Faà di Bruno Hopf algebras

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Abstract

This is a short review on the Faà di Bruno formulas, implementing composition of real-analytic functions, and a Hopf algebra associated to such formulas. This structure allows, among several other things, a short proof of the Lie–Scheffers theorem, and relating the Lagrange inversion formulas with antipodes. It is also the maximal commutative Hopf subalgebra of the one used by Connes and Moscovici to study diffeomorphisms in a noncommutative geometry setting. The link of Faà di Bruno formulas with the theory of set partitions is developed in some detail.

1 The Faà di Bruno formula

Faà di Bruno (Hopf, bi)algebras appear in several branches of mathematics and physics, and may be introduced in several ways. Here we start from the group G of formal exponential power series like

$$f(t) = \sum_{n=1}^{\infty} \frac{f_n}{n!}\, t^n,$$

with $f_1 > 0$. (In view of Borel's lemma, one may regard them as local representatives of orientation-preserving diffeomorphisms of $\mathbb{R}$ leaving 0 fixed. The question of analyticity is considered below.)

On this group of power series we consider the coordinate functions

$$a_n(f) := f_n = f^{(n)}(0), \quad n \geq 1.$$

We wish to compute $h_n = a_n(h)$, where h is the composition $f \circ g$ of two such diffeomorphisms, in terms of the f_n and g_n. Now,

$$h(t) = \sum_{k=1}^{\infty} \frac{f_k}{k!} \biggl(\sum_{l=1}^{\infty} \frac{g_l}{l!}\, t^l \biggr)^k.$$

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To compute the nth coefficient h_n we need only consider the sum up to $k = n$, since the remaining terms contain powers of t higher than n. From Cauchy's product formula,

$$h_n = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{l_i \geq 1,\, l_1+\cdots+l_k=n} \frac{n!\, g_{l_1} \cdots g_{l_k}}{l_1! \cdots l_k!}.$$

If among the l_i there are λ_1 copies of 1, λ_2 copies of 2, and so on, then the sum $l_1+\cdots+l_k = n$ can be rewritten as

$$\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n, \quad \text{with} \quad \lambda_1 + \cdots + \lambda_n = k. \tag{1}$$

Since there are $k!/\lambda_1! \cdots \lambda_n!$ contributions from g of this type, it follows that

$$h_n = \sum_{k=1}^{n} f_k \sum_{\lambda} \frac{n!}{\lambda_1! \cdots \lambda_n!} \frac{g_1^{\lambda_1} \cdots g_n^{\lambda_n}}{(1!)^{\lambda_1}\,(2!)^{\lambda_2} \cdots (n!)^{\lambda_n}} =: \sum_{k=1}^{n} f_k\, B_{n,k}(g_1, \ldots, g_{n+1-k}), \tag{2}$$

where the sum $\sum_\lambda$ runs over the sequences $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^{\mathbb{N}}$ satisfying (1), and the $B_{n,k}$ are called the (partial, exponential) *Bell polynomials*. Usually these are introduced by the expansion

$$\exp\biggl(u \sum_{m\geq 1} x_m \frac{t^m}{m!}\biggr) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \biggl[\sum_{k=1}^{n} u^k B_{n,k}(x_1, \ldots, x_{n+1-k})\biggr],$$

which is a particular case of (2). Each $B_{n,k}$ is a homogeneous polynomial of degree k. (This is a good moment to declare that the scalar field $\mathbb{R}$ may be replaced by any commutative field of characteristic zero.)

Formula (2) can be recast as

$$h^{(n)}(t) = \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1! \cdots \lambda_n!} f^{(k)}(g(t)) \biggl(\frac{g^{(1)}(t)}{1!}\biggr)^{\lambda_1} \biggl(\frac{g^{(2)}(t)}{2!}\biggr)^{\lambda_2} \cdots \biggl(\frac{g^{(n)}(t)}{n!}\biggr)^{\lambda_n}. \tag{3}$$

Expression (3) is the famous formula attributed to Faà di Bruno (1855, 1857), who in fact followed previous authors; his original contribution was a determinant form of it. Apparently (3) goes back to Arbogast (1800); we refer the reader to [1] —and references therein— for these historical matters. Note that if g, f are differentiable up to order n, then h is also differentiable up to order n, and the expressions for its derivatives hold.

The formula shows that the composition of two real-analytic functions is real-analytic. Indeed, by use of (2) or (3) with $f(t) = \sum_{k=1}^{\infty} t^k = t/(1-t)$ and $g(t) = \sum_{l=1}^{\infty} xt^l = xt/(1-t)$ one sees that

$$\sum_{k=1}^{n} \sum_{\lambda} \frac{k!}{\lambda_1! \lambda_2! \cdots \lambda_n!}\, x^k = x(1+x)^{n-1}. \tag{4}$$

Now, a smooth function g on an open interval $I \subseteq \mathbb{R}$ is analytic [2, Chap. 1] if and only if for each $y \in I$ there is an open interval J_y with $y \in J_y \subseteq I$ and constants A, B such that

$$|g^{(j)}(t)| \leq \frac{A\, j!}{B^j} \quad \text{for all} \quad t \in J_y \tag{5}$$

which guarantees local uniform convergence of the Taylor series of g. Assume further that g takes values in an open interval on which the smooth function f is defined. If f is also analytic with $|f^{(m)}(s)| \leq C\, m!/D^m$ for all m at $s = g(t)$, it follows from (3) and (4) that

$$|h^{(n)}(t)| \leq \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1! \cdots \lambda_n!} \frac{C\,k!}{D^k} \left(\frac{A}{B}\right)^{\lambda_1} \cdots \left(\frac{A}{B^n}\right)^{\lambda_n} = n!\, \frac{C}{B^n} \frac{A}{D} \left(1 + \frac{A}{D}\right)^{n-1} = \frac{E\, n!}{F^n},$$

with $E = AC/(A + D)$ and $F = BD/(A + D)$. Hence $f \circ g$ is analytic on the domain of g.

2 Hopf algebras

Introduce the notation, with (1) understood:

$$\binom{n}{\lambda; k} := \frac{n!}{\lambda_1! \cdots \lambda_n!\, (1!)^{\lambda_1} (2!)^{\lambda_2} \ldots (n!)^{\lambda_n}}.$$

A *Hopf algebra* dual to G is obtained when we define a *coproduct* Δ on the polynomial algebra $\mathbb{R}[a_1, a_2, \ldots]$ of coordinate functions by requiring that $\Delta a_n(g, f) = a_n(f \circ g)$, or equivalently, $a_n(f \circ g) = m(\Delta a_n(g \otimes f))$ where m means multiplication. This entails that

$$\Delta a_n = \sum_{k=1}^{n} \sum_{\lambda} \binom{n}{\lambda; k} a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_n^{\lambda_n} \otimes a_k.$$

The unnecessary flip of f and g is traditional. This *Faà di Bruno bialgebra*, so called by Joni and Rota [3], is commutative but not cocommutative. Since a_1 is a grouplike element, it must be invertible for this to be a Hopf algebra. For that, one must either adjoin an inverse a_1^{-1}, or else put $a_1 = 1$, as we do from now on. That is, we consider only the subgroup G_1 of diffeomorphisms tangent to the identity at 0. The first instances of the coproduct are, accordingly,

$$\begin{aligned}
\Delta a_2 &= a_2 \otimes 1 + 1 \otimes a_2, \\
\Delta a_3 &= a_3 \otimes 1 + 1 \otimes a_3 + 3a_2 \otimes a_2, \\
\Delta a_4 &= a_4 \otimes 1 + 1 \otimes a_4 + 6a_2 \otimes a_3 + (3a_2^2 + 4a_3) \otimes a_2, \\
\Delta a_5 &= a_5 \otimes 1 + 1 \otimes a_5 + 10a_2 \otimes a_4 + (10a_3 + 15a_2^2) \otimes a_3 + (5a_4 + 10a_2 a_3) \otimes a_2. \qquad (6)
\end{aligned}$$

The resulting graded connected Hopf algebra $\mathcal{F}$ is called the *Faà di Bruno Hopf algebra*; the degree # being given by $\# a_n = n - 1$.

Consider the graded dual Hopf algebra $\mathcal{F}'$. Its space of primitive elements has a basis $\{a'_n : n \geq 2\}$ defined by $\langle a'_n, a_m \rangle = \delta_{nm}$ and $\langle a'_n, a_{m_1} a_{m_2} \ldots a_{m_r} \rangle = 0$ for $r > 1$. Their product is given by the duality recipe $\langle b'c', a \rangle := \langle b' \otimes c', \Delta a \rangle$, leading to:

$$a'_n a'_m = \binom{m - 1 + n}{n} a'_{n+m-1} + (1 + \delta_{nm})(a_n a_m)'.$$

In particular, taking $b'_n := (n+1)!\, a'_{n+1}$ for $n \geq 1$, we are left with the commutator relations

$$[b'_n, b'_m] = (m-n) b'_{n+m}. \tag{7}$$

The Milnor–Moore theorem implies that $\mathcal{F}'$ is isomorphic to the enveloping algebra of the Lie algebra $\mathcal{A}$ spanned by the b'_n with these commutators.

A curious consequence of (7) is that the space $P(\mathcal{F})$ of primitive elements of $\mathcal{F}$ just has dimension 2. Indeed, $P(\mathcal{F}) = (\mathbb{R}1 \oplus \mathcal{F}'^{\,2}_{+})^{\perp}$, where $\mathcal{F}'_+$ is the augmentation ideal of $\mathcal{F}'$. But (7) entails that there is a basis of $\mathcal{F}'$ made of products, except for its first two elements: therefore, $\dim P(\mathcal{F}) = 2$. A basis of $P(\mathcal{F})$ is given by $\{a_2, a_3 - \frac{3}{2}a_2^2\}$. The second of these corresponds to the Schwarzian derivative, which is known [4] to be invariant under the projective group $PSL(2, \mathbb{R})$. Nonexistence of more primitive elements of $\mathcal{F}$ is related to the affine linear and Riccati equations being the only Lie–Scheffers systems [5, 6] over the real line.

The Faà di Bruno Hopf algebra $\mathcal{F}$ reappears as the maximal commutative Hopf subalgebra of the (noncommutative geometry) Hopf algebra H of Connes and Moscovici [7]. Their description of $\mathcal{F}$ uses a different set of coordinates $\delta_n(f) := [\log f'(t)]^{(n)}(0)$, $n \geq 1$. Since

$$h(t) := \sum_{n\geq 1} \delta_n(f) \frac{t^n}{n!} = \log f'(t) = \log\left(1 + \sum_{n\geq 1} a_{n+1}(f)\, \frac{t^n}{n!}\right),$$

it follows from formula (2), for logarithm and exponential functions respectively, that

$$\delta_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)!\, B_{n,k}(a_2, \dots, a_{n+2-k}) =: L_n(a_2, \dots, a_{n+1}),$$

inverted by

$$a_{n+1} = \sum_{k=1}^{n} B_{n,k}(\delta_1, \dots, \delta_{n+1-k}) =: Y_n(\delta_1, \dots, \delta_n),$$

where the L_n and the Y_n are respectively called the *logarithmic polynomials* and the (complete, exponential) *Bell polynomials*. In this way we get $\delta_1 = a_2$; $\delta_2 = a_3 - a_2^2$; $\delta_3 = a_4 - 3a_2a_3 + 2a_2^3$; $\delta_4 = a_5 - 3a_3^2 - 4a_2a_4 + 12a_2^2a_3 - 6a_2^4$; and so on. Since the coproduct is an algebra morphism, by use of (6) we may obtain the coproduct in the Connes–Moscovici coordinates. For instance,

$$\Delta\delta_4 = \delta_4 \otimes 1 + 1 \otimes \delta_4 + 6\delta_1 \otimes \delta_3 + (7\delta_1^2 + 4\delta_2) \otimes \delta_2 + (3\delta_1\delta_2 + \delta_1^3 + \delta_3) \otimes \delta_1.$$

It is not easy to find a closed formula for $\Delta(\delta_n)$ directly from (6). Fortunately, through $\mathcal{F}'$ another method is available. Using $B_{n,1}(a_2, \dots, a_{n+1}) = a_{n+1}$, one finds that $\langle b'_n, \delta_m \rangle = (n+1)!\, \delta_{n,m}$. Let A be the graded free Lie algebra generated by primitive elements X_n, $n \geq 1$. Its enveloping algebra $\mathcal{U}(A)$ is the *concatenation Hopf algebra*. A linear basis for

$\mathcal{U}(A)$, indexed by all vectors with positive integer components $\bar{n} = (n_1, \dots, n_r)$, is made of products $X_{\bar{n}} := X_{n_1} X_{n_2} \dots X_{n_r}$, together with the unit element $X_\emptyset = 1$. Its coproduct is

$$\Delta(X_{\bar{n}}) := \sum_{\bar{n}^1, \bar{n}^2} \mathrm{sh}_{\bar{n}}^{\bar{n}^1, \bar{n}^2} X_{\bar{n}^1} \otimes X_{\bar{n}^2},$$

with $\mathrm{sh}_{\bar{n}}^{\bar{n}^1, \bar{n}^2}$ denoting the number of shuffles of the vectors $\bar{n}^1$, $\bar{n}^2$ that produce $\bar{n}$. Let $u^{\bar{n}}$ denote a dual basis to $X_{\bar{n}}$; the graded dual of $\mathcal{U}(A)$ is the *shuffle Hopf algebra H* with product and coproduct respectively given by

$$u^{\bar{n}^1} u^{\bar{n}^2} := \sum_{\bar{n}} \mathrm{sh}_{\bar{n}}^{\bar{n}^1, \bar{n}^2} u^{\bar{n}}, \qquad \Delta(u^{\bar{n}}) := \sum_{\bar{n}^1 \bar{n}^2 = \bar{n}} u^{\bar{n}^1} \otimes u^{\bar{n}^2},$$

where $\bar{n}^1 \bar{n}^2$ is the concatenation of the vectors $\bar{n}^1$, $\bar{n}^2$. The surjective morphism $\rho \colon A \to \mathcal{A}$ defined by $\rho(X_n) := b'_n$ extends, by the universal property of enveloping algebras, to a surjective morphism $\rho \colon \mathcal{U}(A) \to \mathcal{F}'$, whose transpose is the injective Hopf map $\rho^t \colon \mathcal{F} \to H$ given by $\delta_n \mapsto \Gamma_n := \delta_n \circ \rho$. We may thus regard $\mathcal{F}$ as a Hopf subalgebra of H, and thereby compute the coproduct of $\mathcal{F}$ from that of H. The argument may look circular, since we seem to need an expression for the Γ_n, which in turn requires computing $\Delta(\delta_n)$. But we can write

$$\langle \Gamma_m, X_{\bar{n}} \rangle = \langle \delta_m, \rho(X_{\bar{n}}) \rangle = \langle \delta_m, b'_{n_1} \cdots b'_{n_r} \rangle = \langle \Delta(\delta_m), b'_{n_1} \otimes b'_{n_2} \cdots b'_{n_r} \rangle. \tag{8}$$

Thus, to compute Γ_n, the only terms we need in the expansion of $\Delta(\delta_n)$ are $\delta_n \otimes 1 + 1 \otimes \delta_n$ and the bilinear terms, namely multiples of $\delta_i \otimes \delta_j$; the remaining terms are of the form (constant) $\delta_{i_1}^{r_1} \cdots \delta_{i_k}^{r_k} \otimes \delta_j$, where $r_1 i_1 + \cdots + r_k i_k + j = n$. The bilinear part $B(\delta_n)$ may be computed by induction [7] to be

$$B(\delta_n) = \sum_{i=1}^{n-1} \binom{n}{i-1} \delta_{n-i} \otimes \delta_i. \tag{9}$$

Substituting (9) repeatedly in (8), one obtains

$$\Gamma_n = n! \sum_{\bar{n} : n_1 + \cdots + n_r = n} C^{\bar{n}} u^{\bar{n}} \quad \text{with coefficients} \quad C^{\bar{n}} := (n_r + 1) \prod_{i=2}^{r} (n_i + \cdots + n_r).$$

For instance, $\Gamma_1 = 2u^1$ and $\Gamma_3 = 12(2u^3 + u^{(2,1)} + 3u^{(1,2)} + 2u^{(1,1,1)})$.

Another calculation of the Γ_n was sketched in [8]; it eventually allows to improve (9) to

$$\Delta(\delta_n) = \delta_n \otimes 1 + 1 \otimes \delta_n + \sum_{\bar{n} \in N_n} \frac{n!}{n_1! \dots n_r!} K_{n_r}^{n_1, \dots, n_{r-1}} \delta_{n_1} \dots \delta_{n_{r-1}} \otimes \delta_{n_r},$$

where $N_n := \{\, \bar{n} : n_1 + \cdots + n_r = n,\ r > 1 \,\}$ and, mindful that $\binom{n_r}{k} = 0$ when $n_r < k$,

$$K_{n_r}^{n_1, \dots, n_{r-1}} = \sum_{k=1}^{r-1} \binom{n_r}{k} \sum_{\bar{n}^1 \cdots \bar{n}^k = (n_1, \dots, n_{r-1})} \frac{1}{r^1! \dots r^k!} \prod_{i=1}^{k} \frac{1}{1 + n_1^i + \cdots + n_{r^i}^i}.$$

For $r = 2$, this becomes $K_i^{n-i} = \frac{i}{1+n-i}$, thus the coefficient of $\delta_{n-i} \otimes \delta_i$ is $\binom{n}{i}\frac{i}{1+n-i} = \binom{n}{i-1}$, as in (9).

From the combinatorial viewpoint, the Faà di Bruno Hopf algebra is the *incidence Hopf algebra* corresponding to intervals formed by partitions of finite sets. This is no surprise, since the coefficients of a Bell polynomial $B_{n,k}$ just count the number of partitions of $\{1, \dots, n\}$ into k blocks. A *partition* $\pi \in \Pi(S)$, of a finite set S with n elements, is a collection $\{B_1, B_2, \dots, B_k\}$ of nonempty disjoint subsets, called *blocks*, such that $\bigcup_{i=1}^k B_i = S$. We simply write $\pi \vdash n$ for such, with $|\pi|$ being the number of blocks in π.

We say that π is of *type* $(\alpha_1, \dots, \alpha_n)$ if exactly α_i of these B_j have i elements; thus $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ and $\alpha_1 + \cdots + \alpha_n = k$ [9]. We say that π *refines* τ, and write $\{A_1, \dots, A_n\} = \pi \leq \tau = \{B_1, \dots, B_m\}$, if each A_i is contained in some B_j. A subinterval $[\pi, \tau] = \{\sigma : \pi \leq \sigma \leq \tau\}$ of the lattice $\mathcal{P}$ of partitions of finite sets is isomorphic to the poset $\Pi_1^{\lambda_1} \times \cdots \times \Pi_n^{\lambda_n}$, where $\Pi_j := \Pi(\{1, \dots, j\})$ and λ_i blocks of τ are unions of exactly i blocks of π. One assigns to each interval the sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ and declares two intervals in $\mathcal{P}$ to be equivalent when their vectors λ are equal. From the matching $\widetilde{[\pi, \tau]} \leftrightarrow \lambda \leftrightarrow \widetilde{\Pi}_1^{\lambda_1}\widetilde{\Pi}_2^{\lambda_2} \cdots \widetilde{\Pi}_n^{\lambda_n}$ of equivalence classes, one may regard the family $\widetilde{\mathcal{P}}$ of equivalence classes as the algebra of polynomials of infinitely many variables $\mathbb{R}[\widetilde{\Pi}_1, \widetilde{\Pi}_2, \dots]$. By means of the general theory of coproducts for incidence bialgebras [10] one then recovers the Faà di Bruno algebra under the identifications $a_n \leftrightarrow \widetilde{\Pi}_n$. The cardinality in the sense of category theory [11] of the groupoid of finite sets equipped with a partition is given by

$$\sum_{n=0}^{\infty} \sum_{k=1}^{n} \frac{1}{n!} B_{n,k}(1, \dots, 1) = e^{e-1}.$$

The *characters* of $\mathcal{F}$ form a group $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ under the convolution operation of Hopf algebra theory. The action of a character f is determined by its values on the a_n. The map $f \mapsto f(t) = \sum_{n=1}^{\infty} f_n t^n/n!$, where $f_n := \langle f, a_n \rangle$, matches characters with exponential power series over $\mathbb{R}$ such that $f_1 = 1$. This correspondence is an *anti-isomorphism* of groups: indeed, the convolution $f * g$ of $f, g \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{F}, \mathbb{R})$ is given by

$$\langle f * g, a_n \rangle := m(f \otimes g)\Delta a_n = \langle g \circ f, a_n \rangle.$$

This is just the nth coefficient of $h(t) = g(f(t))$. Also, the algebra endomorphisms $\mathrm{End}_{\mathrm{alg}}(\mathcal{F})$ form a group under the convolution of the unital algebra $\mathrm{End}(\mathcal{F})$ of linear endomorphisms.

The inverse under functional composition of an exponential series is given by the reversion formula of Lagrange [12], one of whose forms [13] states that if f and g are two such series and if $f_1 = 1$, $f \circ g(t) = g \circ f(t) = t$, then

$$g_n = \sum_{k=1}^{n-1} (-1)^k B_{n-1+k,k}(0, f_2, f_3, \dots). \tag{10}$$

Now, the inverse under convolution of $f \in \mathrm{End}_{\mathrm{alg}}(\mathcal{F})$ is $g = f \circ S$, with S the antipode map of $\mathcal{F}$. The multiplicativity of f forces

$$S(a_n) = \sum_{k=1}^{n-1} (-1)^k B_{n-1+k,k}(0, a_2, a_3, \dots).$$

One may reverse the roles and prove the combinatorial identity (10) from Hopf algebra theory [14]. The real-analytic inverse function theorem is stated in a completely similar way to the standard inverse function theorem for differentiable functions, and it can be proved by use of the formula of Faà di Bruno, with the help of the estimates (5).

Use of partitions with special properties may lead to other incidence algebras: for instance, if we restrict to noncrossing partitions, we obtain a cocommutative Hopf algebra, with the commutative group operation on characters essentially corresponding to Lagrange reversion of the Cauchy product of reverted series [15].

3 Faà di Bruno formulas in several variables

To go to higher Faà di Bruno formulas means to consider exponential N'-series in N' variables ("colours") of the form

$$f(t_1, \dots, t_{N'}) = \left(t_1 + \sum_{|\bar{m}|>1} f^1_{\bar{m}} \frac{t^{\bar{m}}}{\bar{m}!}, t_2 + \sum_{|\bar{n}|>1} f^2_{\bar{n}} \frac{t^{\bar{n}}}{\bar{n}!}, \dots, t_{N'} + \sum_{|\bar{p}|>1} f^{N'}_{\bar{p}} \frac{t^{\bar{p}}}{\bar{p}!} \right), \tag{11}$$

where $\bar{m}, \bar{n}, \dots, \bar{p} \in \mathbb{N}^{N'}$. The simplest way to go about this is to rewrite Eq. (3) in terms of partitions:

$$(f \circ g)^{(n)}(t) = \sum_{\pi \vdash n} f^{(|\pi|)}(g(t)) \prod_{l \in \pi} g^{(|l|)}(t), \tag{12}$$

where the coefficients $\binom{n}{\lambda;k}$ of (3) count partitions yielding equal summands. (Note that $\sum_{|\lambda|=k} \binom{n}{\lambda;k} = \left\{ n \atop k \right\}$ is the Stirling number of the second kind [16, Chap. 6].) For instance, for $n = 4$ one immediately finds:

$$\begin{aligned}(f \circ g)^{(\mathrm{iv})}(t) = f'(g(t))\, g^{(\mathrm{iv})}(t) + 4f''(g(t))\, g'(t) g'''(t) + 3f'''(g(t))\, g''(t)^2 \\ + 6f'''(t)\, g'(t)^2 g''(t) + f^{(\mathrm{iv})}(t)\, g'(t)^4.\end{aligned}$$

The above can be generalized to formal series in several indeterminates as follows. Consider the set of maps from $\{1, \dots, n\}$ to a set of *colours* $\{1', \dots, N'\}$. This allows consideration of coloured partitions of $\{1, \dots, n\}$, with monocoloured partitions being of the same type as before. There are m families of those series, one for each colour, with tangency at the identity being enforced by:

$$\partial_{t_i} f^i(0, \dots, 0) = 1, \quad \partial_{t_j} f^i(0, \dots, 0) = 0 \text{ for } j \neq i; \quad \text{with} \quad i, j \in \{1', \dots, N'\}.$$

Then the very formula (12) is valid, provided we understand now $|l|$ as a vector of colours. A fully multivariate treatment in this vein is provided by [17]. See the simplest case $N' = 2$ in the foreword to the book [18] – this book contains much interesting information besides.

The series (11) can be regarded as characters of "coloured" Faà di Bruno Hopf algebras $\mathcal{F}(N')$ [14]. For any finite set X gifted with a colouring map $\theta\colon X \to \{1', \dots, N'\}$, one considers partitions π whose sets of blocks are also coloured, provided $\theta(\{x\}) = \theta(x)$ for singletons. Such coloured partitions form a poset, with $\pi \le \rho$ if π refines ρ as partitions, and if $\theta_\pi(B) = \theta_\rho(B)$ for each block B of π which is also a block of ρ; this condition entails that ρ induces a coloured partition $\rho \,|\, \pi$ of the set of blocks of π. Coloured partitions π of X with $\theta(X) = r$ (i.e., the one-block partition of X is assigned the colour r) form a poset $\Pi^r_{\bar n}$ where $\bar n \in \mathbb{N}^{N'}$ counts the colours of its elements; their types $\widetilde{\Pi}^r_{\bar n}$ generate the Hopf algebra $\mathcal{F}(N')$, with coproduct given by:

$$\Delta\widetilde{\Pi}^r_{\bar n} := \sum_{\pi \in \Pi^r_{\bar n}} \biggl(\prod_{B \in \pi} \widetilde{\Pi}^{\theta(B)}_{|B|} \biggr) \otimes \widetilde{\Pi}^r_{|\pi|}.$$

A character f of $\mathcal{F}(N')$ is specified by its values on algebra generators $f^r_{\bar n} = f(\widetilde{\Pi}^r_{\bar n})$, which yield coefficients of the N'-series (11). The convolution of two such characters g, f has coefficients

$$g * f\bigl(\widetilde{\Pi}^r_{\bar n}\bigr) = \sum_{\pi \in \Pi^r_{\bar n}} f^r_{|\pi|} \prod_{B \in \pi} g^{\theta(B)}_{|B|} = \sum_{|\bar k| \le |\bar n|} f^r_{\bar k} / \bar k! \prod_{(B_1, \dots, B_{|\bar k|})} g^1_{|B_1|} \cdots g^{N'}_{|B_{|\bar k|}|}$$

where the second product ranges over *ordered* coloured partitions of a set with $|\bar n|$ elements; since there are $\bar n! \big/ \prod_i (\bar m_i!)^{\lambda_{i,\bar m_i}}$ of these with prescribed colours, rearrangement of the right hand side yields a formula for (11). Thus, the character group of $\mathcal{F}(N')$ is anti-isomorphic to the group of N'-series like (11) under composition. Also, the antipode on $\mathcal{F}(N')$ provides Lagrange reversion in several variables [14].

For the applications of Faà di Bruno algebras to real analytic function theory we recommend [2]. The Faà di Bruno algebras (perhaps involving functional derivatives) have applications in quantum field theory. Some elementary ones are described in [10]. Deeper ones related to renormalization theory were broached in [19, 20], and further explored in [21]. A nice, relatively recent work in this respect is [22].

Much remains to be delved into. Noncommutative Bell polynomials were studied in [23]. Faà di Bruno's formulas for operads have been developed in [24]. An application to control theory is found in [25]. A recent comprehensive treatise on Bell polynomials and generalized Lagrange inversion [26] is also recommended.

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