Huygens and π

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Abstract

The Dutch scientist Christiaan Huygens refined Archimedes' celebrated geometrical computation of π to its highest point. Yet the rich content of his beautiful treatise *De circuli magnitudine inventa* (1654) has apparently never been presented in modern form. Here we offer a detailed and contemporary development of several of his most striking results. We also make a historical conjecture concerning Archimedes' trisection figure.

1 Introduction

Over two thousand years ago Archimedes of Syracuse (about 275–212 B.C.), the greatest mathematician of antiquity (and just possibly ever) authored his monograph entitled *The Measurement of a Circle* [5, pp. 91–98], in which he proved the following celebrated inequality:

$$\boxed{3\frac{10}{71} < \pi < 3\frac{1}{7}}. \tag{1.1}$$

(Both bounds are accurate to *two* decimal places.)

We translate Archimedes' own statement of (1.1) since it might surprise the modern reader:

Theorem 1.1. *The circumference of any circle is greater than three times the diameter and exceeds it by a quantity less than a seventh part of the diameter but greater than ten seventy-first parts.*

Observe that there is *no mention of the constant* π. Indeed, although Euclid proved in Proposition XII.2 of *Elements* that the ratio [of areas] of a circle to the square on its diameter is constant, Greek mathematicians never had a special name or notation for it.

We recall that Archimedes proved (1.1) in three steps:

- ◇ First, he "compressed" a circle between inscribed and circumscribed regular *hexagons*.

- ◇ Second, he proved (in geometric guise) the identity

$$\cot\left(\frac{\theta}{2}\right) = \cot\theta + \sqrt{\cot^2\theta + 1}. \tag{1.2}$$

- ◇ Third, he used (1.2) to *recursively* compute the perimeters of inscribed and circumscribed regular 12-gons, 24-gons, 48-gons, and finally 96-gons.

For the circumscribed 96-gon, Archimedes obtained [5]:

$$\frac{\text{perimeter of 96-gon}}{\text{diameter}} < \frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} < 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} = 3\frac{1}{7}\,. \tag{1.3}$$

For the inscribed 96-gon, he obtained

$$\frac{\text{perimeter of 96-gon}}{\text{diameter}} > \frac{6336}{2017\frac{1}{4}} = 3 + \frac{10}{71} + \frac{37}{572899} > 3\frac{10}{71}\,. \tag{1.4}$$

Subsequent generations of mathematicians sought closer bounds on π by increasing the number of sides of the inscribed and circumscribed polygons but, until the advent of symbolic algebra and calculus, Archimedes' procedure remained the paradigm *for almost two thousand years!*

Some 1900 years later, in 1654, the 25-year-old Dutch mathematician Christiaan Huygens applied a bewildering *tour de force* of elementary geometry to the perimeter of an inscribed regular 60-gon to obtain the following spectacular improvement of (1.1):

$$\boxed{3.1415926533 < \pi < 3.1415936538}\,. \tag{1.5}$$

He did so in the last section of his brilliant treatise *De circuli magnitudine inventa* [9] which, for brevity, we shall simply call *Inventa*. In the preface to *Inventa*, Huygens declares that (up to his time) the only theorem in circle quadrature with a proper proof states that the perimeter of a circle is bounded above and below by the perimeters of a circumscribed and inscribed polygon, respectively, and that his treatise will do more.

His assessment is rather modest since *Inventa* not only is epoch-making for circle quadrature – as shown, for example, by (1.5) – but is also one of the most beautiful and important elementary geometric works ever written, and like the *Measurement* of Archimedes, will retain its value even if its results can be now obtained much more quickly using modern analysis.

Inventa not only contains the estimates (1.5), but also the first proofs in the history of mathematics of the famous inequalities:

$$\boxed{\frac{3\sin x}{2+\cos x} \leqslant x \leqslant \frac{2}{3}\sin x + \frac{1}{3}\tan x} \tag{1.6}$$

where $0 \leqslant x \leqslant \frac{\pi}{2}$.

Huygens proves them (geometrically!) as Theorems XII and XIII of his treatise. The first proof of any famous theorem acquires a historical and methodological importance, all the more so if the theorem had remained stated but unproved for some time. The lower bound is apparently due to Nikolaus of Cusa [3] while the upper bound is due to Snell (Willebrord Snellius) [17], but neither gave a rigorous proof. Huygens asserted, correctly, that his treatise presented the proofs which were lacking.

Notwithstanding their antiquity, the inequalities of Cusa and Snell continue to be a source of contemporary research. See, for example, [1, 2, 11].

Moreover, Huygens shows his wonderful originality by transforming Archimedes' exact determination of the barycenter of a parabolic segment into *barycentric inequalities* for circular segments. The latter become even more precise *arc-length inequalities*, namely:

$$\boxed{x < \sin x + \frac{10(4\sin^2\frac{x}{2} - \sin^2 x)}{12\sin\frac{x}{2} + 9\sin x}} \tag{1.7}$$

and

$$\boxed{x > \sin x + \frac{10(4\sin^2\frac{x}{2} - \sin^2 x)}{12\sin\frac{x}{2} + 9\sin x + 8\dfrac{(2\sin\frac{x}{2} - \sin x)^2}{12\sin\frac{x}{2} + 9\sin x}}} \tag{1.8}$$

both valid in the range $0 < x < \pi$.

These two barycentric inequalities are the tools that Huygens uses to prove the results (1.5) at the end of his *Inventa*.[1] This is by no means straightforward, but he carries out in great detail the proof of the first inequality (1.7).

He never published a proof of the second inequality (1.8). It can be proved without difficulty with modern techniques; this was first achieved, to our knowledge, by Iosif Pinelis in a discussion with one of us on *Math Overflow* [13].

We also point out that numerical analysts describe the three formulas (1.6), (1.7), (1.8) as historically the first examples of *Richardson extrapolation*, in which suitable algebraic combinations of simple functions produce highly accurate approximations. These examples appeared some three centuries before the papers of L. Richardson who originated the modern method [14].

It is not easy for the modern reader to find a good presentation of Huygens' beautiful geometrical proofs; one must consult the *Inventa* itself. Sadly, even the *Inventa* presents difficulties to today's reader, for Huygens chose to present *Inventa* in a strict Euclidean–Archimedean format:

- ◇ The proofs are entirely synthetic, without contextual analysis or motivation.
- ◇ There is no algebraic notation; proportions and inequalities are written out as long sentences in words.
- ◇ The main steps are not set out separately. Hence, intricate proofs comprise a single paragraph of several pages of running printed text.

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[1] Huygens' *Inventa* comprises 20 "propositions", consisting of 16 theorems interleaved with 4 "problems". With this dual labeling, the last section is called Problem IV and also Proposition XX.

These criticisms of format also apply to the French [8], German [15], and – somewhat less so – to the English translation [12] of *Inventa*, since they are strictly literal.

A few studies of Huygens' work are available. For instance, the long paper by Schuh [16], from 1914, covers some of the thematic ground. Later, an important paper by Hoffman [7] considered the historical sources of Huygens' *Inventa* and gave a general description of its contents. Both authors correctly asserted that the idea behind Huygens' treatise was to approximate a circular segment by a parabolic one. Hofmann briefly described Huygens' proofs of the Cusa and Snell inequalities and discussed the ideas behind Huygens' barycentric inequalities. However, his purpose differs from ours in that he replaced many of Huygens' proofs with his own, including a calculus proof of what we call the Grossehilfsatz (see Appendix A). In this paper, we resurrect Huygens' own original proofs, in some detail, so that the modern reader can appreciate their geometric beauty.

Our exposition presents Huygens' proofs of (1.6) and makes some of the rich content of *Inventa* available in a modern form.

The fundamental idea in Huygens' tract. is simple and brilliant: Huygens approximates the area of a *circular* segment (and thus, sector) by the area of a suitable *parabolic* segment.

Why?

Because Archimedes exhaustively investigated the metrical properties of a parabolic segment, and so Huygens is able to transform Archimedes' *exact equations* for the area of a parabolic segment into *inequalities* for the area of a circular segment. But the area of a circular segment is a simple function of its radius and its arc length. Therefore, the area inequalities become the aforementioned *arc length inequalities*.

The same idea undergirds Huygens' barycentric inequalities: the exact position of the barycenter of a parabolic segment, as determined by Archimedes, becomes the approximate location of the barycenter of a *circular* segment. Since Huygens determined the exact location of the latter, the relation between the two barycenters yields an arc-length inequality.

Plan of the paper. Section 2 exposits Huygens' proofs of Nikolaus of Cusa's *lower* bound on the circumference of a circle, and Snell's *upper* bound. Both bounds were originally proposed without proof by the mentioned authors.

Section 3 presents Huygens' proofs of his own original barycentric inequalities, the last of which Huygens stated without proof. These inequalities allowed Huygens to obtain upper and lower bounds on π with *nine* decimal figures of accuracy using only an inscribed 60-gon.

In the short Section 4, we offer the conjecture, based on the famous Archimedes' *trisection figure*, that Archimedes may have used (an equivalent of) the Snell–Cusa convergence-improving inequalities to compute more accurate bounds on π.

There are two appendices. In Appendix A, we retrieve the very Archimedean proof of a theorem of Huygens from an earlier work [10] that provides the essential step in locating the barycenter of a circular segment. Indeed, in subsection 3.5 we prove that this discovery of Huygens predates the modern formula for its location. In Appendix B, we determine an area inequality inspired by the work of Schuh and Hofmann.

Some terminology. In what follows, the term *segment* will always refer to the plane figure enclosed by an arc of a curve (either a circle or a parabola) and the chord joining the endpoints

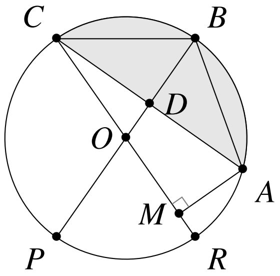

Figure 1: A circular segment less than a semicircle

of the arc. (Line segments are never in question.)

A *maximum triangle* in a circular segment is one whose base and altitude are the same as those of the segment. Such a triangle is isosceles. Its apex is the midpoint of the circular arc, called the *vertex* of the circular segment, i.e., the point on its arc most distant from its chord.

The *diameter* of a circular segment is the perpendicular bisector of the base chord; as such, it is a part of a diameter of the full circle. (The term is not used to denote the maximal distance between two points of the segment.) The length of the diameter is called the *height* of the segment.

The *area* of a plane figure is denoted by parentheses; thus $(\triangle\, ABC)$ and (segment ABC) are the areas of the triangle ABC and the segment ABC, respectively.

The *sine* of a circular arc (less than a semicircle) is the length of the perpendicular from one extremity of the arc to the circle's diameter through the other extremity. (This is Huygens' terminology; in modern language, that would be the sine of the central angle times the radius of the circle.)

In Figure 1, the shaded area is the circular segment ABC, with vertex B, base or chord AC, and diameter BD. The two diameters of the circle BP and CR meet at its center O. The perpendicular AM from A to the diameter CR is the sine of the arc $\overset{\frown}{AC}$.

2 Two early bounds on the circumference

Huygens bases the first part of his tract *Inventa* (up to and including its Theorem XIII) on two propositions giving upper and lower bounds for the area of a *circular segment* (and therefore, sector), clearly inspired by Archimedes' corresponding proposition on *parabolic* segments.

2.1 The first Heron–Huygens lemma

In his essay *The quadrature of the parabola* [5, p. 246], Archimedes famously proved the following area relation.

Theorem 2.1. *Every segment bounded by a parabola and a chord is equal to* ***four-thirds*** *of the triangle which has the same base and height as the segment.*

Archimedes' proof exhibits the first explicit sum of an infinite (geometric) series in the history of mathematics (of ratio $\frac{1}{4}$).

It is of historical interest that Heron of Alexandria *anticipated Huygens by some 1600 years* in section 32 of his treatise *Metrika* [6]. Thought to be irretrievably lost for almost two thousand years, the manuscript *Metrika* was rediscovered in 1896, some 250 years after the writing of *Inventa*. Therefore, Huygens was quite unaware of Heron's work when he did his own investigation. Nevertheless, the statement of Huygens' Theorem I (immediately below) coincides almost word for word with Heron's. For this reason we call the two bounds the *Heron–Huygens lemmas*.

Lemma 2.2. *If in a segment of a circle less than a semicircle a maximum triangle be inscribed, and in the subtended segments triangles be similarly inscribed, the triangle first drawn will be less (in area) than* ***four times*** *the sum of the two which were drawn in the subtended segments.*

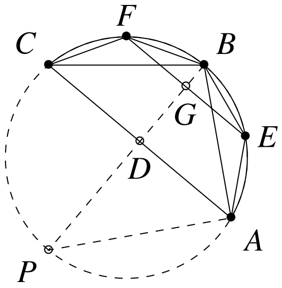

Figure 2: Maximum triangles in circle segments

Proof. Given the segment ABC of a circle, and BD the diameter of the segment, let there be inscribed a maximum triangle $\triangle ABC$ (see Figure 2). Likewise, in the two subtended segments let there be inscribed maximum triangles $\triangle AEB$ and $\triangle BFC$. It is required to prove that the *area* $(\triangle ABC)$ is less than *four times* the sum of $(\triangle AEB)$ and $(\triangle BFC)$.

Let EF be joined, cutting the diameter of the segment at the point G. Then there are *three congruent triangles*, namely $\triangle AEB$, $\triangle BFC$, and the new triangle $\triangle EBF$: see Figure 2. We shall prove that

$$(\triangle ABC) < 8(\triangle EBF), \tag{2.1}$$

which will establish the lemma, since $8(\triangle EBF) = 4(\triangle AEB) + 4(\triangle BFC)$.

We claim that the heights of $\triangle ABC$ and $\triangle EBF$ are related by:

$$BD < 4\,BG. \tag{2.2}$$

To see that, notice that the arc $\overset{\frown}{AB}$ is bisected by the point E. Therefore

$$EA \text{ (or } EB) > \frac{1}{2}AB \quad \text{and thus} \quad AB^2 < 4\,EB^2 \text{ or } 4\,EA^2,$$

which implies

$$\frac{BD}{BG} = \frac{BD \cdot BP}{BG \cdot BP} = \frac{AB^2}{EB^2} < 4,$$

if BP is the diameter of the circle through the point B, since $AB^2 = BD \cdot BP$ and $EB^2 = BG \cdot BP$. The first of these equalities comes from the similar triangles $\triangle\, DBA \sim \triangle\, ABP$, which entails $DB : AB = AB : BP$. The other equality follows likewise from $\triangle\, EBG \sim \triangle\, PBE$.

Next, the bases of $\triangle\, ABC$ and $\triangle\, EBF$ obey:

$$AC < 2\,EF. \tag{2.3}$$

This follows from the triangle inequality:

$$EF = AB = BC \implies 2\,EF = AB + BC > AC.$$

Together, (2.2) and (2.3) yield the desired inequality (2.1). ☐

Lemma 2.3 (Heron–Huygens I: Theorem III of *Inventa*)**.** *The area of a circular segment less than a semicircle has a* ***greater ratio*** *to the area of its maximum inscribed triangle* ***than four to three****.*

Proof. In the circular segments of Figure 2 whose bases are the chords AE, EB, BF, FC, we again draw the maximum isosceles triangles, and then apply the process again, etc., and so fill out the segment. Applying Lemma 2.2, we obtain

$$\boxed{(\text{segment } ABC) > \frac{4}{3}\,(\triangle\, ABC)} \tag{2.4}$$

from:

$$\begin{aligned}(\text{segment } ABC) &= (\triangle\, ABC) + 2(\triangle\, AEB) + 4(\cdots) + \cdots \\ &> (\triangle\, ABC)\Big(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\Big) = \frac{4}{3}\,(\triangle\, ABC).\end{aligned}$$

☐

One can see that this is almost exactly the proof that Archimedes gives for the area of a *parabolic* segment. The only difference is that equality is replaced by *inequality* since the parabolic segment is *smaller* than the corresponding circular segment. Moreover, Heron proves this theorem in almost exactly the same way.

2.2 Huygens' first arc length inequality

Huygens' first inequality is a *lower* bound for the circumference of a circle. Here we see how Huygens proves inequalities about circular and polygonal *areas*, and then translates them into inequalities about *perimeters*.

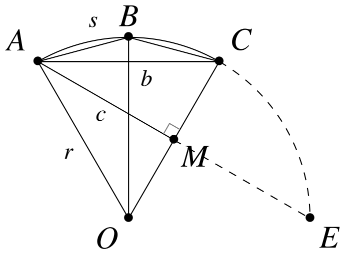

Figure 3: Polygon perimeter comparisons

Theorem 2.4 (First arc length inequality: Theorem VII of *Inventa*)**.** *If C denotes the circumference of a circle, then*

$$\boxed{C > C_{2n} + \frac{1}{3}(C_{2n} - C_n)} \tag{2.5}$$

where C_n denotes the perimeter of the inscribed regular polygon of n sides.

Proof. We first obtain an inequality for a *sector* and its arc. In a circle with center O and radius r, let ACE be a segment, with diameter CM and base AE. We set $b := AC$, $c := AM$ and let s be the length of the circular arc $\overset{\frown}{AC}$, whose midpoint is B. See Figure 3. We shall show that

$$\overset{\frown}{AC} = s > b + \frac{1}{3}(b - c) = AC + \frac{1}{3}(AC - AM). \tag{2.6}$$

We now add the area $(\triangle\, OAC) = \frac{1}{2}cr$ to that of the circular segment ABC to form the circular *sector* of area $(\text{sector}\, OAC) = \frac{1}{2}rs$. Thus

$$(\triangle\, ABC) = 2(\triangle\, OBC) - (\triangle\, OAC) = (b - c)\cdot\frac{r}{2}\,.$$

The first Heron–Huygens lemma (Lemma 2.3), applied to the segment ABC, now implies

$$\frac{4}{3}(b - c)\cdot\frac{r}{2} + \frac{cr}{2} < (\text{segment}\, ABC) + (\triangle\, OAC) = (\text{sector}\, OAC) = \frac{rs}{2}\,,$$

and therefore

$$b + \frac{1}{3}(b - c) = \frac{4b - c}{3} = c + \frac{4}{3}(b - c) < s \tag{2.7}$$

which establishes (2.6).

Now consider the special case where $b = AC$ is a side of an inscribed regular polygon of $2n$ sides; then $2c = 2\,AM = AE$ will be a side of the inscribed regular polygon of n sides. On multiplying (2.6) by $2n$, we arrive at Snell's inequality (2.5). ☐

Error Analysis. We observe, as in a footnote in Huygens' *Oeuvres* [8, p. 128], that if p_n denotes the perimeter of an inscribed regular n-gon in a circle of radius 1, the central angle of the n-gon is $\frac{2\pi}{n}$ and we obtain the following Taylor expansion for *Huygens' first inequality*:

$$p_n = 2n\sin\left(\frac{\pi}{n}\right) \implies p_{2n} + \frac{1}{3}(p_{2n} - p_n) = 2\pi - \frac{\pi^5}{240n^4} + \frac{\pi^7}{80640n^6} - \cdots$$

which shows not only that the approximation is in *defect*, but also that the error does not exceed $\dfrac{\pi^5}{240n^4} \doteq \dfrac{1.275\cdots}{n^4}$, which clearly is quite small for large n.

2.3 Nikolaus of Cusa's lower bound

Theorem 2.5 (Nikolaus of Cusa: Theorem XIII of *Inventa*)**.** *If to the diameter of a circle a radius is added in the same direction, and a line is drawn from the end of the extended line cutting the circle and meeting the tangent to the circle at the opposite extremity of the diameter, this will intercept a part of the tangent **less** than the adjacent intercepted **arc**.*

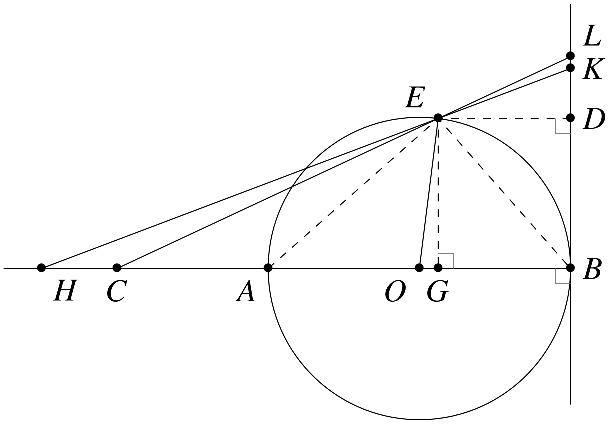

Figure 4: Tangent and arc length comparison

Proof. Given a circle with diameter AB, let AB be produced to the point C such that AC is equal to the radius. Let CL be drawn cutting the circumference the second time in E and meeting at L the tangent to the circle at the extremity B of the diameter. (See Figure 4.) One must prove that

$$\boxed{BL < \overset{\frown}{EB}}. \tag{2.8}$$

Draw AE and EB, and locate H on the prolongation of the diameter AB beyond A so that $AH = AE$. Produce HE to meet the tangent line BL at K. Finally, draw perpendiculars EG from E to the diameter AB and ED from E to the tangent BL.

Now $\angle EHA = \angle HEA$ since $\triangle HAE$ is isosceles; and since $\angle AEB = \frac{\pi}{2}$, it follows that

$$\angle HEA + \angle KEB = \frac{\pi}{2}\,.$$

Now, $\angle HBK = \frac{\pi}{2}$; hence, in the triangle $\triangle HKB$,

$$\angle KHB + \angle BKH = \frac{\pi}{2}\,.$$

Subtracting equals from equals, $\angle HEA$ on one side and $\angle KHB = \angle EHA$ on the other, we conclude that $\angle KEB = \angle BKH = \angle BKE$. Therefore, the triangle $\triangle KEB$ is isosceles, with $BE = BK$.

Moreover, $BD = EG$ as sides of the rectangle $BDEG$, which implies

$$DK = BE - EG.$$

Next,

$$\frac{AG}{AE} = \frac{AE}{AB} \implies \frac{AB + AG}{2} > AE,$$

i.e., the arithmetic mean of AB and AG is greater than their geometric mean; it follows that

$$AH = AE < \frac{1}{2}(AB + AG) = CA + \frac{1}{2}AG,$$

and subtracting CA from both sides[2] yields $CH < \frac{1}{2}AG$.

But $CA = \frac{1}{2}AB > \frac{1}{2}AG$, and on adding CA to AG, one deduces that

$$CG > \frac{3}{2}AG > 3CH. \tag{2.9}$$

But, since

$$\frac{GH}{EG} = \frac{DE}{DK} \quad \text{and} \quad \frac{EG}{CG} = \frac{DL}{DE}$$

by similar triangles $\triangle EHG \sim \triangle KED$ and $\triangle ECG \sim \triangle LED$, then, by multiplying these two ratios, one deduces

$$\frac{GH}{CG} = \frac{DL}{DK} \quad \text{and consequently} \quad \frac{CH}{CG} = \frac{KL}{DK}\,, \quad \text{whereby} \quad \frac{CG}{CH} = \frac{DK}{KL}\,.$$

Thus, on account of (2.9),

$$BE - EG = BK - BD = DK > 3KL, \quad \text{and thus} \quad KL < \frac{1}{3}(BE - EG).$$

Then, invoking the inequality (2.6) from the proof of Theorem 2.4, i.e., Theorem VII of *Inventa*, we obtain the desired inequality (2.8):

$$BL = BK + KL = BE + KL < BE + \frac{1}{3}(BE - EG) < \overset{\frown}{BE}. \qquad \square$$

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[2] This subtraction is valid because $AH > CH$; or equivalently, AE is longer than the radius of the circle. That happens because the point E is chosen as the *second* intersection of the line from C to L with the circumference of the circle. Another consequence is that C lies between B and H, so that K lies between B and L, as shown in Figure 4.

Error Analysis. If we take the radius $OA = 1$, and put $x := \overset{\frown}{BE}$, then by similar triangles $\triangle LBC \sim \triangle EGC$, we get

$$\frac{BL}{3} = \frac{BL}{BC} = \frac{EG}{CG} = \frac{EG}{2 + OG}; \quad \text{in other words,} \quad \frac{BL}{3} = \frac{\sin x}{2 + \cos x}.$$

Now, we just proved that $x > BL$; hence,

$$\boxed{x > \frac{3 \sin x}{2 + \cos x}}. \tag{2.10}$$

The formula (2.10) *is the modern statement of Cusa's inequality.* Its accuracy is given by

$$\frac{3 \sin x}{2 + \cos x} = x - \frac{x^5}{180} - \frac{x^7}{1512} - \cdots$$

which shows Cusa's approximation

$$x \doteq \frac{3 \sin x}{2 + \cos x}$$

to be in *defect*; and if we put $x := \frac{\pi}{2n}$, we conclude that

$$2\pi = 4n \cdot BL + \frac{\pi^5}{1440n^4} + \cdots$$

with an error $\dfrac{\pi^5}{1440n^4} = \dfrac{0.2125\cdots}{n^4}$ which is quite small for large n.

2.4 The second Heron–Huygens lemma

We now seek to prove an *upper* bound for the circumference.

Lemma 2.6. *If a triangle is drawn having the same base as a segment of circle less than a semicircle and having its sides* ***tangent*** *to the segment, and if a line is drawn tangent to the segment at its vertex, this cuts off from the given triangle a triangle greater than* ***one half*** *of the maximum triangle described within the segment.*

Proof. Take a circular segment ABC less than a semicircle with its vertex at B, and let the lines AE and CE, tangents to the segment at the extremities of its base, meet at E. (They will indeed meet, since the segment is less than a semicircle.) Moreover, let the line FG, with F on EA and G on EC, be drawn tangent to the segment at its vertex B; join AB and BC. (See Figure 5.) We must prove:

$$\boxed{(\triangle\, FEG) > \frac{1}{2}(\triangle\, ABC)}.$$

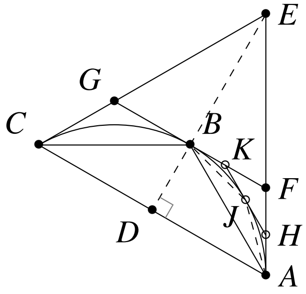

Figure 5: Outer and inner triangles of a circle segment

Clearly $\triangle AEC$, $\triangle FEG$, $\triangle AFB$ and $\triangle BGC$ are all isosceles; and B bisects FG. Therefore,

$$\begin{aligned} EF + EG > FG &\implies EF > CG \text{ or } AF \\ &\implies AE = AF + EF < 2EF. \end{aligned}$$

We claim that

$$(\triangle FEG) > \frac{1}{4}(\triangle AEC). \tag{2.11}$$

To see that, draw the straight line EB cutting AC perpendicularly at D (Figure 5). Then

$$EF > \frac{1}{2}AE \implies BE > \frac{1}{2}DE$$

since these lines are in the same proportion, because $FB \parallel AD$. Also, $\triangle FEG \sim \triangle AEC$, so

$$\frac{1}{2}EF \cdot BE > \frac{1}{8}AE \cdot DE \implies \frac{1}{2}FG \cdot BE > \frac{1}{8}AC \cdot DE$$

which confirms (2.11).

Moreover,

$$\frac{AF}{AE} = \frac{\text{altitude of } \triangle ABC}{\text{altitude of } \triangle AEC} = \frac{BD}{DE}$$

and the two triangles have the same base AC. Hence,

$$AF < \frac{1}{2}AE \implies BD < \frac{1}{2}DE \implies (\triangle ABC) < \frac{1}{2}(\triangle AEC).$$

This last inequality, combined with (2.11), yields the desired result:

$$(\triangle FEG) > \frac{1}{2}(\triangle ABC). \qquad \square$$

Theorem 2.7 (Heron–Huygens II: Theorem IV of *Inventa*)**.** *The area of a circular segment less than a semicircle is* ***less*** *than* ***two thirds*** *of the area of a triangle having its base in common with this segment and its sides* ***tangent*** *to it.*

Proof. Referring again to Figure 5, we obtain:

$$(\triangle FEG) > \frac{1}{2}(\triangle ABC), \quad (\triangle HFK) > \frac{1}{2}(\triangle AJB), \quad \text{and so on.}$$

That is, we repeat the process with smaller and smaller triangles; and we obtain an infinite sequence of inequalities in which the area on the left-hand side is greater that one-half of the area on the right-hand side.

Adding all these inequalities, we obtain

$$(\triangle AEC) - (\text{segment } ABC) > \frac{1}{2}(\text{segment } ABC).$$

In other words,

$$(\text{segment } ABC) < \frac{2}{3}(\triangle AEC),$$

which establishes the theorem. □

2.5 The second Snell theorem

Lemma 2.8 (Theorem VI of *Inventa*)**.** *If A_n is the area of an* inscribed *regular polygon of n sides, S the area of the circle, and A'_n the area of a* circumscribed *regular polygon of n sides, then*

$$\boxed{S < \frac{2}{3}A'_n + \frac{1}{3}A_n}. \tag{2.12}$$

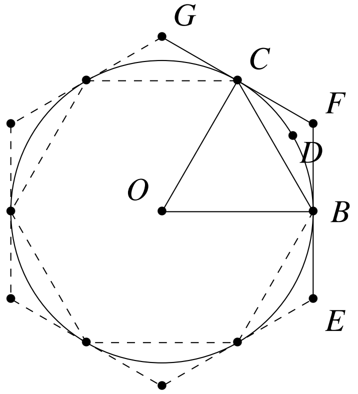

Figure 6: Inscribed and circumscribed regular polygons

Proof. Given a circle with center O; let there be inscribed in it an equilateral polygon[3] (say a hexagon), one of whose sides is BC; and let $EFG\cdots$ be circumscribed and similar to it, with the sides tangent to the circle at the vertices of the first polygon: see Figure 6.

---

[3] An equilateral polygon inscribed in a circle is actually a *regular* polygon.

We claim that the area of the circle is less than two thirds the area of polygon $EFG\cdots$ plus one third of that of polygon $BC\cdots$, that is,

$$S < \frac{2}{3}(\text{polygon } EFG\cdots) + \frac{1}{3}(\text{polygon } BC\cdots). \tag{2.13}$$

To see that, draw the radii OB and OC. Then, since $\triangle BFC$ rests on the base BC of the segment BDC, with its other sides tangent to the segment BDC, Theorem 2.7 (Theorem IV of *Inventa*) shows that

$$(\text{segment } BDC) < \frac{2}{3}(\triangle BFC).$$

It follows that

$$(\triangle OBC) + \frac{2}{3}(\triangle BFC) > (\text{sector } OBC).$$

In other words,

$$\begin{aligned} (\triangle OBC) + \frac{2}{3}\big[(\text{quadrangle } OBFC) - (\triangle OBC)\big] &> (\text{sector } OBC), \\ \text{or} \quad \frac{2}{3}(\text{quadrangle } OBFC) + \frac{1}{3}(\triangle OBC) &> (\text{sector } OBC). \end{aligned}$$

The area estimate (2.13) is then obtained by taking the sum as many times as copies of the sector OBC are contained in the circle. The theorem follows, since (2.13) is just a restatement of the estimate (2.12). ☐

Theorem 2.9 (Theorem VIII of *Inventa*)**.** *Given a circle, if at the extremity of a diameter a tangent is drawn, and if from the opposite extremity of the diameter a line is drawn which cuts the circumference and meets the tangent produced, then* ***two thirds*** *of the intercepted tangent plus* ***one third*** *of the line dropped from the point of intersection perpendicular to the diameter are* ***greater*** *than the adjacent* ***subtended arc****.*

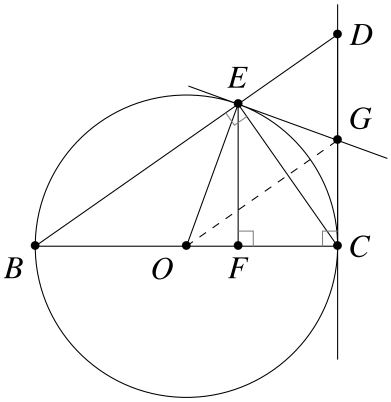

Figure 7: Another arc-length comparison

Proof. Take a circle with center O and diameter BC; and draw from C a line CD tangent to the circle. And let a line BD, drawn from the other extremity of the diameter, meet this line at D and intersect the circumference at E; let EF be the perpendicular from E to the diameter BC (see Figure 7). One must prove that

$$\boxed{\overset{\frown}{CE} < \frac{2}{3}CD + \frac{1}{3}EF}. \tag{2.14}$$

Join OE and CE. At the point E draw a tangent to the circle which meets the tangent CD at G. Then

$$EG = CG = DG.$$

Indeed, if we draw a circle with center G which passes through the points C and E, it will also pass through the point D because $\angle CED$ is a right angle. By the proof of Lemma 2.8,

$$(\text{sector}\, OEC) < \frac{2}{3}(\text{quadrangle}\, OEGC) + \frac{1}{3}(\triangle\, EOC).$$

Now, $(\text{quadrangle}\, OEGC) = 2(\triangle\, OCG)$ equals the area of a triangle with base $2CG = CD$ and an altitude OC; whereas $(\triangle\, OEC)$ equals the area of a triangle with base EF and the same altitude OC. Therefore,

$$\begin{aligned}\frac{2}{3}(\text{quadrangle}\, OEGC) + \frac{1}{3}(\triangle\, EOC) &= \left(\triangle \text{ with base } \frac{2}{3}CD + \frac{1}{3}EF \text{ and altitude } OC\right)\\ &= \frac{1}{2}\left(\frac{2}{3}CD + \frac{1}{3}EF\right)\cdot OC\\ &> (\text{sector}\, OEC) = \frac{1}{2}\overset{\frown}{CE}\cdot OC,\end{aligned}$$

which entails the required inequality (2.14). □

Theorem 2.10 (Second Snell Theorem: Theorem IX of *Inventa*)**.** *The circumference of a circle is less than* ***two thirds*** *of the perimeter of an equilateral polygon* ***inscribed*** *in it plus* ***one third*** *of a similar* ***circumscribed*** *polygon.*

Proof. In symbols, we must prove that

$$\boxed{C < \frac{2}{3}C_n + \frac{1}{3}C'_n} \tag{2.15}$$

where C_n denotes the perimeter of the inscribed polygon of n sides, C'_n is the perimeter of the circumscribed polygon of n sides, and C is the circumference of the circle.

Given a circle with center O, let there be inscribed in it an equilateral polygon, one of whose sides is CD. Let there be circumscribed another polygon with sides parallel to the former; call EF its side which is parallel to CD. (See Figure 8.) *We have to prove that the*

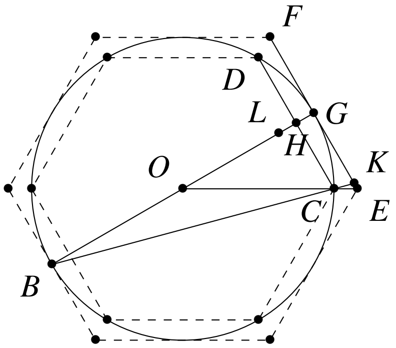

Figure 8: Circumference versus polygon perimeters

circumference of the circle is less than two thirds of the perimeter of polygon $CD\cdots$ plus one third of the perimeter of polygon $EF\cdots$.

Draw the diameter BG of the circle that bisects the side CD of the inscribed polygon at H, and the parallel side EF of the circumscribed polygon at G (it is obvious that G is the point of tangency of the side EF). Choose L on the diameter BG between O and H such that $HL = GH$; draw BC, produced to meet the side EF at K, and extend the radius OC to meet the vertex E of the circumscribed polygon. Then

$$HL = GH \implies BL = 2OH \implies \frac{OG}{OH} = \frac{BG}{BL}.$$

But now

$$\frac{BH}{BL} > \frac{BG}{BH}$$

since $BG > BH > BL$, exceeding one another by the same amount. Therefore,

$$\frac{OG}{OH} = \frac{BG}{BL} = \frac{BG}{BH}\frac{BH}{BL} > \frac{BG^2}{BH^2}.$$

Consequently, by similar triangles,

$$\frac{OG}{OH} = \frac{EG}{CH} \quad \text{and} \quad \frac{BG}{BH} = \frac{KG}{CH} \implies \frac{EG}{CH} > \frac{KG^2}{CH^2}.$$

This in turn implies that

$$\frac{EG}{KG} > \frac{KG}{CH}.$$

Since the arithmetic mean of EG and CH is greater than their geometric mean, we infer

$$EG + CH > 2\sqrt{EG \cdot CH} > 2\sqrt{KG^2} = 2\,KG.$$

So, in particular, $\frac{1}{3}(EG + CH) > \frac{2}{3}KG$.

Hence, finally,

$$\frac{1}{3}EG + \frac{2}{3}CH > \frac{2}{3}KG + \frac{1}{3}CH > \overset{\frown}{CG}, \tag{2.16}$$

where the last inequality follows from (2.14) in the previous Theorem 2.9.

And now one extends the inequality (2.16) for the arc $\overset{\frown}{CG}$ to the entire circle. □

Error Analysis. If p'_n is the perimeter of a circumscribed regular n-gon with central angle $\frac{2\pi}{n}$ in a circle of radius 1, then

$$p'_n = 2n\tan\left(\frac{\pi}{n}\right) \quad \text{and} \quad p_n = 2n\sin\left(\frac{\pi}{n}\right),$$

and we obtain

$$\frac{2}{3}p_n + \frac{1}{3}p'_n = 2\pi + \frac{\pi^5}{10n^4} + \frac{\pi^7}{8n^6} + \cdots$$

which shows the approximation to be in *excess*; and the *error* is $\dfrac{\pi^5}{10n^4} \doteq \dfrac{30.6}{n^4}$ which is small for large n. One also sees that it is somewhat less accurate than the lower bound, although it is of the same order.

2.6 Snell's upper bound

As we pointed out earlier, Huygens proves two famous inequalities. One of them is the Cusa inequality of a previous section. The second one we discuss now. It is fascinating that the figure involved is the *Archimedes trisection figure* – see, for instance, [5] and the last section of this paper.

Theorem 2.11 (Snell's Upper Bound: Theorem XII of *Inventa*)**.** *If between the diameter produced of a circle and its circumference a line equal to the radius is inserted, which when produced cuts the circle and meets a tangent to the circle at the other extremity of the diameter, this line will intercept a part of the tangent* ***greater*** *than the adjacent intercepted* ***arc****.*

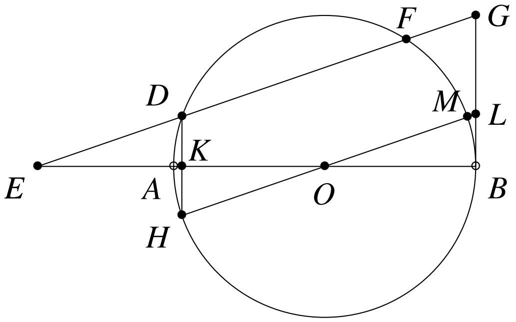

Figure 9: Arc and tangent intercepts

Proof. Given a circle with center O and diameter AB, produce the diameter beyond A to a point E, and take the point D on the circle such that the line ED is equal to the radius OA. When produced, the line ED cuts the circumference again at F and meets at the point G the tangent to the circle at B. (See Figure 9.) We shall prove that

$$\boxed{BG > \overset{\frown}{BF}}\,. \tag{2.17}$$

Draw HL through the center O parallel to EG, meeting the circumference at H and M and the tangent BG at L. Draw the line DH, cutting the diameter AB at K. Then

$$\triangle EDK \sim \triangle OHK$$

since the angles at K are equal and $\angle DEK = \angle HOK$. But $ED = OH$, and these sides are subtended by equal angles; hence, $DK = HK$.

Therefore, the radius OA bisects both the chord DH (perpendicularly) and the arc $\overset{\frown}{DAH}$. Since $HM \parallel DF$ by construction, it follows that

$$\overset{\frown}{FM} = \overset{\frown}{DH} = 2\,\overset{\frown}{AH}.$$

But $\overset{\frown}{AH} = \overset{\frown}{BM}$, and therefore

$$\overset{\frown}{BF} = \overset{\frown}{BM} + \overset{\frown}{FM} = 3\,\overset{\frown}{AH}.$$

Moreover, assuming that the radius OA has length 1,

$$HK = \text{sine of } \overset{\frown}{AH} \quad \text{and} \quad BL = \text{tangent of } \overset{\frown}{AH}.$$

These relations, together with formula (2.15) – or (2.16) – of the second Snell theorem, imply that

$$\frac{2}{3}HK + \frac{1}{3}BL > \overset{\frown}{AH}.$$

Noting that $DHLG$ is a parallelogram, we deduce the required inequality (2.17):

$$\begin{aligned} 2\,HK + BL > 3\,\overset{\frown}{AH} &\implies BL + DH > 3\,\overset{\frown}{AH} \\ &\implies BL + LG > 3\,\overset{\frown}{AH} \implies BG > \overset{\frown}{BF}. \end{aligned}$$ □

Error Analysis. If we now put $x := \overset{\frown}{AH}$, so that $3x = \overset{\frown}{BF}$, then

$$\left.\begin{aligned} LG = DH &= 2\sin x \\ BL &= \tan x \end{aligned}\right\} \implies BG = 2\sin x + \tan x.$$

The estimate $\frac{1}{3}\overset{\frown}{BF} \doteq \frac{1}{3}BG$ is Snell's own approximation:

$$x \doteq \frac{2\sin x + \tan x}{3}\,.$$

The expansion

$$\frac{2\sin x + \tan x}{3} = x + \frac{x^5}{20} + \frac{x^7}{56} + \cdots$$

shows that the approximation is in *excess*; and the error one commits in the approximation is about $-x^5/20$.

If we apply the *first* (Cusa) inequality (2.8) for $n = 6$ and this *second* (Snell) inequality (2.15) for $n = 12$, we obtain

$$3.1411\cdots < \pi < 3.1424\cdots$$

which only give *two*-place accuracy. Archimedes needed a 96-gon to obtain similar bounds. This shows the extraordinary improvement that the Snell–Cusa inequalities achieve over Archimedes' original computation.

If instead we apply the first (Cusa) inequality for $n = 30$ and the second (Snell) inequality for $n = 60$, we arrive at

$$3.1415917\cdots < \pi < 3.141594\cdots$$

which shows that the decimal expansion of π begins with $\pi \doteq 3.14159$. Even with this *five*-place accuracy, we still don't need the full 96 sides that Archimedes used.

3 Huygens' barycentric theorems

3.1 The barycenter

After proving the Cusa–Snell inequalities, Huygens offers his own very elegant approximation which is based on an observation about the barycenter of a circular segment. What is novel and original is how Huygens *transforms the location of a barycenter into an inequality about perimeters*.

Theorem 3.1 (Theorem XIV of *Inventa*)**.** *The barycenter of a circular segment divides the diameter of the segment so that the part near the vertex is* ***greater*** *than the rest, but* ***less*** *than* ***three halves*** *of it.*

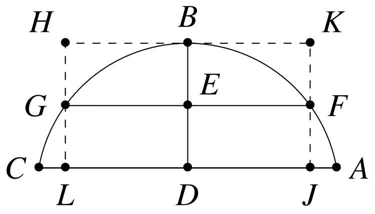

Figure 10: A rectangle framing a circular segment

Proof. Take a segment ABC of a circle (and let it be put less than a semicircle because others do not satisfy the proposition), and let E be the midpoint of its diameter BD.

Step 1. First, we show that the barycenter of the segment ABC lies *below* this midpoint E (viewed from the vertex B).

It is evident from considerations of symmetry that the barycenter lies on the diameter BD.

Draw a line through E parallel to the base AC, meeting the circumference on either side at points F and G. Draw the lines KJ through F and HL through G, perpendicular to AC; these, together with the line KH tangent to the segment at its vertex B, form the rectangle $KHLJ$: see Figure 10. Since, by assumption, the segment ABC is less than a semicircle, the rectangle $FGLJ$, which is one half of the given rectangle, is contained *within* the segment $AFGC$; and the regions AFJ and LGC are left over.

But $KHGF$, the other half of the rectangle $KHLJ$, *includes* the segment FBG; and also includes the regions FBK and GBH. Since those two regions lie wholly *above* the line FG, the *barycenter of their union* will be located above it.

Now the point E, on this same line FG, is the barycenter of the whole rectangle $KHLJ$. Therefore the barycenter of the remaining region $BFJLGB$ will lie *below* the line FG.

But the barycenter of the pair of regions AFJ and LGC lies also *below* FG. Therefore, the barycenter of the magnitude composed of these regions and the region $BFJLGB$, i.e., of the whole segment ABC, must also be found below the line FG, and hence *below the point* E on the diameter.

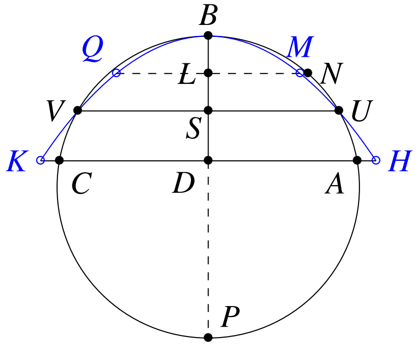

Figure 11: A parabolic segment framing a circular segment

Step 2. Now cut the same diameter BD at S so that BS is *three halves* of the remainder SD. We assert that the barycenter of the segment ABC is closer to B than the point S.

Let BP be the diameter of the whole circle, as in Figure 2. Draw a line through S parallel to the base AC, meeting the circumference in U and V. Next, draw a *parabola* with vertex B, axis BD, and *latus rectum* equal to SP; meeting the base AC of the circular segment at H and K. Then, since

$$US^2 = VS^2 = BS \cdot SP,$$

the parabola will cross the circle at the points U and V. (See Figure 11.)

Now, the arcs $\overset{\frown}{BU}$ and $\overset{\frown}{BV}$ of the parabola will fall *within* the circle, but the remaining arcs $\overset{\frown}{UH}$ and $\overset{\frown}{VK}$ will lie *outside* it. We prove this as follows: through any point L on the diameter between B and S, draw the line NL parallel to the base AC, meeting the circumference of the circle at N and the parabola at points M and Q. Now the relations

$$\left.\begin{aligned} NL^2 &= BL \cdot BP \\ ML^2 &= BL \cdot SP \end{aligned}\right\} \quad \text{together with} \quad BL \cdot BP > BL \cdot SP$$

imply that $NL^2 > ML^2$, and hence $NL > ML = QL$.

The same holds for any such parallel line drawn between B and S; and therefore the arcs $\overset{\frown}{BU}$ and $\overset{\frown}{BV}$ of the circumference must lie entirely outside of the parabola.

Again, since

$$\left.\begin{aligned} AD^2 &= BD \cdot DP \\ HD^2 &= BD \cdot SP \end{aligned}\right\} \implies HD > AD,$$

and the same will hold for any line parallel to the base drawn between S and D. Therefore the arcs $\overset{\frown}{UA}$ and $\overset{\frown}{VC}$ will fall *within* the parabola.

Now we examine the regions $UNBM$, and BQV, as well as HUA and VCK. Since the latter two lie entirely below the line UV, the barycenter of their union will also lie below it. But the barycenter of the parabolic segment HBK lies on UV at the point S.[4] Therefore, the barycenter of the remaining region $AUMBQVC$ will lie *above* the line UV. But the barycenters of the regions $UMBN$ and BVQ that also are above the line UV will likewise lie above that line. Therefore, the region composed of these two together with $AUMBQVC$, i.e., the segment ABC of the circle, will have its barycenter above the line UV. And since that barycenter lies on the diameter BD, it will be *nearer* the vertex B than the point S. □

We now prove a theorem, also due to Huygens in an earlier work *Theoremata* [10], which relates the area of a circular segment to the position of its barycenter.

Theorem 3.2 (Theorem VII of *Theoremata*)**.** *The area of a circular segment is to the area of the inscribed triangle with the same base and height as two-thirds of the diameter of the opposite segment is to the distance from the center of the circle to the barycenter of the [original] segment.*

Proof. Let ABC and $\triangle ABC$ be the given segment and triangle. Let BD be the diameter of the segment and prolong it to the center O of the circle. Let PD be the diameter of the remaining segment. Let L be the barycenter of segment ABC. The theorem may then be restated as:

$$\frac{(\text{segment } ABC)}{(\triangle ABC)} = \frac{2}{3} \frac{PD}{OL}\,. \tag{3.1}$$

To prove (3.1), locate G on the diameter BP of the circle between O and P such that

$$OG^2 = BD \cdot PD \tag{3.2}$$

---

[4] Archimedes, *Equilibrium of Planes*, Book II, Proposition 8 [5]. This theorem states that if AO is the diameter of a parabolic segment with vertex A, and if G is its barycenter, then $AG = \frac{3}{2}\,GO$.

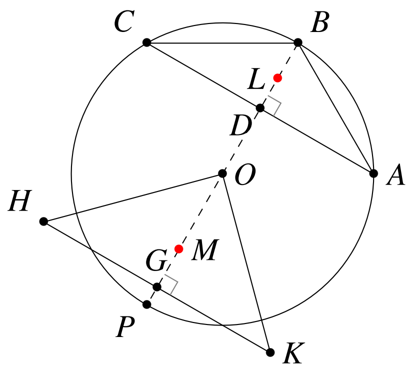

Figure 12: Balancing a circular segment with an isosceles triangle

Draw the line KH through G parallel to the base AC of the first segment, such that $KH = AC$ and G is the midpoint of KH. Let M be the barycenter of the triangle $\triangle KOH$.

We now need the following auxiliary proposition, also taken from *Theoremata* [10]. We defer its long (and very Archimedean) proof to Appendix A.

Proposition 3.3 (Theorem V of *Theoremata*)**.** *The barycenter of the figure that combines the segment ABC of the circle with the triangle $\triangle KOH$ just described lies at the centre of the circle.* ⊟

Now, by a famous theorem of Archimedes,[5]

$$OM = \frac{2}{3} OG. \tag{3.3}$$

Thus,

$$\frac{(\triangle KOH)}{(\triangle ABC)} = \frac{OG}{BD} = \frac{PD}{OG},$$

using (3.2). Combined with (3.3), this yields

$$\frac{(\triangle KOH)}{(\triangle ABC)} = \frac{\frac{2}{3} PD}{OM}.$$

However, the aforementioned barycenters are at M and L, so, by Proposition 3.3, they balance at O. Therefore,

$$\frac{(\text{segment } ABC)}{(\triangle KOH)} = \frac{OM}{OL}$$

and, on multiplying the previous two ratios, (3.1) follows at once. □

Now we transform this result on the barycenter of a circular segment into an inequality on its *area*.

---

[5] Archimedes, *Equilibrium of Planes*, Proposition I.14 [5, p. 201]: the barycenter of a triangle lies at the intersection of the medians. (The ratio 2:1 on each median follows easily.)

Theorem 3.4 (Theorem XV of the *Inventa*)**.** *The area of a circular segment less than a semicircle has a* greater *ratio to its maximum inscribed triangle than* 4:3*, but* less *than the ratio which* $\frac{10}{3}$ *of the diameter of the remaining segment has to the diameter of the circle plus three times the line which reaches from the center of the circle to the base of the segment.*

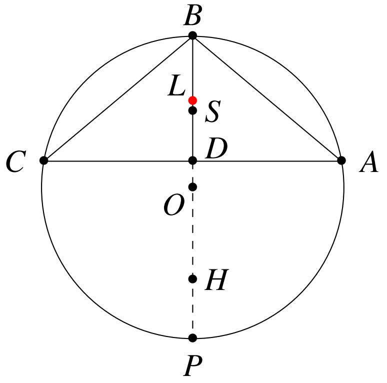

Figure 13: Comparing areas of a circular segment and its inscribed triangle

Proof. Take a circular segment ABC less than a semicircle in which is inscribed the maximum triangle $\triangle ABC$. Extend the diameter BD of the segment be to a diameter BP of the circle, passing through its center O. (See Figure 13.)

We must first prove that

$$\boxed{\frac{(\text{segment } ABC)}{(\triangle ABC)} > \frac{4}{3}}.$$

Again let L be the barycenter of the *segment* ABC. Using Theorem 3.1, we obtain

$$BD < 2\,BL \quad \text{and so} \quad PD = PB - BD > 2\,OB - 2\,BL = 2\,OL,$$

and since $BP = 2\,OB$, we deduce that

$$\frac{BP}{PD} < \frac{OB}{OL}, \quad \text{whereby} \quad \frac{PD}{OL} > \frac{BP}{OB} = 2,$$

and therefore $PD > 2\,OL$.

Now let PD be cut at H such that $HD = 2\,HP$. Then, since $HD = \frac{2}{3}PD$, it follows that

$$HD > \frac{4}{3}\,OL,$$

and also, by (3.1),

$$\frac{HD}{OL} = \frac{(\text{segment } ABC)}{(\triangle ABC)}, \quad \text{whereby} \quad \frac{(\text{segment } ABC)}{(\triangle ABC)} > \frac{4}{3}$$

as required.

Secondly, we must prove that

$$\boxed{\frac{(\text{segment}\,ABC)}{(\triangle\,ABC)} < \frac{10}{3}\cdot\frac{PD}{BP+3\,OD}}\,. \tag{3.4}$$

Choose S on the diameter BD of the segment, as in the proof of Theorem 3.1, so that

$$BS = \frac{3}{2}\,SD.$$

By Theorem 3.1, S falls between L and D since L is the barycenter of segment ABC. This means that $OL > OS$. Then, since

$$\frac{HD}{OL} = \frac{(\text{segment}\,ABC)}{(\triangle\,ABC)} \quad \text{and} \quad OL > OS \implies \frac{HD}{OL} < \frac{HD}{OS}\,,$$

it follows that

$$\frac{(\text{segment}\,ABC)}{(\triangle\,ABC)} < \frac{HD}{OS}\,.$$

Now, since $OS = OD + DS = OD + \frac{2}{5}\,BD$, so that

$$5\,OS = 2\,BD + 5\,OD = 2\,OB + 3\,OD = BP + 3\,OD,$$

we obtain the desired estimate (3.4):

$$\frac{(\text{segment}\,ABC)}{(\triangle\,ABC)} < \frac{5\,HD}{5\,OS} = \frac{10}{3}\,\frac{PD}{5\,OS} = \frac{10}{3}\,\frac{PD}{BP+3OD}\,. \qquad \square$$

Now we are ready to apply the previous inequality on *areas* to obtain an inequality on a *circular arc*.

Theorem 3.5 (Theorem XVI of *Inventa*)**.** *Any* arc *less than a semicircumference is* greater *than its chord plus one-third of the difference between that chord and its sine. But it is* less *than the chord plus the line which has the same ratio to the aforementioned one third that four times the chord plus the sine has to twice the chord plus three times the sine.*

Proof. Take a circle with center O and on it an arc $\overset{\frown}{AC}$, less than a semicircle. The sine of the arc $\overset{\frown}{AC}$ is the perpendicular AM from A to the diameter CR. Let B be the midpoint of the arc $\overset{\frown}{AC}$; the diameter PB of the circle bisects the chord AC (perpendicularly) at D. (See Figure 14.)

We already showed in the proof of Theorem 2.4 (*VII of Inventa*), see formula (2.6), that:

$$\overset{\frown}{AC} > AC + \frac{1}{3}(AC - AM). \tag{3.5}$$

This establishes the first statement of the theorem.

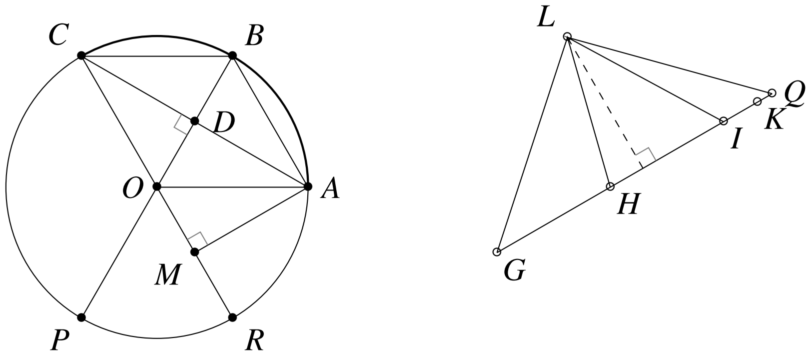

Figure 14: Estimating the arc length of a circular arc

Now take another line GH, produced successively to points I, K and Q, such that:

- ⋄ $GH = AM$ and $GI = AC$, so that $AC - AM = GI - GH = HI$.
- ⋄ $GK = GI + IK$, where $IK = \frac{1}{3}HI$.
- ⋄ $GQ = GI + IQ$, where by construction $\dfrac{IQ}{IK} := \dfrac{4\,GI + GH}{2\,GI + 3\,GH} = \dfrac{4\,AC + AM}{2\,AC + 3\,AM}$.

The sum relation $GQ = GI + IQ$ follows because $IQ > IK$, which in turn comes from $4\,AC + AM > 2\,AC + 3\,AM$ since $AC > AM$. The placement of K also shows that

$$GK = AC + \frac{1}{3}(AC - AM).$$

It follows from (3.5) that $\overset{\frown}{AC} > GK$.

The second statement of the theorem can now be expressed as:

$$\overset{\frown}{AC} < GQ = AC + \frac{1}{3}(AC - AM) \cdot \frac{4AC + AM}{2AC + 3AM}\,. \tag{3.6}$$

In order to prove (3.6), construct the triangles with common vertex L, common altitude equal to the radius OB of the circle, and respective bases GH, HI, and IQ. Join OA, OC, AB and BC (see Figure 14 again).

Then, since

$$\frac{IQ}{IK} = \frac{4\,GI + GH}{2\,GI + 3\,GH} \implies \frac{IQ}{IH} = \frac{IQ}{3\,IK} = \frac{4\,HI + GH}{6\,GI + 9\,GH}\,,$$

it follows that

$$\frac{QH}{IH} = \frac{QI + IH}{IH} = \frac{10\,GI + 10\,GH}{6\,GI + 9\,GH} = \frac{10}{3} \cdot \frac{GI + GH}{2\,GI + 3\,GH} = \frac{10}{3} \cdot \frac{AC + AM}{2\,AC + 3\,AM}\,.$$

Moreover, from the similar triangles $\triangle\,CAM \sim \triangle\,COD$ we obtain

$$\frac{AC}{AM} = \frac{OC}{OD}\,.$$

Therefore,

$$\frac{QH}{IH} = \frac{10}{3} \cdot \frac{OC + OD}{2\,OC + 3\,OD} = \frac{10}{3} \cdot \frac{PD}{PB + 3\,OD}\,.$$

Hence, by (3.4) in the proof of Theorem 3.4 (XV of *Inventa*),

$$\frac{(\text{segment}\,ABC)}{(\triangle\,ABC)} < \frac{QH}{IH} = \frac{(\triangle\,QHL)}{(\triangle\,IHL)}\,.$$

We now assert the equality of areas $(\triangle\,IHL) = (\triangle\,ABC)$.

Indeed, since the base GH of the triangle $\triangle\,GHL$ equals the altitude AM of $\triangle\,OAC$ and vice versa, we conclude that $(\triangle\,GHL) = (\triangle\,OAC)$.

Moreover, since $GI = AC$ we similarly deduce that

$$(\triangle\,GIL) = (\triangle\,OAB) + (\triangle\,OBC) = (OABC),$$

and then $(\triangle\,IHL) = (\triangle\,GIL) - (\triangle\,GHL) = (OABC) - (\triangle\,OAC) = (\triangle\,ABC)$, as claimed. Therefore,

$$\frac{(\text{segment}\,ABC)}{(\triangle\,ABC)} < \frac{(\triangle\,QHL)}{(\triangle\,ABC)} \quad \text{which implies} \quad (\triangle\,QHL) > (\text{segment}\,ABC).$$

Moreover,

$$(\triangle\,QGL) = (\triangle\,QHL) + (\triangle\,GHL) > (\text{segment}\,ABC) + (\triangle\,OAC) = (\text{sector}\,OAC).$$

But the altitude of $\triangle\,QGL$ equals the radius OC, by construction. Therefore the *base* GQ of this triangle will be greater than the *arc* $\overset{\frown}{AC}$. This establishes (3.6) and proves Theorem 3.5. ◻

Corollary 3.6. *Combining Theorems 2.4 and 3.5, we arrive at:*

$$\boxed{C_{2n} + \frac{C_{2n} - C_n}{3} < C < C_{2n} + \frac{C_{2n} - C_n}{3} \cdot \frac{4C_{2n} + C_n}{2C_{2n} + 3C_n}}$$

where C_n is the perimeter of an inscribed regular polygon of n sides and C is the circumference of the circle.

3.2 Huygens' barycentric equation

We return to Huygens' equation (3.1), restating it in modern notation. We follow, to some extent, Hofmann's treatment [7].

In a circle of radius r, consider a circular segment ABC, of area $\Sigma < \frac{1}{2}\pi r^2$. Choose a Cartesian coordinate system with origin at the vertex B of the segment and positive x-axis along the diameter BD (see Figure 15). The equation of the circle is

$$y^2 = 2rx - x^2.$$

The base of the segment lies on a line $x = a$ where $a < r$, and the endpoints of its arc are

$$A = (a, \tfrac{1}{2}b) \quad \text{and} \quad C = (a, -\tfrac{1}{2}b), \quad \text{where} \quad \tfrac{1}{4}b^2 = a(2r - a).$$

Let $L = (\xi, 0)$ be the barycenter of the segment ABC which, by symmetry, lies on the diameter with $\xi < a$. The area of the inscribed triangle may be denoted by

$$\delta := (\triangle\, ABC) = \frac{ab}{2}\,. \tag{3.7}$$

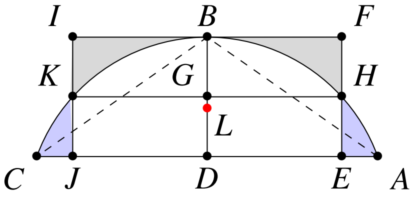

Figure 15: Estimating the segment's barycenter

Lemma 3.7 (Hofmann)**.**

$$\xi > \frac{a}{2}\,. \tag{3.8}$$

Proof. In the circular segment ABC with base AC and diameter BD, let $G = (\frac{1}{2}a, 0)$ be the midpoint of BD and draw the chord HK through G parallel to AD. Then HK is the midline of a rectangle $EFIJ$ whose base EI is part of the base AC of the segment, and whose opposite side FJ is tangent to the circle at B.

The barycenter of this rectangle is G. If we *remove* from the rectangle the regions BFH and BIK outside the circular segment, and *add* to the rectangle the regions HAE and KCJ inside the segment, we recover the segment ABC. Both of these moves displace the barycenter from the midline HK towards the base AC of the segment (i.e., downwards, in Figure 15). Thus L which, by symmetry, lies on the diameter BD, also lies below the chord HK, i.e., nearer the base AC. In other words, $L = (\xi, 0)$ lies on BD between $G = (\frac{1}{2}a, 0)$ and $D = (a, 0)$. In brief, $\frac{1}{2}a < \xi < a$. □

We now restate Huygens' equation (3.1).

Theorem 3.8 (Huygens' barycentric equation)**.**

$$\boxed{\frac{\Sigma}{\delta} = \frac{2}{3} \cdot \frac{2r - a}{r - \xi}}\,. \tag{3.9}$$

This is a very powerful result. For example, it leads quite rapidly to a *new proof* of the inequality (2.4), i.e., of Lemma 2.3 (Theorem III of *Inventa*). Indeed, if we substitute (3.8) in the Huygens barycentric equation we do recover (2.4):

$$\frac{a}{2} < \xi \implies \frac{\Sigma}{\delta} > \frac{2}{3} \cdot \frac{2r - 2\xi}{r - \xi} = \frac{4}{3}\,.$$

3.3 Hofmann's proof of XVI of *Inventa*

In this subsection, another proof of Theorem 3.5 is given, following Hofmann [7], who in turn takes cues from Schuh [16].

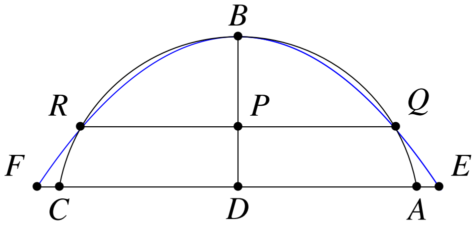

Figure 16: Comparing circular and parabolic segments

Hofmann's proof of Theorem 3.5. Consider a circular segment ABC with vertex B and diameter BD.

Let P be the point that divides the diameter in the ratio $BP : PD = 3 : 2$. Draw the chord QR through P, parallel to the base AC of the segment, with Q between A and B, R between B and C, on the circular are $\overset{\frown}{AC}$. Next, we construct the *parabola* with axis of symmetry BD which passes through B, Q and R, and cuts the prolongation of the line AC at the points E and F. (See Figure 16.)

As already argued in the proof of Theorem 3.1 (see Figure 11), the *parabolic* arcs $\overset{\frown}{BQ}$ and $\overset{\frown}{BR}$ lie inside the segment ABC, while the parabolic arcs $\overset{\frown}{QE}$ and $\overset{\frown}{RF}$ lie outside that segment.

By Proposition II.8 of Archimedes' work *On the Equilibrium of Planes* [5, pp. 214], the point P is the barycenter of the *parabolic* segment EBF. To change this parabolic segment into the *circular* segment ABC, we make two modifications. First, we *add* the slivers BQ and BR bounded by the circular and parabolic arcs, above the line PQ (i.e., on the side nearer B). Second, we *remove* the regions AEQ and respectively CFR bounded by the parabolic arcs $\overset{\frown}{EQ}$ (resp. $\overset{\frown}{FR}$), the circular arcs $\overset{\frown}{QA}$ (resp. $\overset{\frown}{RC}$) and the lines AE (resp. CF); these regions lie below the line PQ (i.e., on the side nearer D). Both modifications shift the barycenter upwards, towards B. In this way, the barycenter L of the circular segment ABC, on the diameter BD, is seen to lie between P and B. Therefore,

$$\xi < \frac{3a}{5}\,.$$

Combining this relation with (3.9), we obtain

$$\frac{\Sigma}{\delta} < \frac{4}{3} \cdot \frac{2r - a}{2r - 6a/5}\,.$$

Now reconsider the circular segment AEB in the notation of Figure 14, with the parameters

$$BD = a, \quad AC = b, \quad AM = c, \quad OA = OB = OC = r, \quad OD = r - a, \tag{3.10}$$

and the arc length s of $\overset{\frown}{AC}$.

The area (3.7) of the inscribed triangle $\triangle ABC$ is $\delta = \frac{1}{2}ab$. On the other hand, the similarity $\triangle COD \sim \triangle CAM$ implies that

$$\frac{r}{r-a} = \frac{b}{c}\,.$$

Therefore, Huygens' barycentric equation (3.9) gives us

$$\Sigma < \frac{4\delta}{3} \cdot \frac{2r-a}{2r-6a/5} = \frac{2ab}{3} \cdot \frac{b+c}{\frac{2}{5}(2b+3c)} = \frac{10}{3} \cdot \frac{b^2-c^2}{2b+3c} \cdot \frac{r}{2}\,.$$

Adding the area $\frac{1}{2}rc$ of the triangle $\triangle OAC$, we obtain

$$\frac{rs}{2} = (\text{sector}\,OAC) = (\triangle\,OAC) + \Sigma < \frac{rc}{2} + \frac{10}{3} \cdot \frac{b^2-c^2}{2b+3c} \cdot \frac{r}{2}$$

whereby, on dividing by $r/2$,

$$s < c + \frac{10}{3} \cdot \frac{b^2-c^2}{2b+3c}\,. \tag{3.11}$$

This inequality (3.11) is easily seen to be equivalent to

$$s < b + \frac{b-c}{3} \cdot \frac{4b+c}{2b+3c}$$

which is precisely Huygens' (3.6). □

To compare (3.11) with the one stated in the introduction, the correspondence is:

$$s = x, \quad c = \sin x, \quad b = 2\,AD = 2\sin\frac{x}{2}\,. \tag{3.12}$$

Thereby, (3.11) exactly recovers the modern formula (1.7).

Error Analysis. If we apply Corollary 3.6, consistent with (3.11), to a circle of diameter 1, we obtain

$$\left\{C_{2n} + \frac{C_{2n}-C_n}{3} \cdot \frac{4C_{2n}+C_n}{2C_{2n}+3C_n}\right\} - \pi < \frac{\pi^7}{22400} \cdot \frac{1}{n^6} + O\Big(\frac{1}{n^8}\Big)$$

which shows the high accuracy of Huygens' approximation.

Remark 3.9. Although not needed for the proof, the diagram in Figure 16 poses an intriguing question: which area is larger, the sliver BQ or the triangular region AEQ? It turns out that they are almost equal, but the latter is indeed larger than the former in all cases. We examine this relation in Appendix B.

3.4 Huygens unproved barycentric theorem, at the end of *Inventa*

Huygens wanted a *lower* bound with the same accuracy $O(1/n^6)$ as his upper bound (3.6). All he says, in the final section (Problem IV, alias Proposition XX) of *Inventa*, is this [12, p. 63]:

> "[. . .] another lower limit may be obtained more accurate than the first, if we use the following rule which depends upon a more careful investigation of the center of gravity."

To achieve it he announced, *without proof*, the following approximation:

> *Let four-thirds of the difference between the limits found be added to twice the chord plus three times the sine, and let the difference between the chord and the sine have the same ratio to another line that the line thus made up has to three and one-third or ten-thirds times the sum of the two; this other line added to the sine makes a line which is less than the arc.*

To parse this in terms of our parameters (3.10), where the chord AC has length b and the sine AM has length c, the lower bound (3.5) is expressed by (2.7):

$$s > b + \frac{1}{3}(b - c) = c + \frac{4}{3}(b - c),$$

so the two bounds differ by

$$\frac{10(b^2 - c^2)}{3(2b + 3c)} - \frac{4(b - c)}{3} = \frac{2(b - c)^2}{3(2b + 3c)}\,.$$

Thus, if

$$d := 2b + 3c + \frac{8(b - c)^2}{9(2b + 3c)}$$

then the new putative lower bound is $c + e$, where

$$\frac{b - c}{e} = \frac{d}{\frac{10}{3}(b + c)}$$

and hence, according to Huygens,

$$s > c + e = c + \frac{10}{3} \cdot \frac{b^2 - c^2}{2b + 3c + \dfrac{8(b - c)^2}{9(2b + 3c)}}\,. \tag{3.13}$$

We may combine these upper and lower bounds, showing their structural similarity.

Theorem 3.10. *The arc length s of a circle segment with chord b and sine c satisfies the bounds:*

$$c + \frac{10}{3} \cdot \frac{b^2 - c^2}{2b + 3c + \dfrac{8(b-c)^2}{9(2b+3c)}} < s < c + \frac{10}{3} \cdot \frac{b^2 - c^2}{2b + 3c}\,. \tag{3.14}$$

On converting the parameters of the inequalities (3.14) with the correspondence (3.12), we recover the formulas (1.7) and (1.8) stated in the introduction. The theorem also gives a range of validity of those formulas as $0 \leqslant x < \pi$. (The limiting case $x = \pi$ gives the unremarkable estimates $\frac{30}{11} < \pi < \frac{10}{3}$.) Naturally, the method of approximation by regular polygons shows that the power of the inequalities lies in their application to *small* angles.

Corollary 3.11. *The circumference C of a circle may be estimated in terms of the perimeters C_n, C_{2n} of inscribed regular polygons of n and $2n$ sides, respectively, as follows:*

$$\boxed{C_n + \frac{10}{3} \cdot \frac{C_{2n}^2 - C_n^2}{2C_{2n} + 3C_n + \dfrac{8}{9} \cdot \dfrac{(C_{2n} - C_n)^2}{2C_{2n} + 3C_n}} < C < C_n + \frac{10}{3} \cdot \frac{C_{2n}^2 - C_n^2}{2C_{2n} + 3C_n}}\,. \tag{3.15}$$

Proof. If the chord AC of Figure 14 is the side of a regular polygon of $2n$ sides, its perimeter is $C_{2n} = 2nb$. The sine AM is *half* the side length of a regular polygon of n sides (see Figure 3); hence its perimeter is $C_n = 2nc$. On multiplying all terms of (3.14) by $2n$, we obtain the global estimate (3.15). □

For $n = 30$, Huygens used the following approximations in (3.15):

$$3.13585389802979 < C_{30} < 3.13585389802980,$$
$$3.14015737457639 < C_{60} < 3.14015737457640,$$

which gives

$$3.14159265339060 < \pi < 3.14159265377520.$$

This rounds to the now famous result, declared in *Inventa*:

$$\boxed{3.1415926533 < \pi < 3.1415926538}\,. \tag{3.16}$$

Apparently, Huygens never wrote down a geometric proof of his new lower bound (3.13). To this day, such a *geometric* proof remains unknown.

However, in 1914 Frederik Schuh [16] replaced a circular segment by the cleverly chosen parabolic segment of Figure 16, used Archimedes' determination of the barycenter of the latter to prove the following inequality:

$$\xi > \frac{3}{5}a - \frac{3a^2}{25(r - \frac{3}{5}a)}\,. \tag{3.17}$$

Then Schuh used Huygens' barycentric equation (3.9) to transform (3.17) into the version of Huygens' final inequality with a constant 27 instead of 8 in the denominator of Huygens' original inequality (3.13). It is worth pointing out, as Schuh himself noted [16, p. 247], that if we replace the 3 in the numerator of (3.17) by $\frac{8}{9}$, then his transformation would produce Huygens' original lower bound. This is the best result known, in a geometric formulation.

Of course, with differential calculus it is not difficult to verify the inequality (1.8), equivalent to Huygens' lower bound. Pinelis [13] suggested using the change of variable $t := \tan(x/4)$ to transform the right-hand side into a rational approximation $f(t)$ to $4\arctan t$, which he then showed to be an underestimate. Pinelis' argument seems to be the first available proof of Huygens' lower bound.

3.5 The location of the barycenter

The distance from the center of a circle to the barycenter of a circular segment is

$$\frac{4}{3}\cdot\frac{r\sin^3\frac{\theta}{2}}{\theta-\sin\theta} \tag{3.18}$$

where r is the radius of the circle and θ is the central angle of the segment. This formula can readily be obtained nowadays by standard calculus techniques. Whether Huygens was cognizant of it is not clear to us; but we now show that it is *equivalent* to Huygens' Theorem V in *Theoremata*, using only classical geometrical arguments and the law of the lever.

That theorem, that we refer to as Huygens' Grossehilfsatz (quoted here as our Proposition 3.3) is the *Archimedean-style statement of the location of the barycenter of a circular segment*. It is an *implicit* statement, in that it declares that the circular segment balances a certain triangle whose barycenter is known (Archimedes located it at the intersection of the medians), if the fulcrum is the center of the encompassing circle. It does not explicitly give a formula for the barycenter; rather, it locates it so that the equilibrium takes place.

We now show how to transform that implicit statement into the explicit formula above.

Theorem 3.12. *According to Huygens' Proposition 3.3, the distance from the center of the circle of radius r to the barycenter of the segment with central angle θ is given by* (3.18).

Proof. Let $\bar{x} := OL$ be the distance from the center of the circle to the barycenter of the segment ABC, see Figure 12, and let $a = BD$ be the diameter and $b = AB$ the base of the segment, as in (3.10). The Huygens associated triangle $\triangle KOH$ has base b and height $\sqrt{a(2r-a)} = \frac{1}{2}b$, in view of (3.2). Therefore the distance from the center O of the circle to the barycenter M of that triangle is $\frac{2}{3}\cdot\frac{1}{2}b = \frac{1}{3}b$. Its area is $(\triangle KOH) = \frac{1}{4}b^2$.

Now, since $\frac{1}{2}b = r\sin\frac{\theta}{2}$ and $r - a = r\cos\frac{\theta}{2}$, the area of the circular segment ABC is

$$\frac{1}{2}r^2\theta - \frac{1}{2}b(r-a) = \frac{1}{2}r^2\theta - r^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{r^2}{2}(\theta-\sin\theta).$$

Therefore, by the law of the lever,

$$OM\cdot(\triangle KOH) = OL\cdot(\text{segment } ABC),$$

that is,

$$\frac{b}{3}\cdot\frac{b^2}{4} = \bar{x}\cdot\frac{r^2}{2}(\theta - \sin\theta), \tag{3.19}$$

which is to say,

$$\frac{b^3}{12} = \frac{2}{3}r^3\sin^3\Big(\frac{\theta}{2}\Big) = \bar{x}\cdot\frac{r^2}{2}(\theta - \sin\theta).$$

Solving for $\bar{x}$, we arrive at (3.18):

$$\bar{x} = \frac{4}{3}\,\frac{r\sin^3\frac{\theta}{2}}{\theta - \sin\theta}$$

which is the known modern formula. □

Clearly, the steps are reversible: since the last formula is equivalent to (3.19), it expresses that the triangle $\triangle KOH$ and the segment ABC balance at the coordinate origin, i.e., at the center O of the circle.

Huygens solved the general problem of locating the barycenter of a segment of *any* conic section with a center (namely, an ellipse or a hyperbola) in his earlier work *Theoremata*. In this paper, we adapt his results only to the special case of a circle. We surmise that he was trying to complete the earlier work of Archimedes on *parabolic* segments.

4 A historical conjecture

Both the Cusa inequality and Snell's own inequality may be termed *convergence-improving* inequalities, since they produce at least twice the accuracy as the original procedure of Archimedes.

We make the following historical conjecture: *Archimedes was fully cognizant of the Snell–Cusa convergence-improving inequalities and may have used them to obtain closer bounds on π.*

We base our suggestion on the following considerations.

(1) It is well known that the extant text of *Measurement* is a damaged and corrupt extract from a more comprehensive study by Archimedes of the metric properties of circular figures. Unfortunately, his full tract was lost due to the willy-nilly of history.

(2) Some three centuries later, Heron [6] quotes closer bounds on π computed by Archimedes, but they are corrupted (the quoted lower bound is actually a quite good *upper* bound). Moreover, he gives no indication as to how they were calculated.

(3) Heron also cites a proposition on circular segments from *Measurement* (namely, the *first Heron–Huygens lemma*) which is *not* in the extant text of today; indeed Eutokios (6th century A.D.) had a version of *Measurement* [4] which virtually coincides with today's variant. So by then the original full treatise had already been lost.

(4) Now we submit the major piece of evidence for our conjecture. The treatise *The Book of Lemmas*, transmitted by Arab mathematicians, contains as its Proposition 8 Archimedes' famous *trisection of an angle* [5, pp. 309]. The trisection construction creates a diagram (see Figure 9 above) *which coincides exactly with the Snell–Cusa figures in Huygens' treatise*. We submit that this is no mere coincidence. Rather, it seems to us that Archimedes worked assiduously on finding approximations to π and *as an incidental by-product* encountered the trisection construction. His familiarity with the trisection figure and his investigations of such approximations suggests to us that he must have been *fully aware of the applications of that figure to convergence-improving approximations*. To suggest otherwise is to say that the most brilliant mathematician of antiquity, working on geometric approximations to π, *was unaware of that figure's applications to his researches*. We consider this latter conclusion to be inconceivable.

(5) Assuming his familiarity with the Snell–Cusa improvements, it appears quite likely that he used them to compute the closer bounds cited by Heron. Our current knowledge precludes rigorous conclusions, but the argument outlined above seems very plausible.

A Huygens' barycentric *Grossehilfsatz*

The following argument establishes the auxiliary Proposition 3.3, needed for Theorem 3.2.

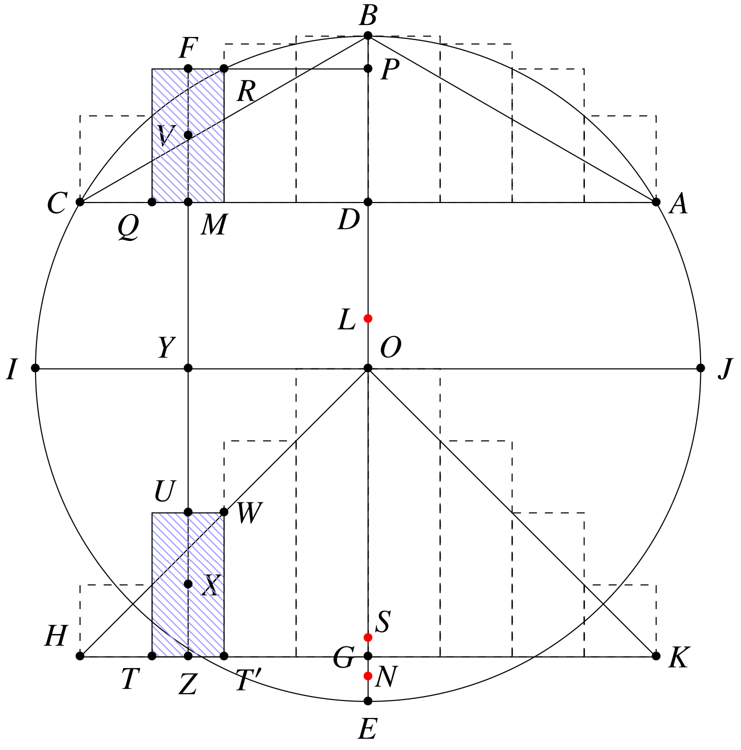

Figure 17: Balancing a circular segment and a triangle

Figure 17, adapted from [10], reproduces most of Figure 12 – except for the barycentric points L and M of that diagram – plus some further detail. The circle segment ABC and the triangle KOH are as before; they do not overlap. The emphasis now is on their disjoint union:

$$\mathcal{H} := \text{segment}\, ABC \cup \triangle\, KOH.$$

The segment ABC, less than a semicircle, is given; the diameter BE of the circle extends the diameter BD of the segment ABC. The triangle $\triangle\, KOH$ is constructed by first locating G on the radius OE so that

$$OG^2 = BD \cdot DE. \tag{A.1}$$

The base KH of the triangle $\triangle\, KOH$ is drawn through G, equal and parallel to AC, so that G is its midpoint. The half-chord CD is (by similar triangles, $\triangle\, BCD \sim \triangle\, CED$) the mean proportional of BD and DE,

$$BD \cdot DE = CD^2,$$

consequently, $CD = OG$.

Let L now denote the barycenter of the combined figure segment $ABC \cup \triangle\, KOH$. Proposition 3.3 can now be restated as: $L = O$.

Proof. Suppose, *arguendo*, that $L \neq O$. Then either L is “above” O, that is, L is an interior point of the radius OB; or L is “below” O, i.e., L is an interior point of the radius OE. We shall show that the first case is impossible; and by a similar argument, the second case will also be ruled out.

Assume, then, that L lies on OB with $L \neq O$. Choose a magnitude of area $\mathcal{M}$ such that

$$\frac{OG}{OL} = \frac{(\triangle\, ABC \cup \triangle\, KOH)}{\mathcal{M}},$$

where we view the triangles as laminas of uniform density. Then cover the combined figure $\mathcal{H}$ with parallelograms of equal width, symmetric with respect to the diameter BE, such that if $\mathcal{R}$ is the total area of the *excess parts* of the cover, then $\mathcal{R} < \mathcal{M}$. That is possible since, by taking thinner parallelograms, *we can make the total excess area as small as we please*. Hence

$$\frac{(\triangle\, ABC) + (\triangle\, KOH)}{\mathcal{R}} > \frac{(\triangle\, ABC) + (\triangle\, KOH)}{\mathcal{M}} = \frac{OG}{OL}$$

and therefore also:

$$\frac{(\text{segment}\, ABC) + (\triangle\, KOH)}{\mathcal{R}} > \frac{OG}{OL}.$$

Now define the point N on the prolongation of BG *below* the base KH such that

$$\frac{ON}{OL} = \frac{(\text{segment}\, ABC) + (\triangle\, KOH)}{\mathcal{R}}.$$

Next, draw the diameter IJ through O, parallel to the bases AC and KH. Take two *corresponding parallelograms* RQ and WT in the cover with *respective barycenters*

$$V \text{ in } RQ \quad \text{and} \quad X \text{ in } WT,$$

and draw their midline $ZUMF$, cutting the line IJ at Y. Finally, on the diameter BE take the point S such that $SE = BP$.

Lemma A.1. *Y is the barycenter of the combined parallelograms $RQ \cup WT$.*

Proof. By similar triangles,

$$\frac{CD}{PR} = \frac{GH}{GT'} = \frac{OH}{OW} = \frac{ZY}{UY}\,.$$

The numerator and denominator on the first of these ratios are mean proportionals:

$$BD \cdot DE = CD^2 \quad \text{and} \quad BP \cdot PE = PR^2,$$

and therefore

$$\frac{BD \cdot DE}{BP \cdot PE} = \frac{CD^2}{PR^2} = \frac{ZY^2}{UY^2}\,.$$

It follows immediately that

$$\frac{BD \cdot DE}{BD \cdot DE - BP \cdot PE} = \frac{ZY^2}{ZY^2 - UY^2}\,. \tag{A.2}$$

The denominator on the left-hand side can be simplified thus:

$$\begin{aligned} BD \cdot DE - BP \cdot PE &= (BP + PD) \cdot DE - BP \cdot (PD + DE) = PD \cdot (DE - BP) \\ &= (DE - SE) \cdot DP = DS \cdot DP, \end{aligned} \tag{A.3}$$

by the location of the point S.

Next, notice that

$$ZY^2 - UY^2 = ZU^2 + 2\,ZU \cdot UY \tag{A.4}$$

as an instance of the algebraic identity $(r+s)^2 - r^2 = s^2 + 2rs$. Since X is the midpoint of the line ZU, hence $ZX = XU = \frac{1}{2}ZU$, the right-hand side of (A.4) becomes

$$4\,XU^2 + 4\,XU \cdot UY = 4\,XU \cdot (XU + UY) = 4\,XU \cdot XY,$$

or equivalently, $ZU^2 + 2\,ZU \cdot UY = 2ZU \cdot XY$. Then (A.4) simplifies to

$$ZU^2 - UY^2 = 2ZU \cdot XY. \tag{A.5}$$

On substituting the relations (A.3) and (A.5), the equality (A.2) reduces to

$$\frac{BD \cdot DE}{DS \cdot DP} = \frac{ZY^2}{2ZU \cdot XY}. \tag{A.6}$$

Now, by the definition of the point G using (A.1),

$$BD \cdot DE = OG^2 = ZY^2. \tag{A.7}$$

Moreover, since $SE = BP$ implies $OS = OP = OD + DP$, it follows that

$$DS = OD + OS = 2\,OD + DP = 2\,YM + FM = 2(YM + MV) = 2\,YV.$$

Therefore,

$$DS \cdot DP = 2\,YV \cdot DP = 2YV \cdot FM.$$

This, together with (A.6) and (A.7), allows us to conclude that $YV \cdot FM = XY \cdot ZU$.

And that is equivalent to

$$\frac{YV}{XY} = \frac{ZU}{FM} = \frac{(\text{parallelogram}\,WT)}{(\text{parallelogram}\,RQ)}\,.$$

By the law of the lever, this means that *Y is the barycenter of the two parallelograms $RQ \cup WT$ taken together*. This completes the proof of the lemma. □

The same proof shows that the barycenter of *every* pair of corresponding parallelograms in the covering has its barycenter on the line IJ. Since the barycenter of the covering also lies on the diameter BE, it coincides with the center O of the circle.

But now L is the barycenter of the combined figure segment $ABC \cup \triangle\, KOH$. Thus, the barycenter of just the excess parts, of area $\mathcal{R}$, is necessarily on the prolongation of the line LO such that

$$\frac{\text{the added part}}{OL} = \frac{(\text{segment}\,ABC) + (\triangle\, KOH)}{\mathcal{R}} = \frac{ON}{OL}\,.$$

The endpoint N of this added part is therefore *the barycenter of the added part.*

But, *that cannot be so!* Indeed, a line drawn through N parallel to the base KH of $\triangle\, KOH$ leaves all of the area portions which form $\mathcal{R}$ on the opposite side of that line to N, which contradicts the nature of a barycenter.

This rules out the first case, namely, that L is a point of the radius OB, with $L \neq O$.

Were L to be a point of the opposite radius OE, with $L \neq O$, a similar proof shows that the barycenter of $\mathcal{R}$ would lie *above the segment ABC*, which again is absurd.

Therefore $L = O$, as claimed. □

B An area inequality

We may reconsider the comparison of the circular and parabolic segments in the second proof of Theorem 3.5. For that, we reproduce Figure 16, at a larger scale, in Figure 18.

We find it convenient to introduce Cartesian coordinates, as follows. Place the origin $(0, 0)$ at the midpoint D of the chord AC and extend that chord to the x-axis; thus, $A = (a, 0)$ and $C = (-a, 0)$ for some $a > 0$. The vertex of the circle segment will lie at $B = (0, b)$ for some $b > 0$.

If the circle has radius r, its center lies at $(0, b - r)$. Hence,

$$a^2 + (r - b)^2 = r^2.$$

The barycenter P of the parabolic segment lies at $(0, \frac{2}{5}b)$ by Archimedes' theorem. The parabola and circle intersect, by construction, at the points $Q = (p, \frac{2}{5}b)$ and $R = (-p, \frac{2}{5}b)$. The parabola cuts the x-axis (the extended chord AC) at $E = (c, 0)$ and $F = (-c, 0)$.

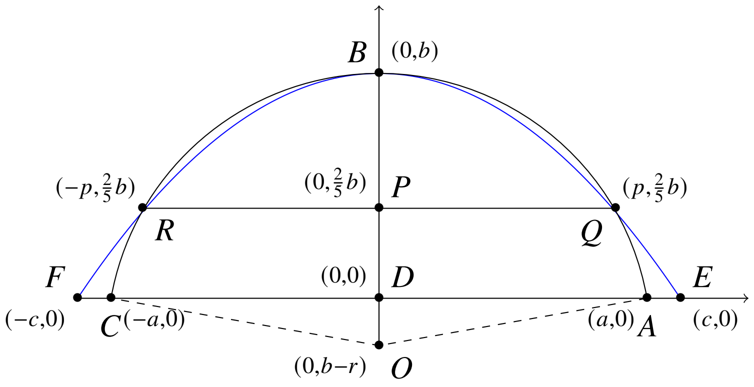

Figure 18: Circular versus parabolic segments, again

The equation of the circle is $x^2 + (y + r - b)^2 = r^2$, or better, in the semicircle including the arc ABC,

$$y = \sqrt{r^2 - x^2} - r + b.$$

The parabola with vertex at B, passing through Q and R, has the equation

$$y = b - \frac{bx^2}{a^2 + \frac{2}{5}b^2} \equiv b - \frac{bx^2}{2rb - \frac{3}{5}b^2}\,. \tag{B.1}$$

Since $(\pm p, \frac{2}{5}b)$ are the intersections of the parabola and the circle, we obtain

$$p = \sqrt{\frac{3}{5}\left(a^2 + \frac{2}{5}b^2\right)} =: \frac{\sqrt{3}}{5}\sqrt{5a^2 + 2b^2}.$$

The parabola (B.1) cuts the x-axis at $(\pm c, 0)$, where

$$c = \frac{1}{\sqrt{5}}\sqrt{10rb - 3b^2}.$$

Here is a question raised by Figure 18: which area is greater, the sliver BQ between parabola and circle, or the curved triangle QAE? If we also compare their mirror images, the sliver BR and the curved triangle RCF, we see that twice their difference equals the difference in areas between the circular segment ABC and the parabolic segment EBF. Let us calculate those areas.

In his *Quadrature of the Parabola*, Proposition 17, Archimedes already computed the area of such a parabolic segment [5, pp. 246] as $4/3$ the area if the inscribed triangle with the same base and height. Therefore,

$$(\text{segment}\, EBF) = \frac{4}{3}\,(\triangle\, EBF) = \frac{4}{3}\, bc = \frac{4b}{3\sqrt{5}}\,\sqrt{10rb - 3b^2}. \tag{B.2}$$

For the circular segment ABC, we subtract the triangle $\triangle\, OAC$ from the *sector* OAC. This triangle has base $AC = 2a$, and altitude $OD = r - b$, so

$$(\triangle\, OAC) = a(r-b) = (r-b)\sqrt{2rb - b^2}.$$

The sector has angle $\angle AOC =: \theta$ with $0 < \theta < \pi$. From $\triangle\, DOA$ one sees that $\cos(\frac{1}{2}\theta) = (r-b)/r$, so that

$$\frac{\theta}{2} = \frac{\pi}{2} - \arcsin\frac{r-b}{r}\,.$$

Now the area of the sector OAC is

$$(\text{sector}\, OAC) = \frac{1}{2}\, r^2\theta = \frac{\pi r^2}{2} - r^2 \arcsin\frac{r-b}{r}$$

and thus:

$$(\text{segment}\, ABC) = \frac{\pi r^2}{2} - r^2 \arcsin\frac{r-b}{r} - (r-b)\sqrt{2rb - b^2}. \tag{B.3}$$

The difference in areas between the sliver BQ and the wedge QAE is half the difference between (B.3) and (B.2), namely:

$$\frac{\pi r^2}{4} - \frac{1}{2} r^2 \arcsin\frac{r-b}{r} - \frac{r-b}{2}\,\sqrt{2rb - b^2} - \frac{2b}{3\sqrt{5}}\,\sqrt{10rb - 3b^2}.$$

We claim that this quantity is *negative* for all $0 < b \leqslant r$. Since all its terms are areas, we may rescale by setting $x := b/r$, $0 < x \leqslant 1$.

Lemma B.1. *The real function*

$$f(x) := \frac{\pi}{4} - \frac{1}{2}\arcsin(1-x) - \frac{1}{2}\,(1-x)\sqrt{2x - x^2} - \frac{2x}{3\sqrt{5}}\,\sqrt{10x - 3x^2}$$

satisfies $f(x) < 0$ *for* $0 < x \leqslant 1$.

Proof. Notice first that $f(0) = 0$. So it is enough to show that f is strictly decreasing in the interval $0 \leqslant x \leqslant 1$; which will follow from $f'(x) < 0$ for $0 < x < 1$.

The derivative is easily computed:

$$f'(x) = \sqrt{2x - x^2} - \frac{2x(5-2x)}{\sqrt{5}\,\sqrt{10x - 3x^2}}\,.$$

This is a difference of two positive terms; so we must now check the following equivalent inequalities:

$$\begin{aligned} \sqrt{2x - x^2} &< \frac{2x(5-2x)}{\sqrt{5}\,\sqrt{10x - 3x^2}} &&\text{for}\quad 0 < x < 1, \\ \sqrt{5(2x - x^2)(10x - 3x^2)} &< 2x(5 - 2x) &&\text{for}\quad 0 < x < 1, \\ \sqrt{5(2 - x)(10 - 3x)} &< 10 - 4x &&\text{for}\quad 0 < x < 1, \\ 5(2 - x)(10 - 3x) &< (10 - 4x)^2 &&\text{for}\quad 0 < x < 1. \end{aligned}$$

But the last inequality comes immediately from

$$(10 - 4x)^2 - 5(2 - x)(10 - 3x) = (100 - 80x + 16x^2) - 5(20 - 16x + 3x^2) = x^2 > 0. \qquad \square$$

Therefore, $0 > f(x) > f(1)$ whenever $0 < x < 1$. Actually, the lower bound is quite small:

$$f(1) = \frac{\pi}{4} - \frac{2\sqrt{35}}{15} \doteq 0.785398 - 0.788811 = -0.003413.$$

Thus, in a circular segment less than a semicircle, the difference in areas between wedge and sliver is less than about $r^2/300$.

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We thank Iosif Pinelis for discussions on MathOverflow [13] in regard to Huygens' final inequality. The authors acknowledge the good offices of the Centro de Investigación en Matemática Pura y Aplicada via a research project at the University of Costa Rica.

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