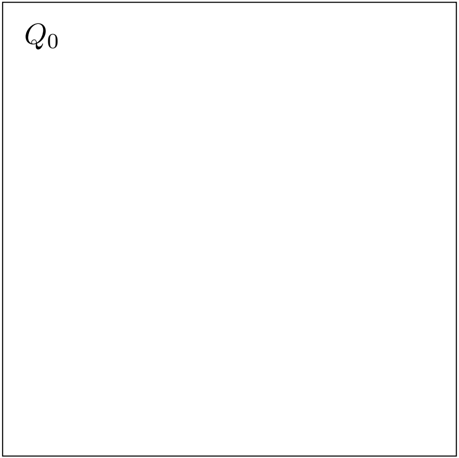
Sometimes the second condition is replaced by requiring that the sets E_S are disjoint.

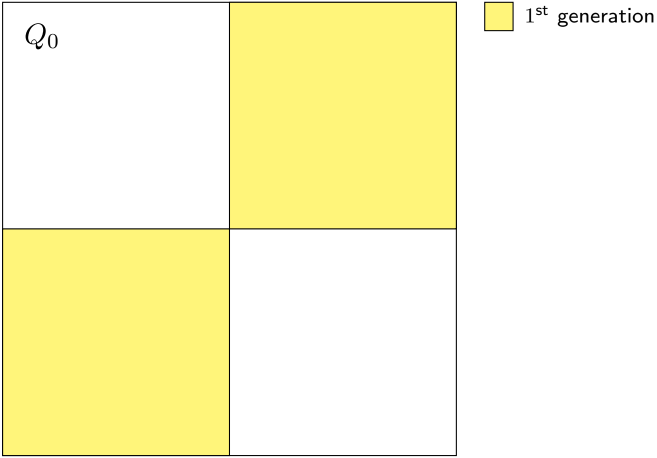
Usually we take $\mathcal{S}$ a subcollection of a dyadic grid such that

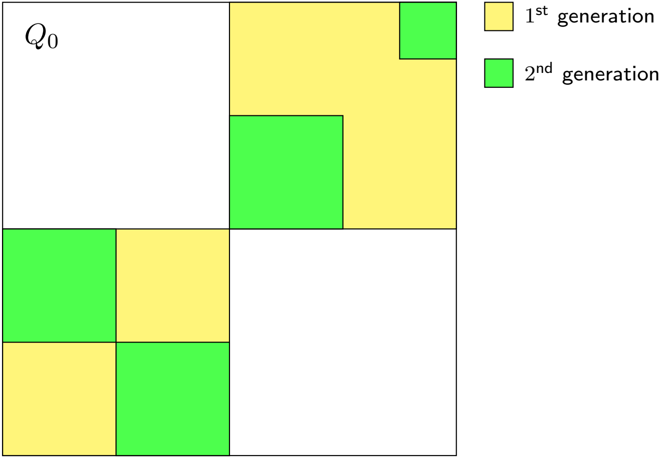
$$\sum_{S' \in \mathrm{Ch}_{\mathcal{S}}(S)} |S'| \leq \frac{1}{2}|S|.$$

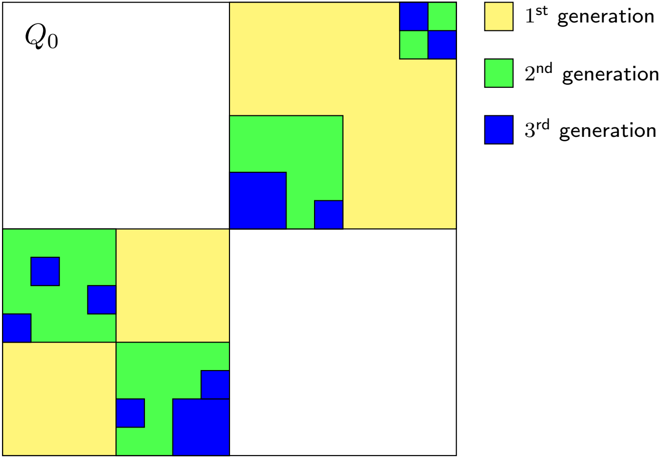
Here, $\mathrm{Ch}_{\mathcal{S}}(S) = \{ S' \in \mathcal{S}$ maximal : $S' \subsetneq S \}$.

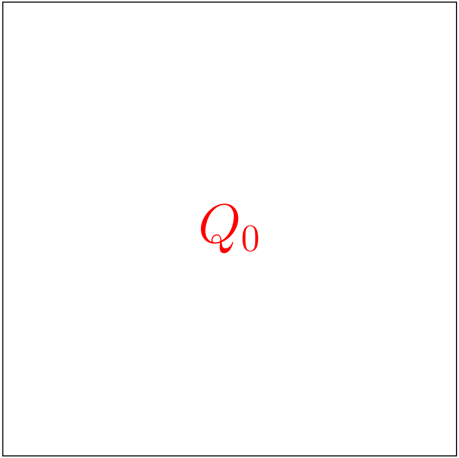
Then take $E_S = S \setminus \bigcup_{S' \in \mathrm{Ch}_{\mathcal{S}}\, S}$ S'

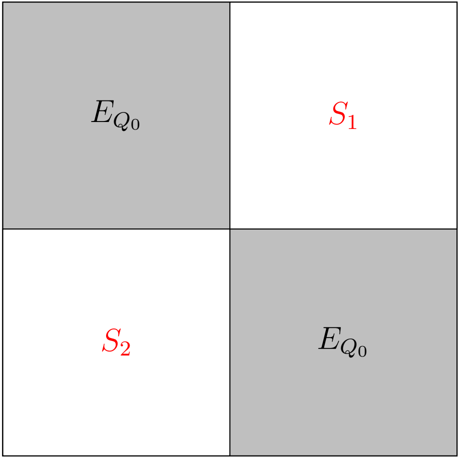

Q_0


Q_0
1^{st} generation


Q_0
1st generation
2nd generation


Q_0
1st generation
2nd generation
3rd generation


Q_0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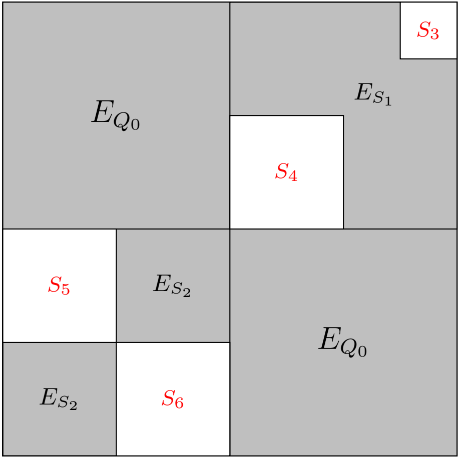
E_{Q_0}
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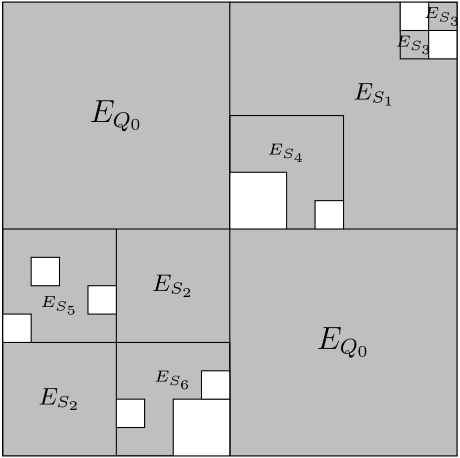
S_6


E_{S_3}
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E_{S_5}
E_{S_2}
E_{Q_0}
E_{S_6}
E_{S_2}

Given a sparse collection $\mathcal{S}$, a sparse operator is defined by

$$\Lambda_{\mathcal{S}} f(x) = \sum_{S \in \mathcal{S}} \langle f \rangle_S \, \mathbb{1}_S(x).$$

This operator satisfies a weak 1-1 bound and are bounded (strongly) on L^p for $p > 1$. They are also bounded on $L^p(w)$ for an A_2 weight w.

To exemplify the convenience of working with these operators we look at the proof of the boundedness on L^p.

If $f \in L^p$ and $g \in L^q$ with $\|g\| = 1$,

$$\begin{aligned}\langle \Lambda f, g\rangle &= \left\langle \sum_{S\in\mathcal{S}} \langle f\rangle_S \mathbb{1}_S, g \right\rangle = \sum_{S\in\mathcal{S}} \langle f\rangle_S \langle \mathbb{1}_S, g\rangle = \sum_{S\in\mathcal{S}} \langle f\rangle_S \langle g\rangle_S |S| \\ &\leq 2\sum_{S\in\mathcal{S}} \langle f\rangle_S \langle g\rangle_S |E_S| = 2\sum_{S\in\mathcal{S}} \int \langle f\rangle_S \langle g\rangle_S \mathbb{1}_{E_s}(x)\, dx \\ &\lesssim \int \sum_{S\in\mathcal{S}} \langle f\rangle_S \mathbb{1}_{E_S} \langle g\rangle_S \mathbb{1}_S(x)\, dx \leq \int \mathcal{M}g(x) \sum_{S\in\mathcal{S}} \langle f\rangle_S \mathbb{1}_{E_S}\, dx \\ &\leq \int \mathcal{M}g(x)\mathcal{M}f(x)\, dx \leq \|\mathcal{M}f\|_{L^p} \|\mathcal{M}g\|_{L^q}\end{aligned}$$

Let $\langle f \rangle_{S,r} = \left(\frac{1}{|S|} \int_S |f(x)|^r \, dx \right)^{1/r}$; For $r, s > 1$, a bilinear (r, s)-sparse form is defined by

$$\Lambda_{\mathcal{S},r,s}(f, g) = \sum_{S \in \mathcal{S}} \langle f \rangle_{S,r} \langle g \rangle_{S,s} |S|.$$

We say that an operator T is in Sparse(r, s) if there exists a sparse collection $\mathcal{S}$ such that

$$\langle Tf, g \rangle \lesssim \Lambda_{\mathcal{S},r,s}(f, g),$$

for all compactly supported functions f, g. We will make this more precise later in terms of a sparse norm.

Boundedness:

- If T is in Sparse(r, s) with $1 \leq r < s'$, then T is bounded on L^p for every $r < p < s'$.

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- If T is in Sparse(r, s), for $1 \leq r < s'$, then, for every $p \in (r, s')$, then T is bounded on $L^p(w)$, for weights w belonging to an intersection of a special class A_{p_0} and a reverse-Hölder class (Bernicot, Frey and Petermichl '16).

- We define a sparse form $\Lambda'_{\mathcal{S},r,s}$. by using the non-local average:

$$\langle\langle f \rangle\rangle_{Q,r} = \left[|Q|^{-1} \int |f(x)|^r [1 + \mathsf{dist}(x,Q)/|Q|]^{-(n+1)} dx \right]^{\frac{1}{r}}.$$

Lemma (Culiuc, Kesler, Lacey)

Lemma 2.8 For bounded and compactly supported functions f, g, and $1 \leq r, s < \infty$, we have

$$\sup_{\mathcal{S}} \Lambda'_{\mathcal{S},r,s}(f,g) \lesssim \sup_{\mathcal{S}} \Lambda_{\mathcal{S},r,s}(f,g).$$

Universal domination:

There is one sparse form *"to rule them all"*...

Lemma (Lacey, M.)

Given finitely supported functions f, g, there is a sparse form Λ^* and a constant $C > 0$ such that for any other sparse operator Λ we have

$$\Lambda(f, g) \leq C\Lambda^*(f, g).$$

Sparse bounds for Bochner-Riesz multipliers

Sparse bounds for Bochner-Riesz multpliers

(joint work with Michael Lacey and María Carmen Reguera)

We define the Bochner-Riesz operator by the multiplier

$$\widehat{B_\delta^R f}(\xi) = (1 - |\xi|^2/R^2)_+^\delta \widehat{f}(\xi).$$

We have that $\lim_{R\to\infty} B_\delta^R f = f$ in L^p if and only if

$$\|B_\delta^R f\|_p \lesssim_p \|f\|_p,$$

where the implied constant doesn't depend on R. By dilation arguments, it is enough to consider the case $R = 1$, that is

$$\widehat{B_\delta f}(\xi) = (1 - |\xi|^2)_+^\delta \widehat{f}(\xi).$$

Conjecture (Bochner-Riesz conjecture)

The operator B_δ is bounded on $L^p(\mathbb{R}^n)$ if

$$n\left|\tfrac{1}{p}-\tfrac{1}{2}\right| < \tfrac{1}{2}+\delta, \quad 0<\delta<\frac{n-1}{2}.$$

The condition is the same as

$$\underbrace{\frac{2n}{n+1+2\delta}}_{p_0'(\delta)} < p < \underbrace{\frac{2n}{n-1-2\delta}}_{p_0(\delta)}.$$

$\delta_n = \frac{n-1}{2}$ is called the critical index for summability.

Previous Results

- **Carleson-Sjölin:** The conjecture is true for $n = 2$.
- **Herz:** The restrictions on p and λ are necessary.
- **Bourgain, Guth:** For $n \geq 3$, the conjecture is true with the added restrictions

$$\max(p, p') > \begin{cases} \frac{2(4n+3)}{4n-3} & n \equiv 0 \mod 3 \\ \frac{2n+1}{n-1} & n \equiv 1 \mod 3 \\ \frac{4(n+1)}{2n-1} & n \equiv 2 \mod 3 \end{cases}$$

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- **Benea, Bernicot, Luque:** Sparse bounds for Bochner-Riesz for certain indexes.

Theorem (Lacey, M., Reguera)

Let $n = 2$, *and* $0 < \delta < \frac{1}{2}$. *Let* $\mathbf{R}(2,\delta)$ *be the open trapezoid with vertices*

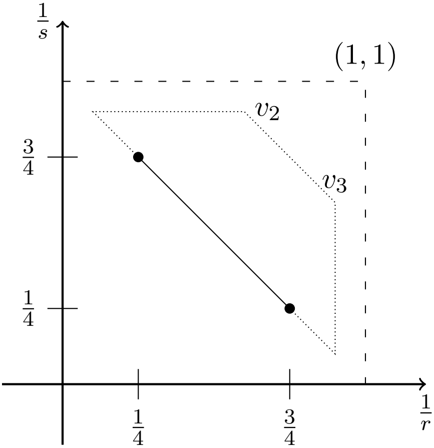
$$v_{2,\delta,1} = (\tfrac{1-2\delta}{4}, \tfrac{3+2\delta}{4}), \quad v_{2,\delta,2} = (\tfrac{1+6\delta}{4}, \tfrac{3+2\delta}{4}),$$
$$v_{2,\delta,3} = (\tfrac{3+2\delta}{4}, \tfrac{1+6\delta}{4}), \quad v_{2,\delta,4} = (\tfrac{3+2\delta}{4}, \tfrac{1-2\delta}{4}).$$

There holds,

$$\|B_\delta : (r,s)\| < \infty, \qquad (\tfrac{1}{r}, \tfrac{1}{s}) \in \mathbf{R}(2,\delta).$$

Moreover, the inequality above fails for $\frac{1}{r} + \frac{1}{s} > 1$, *with* $(\frac{1}{r}, \frac{1}{s})$ *not in the closure of* $\mathbf{R}(2,\delta)$.

The region described in the theorem is the following



Formulation in terms of annuli: Let χ be a Schwartz function such that $\mathbb{1}_{[-1/4,1/4]} \le \chi \le \mathbb{1}_{[-1/2,1/2]}$. Set S_τ to be the Fourier multiplier with symbol $\chi((|\xi|-1)/\tau)$.

Conjecture

Subject to the condition $n\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{2}$, *there holds*

$$\|S_\tau\|_{L^p \mapsto L^p} \lesssim_\epsilon 1, \qquad 0 < \tau < 1.$$

$A(\tau) \lesssim_\epsilon B(\tau)$ means that for all $0 < \epsilon < 1$, there is C_ϵ so that uniformly in $0 < \tau < 1$, there holds $A(\tau) \le C_\epsilon \tau^{-\epsilon} B(\tau)$.

For $n \geq 3$, and $0 < \delta < \frac{n-1}{2}$, let $\mathbf{R}(n, p_0, \delta)$ be the open trapezoid with vertexes

$$v_{n,\delta,1} = (\tfrac{1}{p_0}(1 - \tfrac{2\delta}{n-1}), \tfrac{1}{p_0'} + \tfrac{1}{p_0}\tfrac{2\delta}{n-1}),$$
$$v_{n,\delta,2} = (\tfrac{1}{p_0} + \tfrac{1}{p_0'}\tfrac{2\delta}{n-1}, \tfrac{1}{p_0'} + \tfrac{1}{p_0}\tfrac{2\delta}{n-1}),$$
$$v_{n,\delta,3} = \overline{v_{n,p_0,2}}, \quad v_{n,\delta,4} = \overline{v_{n,p_0,1}}, \quad \text{where } \overline{(a,b)} = (b,a).$$

The region is similar to the previous one.

Theorem (Lacey, M., Reguera)

Assume dimension $n \geq 2$. *And let* $1 < p_0 < 2$ *be such that the* L^{p_0} *boundedness holds. Then, for* $0 < \delta < \frac{n-1}{2}$, *the following sparse bound hold.*

$$\|B_\delta : (r, s)\| < \infty, \qquad (\tfrac{1}{r}, \tfrac{1}{s}) \in \mathbf{R}(n, p_0, \delta).$$

Moreover, for the critical value of $p = p(\delta)$ *given by* $\frac{n}{p_\delta} = \frac{n+1}{2} + \delta$, *the inequality above fails for* $\frac{1}{r} + \frac{1}{s} > 1$, *with* $(\frac{1}{r}, \frac{1}{s})$ *not in the closure of* $\mathbf{R}(n, p_\delta, \delta)$.

In general, interpolation is an issue when dealing with sparse bounds, but here, we only need a "single scale" version of the sparse form

$$\tilde{\Lambda}_{\tau,r,s}(f,g) = \sum_{\substack{Q \in \mathcal{D} \\ 1 \le \ell Q \le \frac{1}{\tau^{1+\eta}}}} \langle\langle f \rangle\rangle_{Q,r} \langle\langle g \rangle\rangle_{Q,s} |Q|, \qquad 0 < \eta < 1.$$

Note that here, we are using all dyadic cubes with side length between 1 and $\tau^{-1-\eta}$.

Lemma

Let $1 \le r_j, s_j \le \infty$ *for* $j = 0, 1$ *and fix* $0 < \tau < \infty$. *Suppose that for some linear operator* T *we have*

$$|\langle Tf, g\rangle| \le C_j \tilde{\Lambda}_{\tau, r_j, s_j}(f, g), \qquad j = 0, 1,$$

for all smooth compactly supported functions f, g. *Then, for* $0 < \theta < 1$, *we have*

$$|\langle Tf, g\rangle| \le C_0^{\theta} C_1^{1-\theta} \tilde{\Lambda}_{\tau, r_\theta, s_\theta}(f, g),$$

where $\frac{1}{r_\theta} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}$, *and similarly for* s_θ.

Proof of the sparse bound

For each $0 < \delta < \frac{n-1}{2}$, we have

$$B_\delta = T_0 + \sum_{k=1}^{\infty} 2^{-k\delta} \operatorname{Dil}_{1-2^{-k}} S_{2^{-k}},$$

Proof of the sparse bound

For each $0 < \delta < \frac{n-1}{2}$, we have

$$B_\delta = T_0 + \sum_{k=1}^{\infty} 2^{-k\delta} \operatorname{Dil}_{1-2^{-k}} S_{2^{-k}},$$

- T_0 is a Fourier multiplier supported near the origin.
- $\operatorname{Dil}_s f(x) = f(x/s)$.
- Each $S_{2^{-k}}$ has multiplier $\chi_k(2^k \big||\xi| - 1\big|)$, the χ_k satisfy a uniform class of derivative estimates.

With this decomposition, which is present in several places in the literature, and in different variants (e.g. Córdoba or Duoandikoetxea), it is enough to prove the sparse bound for the annuli operator.

Theorem (Lacey, M., Reguera)

Assume dimension $n \geq 2$*. And let* $1 < p_0 < 2$ *be such that the* L^{p_0} *boundedness holds. Then, the following sparse bounds hold. For all* $(\frac{1}{r}, \frac{1}{s}) \in \mathbf{R}(n, p_0, \delta)$*, there is a* $\kappa = \kappa(r, s) > 0$ *so that*

$$\|S_\tau : (r, s)\| \lesssim_\epsilon \tau^{-\delta+\kappa}, \qquad 0 < \tau < 1.$$

Let K_τ be the kernel of the operator S_τ. We have the following lemma, which proof considers standard estimates over the Haar measure on $\mathbb{S}^{n-1}$.

Lemma

For $0 < \tau < \frac{1}{2}$, *these properties hold.*

1. *For all* $0 < \eta < 1$ *and* $N > 1$,

$$|K_\tau(x)| \lesssim \tau \cdot \begin{cases} [1+|x|]^{-\frac{n-1}{2}} & |x| < C\tau^{-1-\eta} \\ |x|^{\frac{1-n}{2}}[\tau|x|]^{-N} & \text{otherwise.} \end{cases}$$

   The implied constants depend upon $0 < \eta < 1$, *and* $N > 1$.

2. $\|S_\tau\|_{1\mapsto 1} \lesssim \tau^{-\frac{n-1}{2}}$.

The decay condition allows us to work with the "single scale" version of the sparse form, $\tilde{\Lambda}_{\tau,r,s}$. We define $\|T : (r, s, \tau)\|$ to be the best constant C in the inequality

$$|\langle Tf, g\rangle| \leq C\tilde{\Lambda}_{\tau,r,s}(f, g),$$

the inequality holding uniformly over all bounded and compactly supported functions f, g.

The following lemma provides us with the key points on the "sparse plane" to carry over the needed interpolation.

Lemma

Assume dimension $n \geq 2$. And let $1 < p_0 < 2$ such that the L^{p_0} boundedness holds. These sparse bounds hold, for all $0 < \tau, \eta < 1$.

$$\|S_\tau : (1, 1, \tau)\| \lesssim \tau^{-\frac{n-1}{2} - n\eta},$$
$$\|S_\tau : (1, \infty, \tau)\| \lesssim \tau^{-\frac{n-1}{2} - n\eta},$$
$$\|S_\tau : (p_0, p_0', \tau)\| \lesssim \tau^{-\eta}$$

The implied constants depend upon $0 < \eta < 1$.

"Proof by picture" of the sparse bound for the annuli:

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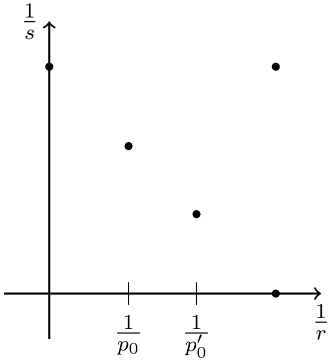

Interpolation along the dotted lines gives us

$$\|S_\tau : (r, s, \tau)\| \lesssim \tau^{-\delta-\eta}, \qquad 0 < \tau, \eta < 1.$$

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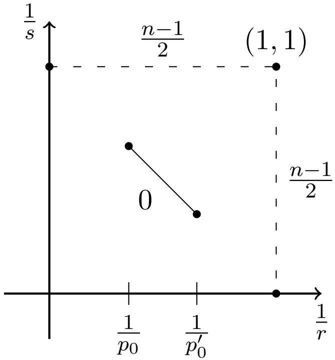

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$$\|S_\tau : (r,s,\tau)\| \lesssim \tau^{-\delta-\eta}, \qquad 0 < \tau, \eta < 1.$$

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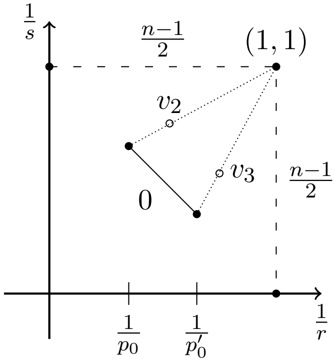

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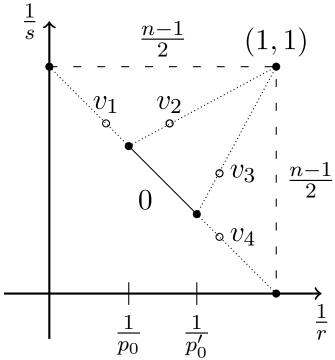

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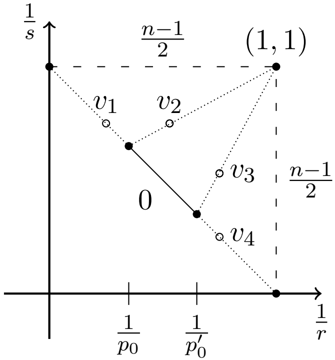

Interpolation along the dotted lines gives us

$$\|S_\tau : (r, s, \tau)\| \lesssim \tau^{-\delta-\eta}, \qquad 0 < \tau, \eta < 1.$$

Sparse bounds for Maximal Bochner-Riesz

Sparse Maximal Bochner-Riesz
(joint work with Robert Kesler)

The Bochner-Riesz operators on $\mathbb{R}^n$ with smoothing parameter $\delta \in \mathbb{R}$, are given by the family of Fourier multiplier $\left(1 - |\xi|^2/R^2\right)_+^\delta$, that is,

$$\widehat{B_\delta^R f}(\xi) = \left(1 - |\xi|^2/R^2\right)_+^\delta \hat{f}(\xi),$$

where $\widehat{f}$ represents the Fourier transform of f, $\delta \in \mathbb{R}$ and $R > 0$. Define the maximal Bochner-Riesz operator by

$$B_\delta^* f(x) = \sup_{R>0} B_\delta^R f(x).$$

Some previous work

- Carbery (1983): B_δ^* is bounded on $L^p(\mathbb{R}^2)$ for $0 < \delta < \frac{1}{2}$ and $\frac{2}{1+2\delta} < p < \frac{4}{1-2\delta}$.
- Christ (1985): B_δ^* is bounded on $L^p(\mathbb{R}^n)$ for all $n \geq 3, \delta > \frac{1}{2} \cdot \frac{n-1}{n+1}$ and $2 \leq p < \frac{2n}{n-1-2\delta}$.
- Tao (1998 and 2002): For $p \leq 2$ you need the restriction $\delta \geq \frac{2n-1}{2p} - \frac{n}{2}$ and further improvements.
- Lee (2004): Boundedness in L^p for $n \geq 3$ with the restrictions $p > \frac{2n+4}{n}$, $\delta > \max\{n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$.
- Several endpoint estimates.

Main theorem

Theorem (Kesler, M.)

Let $n \geq 2$ *and* $0 < \delta < \frac{n-1}{2}$. *Let* $1 < p_n < 2$ *such that the* L^{p_n} *boundedness holds. Then for every* $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{R}(n, \delta)$

$$||B_\delta^* : (p, q)|| < \infty.$$

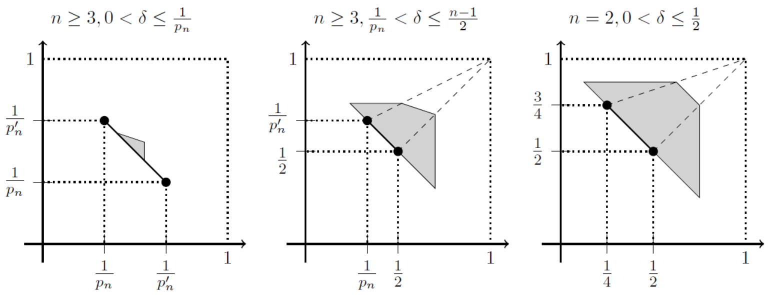

FIGURE 1. The region $\mathcal{R}(n, \delta)$ for the different values of n and δ.

There is a decomposition

$$B_\delta^* f(x) \leq A \left[M_{HL} f(x) + \sum_{k=1}^{\infty} 2^{(1/2-\delta)j} S_{\delta,j} f(x) \right]$$

Where $S_{\delta,j}$ is a square-like function

$$S_{\delta,j} f(x) := \left(\int_0^\infty |f * \check{a}_{\delta,j,t}^{(2^{-j})}(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Using restriction etimates

Lemma

Let $n \geq 2$ *and* $p_0 = 2 \cdot \frac{n+1}{n-1}$. *Then the scale-*$1$ *Bochner-Riesz square function*

$$S_{\delta,j} f(x) := \left(\int_{2^{-1}}^{4} \left| f * \check{a}_{\delta,j,t}^{(2^{-j})}(x) \right|^2 \frac{dt}{t} \right)^{1/2}$$

satisfies $||S_{\delta,j}^n : L^{p_0}(\mathbb{R}^n) \to L^{p_0}(\mathbb{R}^n)|| \leq A 2^{-\frac{j}{2}} 2^{\frac{j}{2} \frac{n-1}{n+1}}$ *for all* $j \geq 1$ *and* $0 < \delta < \frac{n-1}{2}$.

After some lemmas...

Lemma

Let $\bar{p} < 2 \leq p$ *and* $\bar{q} < q$. *Let* $\{S_l\}_{l\in\mathbb{Z}}$ *be any collection of pairwise disjoint sets. Then for every measurable* $f, g : \mathbb{R}^n \to \mathbb{C}$ *bounded and compactly supported there is a sparse collection* $\mathcal{S}$ *of cubes such that for every cube* $S \in \mathcal{S}$ *there exists* $Q \in \mathcal{C}$ *such that* $Q \subset S$ *and for which*

$$\int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \left[M_{\mathcal{C}}^{\bar{p}} \left(f * \check{\rho}_k \right) \right]^p \right)^{1/p} \cdot \left(\sum_{l\in\mathbb{Z}} \left| M_{\mathcal{C}}^{\bar{q}} \left(g 1_{S_l} \right) \right|^q \right)^{1/q} \quad dx$$

$$\leq A_{p,q} \sum_{S\in\mathcal{S}} \langle f \rangle_{S,p} \langle g \rangle_{S,q} |S|.$$

Further remarks

- Kerler and Lacey proved sparse bounds for a section of the boundary, which imply endpoint weak-type estimates.
- Newer techniques can be applied to the maximal Bochner-Riesz operator.
- Using the techniques of this results, it is possible to derive sparse bounds for Bochner-Riesz multipliers with negative index (more details can be found in the separate works of Bak and Gutiérrez).
- It is worth exploring a version of sparse form that exploits the geometry of the problem, by involving for example, Kakeya's maximal function (Carbery or Carbery-Seeger).

Thank you!