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► **To cite this version:**

Helen Alfaro. Unpacking the reasoning task solution approaches of pre-service teachers in the Teacher Education and Development Study in Mathematics. Thirteenth Congress of the European Society for Research in Mathematics Education (CERME13), Alfréd Rényi Institute of Mathematics; Eötvös Loránd University of Budapest, Jul 2023, Budapest, Hungary. hal-04404282

HAL Id: hal-04404282

<https://hal.science/hal-04404282v1>

Submitted on 18 Jan 2024

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Unpacking the reasoning task solution approaches of pre-service teachers in the Teacher Education and Development Study in Mathematics

Helen Alfaro¹

¹University of Costa Rica, Mathematics Education Department, Costa Rica; helen.alfaro@ucr.ac.cr

The knowledge that teachers develop through their academic training greatly influences their performance. It is very difficult for them to promote skills and knowledge in students if they do not have them. Future Costa Rican teachers should be prepared to develop reasoning and argumentation skills in their future students, but there is evidence of low performance in reasoning tasks. In this study, the solutions of 79 Costa Rican future mathematics teachers to two reasoning tasks of the Teacher Education and Development Study in Mathematics (TEDS-M) are analysed in depth using qualitative content analysis. The results showed, among others, that future teachers have deficiencies in retrieving and recognizing key information from statements, and do not monitor whether their justifications are valid. In addition, they evidenced similar reasoning approaches in the solutions.

Keywords: Preservice mathematics teachers, solution approaches, reasoning skills, TEDS-M.

Introduction

There is a worldwide interest to improve the learning and performance of students in mathematics. Given that the “teaching quality” has been considered as the school-related factor that has the greatest influence on students’ achievement (e.g., Hsieh et al., 2011), attention should be paid to the quality of teacher education and the knowledge gained by preservice teachers during training programs. In Costa Rica, however, there is a lack of control over the quality and differences of mathematics teacher education programs (TEP) (Roman & Lentini, 2018). A diagnostic test for in-service teachers in 2010 showed low performance in school mathematics and significant differences in grades depending on the TEP from which the teachers came (Ministerio de Educación Pública, MEP, 2011). Considering these issues, in previous research the opportunities to learn, the beliefs and the knowledge for teaching mathematics of 79 Costa Rican preservice teachers were studied using the instruments of the TEDS-M study (Brese & Tatto, 2012). The data collected from the participants of four public universities showed that the TEPs propose a high percentage of courses in tertiary level mathematics and problem solving classes (Alfaro & Joutsenlahti, 2020) that could favour a solid training for the success of future teachers in solving mathematical tasks of different levels of cognitive complexity. However, the participants showed difficulties in solving the mathematical knowledge items of the cognitive reasoning domain (Alfaro, 2022). According to the Costa Rican school math curriculum (MEP, 2012), developing reasoning skills in future students is a crucial task of mathematics teachers; thus, it is insufficient to assess preservice teachers based solely on task correctness. It is important to carry out an in-depth analysis of their answers to understand their way of thinking, identify strengths and weaknesses that allow to propose relevant improvement strategies.

Therefore, in this article I propose to disaggregate the solution approaches of two TEDS-M mathematical knowledge tasks of Costa Rican preservice teachers, to obtain information about their

ways of thinking, their conceptual and reasoning limitations, as well as their successes. The results can inform policy makers about methodological or content changes that need to be made in their TEPs, to improve the reasoning skills of future teachers.

Theoretical grounds

Given that teachers' knowledge significantly shapes teaching practices and, consequently, what, and how students learn (Hill et al., 2005), it's reasonable to expect that future teachers' reasoning and argumentation skills will similarly impact students' acquisition of mathematical abilities. The proficiency in reasoning and argumentation skills is a desired competence in future teachers as stated in some knowledge for teaching mathematics frameworks (e.g., Carrillo et al., 2018). The models recognize mathematical reasoning as a fundamental aspect of mathematical practices, essential for tasks such as exploring, justifying, generating, understanding, guiding, refuting, and refining mathematical knowledge, as well as students' work and thought processes. Given its significance, the TEDS-M study assessed it as one of the key cognitive subdomains.

The TEDS-M study is a large-scale international survey designed to assess the effectiveness of TEPs around the world as well as the readiness to teach of preservice mathematics teachers. The knowledge for teaching mathematics framework informed the study and was developed by TEDS-M team (Tatto et al., 2008) taking into consideration previous works developed about mathematics teachers' knowledge (e.g., Ball et al., 2008) and the teacher education standards of the countries that participated in the study. The framework includes the knowledge categories of mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MPCK), and curricular knowledge, which according to Tatto et al. (2008), have an approximate correspondence with the categories of Shulman (1986). The MCK considers crucial definitions, concepts, algorithms, and procedures related to what teachers are expected to teach in secondary school and at least two years beyond (Tatto et al., 2008).

For assessing the preservice teachers MCK, content and cognitive domains were used (Tatto et al., 2008). For the cognitive domains, named *knowing*, *reasoning*, and *applying* certain skills were described according to the level of the cognitive demand. The skills for the reasoning subdomain were *analyze* (e.g. determine and use relationships between variables, make valid inferences from given information), *generalize* (e.g. extend the domain to which the result of mathematical thinking and problem-solving is applicable by restating results in more general and more widely applicable terms), *synthesize or integrate* (e.g. combine mathematical procedures to establish results, and combine results to produce a further result), *justify* (e.g. provide a justification for the truth or falsity of a statement by reference to mathematical results or properties) and *solve non-routine problems* (Tatto et al., 2008).

The highlighted reasoning skills demand substantial cognitive effort and can become apparent through various means within solution processes. This study analyses how these skills were observed in the solutions.

Methods

This study is descriptive in nature and employs the qualitative content analysis method. Data were collected using the TEDS-M test, from which we extracted solutions to tasks MFC711 and MFC814 (Brese & Tatto, 2012). These tasks were completed using paper and pencil by 79 preservice teachers from four public universities in Costa Rica. These participants were enrolled in secondary-level mathematics teaching programs and were in their fourth or fifth year of study. The participants were informed about the voluntary nature of their participation and the anonymization of their data.

The TEDS-M tasks are tailored for preservice teachers, as they encompass knowledge taught in secondary school and about two years beyond. The choice of tasks MFC711 and MFC814 (see figure 1) was informed by several considerations. Firstly, given that Costa Rican participants evidenced poor performance in the reasoning domain tasks, unpacking the conceptual and reasoning constraints they faced will aid in identifying aspects that require reinforcement in TEPs. Secondly, these tasks required a complex response, which means participants had to articulate procedures or proofs; such written responses offer insight into reasoning chains. Finally, since the reasoning domain is more cognitively demanding, it inherently encompasses skills from both the knowing and applying subdomains, which are also relevant to solutions. These tasks, by nature, call for the utilization of reasoning skills such as analysis, justification, generalization, and integration. While evaluating preservice teachers' reasoning abilities based on just two tasks might be insufficient, these tasks still yield pertinent information.

MFC711	MDC814
Prove the following statement: If the graphs of the linear functions $f(x) = ax + b$ and $g(x) = cx + d$ intersect at a point P on the x-axis, the graph of their sum function $(f + g)(x)$ must also go through P.	Let $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $B = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$. Then $A \otimes B$ is defined to be $\begin{bmatrix} pt & qu \\ rv & sw \end{bmatrix}$. Is it true that if $A \otimes B = O$, then either $A = O$ or $B = O$ (where O represents the zero matrix)? Justify your answer.

Figure 1. MFC711 and MFC814 tasks, taken from the TEDS-M released items (Brese & Tatto, 2012)

Given the substantial data volume of 79 complex solutions across two tasks, we employed the Item Scoring Guide (ISG) supplied by the TEDS-M team (Brese & Tatto, 2012) alongside the released items. The ISG not only furnishes guidelines for grading tasks as correct, partially correct, or incorrect, but it also encompasses diverse solution approaches that could have led to each score. For example, the ISG outlines three potential approaches for accurately tackling task MFC711. Consequently, we assigned codes to the solutions of the 79 participants, accounting for these approaches; then, solutions bearing the same code were examined based on the provided description. This systematic analysis facilitated the identification of reasoning patterns and conceptual limitations, intricately connected to the reasoning skills expounded upon in the theoretical framework section.

Results and discussion

The analysis outcomes are organized by task and categorized into correct, partially correct, and incorrect responses, along with the corresponding approaches for each category, as defined in the scoring guide.

Task MFC711

Task MFC711 (see Figure 1) requires participants to know how to prove a mathematical statement. Furthermore, they are expected to have conceptual knowledge of linear functions, sum of functions and function intersection. Additionally, it is necessary to carefully analyse the statement for making valid inferences, integrate results and provide valid justifications. There were three distinct approaches for a correct response, three for a partially correct response and one for incorrect response.

Out of 79 respondents, 27 had a correct answer. 11 preservice teachers provided a correct response that shows the steps for the proof without using the given formulas of the functions $f(x)$ or $g(x)$, which corresponds with the first approach of the ISG (Brese & Tatto, 2012). The primary argument used involved the fact that both $f(x)$ and $g(x)$ intersect the x -axis at the point $P = (p, 0)$, implying that $f(p)=0$ and $g(p)=0$. Additionally, by adding the functions, $(f + g)(p) = f(p) + g(p) = 0 + 0 = 0$. As a result the function $f+g$ also passes through the point P . The proofs did not follow the same order, neither all use words for connecting the ideas. Nevertheless, the performance of the participants demonstrates their recognition that employing the provided formulas and algebraic procedures was unnecessary for validating the statement.

On the other hand, there were 16 participants that used the given formulas of the function to perform their proof. In this second correct approach, participants posed several equations to obtain p in terms of a and b , c and d or a, b, c , and d , depending on the equality from which they began. Then the participants evaluated the sum function with the obtained value, and after some algebraic procedures, they obtained $(f + g)(p) = 0$ (P50, figure 2). Another path of reasoning, under this approach was to find the formula for $(f + g)(x)$, using the given formulas of $f(x)$ and $g(x)$ and then using the fact that in $x = p$, each function was zero, then the sum was zero, like the first route but with the formulas. However, certain solutions initiated by adding $f(x)$ and $g(x)$, equating the sum to zero, and subsequently applying algebraic manipulations to get $(a + c)x + b + d$, thereby establishing connection to $(f + g)(x)$. It is worth noting that none of the participants with correct answers employed graphical approach, constituted the foundation of the third approach.

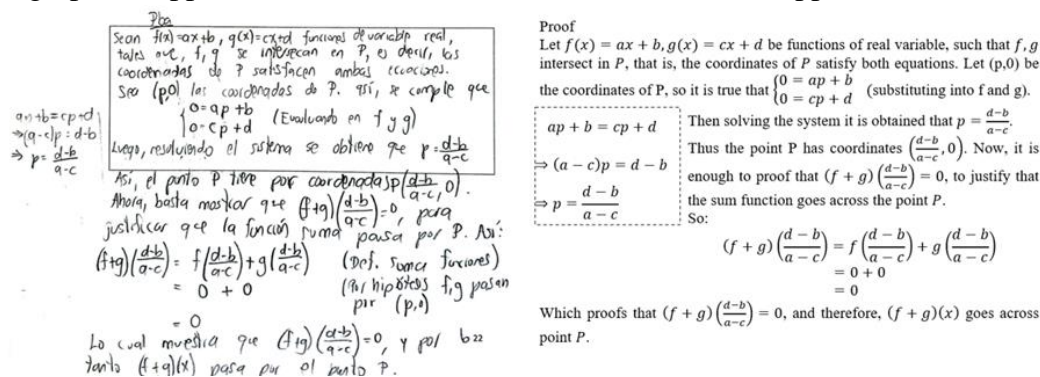


Figure 2. Example of a solution that uses the second route, P50

Among the participants, 26 offered responses that were partially correct. The initial two approaches for partially correct answers align closely with the ideas of the first two approaches of correct responses. Hence, the first partially correct answer approach evidences a chain of reasoning about the functions without using the formulas, but some mistakes were made, or the proof is not complete. There were no participants with responses of this nature. In contrast, there were 24 preservice teachers with responses from the second route in this category.

In most solutions following this approach, participants equated functions f and g , t to determine their point of intersection. However, although they substituted this point into the function $f + g$, they omitted equation like $f(p) = 0$, $g(p) = 0$ or $(f + g)(p) = 0$. Consequently, they failed to conclude the proof. This line of reasoning indicates that participants overlooked a crucial concept from the task statement: the y-coordinate of the intersection point being zero. That mistake resulted in long algebraic procedures without a clear objective.

Finally, there were 13 participants with incorrect responses and 13 left the task blank. The responses with wrong arguments or erased or crossed out marks were categorized as incorrect. From the analysis we found that the responses presented different reasoning and conceptual mistakes. For instance, P26 confused the addition of functions with the composition of functions. Participant P21 intended to use the monotonicity of the functions for proving, assigning a positive value to d in one case and a negative value to d in another, which does not make sense since the monotonicity would be reflected in c not in d .

Another approach included assigning specific values to constants a , b , c , and d to test the statement's validity, as seen with participants P13 and P63. However, even if they had achieved accurate results, this method doesn't constitute a conclusive proof for the statement's validity.

In general, most responses, whether correct, partially correct, or incorrect, lacked mathematical rigor in terms of proof structure and coherence. Another common error was evaluating functions using x instead of the coordinates of point P , possibly stemming from misunderstanding the statement's limited applicability. Some participants also assumed the domain as the set of real numbers \mathbb{R} without proper justification in their proofs.

Task MFC814

For the MFC814 task (see Figure 1), the participants did not require knowledge of linear algebra, despite it includes matrices, because the statement shows an example of how the operation \otimes is defined. Hence, to establish the statement's validity, participants needed to grasp the zero-divisor absence property and propose suitable counterexamples. There were three coding approaches for correct responses, one approach for partially correct, and three for incorrect response.

Out of the total, 36 responses were correct. Among these, 34 fell into the first approach, which involved presenting an accurate and specific counterexample to demonstrate the statement's falsehood. Within this approach, three solution patterns emerged. Some participants provided counterexamples using numerical values for all matrix entries, while others employed letters and zeros in their counterexample entries. In both patterns, the solutions were direct, with participants explicitly stating the statement's inaccuracy and providing counterexamples involving matrices A and

B distinct from the zero matrix. Several answers showed further elaboration, with participants who, in addition to the counterexample, included conclusions such as Q36, who stated: "Note that neither is the null matrix, however, $A \otimes B = 0$ ". The third pattern of response evidenced a different line of reasoning. Here, participants began by assuming that $A \otimes B = 0$ for any matrices A and B . Subsequently, they used the absence of zero-divisors property, either implicitly or explicitly, to justify that not necessarily one matrix must be the zero matrix. An example of this pattern is shown in Figure 3.

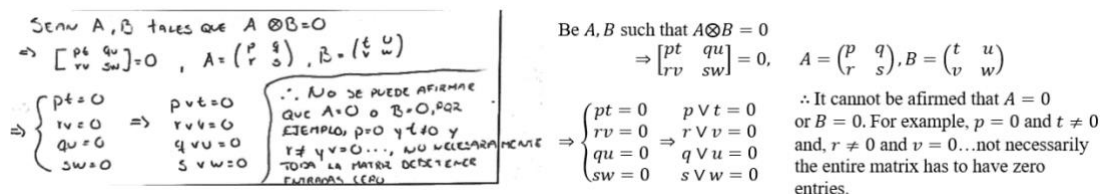


Figure 3. Example of a solution that uses the third pattern in first route, P20

Only two responses were coded as correct using the second approach, which involved stating the statement's falsity and offering a general description of the counterexample through words. For instance, P3 stated "No, for $A \otimes B = 0$ it is only needed that pt, qu, rv, sw be 0 for that, it is enough that at least one of each pair is zero, which could be $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ". This type of response evidences good mathematical communication skills and a thorough understanding of the task. Finally, no responses were classified under the third approach, which pertained to other correct responses. Other responses involved participants asserting the statement's falsehood, emphasizing that only four of the eight entries in both matrices needed to be null. Nonetheless, they omitted to detail the specific arrangement of these null entries required for $A \otimes B = 0$ to hold true. These two ways of thinking reveal that the participants did not verify whether the arguments they used to justify were valid.

There were 25 incorrect responses and 11 were left in blank. A surprising result was that 21 preservice teachers answered that the statement was true, following three patterns. Some started from the conclusion and verified the hypothesis. That is, they took a matrix A or B equal to zero and check that $A \otimes B = 0$. Other way of thinking was recognizing that for the result of a multiplication to be zero, one of the factors must be zero, and then A or B must be the zero matrix, as it is explained in the following response.

P9: Yes, since the first entry of A is multiplied by the first of B (and so on with its entries), for the resulting matrix to be the null matrix, one matrix, either A or B must have 0 as entries so that the result of the product is zero.

In this scenario, participants are overlooking the fact that the zeros don't necessarily have to be present in the same matrix for the statement to hold true. In a different pattern, participants employed verbal explanations to assert the statement's validity, drawing on the concept of zero as the absorbing element in multiplication. For instance, P26 wrote "Yes, it's correct. Since when multiplying always by a null matrix the result will be a null matrix". In this case, participants failed to consider that the resulting matrix's entries were the outcomes of multiplying two factors from distinct matrices. As a result, asserting the zero element of multiplication for the entire matrix is invalid; instead, each entry must be considered individually.

Conclusions

The Costa Rican mathematics curriculum for primary and secondary education outlines the promotion of five mathematical processes, which encompass reasoning and argumentation (MEP,2012). Consequently, fostering reasoning skills among preservice teachers becomes imperative, enabling them to effectively impart these skills to their future students. The analysis of the solutions related to these two reasoning skills, revealed important findings. For example, in both tasks, participants with correct responses choose approaches with particular examples or expressions rather than more general ones. This suggests, that despite nearing the completion of their major, the participants still rely on working with particular cases, which, in some of the solutions, led them to overgeneralizations. The participants exhibit a procedural reproduction level, in such a way that they pose several equations and solve unknowns mechanically, with difficulties to make pertinent connections. On the other hand, deficiencies were observed in participants' capacity to draw pertinent inferences from the provided information. Additionally, other patterns of reasoning errors emerged, such as initiating a proof by assuming the conclusion and then verifying the hypothesis, which has been observed in other studies with preservice teachers (Demir et al., 2018) and is common in students. This last reasoning error was the explanation why 21, out of 79 participants, validate a false statement in task MFC814.

Another finding was that despite there were several approaches for getting an answer as correct, partially correct or incorrect, the Costa Rican preservice teachers' solutions consistently fell into the same approaches in each category. This suggest that participants had similar reasoning paths, which can be interpreted as a limitation regarding knowing different reasoning strategies and was also presented by preservice teachers of the four TEPs involved.

While these results cannot be generalized due to limitations such as population scope and the impossibility to access to participants thinking by means of written tasks, the identified reasoning limitations and errors hold implications for TEPs. The present focus of the TEPs on tertiary level mathematics topics and problem solving might not effectively cultivate high-cognitive-demand reasoning skills. Improvements in methodological and assessment strategies should be made to allow preservice teachers to develop reasoning skills. For example, opportunities to defend mathematical arguments orally, or class simulations where teacher educators model the reasoning carried out to solve a problem, expressing out loud the connections and heuristics that come to mind, as well as the reasons why they choose or reject some of them. In this way, pre-service teachers will be able to incorporate reasoning strategies into their training that can also be carried over to high school classrooms. Finally, it is important to perform studies with in-service teachers to understand how teachers are dealing with these deficiencies in practice.

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