UNIVERSIDAD DE COSTA RICA
SISTEMA DE ESTUDIOS DE POSGRADO

EXACT COSMOLOGICAL SOLUTIONS AND THEIR STABILITY FOR A NON-LINEAR SCALAR FIELD EQUIVALENT TO A MIXTURE OF DARK ENERGY, DUST, AND STIFF MATTER IN A SYMMETRY OF PETROV TYPE D.

Tesis sometida a la consideración de la Comisión del Programa de Estudios de Posgrado de Física para optar al grado y título de Maestría Académica en Física

ANDRÉS ALEXIS ANGULO SIBAJA

Ciudad Universitaria Rodrigo Facio, Costa Rica

2024

Dedication

To my family—Noilyn, Alexis, and Gabriel—I am deeply grateful for your unwavering support throughout my academic years. Words cannot fully capture the depth of my appreciation, but please know that my love for you all is immeasurable.

Acknowledgements

I would like to thank my advisor, Prof. Rodrigo Alvarado, for his guidance and kindness, which were instrumental in the successful completion of this thesis.
I also wish to express my gratitude to Professors Francisco Frutos and David Solano for their valuable recommendations.
Finally, I am grateful to Prof. Manuel Ortega for sharing his knowledge with me, which has greatly inspired and motivated me to continue my academic career in physics.

Esta tesis fue aceptada por la Comisión del Programa de Posgrado en Física de la Universidad de Costa Rica, como requisito parcial para optar al grado y título de Maestría Académica en Física.

Dra. Heidy Gutiérrez Garro
Representante de la Decana
Sistema de Estudios de Posgrado

Dr. Rodrigo Alvarado Marín
Director de Tesis

Dr. Francisco Frutos Alfaro
Asesor

M.Sc. David Solano Solano
Asesor

Dr. Jorge Gutiérrez Camacho
Director
Programa de Posgrado en Física

Andrés Alexis Angulo Sibaja
Sustentante

Contents

Resumen

En este trabajo se encuentran soluciones cosmológicas exactas en una simetría de Petrov tipo D para un campo escalar equivalente a una mezcla de tres fluidos perfectos: energía oscura, polvo y materia rígida. Se estudia la evolución dinámica del universo y determinan los parámetros cosmológicos y singularidades con el escalar de Kretschmann. Finalmente, con la teoría de Kosambi-Cartan-Chern, se calcula la estabilidad de Jacobi de la cosmología con el campo escalar para analizar la robustez del modelo ante desviaciones exponenciales de trayectorias cercanas.

Abstract

In this work, we found exact cosmological solutions in a Petrov type D symmetry for a scalar field equivalent to a mixture of three perfect fluids: dark energy, dust, and stiff matter. We study the dynamical evolution of the universe and determine the cosmological parameters and singularities with the Kretschmann scalar. Finally, by applying the Kosambi-Cartan-Chern theory, we calculate the Jacobi stability of the cosmological model with a scalar field and determine its robustness to exponential deviations from nearby trajectories.

List of Tables

List of abbreviations

FLRW: Friedman-Lemaître-Robertson-Walker
ΛCDM: Lambda-Cold Dark Matter
CMB: Cosmic Microwave Background
EoS: Equation of State
GR: General Relativity
SET: Stress-Energy Tensor
EFE: Einstein's Field Equations
QFT: Quantum Field Theory
NP: Newman-Penrose
PND: Principal Null Direction
KCC: Kosambi-Cartan-Chern

Chapter 1

Introduction

1.1 Introduction

The advances in cosmology in the last centuries, such as the discovery of the existence of other galaxies, the expansion of the universe first seen by Hubble, the invention of a successful theory of gravitation by Albert Einstein, and the possible existence of dark matter, have positioned the field as a fruitful area of physics research.

In this work, we study the exact cosmological solutions of an anisotropic and homogeneous universe with a scalar field representing a mixture of three perfect fluids: dark energy, dust, and stiff matter, also known as the Zeldovich fluid. The scalar field dynamics in cosmological models have been powerful tools in developing theories like inflation [36]. On the other hand, there are extensive studies related to the presence of perfect fluids that filled the universes [8, 14, 52, 62], which represent an essential source of matter in cosmological models.

The solution under study is a Petrov type D symmetry; the derivation of this line element and the exact solutions for relevant perfect fluids are present in [5]. In the case of scalar fields, the author in [8] found two exact solutions and their stability for a fluid with constant pressure in a universe with the Petrov type D space-time and another one with a flat FLRW universe. Using the same Petrov type D symmetry, [7] analyzes a scalar field with a type +cosh potential that causes the universe to be dominated by dark energy at early times and stiff matter at late times. The work in [9] found solutions for an interaction of a scalar field with a spinorial field, equivalent to a model with dark energy and a primordial magnetic field.

In the case of anisotropic and homogeneous universes with scalar fields, for example, the study of Bianchi I symmetry is developed in [2, 38, 48, 72, 74], and for other types of Bianchi in [17, 20, 31, 32, 66, 68]. Particularly, when studying scalar field and fluids, the authors in [74] first defined a scalar field in a Bianchi I model; by setting the potential to zero they obtained a universe filled with stiff matter. This procedure to find solutions is very common and helpful to study a wide range of models with distinct potentials. However, when it comes to studying the universe's matter content, it is harder to interpret which fluids are present, so we first establish the mixture of fluids to know the matter content in the universe and then find the scalar field and the potential.

For a mixture of three fluids in our Petrov D symmetry, the work in [10] found the solutions with quintessence, dust, and radiation, in [12] with a mixture of radiation and dark energy, and in [11] with a mixture of dark energy, dust, and Zeldovich with a primordial and non-linear magnetic field. In the case of this thesis, we find the solution of a scalar field equivalent to a mixture of three fluids: dark energy, dust, and stiff matter.

Due to the non-linearity of the dynamical system, it is important to determine the stability of the system [31], so we are going to dedicate a part of the theoretical framework to comprehend the tools used to evaluate the stability, this kind of work has already been done in [8].

1.1.1 Challenges of the CP and other cosmological problems

The model of cosmology ΛCDM successfully explains a wide range of cosmological observations and has become the standard theoretical framework in physics to explain the composition and evolution of the Universe. However, several observational discrepancies are beginning to challenge the paradigm set by ΛCDM of a homogeneous and isotropic universe.

The CP requires that the largest structures of the Universe such as clusters, filaments, and voids have an upper limit to obey large-scale homogeneity. This limit is currently estimated by ΛCDM simulations to be $\simeq 370$ Mpc [4, 85]. Structures that are exceptionally large and exceed this upper bound have already been discovered. The authors in [25] found a large group of quasars called U1.27 that measures about ~ 500 Mpc at a distance of $z \sim 1.3$. More recently [55] noticed a large filamentary and crescent-shaped structure of about ~ 1 Gpc at a redshift of $z \sim 0.8$ validated by three different statistical analyses [55]. Other more well-known structures like the Hercules–Corona Borealis Great Wall, the largest one so far, have been hard to reconcile with large-scale homogeneity, and there is expected for more large-structure to be found at higher z with the improvements in precision cosmology [4].

The so-called "axis of evil" is a mysterious alignment on the distribution of temperatures anisotropies of the CMB, confirmed with data from the rotations of galaxies [73] and observations of quasar polarizations [43]. The dipole anomaly $l = 1$ is caused by the motion of the solar system with respect to the CMB; coincidentally there are approximately about 10 degrees between its direction and the ecliptic plane. To add to this, the quadrupole $l = 2$ and the octupole $l = 3$ seem to align in a plane that is perpendicular to the ecliptic plane [4]; these observations show that there seems to be a preference reference frame that favors our position in the Universe, which is against the Copernican Principle and the CP.

Recent works [58] using new methods with X-rays from galaxy clusters as standard candles have allowed to estimate values of the expansion rate H_0 from the calibration of the luminosity vs. temperature curve and the observation of the brightness in the galaxies, by obtaining the distance to these galaxies. This method is another way to determine H_0 and compare it with the values obtained in studying different galaxy clusters. The [59] is one of the most recent studies. They estimated the redshift value by measuring the properties of 570 galaxy clusters, and with the help of X-Ray measurements for the distance to the galaxies a variation of 9% was obtained for H_0 around $(l, b) \sim (250^{\circ}{}^{+35^{\circ}}_{-35^{\circ}}, -15^{\circ}{}^{+25^{\circ}}_{-25^{\circ}})$, with a statistical significance of $> 5\sigma$.

Other pieces of evidence that challenge the CP include statistical anomalies between the southern and northern hemispheres of the CMB, the mirror parity anomaly of the CMB, bulk flows, observed dipoles in the distributions of radio galaxies, quasars, and type Ia supernovae, alignments in the distribution of large quasars and preference in the direction of rotation of galaxies. The authors in a recent work [4] review extensively all these increasing numbers of observations and other relevant signatures related to deviations from the CP.

Studies of cosmological models with different fluids have failed to satisfactorily explain how the universe's initial conditions have led to the structures observed today. Especially we can mention the problems in the apparent curvature of space-time; according to observations [27, 30], the spatial curvature of the universe is approximately flat, which means that if we take the Friedmann equation $\frac{k^2}{a^2H^2} = 1 - \Omega$, the density ρ in $\Omega = \rho/\rho_c$ must be very close to the critical density ρ_c, since otherwise there would be significant discrepancies in the properties of the universe, such as its accelerated expansion or the distribution of matter. Given this apparent coincidence, some cosmologists argue that the classical model needs a very accurate fine-tuning of the fluid energy densities. However, this position is highly debated today to the point of discussing whether this problem exists or not [41, 81].

The standard model also fails to explain why if we try to calculate the comoving distance of two points of the CMB that are sufficiently separated, we can obtain distances larger than the cosmological horizon [44]. Which infers that there were regions of the universe that have always been casually disconnected, an argument that contradicts the thermal equilibrium of the entire universe and, therefore, the homogeneity in the temperature distribution of $T = (2.7 \pm 10^{-5})$K of the CMB [50]. In the face of these and other problems, such as the existence of magnetic monopoles, the best known and accepted theory today is to propose an epoch of inflation [36] in the early universe where an exponential expansive growth occurred driven by the vacuum energy of a scalar field [77].

Scalar fields have proven to be quite versatile and viable. They are crucial in developing particle physics and cosmology since they allow us to explain several necessary mechanisms, such as the spontaneous symmetry breaking that gives mass to [65] particles. In addition, dark energy can be modeled as a dynamic, time-varying scalar field known as quintessence [75], and [56, 78] proposed to study dark matter by considering different scalar potentials. Its use has spread across several areas of cosmology and has served to study mechanisms that can explain the possible early inflation, late acceleration of the universe, and the dark sector in cosmology.

We also have to notice that the presence of the scalar field causes non-linearity in the equations of motion. This non-linearity means that the solutions might predict considerable deviations in the face of small perturbations. To address this issue, we present the theory of Jacobi stability that allows us to know the robustness of the model against chaotic trajectories throughout its evolution [15]. The usefulness of studying the stability of the model lies in the fact of being able to determine how the dynamic system evolves under particular initial conditions and its viability for observational studies [28].

Finally, it is worth mentioning that efforts to investigate these cosmological models provide vital information about a more fundamental nature of the Universe within the limitations of our system of knowledge. Because of the number of questions in this field, current theoretical research aims to answer as many of these questions as possible in a single general model. Hence, exploration of novel cosmological scenarios capable of elucidating observational findings is paramount for advancing scientific knowledge and understanding of our Universe.

1.2 Chapter Summaries

In Chapter 2, we introduce modern cosmology through the most accepted theory of gravity, published in 1916 by Albert Einstein. The first section introduces GR, its assumptions, and the most critical aspects of its formalism for this work. Sections 2.2 and 2.3 are in charge of developing the theory with the EoS for the three fluids studied here: stiff matter, dark energy, and dust. Once the present matter of the universe is known, an intuition for analyzing the dynamical evolution is developed in Sections 2.4 and 2.5. Section 2.6 introduces a scalar field minimally coupled with gravity with some possible physical interpretations. In Section 2.7 we briefly review the Hubble, the deceleration parameters, and their importance for cosmological observations.

Chapter 3 discusses the symmetry of the metric studied for the rest of this work. The first two sections summarize the theory necessary to understand the Petrov classification that describes the symmetries of the solutions to the EFE. We choose the NP formalism to establish the classification, so this formalism is introduced in Section 3.2. The symmetries of the Weyl Tensor that allow the Petrov type to be classified are explained in Section 3.3, and finally, the introduction of our Petrov D symmetry is developed in Section 3.4.

For Chapter 4, Section 4.1 consists of deriving some relevant results from the Petrov Type D metric with the theory introduced in Chapter 2 of cosmology. Afterward, in Section 4.2, we calculate the Hubble and deceleration parameters for the Bianchi type I universe and make a simple change of variables to get the results with our Petrov type D symmetry. Finally, in Section 4.3, the study of the Kretschmann invariant is developed, which is necessary to find the possible singularities of the model.

In Chapter 5, the first section introduces the Jacobi stability with a proper generalization through the theory of KCC. The last two sections will treat the particular cases of FLRW and the solution with Petrov type D symmetry of this work.

Chapter 6 presents the solutions with an analysis of the total mixture of fluids. The first section contains the solutions for the scale factors and the scalar field in terms of the fluid parameters. We compare the results with the ones obtained in early works with similar scenarios. Section 6.2 concerns the cosmological parameters already presented in the previous chapter, and Section 6.3 covers the Jacobi Stability for the model. In Chapter 7 we conclude.

Chapter 2

Standard Cosmology through Classical Field Theory

2.1 Einstein field equations

The theory we use to study the cosmological properties of the proposed universe is classical GR. The first fundamental assumption the theory must satisfy is that the laws of physics are invariant to the choice of any reference frame. Hence, tensor equations are the mathematical framework of the theory. Second is Einstein's great intuition of establishing a theory that complies with the Equivalence Principle, commonly divided into three principles: Einstein's Equivalence Principle, the Weak Principle, and the Strong Principle. Third, the theory has to satisfy Newtonian gravity in the low energy limit, and finally, the SET has to be locally conserved.

The EFE are a generalization to the Poisson Equation. However, the gravitational field is now an effect of the tidal forces in the geodesic trajectories of particles in a space-time represented by a pseudo-Riemannian variety. Like the Classical Poisson Theory ($\nabla\phi = 4\pi G\rho$), the metric tensor in the EFE must be coupled to the matter present in the universe, distorting the geometry of space-time, such equations come out of the following action [29]

$$S = \int_\Omega (\mathcal{L}_g + \kappa\mathcal{L}_m)d\Omega, \tag{2.1}$$

where $\mathcal{L}_g$ represents the Lagrangian density that describes the geometry of space-time for a region Ω of the manifold and $\mathcal{L}_m$ the Lagrangian density of matter coupled by a constant κ. One way to obtain the field equations is through the variational principle, by considering a variation of the action given by

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.2}$$

where $g_{\mu\nu}$ is the metric tensor that describes the geometry of the manifold. If $S \to S + \delta S$ and assuming Hamilton's Principle with the restriction that $g_{\mu\nu}$ vanishes at the boundary $\partial\Omega$, we have

$$\delta S = \int_{\Omega} \left(\frac{\delta \mathcal{L}_g}{\delta g_{\mu\nu}} + \kappa \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}} \right) \delta g_{\mu\nu} d\Omega = 0, \tag{2.3}$$

The derivatives inside the parentheses are the functional derivatives, equal to the Euler-Lagrange equations for $g_{\mu\nu}$ as a functional.

The geometric Lagrangian density is given by $\mathcal{L}_g = (-g)^{1/2} R$, where g is the determinant of the metric tensor and R the Ricci scalar that quantifies the curvature in the manifold. The integral of this lagrangian is also known as the Einstein-Hilbert action. Usually, an extra term Λ is added to $\mathcal{L}_g$ to include the expansion of the universe given by dark energy; for now, we ignore it, but we will see later how this term is equivalent to a fluid with negative pressure, and this will be more useful when understanding our mixture of the three fluids. The functional derivative of this Lagrangian is given by

$$\frac{\delta \mathcal{L}_g}{\delta g_{\mu\nu}} = -(-g)^{1/2} G^{\mu\nu}, \tag{2.4}$$

where $G_{\mu\nu}$ is the Einstein tensor with the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.5}$$

Here $R_{\mu\nu}$ is the Riemann tensor connected directly to the thermodynamic variables describing the matter in the SET, as shown below.

The SET specifies the matter present in the universe and is defined as

$$T^{\mu\nu} = (-g)^{-1/2} \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu}}, \tag{2.6}$$

which ultimately leads to the EFE

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{2.7}$$

From (2.5), for an empty-space solution ($\mathcal{L}_m = 0$), the Einstein tensor has to be equal to zero $G^{\mu\nu} = 0$. By manipulating the previous expressions with the metric tensor $g_{\mu\nu}$ we can arrive to

$$R^{\nu}{}_{\gamma} - \frac{1}{2}\delta^{\nu}{}_{\gamma}R = \kappa T^{\nu}{}_{\gamma}, \tag{2.8}$$

where we used that $g_{\gamma\mu}g^{\mu\nu} = \delta^{\nu}{}_{\gamma}$ and taking the trace with $R = R^{\gamma}{}_{\gamma}$ and $T = T^{\gamma}{}_{\gamma}$ we have that $R = -\kappa T$, so we can write the Ricci tensor in terms of only the SET

$$R^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} - \frac{1}{2}\kappa\delta^{\mu}{}_{\nu}T. \tag{2.9}$$

To determine the coupling constant, we use the condition that the theory has to be reduced to Newton's gravity in the low energy limit, obtaining $8\pi G/c^4 \sim 10^{-41}N^{-1}$; notably, this value is much smaller than the other coupling constants present in the standard model.

For the last condition, the conservation of energy, by Noether's theorem, we can infer that the invariance of the action under space-time translations will require the conservation of a Noether current. This current can be deduced by analyzing the equation (2.3) and noting that in zero curvature (Minkowski space-time), the Einstein tensor is zero, performing the standard trick of integration by parts with the cancellation of the surface integral, we have

$$\nabla_{\nu}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}} = 0, \tag{2.10}$$

and with the equation (2.6)

$$\nabla_{\nu}T^{\mu\nu} = \partial_{\nu}T^{\mu\nu} = 0, \tag{2.11}$$

which means that the conserved current is the SET. On the other hand, if we do not consider the Lagrangian density of matter, by the same procedure above, we obtain that $\nabla_{\nu}G^{\mu\nu} = 0$, which would be the contracted Bianchi Identities. Here, we can already observe an essential fact, if we consider the complete equation (2.3), due to the fundamental property of the theory given by the diffeomorphism invariance, the covariant derivative of the SET is zero

$$\nabla_{\nu}T^{\mu\nu} = 0. \tag{2.12}$$

This relationship must not be understood in the same way as the conserved quantity observed in special relativity (2.11), since the covariant derivative includes an extra term of a geometric nature

$$T^{a}_{\ b;a} = \nabla_{\nu} T^{\mu}_{\ \nu} = \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g} T^{a}{}_{b}\right)}{\partial x^{a}} - \frac{1}{2} \frac{\partial g_{ac}}{\partial x^{b}} T^{ac}, \tag{2.13}$$

represented as the second term of this last equation. One of the attempts to solve this unknown is to suppose that there exists a (pseudo)tensor that takes into account the energy of the gravitational field [79]; the best known is the Landau-Lishiftz pseudotensor $t^{\mu\nu}$, when added to $T^{\mu\nu}$ yields the desired conserved quantity $\partial_{\mu}\left[(-g)(T^{\mu\nu} + t^{\mu\nu})\right]$. However, this modification has several difficulties, such as the fact that it is a pseudotensor, unknowns about interpretations in energy conservation locally, or even negative densities in Schwarzschild space-time. After all, the result in (2.12) can be approached from an interpretive perspective by questioning the nature of space-time in Noether conservation currents, but the equation does not have any fundamental inconsistency about the theory itself. Although, it may be of some use in efforts to understand mysteries such as dark matter or even dark energy.

2.2 Perfects Fluids that filled the Universe

Perfect fluids are idealizations of real fluids. The first simplification is that particles do not interact with each other, so phenomena such as viscosity and conductivity are not present. There are also no shear stresses or anisotropic pressures, so the SET only has diagonal components since if the fluid is isotropic, rotations of the coordinate system will generate more than one solution in each non-diagonal component.

A perfect fluid moves through spacetime described by a four-velocity vector $\mathbf{U} = (1, 0, 0, 0)$ with an energy density μ and an isotropic pressure P in the fluid reference frame. These three quantities are sufficient to specify the diagonal structure of the tensor, which takes the following general form [29]

$$T^{\mu\nu} = \mu U^{\mu} U^{\nu} + P S^{\mu\nu}, \tag{2.14}$$

where the components of the four-velocity vector are $U^{a} = dx^{a}/d\tau$ such that $U_{\mu}U^{\mu} = -1$, $S^{\mu\ nu}$ is an arbitrary tensor to be determined

$$S^{\mu\nu} = a U^{\mu} U^{\nu} + b g^{\mu\nu}, \tag{2.15}$$

with a and b as arbitrary constants. To find these constants, the SET must satisfy the conservation law (2.12), the conservation of energy-momentum in the limit for Mikowski space-time (2.11), and the Navier-Stokes equations. These calculations give that $a = 1$ and $b = -1$, so

$$T^{\mu\nu} = (\mu + P)U^{\mu}U^{\nu} - Pg^{\mu\nu}. \tag{2.16}$$

The description of each of the components in the rest frame of this tensor is as follows: T^{00} refers to the energy or matter density μ, $T^{0i} = T^{i0}$ is the moment density, T^{ij} (spatial components) is the tangential stress and is the one that includes terms such as viscosity, finally the components T^{ii} are the ones that include the pressures in the fluid. In the case of comoving coordinates, we replace the line element with the Mikowski metric, and in matrix form, we have

$$T^{\mu\nu} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{2.17}$$

The most straightforward perfect is a dust fluid (sometimes called just matter), where the particles are at rest with each other, so there is no random motion, and the pressure is zero. Since we consider the fluid to have a four-vector velocity U^{μ} and a particle density n in the frame of reference at rest, the particle density flux is equal to $n^{\mu} = nU^{\mu}$, so with the four-momentum vector p^{μ}, the particle density of the fluid is given by the zero-zero component of the following tensor

$$T_{\mathrm{p}}^{\mu\nu} = \mu U^{\mu}U^{\nu}. \tag{2.18}$$

The density parameter is enough to describe the dust fluid. This tensor is the case of (2.16) for $P = 0$, hence the notation with a subscript p. To study the other fluids, we must know the EoS obeyed by the fluids to have a better understanding of the origin and nature of stiff matter and dark energy. Likewise the equation (2.16) is the fundamental form of the SET that connects the perfect fluid dynamics with the geometry of space-time through the EFE.

2.3 Equations of state

Under the simplest assumptions, the equation of state in cosmology for each fluid relates its pressure to its density $P = P(\mu)$ using a constant w of the form $P = w\mu$. In the case of dust for a non-zero energy density, we obtain that $w = 0$. Now if we consider the equation (2.16) another simple case is when $\mu = -p$, which does not seem very reasonable since it has negative pressure, however, the important thing is to see how the SET is reduced only to the geometric part

$$T^{\mu\nu}_{EO} = P g^{\mu\nu}, \tag{2.19}$$

which is the contribution of the vacuum, also called dark energy. To understand this better, we must first see that there is an accelerated expansion of space-time when we add a parameter known as the cosmological constant Λ to the Einstein-Hilbert action (S_G of the equation 6.5) for the expansion of the universe

$$S_g = \int_\Omega \sqrt{-g}(R - 2\Lambda) d\Omega. \tag{2.20}$$

By the same procedure explained in Section 2.1, the EFE with the dark energy term is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.21}$$

In the absence of matter fields, the constant Λ is equivalent to a fluid of the form

$$-\frac{\Lambda}{8\pi G} g^{\mu\nu} = P g^{\mu\nu}, \tag{2.22}$$

then the energy density is

$$\mu = -P = \frac{\Lambda}{8\pi G}. \tag{2.23}$$

Therefore, if there is no cancellation of pressure and density in empty space, there will be a fluid that plays the role of accelerating the universe's expansion with a constant energy density. This term might corresponds to the zero point energy in Quantum Field Theory (QFT) since this field, represented as the ground state, has a non-zero contribution to the vacuum. According to cosmological observations, the energy density of vacuum is around $\mu_\Lambda \sim 10^{-10}\text{erg/cm}^3$, but for QFT if we ignore higher momentums of the ground state above the ultraviolet cutoff value we have $\mu_\Lambda \sim 10^{110}\text{erg/cm}^3$, this famous discrepancy of 120 order of magnitude is known as the cosmological constant problem.

All that remains to talk about is the stiff fluid. Zeldovich originally introduced it in [87] to explain the initial conditions of the Universe near the singularity by considering a cold gas of baryons with an equation of state $P = \mu$ ($w = 1$). This EoS is also present in properties of dark matter modeled as a self-gravitating Bose-Einstein condensate partially relativistic [23]. Specifically, in the dense cores of these condensates the EoS can follow a stiff fluid behavior $P \sim \mu$ [23].

This kind of theory [23] is within what is known as Ultra-light dark matter (see, for example, a review in [33]), which are commonly very light bosons in a general range of 10^{-24} eV $< m <$ eV. Particles form condensates or superfluids on galactic scales due to balances between gravitational forces, self-interactions, and quantum pressure given by the Heisenberg Principle, coming from the equivalent Gross Pitaevskii-Poisson hydrodynamic equations that govern the dynamics of the condensate. These condensates have a rich phenomenology that explains significant dark matter problems, such as the core-cusp problem in the center of galaxies and galaxy rotation curve data.

According to the classical Gross-Pitaevskii equation, the non-relativistic EoS at $T = 0$ for a Bose-Einstein condensate with short-range interactions in a gravitational potential has the following polytropic form [23]

$$P = \frac{2\pi a_s \rho^2}{m^3}, \tag{2.24}$$

where a_s is the scattering length and ρ the rest-mass density. The work in [21, 39] solved the Friedman equations for this type of equation; however, in the early universe, we must include the relativistic regime. A semi-relativistic approach is to consider the same pressure of the condensate in the total energy density μ in the following way

$$\mu = \rho + P \tag{2.25}$$

This equation leads to a FLRW universe dominated by stiff matter in the early universe, as will be seen later.

To obtain the equation (2.25) and, subsequently, the EoS of stiff matter, we must perform a thermodynamic analysis [22]. Assuming that the fluid is at zero temperature, by the first law of thermodynamics, we have that

$$d\left(\frac{\mu}{\rho}\right) = -P\left(\frac{1}{\rho}\right). \tag{2.26}$$

If we perform the differentiation, this last equation takes the following form

$$d\mu = \frac{P + \mu}{\rho} d\rho. \tag{2.27}$$

For the equation of state $P = P(\mu)$, the total energy density is given as

$$\mu = \rho\left(A + \int_0^\rho \frac{P(\rho')}{\rho'^2} d\rho'\right), \tag{2.28}$$

where A is a constant of integration, which is $A = 1$ for the limit of $\mu \approx \rho$ when $\rho \to 0$. Following the form of the EoS of the condensate (2.24), we considered $P = \alpha\rho^2$ for a constant α, then we obtain that

$$\mu = \rho + \alpha\rho^2. \tag{2.29}$$

Solving this equation for ρ and using the EoS we arrive at

$$P = \frac{1}{4\alpha}\left(\sqrt{1 + 4\alpha\mu} - 1\right)^2. \tag{2.30}$$

In the ultra-relativistic limit ($\mu \to \infty$), we arrive at the EoS for stiff matter $P \approx \mu$. The term "stiff" comes from the fact that the speed of sound c_s in $\mu \to \infty$ is equal to that of light

$$c_s = \sqrt{\frac{\partial P}{\partial \mu}} \approx 1. \tag{2.31}$$

In the case of the Bose-Einstein condensate with self-interactions between bosons, the polytropic constant is

$$\alpha = \frac{2\pi a_S}{m^3}. \tag{2.32}$$

There are other ways to obtain the EoS for stiff matter, Zeldovich [87] assumed a cosmological model where the early universe was composed of a gas of baryons interacting through a vector field of bosons to demonstrate how the speed of sound can approach the speed of light, resulting in

$$K = \frac{2\pi g^2}{m_m m_b}, \tag{2.33}$$

where g is the charge of the baryon, m_b its mass, and m_m the mass of the meson.

In conclusion, the stiff matter fluid is a very interesting option to understand the early universe when density and pressure are very high, just like in the very early universe. It is essential to mention that among other results, it has also served to explain the abundance of particle species, the baryon asymmetry, and the density of perturbations for the formation of large structures in the universe [51].

2.4 The dynamic evolution of isotropic and homogeneous universes with perfect fluids

There are different stages that a universe can go through, where a certain fluid dominates over the others or even where all the fluids behave as a single particular fluid. To see this in the case of our mixture of fluids, we will analyze the FLRW universe with dust, stiff matter, and dark energy. Consider a homogeneous and isotropic universe in spherical coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ given by the FLRW metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{2.34}$$

where $a(t)$ is the scale factor considered positive and only time-dependent, k is a constant that describes the curvature with $k = +1$ spherical, $k = -1$ hyperbolic, and $k = 0$ flat. The solutions for the scale factor and the thermodynamic parameters for the different fluids are given by the EFE (2.7) for the FLRW metric (2.34) and the SET (2.17).

To calculate the geometric part, i.e., the Einstein tensor G, we have to take into account the form of the Riemann tensor that encodes the information about the intrinsic curvature of space-time

$$R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho{}_{\nu\mu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\sigma\lambda}\Gamma^\lambda{}_{\nu\mu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\sigma\mu}. \tag{2.35}$$

The Ricci tensor is the contraction of the first and third index of the curvature tensor

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \tag{2.36}$$

while the Ricci scalar is the trace of the Ricci tensor

$$R = R^\mu{}_\mu = R_{\mu\nu}g^{\mu\nu}. \tag{2.37}$$

Affine connections connect the tangent spaces on a manifold; in the case of a pseudo-Riemannian manifold, these are the Levi-Civita connections, also known as the covariant derivative. The Christoffel symbols are the coefficients of the Levi-Civita connection that give the correction to the ordinary derivative, defined as the solutions to the metric compatibility ($\nabla_\rho g_{\mu\nu} = 0$) in the case that there is no torsion ($\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$), these are

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}). \tag{2.38}$$

Therefore, in the FLRW universe, the Christoffel symbols are

$$
\begin{gathered}
\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^0_{22} = a\dot{a}r^2, \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta, \\
\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a}, \\
\Gamma^1_{11} = \frac{kr}{1-kr^2}, \quad \Gamma^1_{22} = -r(1-kr^2), \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta, \\
\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \cot\theta.
\end{gathered} \tag{2.39}
$$

From (2.38) we can see that the lower indices are symmetric $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, since there is no torsion. Now, the computation of the Einstein tensor G can be carried out with the help of (2.35), (2.36) and (2.37), obtaining the following non-zero components

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2}, \tag{2.40}$$

$$G_{ii} = g_{ii}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{2.41}$$

The equations that govern the dynamics of a universe filled with fluids come from equating the components of G with the SET (2.16) hence we can arrive at the well-known Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\mu, \tag{2.42}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G P. \tag{2.43}$$

From here the dependence of the scale factor can be calculated for each fluid individually or mixed. As already mentioned, the observations to determine the curvature of the universe conclude that is approximately flat, so it will be assumed that $k = 0$.

In the case of dust $P = 0$, solving the Friedmann equations gives $\mu_D \propto 1/a^3$, which shows that the density falls inversely proportional to the volumetric expansion. In the case of stiff matter, the EoS is $P = \mu$ and the energy density falls like dust but squared $\mu_Z \propto 1/a^6$, while in the case of dark energy with $P = -\mu$, the energy density remains constant and is given by the equation (2.23), so $\mu_\Lambda = \Lambda/8\pi G$.

2.5 FLWR with dark energy, dust, and stiff matter

Now, it is interesting to see how the flat FLRW universe behaves with these three fluids. The solution presented here was first obtained by [22]. We start by solving the continuity equation, which gives $\mu = \mu_0(a_0/a)^{3(1+w)}$, where μ_0 and a_0 is the energy density and the scale factor for the present moment, respectively. Using Friedman's equations, the analytical solution for the scale factor is of the following form [22]

$$a = a_0 \left[\alpha_{ZD\Lambda} \sinh^2 \left(\frac{3}{2}\beta_\Lambda H_0 t\right) + \gamma_{Z\Lambda} \left(1 - e^{-3\beta_\Lambda H_0 t}\right)\right]^{\frac{1}{3}}, \tag{2.44}$$

here $\alpha_{ZDA}, \beta_\Lambda$ and $\gamma_{Z\Lambda}$ are constants where the subscripts Z, D and Λ refer to the stiff matter, dust, and dark energy fluid respectively, H_0 is the Hubble parameter for the present moment. In the case of $t = 0$ the scalar factor is zero $a = 0$, while at $t \to \infty$

$$a \sim a_0 \alpha_{ZD\Lambda} e^{\beta_\Lambda H_0 t}, \tag{2.45}$$

which means that in the late universe, the dark energy term of the exponent dominates the expansion, and the energy density is the dark energy density $\mu_\Lambda = \Lambda/8\pi G$, which remains constant.

We can also check how this solution behaves near $t = 0$, in this case, the equation (2.44) is approximated around zero, and because $\beta_\Lambda = \sqrt{\Omega_{\Lambda,0}}$ and $\gamma_{Z\Lambda} = \sqrt{\Omega_{Z,0}/\Omega_{\Lambda,0}}$ then

$$a \sim a_0 \left(3\Omega_{Z,0} t\right)^{\frac{1}{3}}. \tag{2.46}$$

This last result shows that the only fluid present at first order is the stiff matter fluid, which dominates in this limit, as previously discussed. In addition, the total energy density diverges because the scale factor depends on the stiff fluid and dust parameters.

It is worth mentioning, that we can also try to solve a universe with radiation ($P = 1/3\mu$) instead of the stiff fluid, however, to our knowledge there is no analytical solution in the literature and the case of the Petrov D symmetry studied here is no exception. Therefore, we do not consider radiation in the mixture.

2.6 Scalars field coupled with gravity

An inflationary stage characterized by an exponential growth of space has been considered essential in the dynamic evolution of the Big Bang Theory. The inflationary scenario in the early universe has been studied extensively with the development of Λ

in (2.21) as a dynamic term. In addition to the relative simplicity of including dark energy in explaining an inflationary universe, phenomenological theories attempting to explain the observations have been varied.

This kind of model, normally called quintessence, is a well-studied case in cosmology and one of the simplest possible extensions of ΛCDM [45]. Here Λ is replaced with a dynamical variable by minimally coupling a scalar field ϕ and a potential $V(\phi)$ with gravity and other relevant fields.

We can consider the parameter Λ equal to zero or be included in the potential $V(\phi)$, so the role it has in the universe can now be explained by the potential energy of the field ϕ. The action of the scalar with gravity is of the form

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} + \frac{1}{2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} - V(\phi)\right). \tag{2.47}$$

Here the field is minimally coupled to gravity, in the non-minimally coupled case, it would include additional terms that depend on R. In this case, the Lagrangian density of the field is $\mathcal{L} = \mathcal{L}(\phi, \phi_{,\mu}, x^\mu)$ and by the Principle of Least Action, the Euler-Lagrange equations are

$$\frac{d}{dx^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{2.48}$$

Now performing the total derivative of $\mathcal{L}$ with respect to x^ν and using the previous equations we have

$$\frac{d}{dx^\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\phi_{,\mu} - \mathcal{L}\delta^\nu_\mu\right) = -\frac{\partial\mathcal{L}}{\partial x^\mu}. \tag{2.49}$$

If the field does not interact with any external source, it is a free field that does not depend on x^μ, then the quantity that is conserved in the parentheses is the SET

$$T^\nu_\mu = \frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\phi_{,\mu} - \mathcal{L}\delta^\nu_\mu. \tag{2.50}$$

The scalar field will be considered spatially homogeneous and only time-dependant $\phi = \phi(t)$, so terms that include $\nabla\phi$ are zero. To observe some of the properties of these quintessence models, we must first take into account that a fluid or mixture of fluids can be equivalent to a scalar field, with the help of (2.16) in the FLRW universe we have

$$\mu = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.51}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{2.52}$$

In such a way that by the EoS $P = w\mu$

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{2.53}$$

We include the subscript of the scalar field in w to indicate that we refer to the EoS of a perfect fluid that behaves according to the field ($P_\phi = w_\phi \mu_\phi$), i.e., not only will the universe be composed of several fluids, but there will be moments when it behaves as one. For example, a dark energy fluid must obey $w_\phi \backsimeq -1$, which requires a very slowly rolling field $\dot{\phi}^2 \ll V(\phi)$, while for stiff matter is the opposite, the kinetic energy must be much greater than the potential energy such that $w_\phi \backsimeq 1$, this is why a scalar field with a stiff behavior is called kination [34].

The model can explain other behaviors besides dark energy or stiff matter. For example, the Ratra-Peebles potential was developed to solve the coincidence problem [80] and the "unnatural" Λ energy scale that is part of fine-tuning problems [67,84]. In this type of potential, the energy of the scalar field presents an evolutionary behavior that follows the energy of radiation ($w = -1$) and dust ($w = 0$) in the early moments of the universe to eventually exceed the density of matter and caused acceleration [45,76].

In the first inflationary models, the basic idea was that the universe began in a high-energy state and then slowly decreased to a lower-energy state. This first state was called the false vacuum since it was not the lowest energy state. To move to lower energy states, several models were presented with different mechanisms, such as quantum tunneling [26], and various potentials associated with a scalar field called inflaton [54] were proposed.

The inflaton model considered a potential in the form of a "hat" with a false vacuum at its top representing the initial potential value, where it was later shown that only a sufficiently high initial value of the field with a small enough relaxation was needed for inflation to occur [37]. This last type of potential is part of what is known as chaotic inflation with a set of potentials following the most commonly scaling relations $V = \frac{1}{2}m^2\phi^2$ and $V = \frac{1}{4}\lambda^4\phi^4$, the fact that is called chaotic is because usually the initial conditions of the scalar field are set to be chaotic.

The constraints of a slowly rolling potential $\dot{\phi}^2 \ll V$ using the FLRW equations with the scalar field sets an upper bound to the mass for the inflaton with $m \leq 10^{-33}$ eV, which is lighter than, for example, Ultra-light dark matter particles (10^{-24} eV $<$ $m < 1$ eV) [33].

The equations governing the dynamics of the scalar field in the FLRW universe (2.34) are obtained from the EFE (2.7). Since the scalar field defines an SET, the conservation of this quantity ($T^{\nu}_{\mu;\nu} = 0$) gives the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2.54}$$

where the therm $3H\dot{\phi}$ produces the damping of the field in an expanding universe. This last equation (2.54) together with the equations that connect the thermodynamic variables with the scalar field (2.51) and (2.52) are sufficient to determine the field and the potential. Note that μ and P must first be obtained using the Friedmann equations. We can proceed in reverse and determine the mixture of fluids from different scalar fields, however this may be much more difficult.

2.7 Cosmological parameters

Cosmological parameters allow theoretical results to be quantified and corroborated with observations. One of the most important is the Hubble parameter, which describes universe's expansion rate. Its measurement is of great vitality since other parameters depend on it. To understand where this quantity comes from, we can first expand the scale factor around $t = t_0$

$$a \simeq a(t_0)\left(1 + H_0(t - t_0) + \ldots\right), \tag{2.55}$$

where the Hubble parameter at t_0, commonly assigned at the present time is

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)}. \tag{2.56}$$

The universe is expanding, so distant objects will present shifts in the frequency of light when measuring them. If we assumed that observations are made at the origin of the coordinate system, for light rays $ds^2 = 0$ and using the FLRW metric (2.34)

$$\frac{dt}{a(t)} = -\frac{dr}{1 - kr^2}. \tag{2.57}$$

Here we chose the minus sign of the root since if the rays are directed towards the origin, the time increases as the distance decreases. Because r is the comoving radial distance, the right part of the equation does not depend on time, so for the time at emission t_e and the time at observation t_o

$$\frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)}. \tag{2.58}$$

In terms of the frequency of light $\Delta t = 1/f$

$$\frac{f_e}{f_o} = \frac{a(t_o)}{a(t_e)}, \tag{2.59}$$

Here, if the scale factor increases (decreases) with time, the frequency decreases (increases), so there will be a red (blue) shift. One way to quantify this measurement is by introducing the factor $1+z$

$$1 + z \equiv \frac{a(t_o)}{a(t_e)} \tag{2.60}$$

where for a negative (positive) value of z, the universe contracts (expands). Commonly the distances are in terms of this parameter, since by (2.55)

$$z = H_0(t_0 - t_e) + \dots, \tag{2.61}$$

the first term works very well for sufficiently close galaxies.

The redshift is given by the Hubble parameter multiplied by the proper distance $d = c(t_0 - t_e)$. We have to notice that d is the distance for two events that occur simultaneously and that changes as the universe expands, unlike the comoving distance, which does not change (the relationship for FLRW is $d = a(t)l$ where l is the comoving distance).

We must also take into consideration that there are peculiar motions of galaxies caused by gravitational effects of the surroundings, so to observe the linear relationship, we have to consider values of cosmological velocities given by the accelerated expansion of the universe much larger (distances where $|z| \gg 10^{-3}$) [83].

For even greater distances, the following term can be included in the series of (2.61)

$$z = H_0(t_e - t_0) + \frac{1}{2}(q_0 + 2)H_0^2(t_0 - t) \tag{2.62}$$

where q_0 is the deceleration parameter, given by

$$q_0 \equiv \frac{-1}{H_0^2 a\,(t_0)} \frac{d^2 a(t)}{dt^2}\bigg|_{t=t_0}. \tag{2.63}$$

The sign of q_0 allows us to know if the universe is accelerating ($\ddot{a_0} > 0$) or decelerating ($\ddot{a_0} < 0$). The minus in front was because, at the beginning of the formulation of q_0 it was believed that the universe was decelerating. Therefore, a positive (negative) value of q_0 corresponds to the deceleration (acceleration) of space-time. The first evidence for the accelerated expansion of the universe was obtained in [69] by observing Type Ia supernovae in the redshift range of $0.16 \leq z \leq 0.62$, the restriction for the deceleration parameter was a negative value $q_0 < 0$ with 2.8σ, indicating the acceleration.

The value of the Hubble parameter at the present moment is

$$H_0 = 100h \text{ km}s^{-1}\text{Mpc}^{-1}, \tag{2.64}$$

where h is a constant that is measured observationally. Significant tension exists between the local measurement and its value inferred by the CMB. This tension is because the Hubble Space Telescope measurements of Cepheids in galaxies with Type Ia Supernovae obtained by the Supernovae and H_0 for the Equation of State project (SHOES) [71], and the CMB data of the early universe collected by the Plank Satellite mission [27] differ in values of H_0 with $h = 0.72 \pm 0.08$ and $h = 0.673 \pm 0.012$ respectively, although the values may vary depending on the model.

Because of the mixture of fluids, it is helpful to see the distribution of their densities. The critical *mass* density of the universe μ_c is the density necessary for the geometry of the universe to be flat, that is, according to the Friedmann equations (2.42)

$$\mu_c = \frac{3H^2}{8\pi G}. \tag{2.65}$$

The value for the critical density is $\mu_c = 2.78h^{-1} \times 10^{11}\text{M}_\odot/(h^{-1}\text{Mpc})^3$, which is the average mass of a galaxy $(10^{10} - 10^{11})$ divided by the average volume that contains it, approximately 1 Mpc^3.

As already explained, these values are so finely tuned (in standard units, this is $\mu_c = 10^{-26}\text{kg m}^{-3}$) that it is difficult to conceive such initial conditions without entering into philosophical arguments, such as the Anthropic Principle and theories that try to get rid of this problem by searching for more "natural" values. On the other hand, some arguments explain how this fine-tuning conspiracy comes from probability distributions assumed without a grounded theoretical reason and then used to describe how likely it is to obtain the value of a parameter or not, especially in cosmology and particle physics [42].

Once the critical density is defined, the mass density of each fluid (μ) is scaled with respect to the critical density (μ_c) to obtain the density parameter for each fluid, which is dimensionless

$$\Omega = \frac{\mu}{\mu_c}. \tag{2.66}$$

In the case that there is a mixture of fluids, all the density parameters must add up to the total density parameter, for this work, it would be as

$$\Omega_Z + \Omega_D + \Omega_\Lambda = \Omega. \qquad (2.67)$$

By the equation (2.42), in a universe with the critical density value $\Omega = 1$.

Chapter 3

Symmetries of the Petrov classification

3.1 The tetrad formalism

As already mentioned, this section consists of introducing the tetrad formalism necessary to develop the NP formalism, the main reference used is [19]. This formalism has been very useful in looking for solutions to the EFE by using a system of four independent basis or tetrads in the tangent space of a Lorentzian manifold that manages to simplify the calculations and find new solutions. The choice of the tetrad will depend on the symmetries of space-time. Although the number of new terms can be overwhelming, this formalism has the great advantage of expressing the differential equations from second order to first order.

Let us consider an orthonormal basis with one temporal component and three spatial components, expressed with the following expansion coefficients

$$e_a{}^i, \tag{3.1}$$

where $a \in \{1, 2, 3, 4\}$, following the notation in [19] $a, b,$ ect., will represent the tetrad indices and $i, j,$ ect., the tensor indices. The tetrad is describe by vectors in the tangent space of the form

$$e_a = e_a{}^i \partial_i. \tag{3.2}$$

The condition for local orthonormal basis is

$$\eta_{ab} = g_{ij} e_a{}^i e_b{}^j. \tag{3.3}$$

As an example, consider the following non-holonomic basis in polar spherical with $a = (1,2,3,4) = (t,r,\theta,\phi)$ and $i = (1,2,3,4) = (t,r,\theta,\phi)$

$$e_t = \partial_t, \quad e_r = \partial_r, \quad e_\theta = \frac{1}{r}\partial_\theta, \quad e_\phi = \frac{1}{r\sin\theta}\partial_\phi. \tag{3.4}$$

Then one can recognize the non-zero components of ${e_a}^i$ as

$${e_t}^t = 1, \quad {e_r}^r = 1, \quad {e_\theta}^\theta = \frac{1}{r}, \quad {e_\phi}^\phi = \frac{1}{r\sin\theta}\partial_\theta. \tag{3.5}$$

By the condition (3.3) it follows that g_{ij} must be the Minkowski metric in spherical polar coordinates

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta^2 \end{pmatrix}. \tag{3.6}$$

The components ${e_a}^i$ are also known as the tetrad, just as a vector $V = V^a\partial_a$, the components V^a are also called vectors, if we ignore the direction of the field for practical purposes when computing tensor calculus. Finally, the definition of the inverse matrix of the orthonormal tetrad ${e_a}^i$ is carried out as ${e^b}_i$, in such a way that

$${e_a}^i {e^b}_i = {\delta^b}_a, \tag{3.7}$$

which can also be written as $\eta_{ab} = {e_a}^i e_{bi}$. So in the same way as ordinary vectors, the one-forms σ with basis $w^\mu = dx^\mu$, which live in the dual vector space to the ordinary vectors and map the vectors V^a to the scalars (that is, for $dx^\mu(\partial_\nu) = {\delta^\mu}_\nu$ we have $\sigma \cdot V = \sigma_\mu V^\mu$), can be written as follows

$$dx^a = {e^a}_\mu dx^\mu \tag{3.8}$$

The matrix η_{ab} and its inverse η^{ab} act as a Minkowski metric but in tetrad space, so it is not difficult to derive the following relations

$$\eta^{ab}\eta_{bc} = {\delta^a}_c, \tag{3.9}$$

$$\eta_{ab}{e^a}_i = e_{bi}, \quad \eta^{ab}e_{ai} = {e^b}_i. \tag{3.10}$$

Here, we can see that the metric η_{ab} allows the tetrad indices to be lowered and raised.

The projection in the tetrad frame of a vector A^j is carried out in the following way

$$A_a = e_{aj}A^j = e_a{}^j A_j. \tag{3.11}$$

Following the same reasoning for tensors of higher dimensions, as will be seen later, the components in the tetrad space can be obtained for the most relevant tensor quantities in GR. Meanwhile, we can show that any tensor transforms from general coordinates to tetrad coordinates in the following way

$$T^{b_1 \ldots b_m}{}_{a_1 \ldots a_n} = e_{a_1}{}^{i_1} \ldots e_{a_n}{}^{i_n} e^{b_1}{}_{j_1} \ldots e^{b_m}{}_{j_m} T_{i_1 \ldots i_n}{}^{j_1 \ldots j_m}. \tag{3.12}$$

The directional derivative is defined as $A_{a,b} = e_b{}^i \partial_i A_a$, or using the relation (3.11) it is written as follows

$$A_{a,b} = e_a{}^j A_{j;i} e_b{}^j + \gamma_{cab} A^c, \tag{3.13}$$

where $\gamma_{cab} = e_c{}^k e_{ak;i} e_b{}^i$ are the Ricci rotation coefficients. It can be shown that because η_{ab} is constant in all its entries, the first two indices are antisymmetric $\gamma_{cab} = -\gamma_{acb}$. Now we define the intrinsic derivative of A_a in the direction of e_b as

$$A_{a|b} = e_a{}^i A_{i;j} e_b{}^j \tag{3.14}$$

which by means of (3.13) can be written as

$$A_{a|b} = A_{a;b} - \eta^{nm} \gamma_{nab} A_m. \tag{3.15}$$

The calculation of the rotation coefficients can be computed more efficiently without the need to evaluate the covariant derivative of the tetrad by using the following definition

$$\lambda_{abc} = e_{bi,j} \left(e_a{}^i e_c{}^j - e_a{}^j e_c{}^i \right) \tag{3.16}$$

and because $\lambda_{abc} = \gamma_{abc} - \gamma_{cba}$, the indices can be permuted. Then just as the Christoffel symbols are calculated in terms of the metric, we can obtain that

$$\gamma_{abc} = \frac{1}{2} \left(\lambda_{abc} + \lambda_{cab} - \lambda_{bca} \right). \tag{3.17}$$

Therefore, to obtain the Ricci rotation coefficients, only the lambdas must be known, which will depend on the tetrad and its ordinary derivatives given by the equation

(3.16). The projection of the Riemann tensor can be put in terms of these coefficients, first by noting that by definition $R^i_{jkl}A_i = A_{j;k;l} - A_{j;k;l}$, then

$$R_{mnkl}e_a{}^m = e_{ai;k;l} - e_{ai;l;k}. \tag{3.18}$$

Projecting the other components into the tetrad space and using the fact that $e_a{}^k{}_{;i} = -\gamma_a{}^k{}_i$ we have

$$R_{abcd} = R_{mikl}e_a{}^m e_b{}^i e_c{}^k e_d{}^l, \tag{3.19}$$

$$= \left[(\gamma_{afg}e^f{}_i e^g{}_k)_{;l} + (\gamma_{afg}e^f{}_i e^g{}_l)_{;k}\right] e_b{}^i e_c{}^k e_d{}^l, \tag{3.20}$$

$$= -\gamma_{abc,d} + \gamma_{abd,c} + \gamma_{baf}(\gamma_c{}^f{}_d - \gamma_d{}^f{}_c) + \gamma_{fac}\gamma_b{}^f{}_d - \gamma_{fad}\gamma_b{}^f{}_c. \tag{3.21}$$

So to know the Riemann tensor, it is only necessary to calculate the Ricci rotation coefficients.

If we go specifically into GR, the EFE in tetrad space can be derived from the following action [61]

$$S = \int_\Omega (\mathcal{L}_g + \kappa\mathcal{L}_M)d\Omega, \tag{3.22}$$

$$= \int_\Omega \left(e e_a{}^i e_b{}^j R^{ab}{}_{ij} + \kappa\mathcal{L}_M\right) d\Omega, \tag{3.23}$$

where $e \equiv \det(e^a{}_i) = (-g)^{1/2}$ since $g_{ij} = \eta_{ab}e^a{}_i e^b{}_j$ represents the same coupling constant as (2.7). Now performing the functional derivative of the action with respect to $e_a{}^i$ we have

$$\frac{\delta\mathcal{L}_g}{\delta e_a{}^i} = -(-g)^{1/2}\left(2R^{ab}{}_{ij}e_b{}^j - R^{bc}{}_{jk}e_b{}^j e_c{}^k e^a{}_i\right). \tag{3.24}$$

While the SET is defined as

$$T^a_{\ i} \equiv \frac{1}{2}(-g)^{-1/2}\frac{\delta\mathcal{L}_M}{\delta e_a{}^i} = \frac{1}{2}(-g)^{-1/2}\frac{\delta g^{jk}}{\delta e_a{}^\mu}\frac{\delta\mathcal{L}_M}{\delta g^{jk}}, \tag{3.25}$$

$$= (-g)^{-1/2}e^{ij}\frac{\delta\mathcal{L}_M}{\delta g^{ij}} = e^{aj}T_{ij}, \tag{3.26}$$

where T_{ij} has the usual definition given by (2.6). Therefore we obtain

$$R^{ab}{}_{ij}e_b{}^j - \frac{1}{2}R^{bc}{}_{jk}e_b{}^j e_c{}^k e_i{}^a = \kappa T^a_{\ i}. \tag{3.27}$$

The EFE (2.7) can be recovered by contracting with e_{aj}, while in tetrad formalism, the EFE are obtained by contracting with ${e_b}^i$

$$R_{ab} - \frac{1}{2}\eta_{ab}R = 8\pi G T_{ab}. \tag{3.28}$$

3.2 NP formalism

Now with the theory of tetrads, the next task is to specify a particular basis. The NP Formalism establishes a tetrad of null vectors, which also makes possible the introduction of spinor basis [3]. These vectors describe light rays moving through space-time and reveal an intrinsic light cone structure that is very useful in approximating some problems and solutions in GR.

The basis are formed by two real vectors l^i and n^i plus two imaginary vectors m^i and $\bar{m}^i$, the condition of being null vectors implies that

$$l^i l_i = n^i n_i = m^i m_i = \bar{m}^i \bar{m}_i = 0. \tag{3.29}$$

While the orthogonalization condition is

$$n^i m_i = l^i m_i = l^i \bar{m}_i = n^i \bar{m}_i = 0. \tag{3.30}$$

Finally, the normalization condition implies that

$$l^i n_i = 1 \quad \text{y} \quad m^i \bar{m}_i = -1. \tag{3.31}$$

The matrix for the metric tensor in the basis of the NP formalism has the following form

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.32}$$

We can put these null tetrads in terms of a set of orthonormalized tetrads $({e_t}^i, {e_x}^i, {e_y}^i, {e_z}^i)$ as follows

$$l^i = \frac{1}{\sqrt{2}}({e_t}^i + {e_z}^i), \quad n^i = \frac{1}{\sqrt{2}}({e_t}^i - {e_z}^i), \tag{3.33}$$

$$m^i = \frac{1}{\sqrt{2}}({e_x}^i + i{e_y}^i), \quad \bar{m}^i = \frac{1}{\sqrt{2}}({e_x}^i - i{e_y}^i). \tag{3.34}$$

These expressions satisfy the relations established in (3.29), (3.30) and (3.31). By the condition (3.3), the metric tensor can be expressed in terms of the null tetrad as

$$g^{ij} = e_a{}^i e_b{}^j \eta^{ab} = l^i n^j + n^i l^j - m^i \bar{m}^j - \bar{m}^i m^j. \tag{3.35}$$

In the case that we have an orthonormal one-form tetrad dx^a for $a \in (1,2,3,4)$, one way to express the null tetrad in terms of these one-forms is in the following way

$$\begin{pmatrix} l \\ n \\ m \\ \bar{m} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \\ dx^4 \end{pmatrix}. \tag{3.36}$$

To calculate the components of the tetrads, we have to use the relations (3.34) in their covariant form, which we can obtain by the metric tensor. Continuing with the same example of spherical coordinates in flat space-time, the metric [57] is considered

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{3.37}$$

If we recognize the tetrad as

$$e^a{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 1 & r\sin\theta \end{pmatrix}, \tag{3.38}$$

by means of the matrix relation (3.36), considering $dx^\mu = (dt, dr, d\theta, d\phi)$, the null tetrad can be written as follows

$$l = \frac{1}{\sqrt{2}}(dx^1 + dx^2) = \frac{1}{\sqrt{2}}(e^1{}_\mu + e^2{}_\mu)dx^\mu = \frac{1}{\sqrt{2}}(dt + dr), \tag{3.39}$$

$$n = \frac{1}{\sqrt{2}}(dx^1 - dx^2) = \frac{1}{\sqrt{2}}(e^1{}_\mu - e^2{}_\mu)dx^\mu = \frac{1}{\sqrt{2}}(dt - dr), \tag{3.40}$$

$$m = \frac{1}{\sqrt{2}}(dx^3 + idx^4) = \frac{1}{\sqrt{2}}(e^3{}_\mu + ie^4{}_\mu)dx^\mu = \frac{1}{\sqrt{2}}(rd\theta + ir\sin\theta d\phi), \tag{3.41}$$

$$\bar{m} = \frac{1}{\sqrt{2}}(dx^3 - idx^4) = \frac{1}{\sqrt{2}}(e^3{}_\mu - ie^4{}_\mu)dx^\mu = \frac{1}{\sqrt{2}}(rd\theta - ir\sin\theta d\phi). \tag{3.42}$$

3.2.1 The Weyl tensor and the NP formalism

The Riemann tensor encodes the curvature of space-time through the metric tensor $g_{\mu\nu}$, making it of great importance for interpreting the behavior of the gravitational field and its evolution. The Weyl tensor has the same symmetries as the Riemann tensor and is used in the NP Formalism to establish a classification criterion for the Petrov symmetries.

The Weyl tensor is the traceless part of the Riemann tensor, so in terms of the tetrad components, it is written as [19]

$$R_{abcd} = C_{abcd} - \frac{1}{2}\left(\eta_{ac}R_{bd} - \eta_{bc}R_{ad} - \eta_{ad}R_{bc} + \eta_{bd}R_{ac}\right) + \frac{1}{6}\left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right)R. \tag{3.43}$$

Unlike the Riemann tensor, which has 20 components, the Weyl tensor has 10. Also, the former reduces to the latter in empty space. Similar to the symmetries of R_{abcd}

$$R_{ab(cd)} = 0, \tag{3.44}$$

$$R_{(ab)cd} = 0, \tag{3.45}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{3.46}$$

where the parentheses in the subscripts represent the symmetrization operator, the Weyl tensor satisfies the following relations

$$C_{ab(cd)} = 0, \tag{3.47}$$

$$C_{(ab)cd} = 0, \tag{3.48}$$

$$C_{abcd} + C_{adbc} + C_{acdb} = 0. \tag{3.49}$$

The contraction with a pair of indices is zero since its trace is null

$$C^{a}{}_{bad} = 0. \tag{3.50}$$

Using these relations, we can obtain the following components of the Weyl tensor in terms of the Riemann and Ricci tensor [19]

$$\begin{gathered}C_{1212} = R_{1212} - R_{12} + \frac{1}{6}R, \quad C_{1213} = R_{1213} - \frac{1}{2}R_{13}, \\ C_{1223} = R_{1223} + \frac{1}{2}R_{23}, \quad C_{1234} = R_{1234}, \quad C_{1313} = R_{1313}, \\ C_{1324} = R_{1324} - \frac{1}{12}R_{12}, \quad C_{1334} = R_{1334} - \frac{1}{2}R_{13}, \quad C_{2323} = R_{2323}, \\ C_{2334} = R_{2334} - \frac{1}{2}R_{23}, \quad C_{3434} = R_{3434}.\end{gathered} \tag{3.51}$$

These components are linearly independent since the rest are zero or expressed as a linear combination of the above. The ten components can be written in terms of five complex scalars

$$\Psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \tag{3.52}$$

$$\Psi_1 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta, \tag{3.53}$$

$$\Psi_2 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta, \tag{3.54}$$

$$\Psi_3 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta, \tag{3.55}$$

$$\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta, \tag{3.56}$$

called the Weyl scalars. To obtain the other five components, it is enough to calculate the complex conjugate. As a simple example, consider the tetrad basis $l = e_0$, $n = e_1$, $m = e_2$ and $\bar{m} = e_3$, then

$$\Psi_0 = -C_{0202}, \tag{3.57}$$

$$\Psi_1 = -C_{1213}, \tag{3.58}$$

$$\Psi_2 = -C_{1342}, \tag{3.59}$$

$$\Psi_3 = -C_{1242}, \tag{3.60}$$

$$\Psi_4 = -C_{2424}, \tag{3.61}$$

Depending on the metric, a set of null tetrad basis can be proposed using the relation (3.36), and the Weyl scalars can be computed. These scalars are necessary to establish the space-time symmetry; in the next section, we will study the classification and delve into the one used in this work.

3.3 Algebraic symmetries of Petrov

Several types of Petrov classification have been developed [18,46,63,64], we introduced in this section the classification by the tensor method, developed in more detail in [19], which is the primary reference. The Weyl tensor (3.43) is entirely specified by the five Weyl scalars, but these, in turn, are specified by the six parameters of the Lorentz group (3 impulses + 3 rotations). Petrov's classification comes from finding which scalars go to zero under an appropriate system orientation.

First, the rotations of each element of the tetrad must be developed through the following classes of rotation: class I) rotation around the axis l, class II) rotation around the axis n, and class III) rotation of the vectors of the plane m-$\bar{m}$ without rotating l and n. Specifically, the rotations for each class are expressed as follows

$$\begin{aligned} &\text{I)} \quad l' = l, m' = \bar{m} + a^* l \text{ y } n' = n + a^* m + a\bar{m} + aa^* l, \\ &\text{II)} \quad n' = n, m' = m + bn, \bar{m}' = \bar{m} + b^* n \text{ y } l' = l + b^* m + b\bar{m} + bb^* n, \\ &\text{III)} \quad l' = A^{-1} l, n' = An, m' = e^{i\theta} m \text{ y } \bar{m}' = e^{-i\theta}\bar{m}, \end{aligned} \tag{3.62}$$

where a and b are two complex variables and A and θ are two real variables. Furthermore, this new tetrad satisfies the orthonormalization relations (3.30) and (3.31). Therefore, the Weyl scalars with a rotation of class I are

$$\begin{aligned} &\Psi_0' = \Psi_0, \Psi_1' = \Psi_1 + a^*\Psi_0, \Psi_2' = \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0, \\ &\Psi_3' = \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a*)^3\Psi_0, \\ &\Psi_4' = \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0, \end{aligned} \tag{3.63}$$

while with a rotation of class 2

$$\begin{aligned} &\Psi_0' = \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4, \\ &\Psi_1' = \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4, \Psi_2' = \Psi_2 + 2b\Psi_3 + b^2\Psi_4, \\ &\Psi_3' = \Psi_3 + b\Psi_4, \Psi_4' = \Psi_4, \end{aligned} \tag{3.64}$$

and finally with a rotation of class 3

$$\begin{aligned} &\Psi_0' = A^{-2}e^{2i\theta}\Psi_0, \Psi_1' = A^{-1}e^{i\theta}\Psi_1, \Psi_2' = \Psi_2, \Psi_3' = Ae^{-i\theta}\Psi_3, \\ &\Psi_4' = A^2 e^{-2i\theta}\Psi_4. \end{aligned} \tag{3.65}$$

Now, we can adequately introduce Petrov's classification. The idea is to perform successive rotations and find which values of the Weyl scalars can be zero by manipulating the variables that give the rotations. As an example, consider first performing a class II rotation and assume the condition that Ψ_0' is zero, then by the transformation (3.64) we have

$$\Psi_0' = \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0. \tag{3.66}$$

The roots of this equation are four b_1, b_2, b_3 and b_4. If now we assume that all these roots are distinct $b_1 \neq b_2 \neq b_3 \neq b_4$ and perform a class I rotation, then by the transformation in (3.63) we have that for Ψ_4''

$$\Psi_4'' = \Psi_4' + 4a^*\Psi_3' + 6(a^*)^2\Psi_2' + 4(a^*)^3\Psi_1' + (a^*)^4\Psi_0'. \tag{3.67}$$

If $b = -1/a^*$ for any of the roots and substitute all the prime Weyl scalars with the help of (3.64) in (3.67), we arrive precisely at the equation (3.66), concluding that $\Psi_4'' = 0$. Class I rotation does not affect Ψ_0' which is zero, and a class III rotation will not affect Ψ_4'' and Ψ_0'', while the other scalars are non-zero, we then have that for a last rotation of class III

$$\begin{aligned}
\Psi_0''' &= A^{-2}e^{2i\theta}\Psi_0'' = 0,\\
\Psi_1''' &= A^{-1}e^{i\theta}\Psi_1'' \neq 0,\\
\Psi_2''' &= \Psi_2'' \neq 0,\\
\Psi_3''' &= Ae^{-i\theta}\Psi_3'' \neq 0,\\
\Psi_4''' &= A^2e^{-2i\theta}\Psi_4'' = 0.
\end{aligned} \tag{3.68}$$

This configuration of the system is known as a Petrov type I symmetry.

Now if we consider that two of the roots of (3.66) are equal, $b_1 = b_2$, then both the equation (3.66) and its derivative, which would be equal to $4\Psi_1'$ for a class II rotation, go to zero. Then, for a class I rotation, Ψ_4'' also goes to zero for the same reason as in the case of a type I system. So the only non-zero Weyl scalars are going to be Ψ_2 and Ψ_3, this classification is known as Petrov type II.

By establishing the relationship between the roots of b, we can obtain the different Petrov symmetries. If two pairs of roots are different from each other, all the scalars cancel out except Ψ_2, this is a Petrov type D symmetry. In the case three roots coincide, the only non-zero scalar is Ψ_3, this is a Petrov type III symmetry. If all roots are equal, the only scalar that does not cancel is Ψ_4, and we reach a Petrov type N symmetry. Finally, if all Weyl scalars are zero, the Weyl tensor is zero, so the space is flat. The Petrov type I, where all the roots are distinct, is algebraically general, while the rest are said to be algebraically special and can be obtained from this Petrov type I symmetry.

There is another classification with much connection with the roots method. It is through classifying the relationship of the null vectors with the Riemann tensor for empty space, or in the case of non-empty space with the Weyl tensor, using the following theorem [29]

Theorem 1 *Each non-empty space-time possesses at least one and no more than four distinct null vectors $k^u l_u = 0$ with $k^u \neq 0$, which satisfy the following relation*

$$k_{[u}C_{p]qr[s}k_{t]}k^q k^r = 0. \tag{3.69}$$

In this last equality, the brackets in the subscripts represent the antisymmetrization operator. The vector that satisfies this theorem is said to be a principal null direction (PND). For example, in the case of a Petrov type I symmetry, it can be shown that the theorem for $k^u = l^u$ holds if and only if Ψ_0=0, so l is a principal null vector. Although there are four main null directions, there can also be two or more that coincide; in such cases, we can establish a stronger restriction for these degenerate vectors. The following theorem puts a condition for vectors with a multiplicity of 2 [35].

Theorem 2 *Each non-empty space-time possesses at least one principal null vector $k^u k_u = 0$ with $k^u \neq 0$ with a multiplicity of 2 if it satisfies the following relation*

$$C_{pqr[s}k_{t]}k^q k^r = 0. \tag{3.70}$$

In the case of a Petrov D type solution, we can show that the double degenerate principal null vectors are $k^u = l^u$ for $\Psi_0 = \Psi_1$ and $k^u = n^u$ for $\Psi_3 = \Psi_4 = 0$ [19]. For the rest of the multiplicities, the theorem is the same only that for a multiplicity of 3 the condition is $C_{pqr[s}k_{t]}k^r = 0$, in this case, the PND is $k^u = l^u$ when $\Psi_0 = \Psi_1 = \Psi_2 = 0$ and we are talking about a Petrov type III symmetry; for a multiplicity of 4 the condition is $C_{pqrs}k^r = 0$, the PND is also $k^u = l^u$ when $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and we are talking about a Petrov type N symmetry. All these results are summarized in Table 1.

3.4 The Petrov type D symmetry

We now introduce the metric used in this work as well as the conditions for its symmetry with the help of the results of the previous sections. Consider a more general form of an anisotropic and homogeneous Bianchi type-I universe given as follows

$$ds^2 = d^2dt^2 - a^2dx^2 - b^2dy^2 - c^2dz^2 \tag{3.71}$$

Table 3.1: Values for the multiplicity of the null vectors and roots of the equation (3.66) with the null values of the Weyl Scalars for each Petrov type.

Type	**Multiplicity for the PNDs/Roots**	**Null Weyl scalars**
I	1 for the four different PNDs/roots.	$\Psi_0 = \Psi_4 = 0$
II	2 for a pair of equal PNDs/roots.	$\Psi_0 = \Psi_1 = \Psi_4 = 0$
D	2 for two pairs of equal PNDs/roots.	$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$
III	3 for a trio of PNDs/equal roots.	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$
N	4 for all equal PNDs/roots.	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$

where a, b, c and d are functions of time. We can analyze under what criteria of these functions the Petrov D type is obtained. We proceed with the NP formalism introduced in previous sections. For this, we can use the transformations from (3.36), similar to what was done in [5], but another null tetrad equally valid in covariant form is the following

$$l = \left(d^2, da, 0, 0\right), \tag{3.72}$$

$$n = \left(\frac{1}{2}, -\frac{a}{2d}, 0, 0\right), \tag{3.73}$$

$$m = \left(0, 0, -\frac{ib}{\sqrt{2}}, -\frac{c}{\sqrt{2}}\right), \tag{3.74}$$

$$\bar{m} = \left(0, 0, \frac{ib}{\sqrt{2}}, -\frac{c}{\sqrt{2}}\right). \tag{3.75}$$

The Weyl scalars are given by the relations (3.52)-(3.56), therefore

$$\Psi_0 = -\frac{1}{2bc}\left(b\ddot{c} - c\ddot{b}\right) - \frac{1}{2abcd}\frac{d}{dt}\left(ad\right)\left(c\dot{b} - b\dot{c}\right), \quad \Psi_1 = 0, \tag{3.76}$$

$$\Psi_2 = \frac{1}{6abd^2}\left(a\ddot{b} - b\ddot{a}\right) + \frac{1}{6abcd^3}\left(d\dot{c} + c\dot{d}\right)\left(\dot{a}b - a\dot{b}\right) - \frac{\Psi_0}{6d^2}, \tag{3.77}$$

$$\Psi_3 = 0, \quad \Psi_4 = \frac{\Psi_0}{4d^4}. \tag{3.78}$$

A simple way is to analyze how many roots the polynomial (3.66) has, or equivalently, the polynomial Ψ_4' for a class I rotation (3.63) given by

$$\Psi_4 + 4z\Psi_3 + 6z^2\Psi_2 + 4z^3\Psi_1 + z^4\Psi_0 = 0. \tag{3.79}$$

Here, we changed the notation of a^* by z to avoid confusion with the scale factor a of (3.71). In the case that $a = b$ we have that $\Psi_2 = -\frac{1}{6d^2}\Psi_0$ and then the previous polynomial would be of the form $4d^4z^4 - 4d^2z^2 + 1 = 0$ with two repeated roots $-1/\sqrt{2}d$ and $1/\sqrt{2}d$, from the Table 1 this symmetry is of a Petrov D solution. In the case that $a = c$ the polynomial would be $4d^4z^4 + 4d^2z^2 + 1 = 0$ with two repeated roots $-i/\sqrt{2}d$ and $i/\sqrt{2}d$, so it is also of Petrov D. Finally, if $c = b$ all the Weyl scalars go to zero except Ψ_2 and by looking at the Table 1 we also conclude that this a Petrov type D solution. Then we can conclude that if two of any of the three functions (a, b and c) are equal, the symmetry is of Petrov type D.
For this work, a particular case of (3.71) will be considered with the following anisotropic and homogeneous metric [5]

$$ds^2 = Fdt^2 - t^{2/3}K\left(dx^2 + dy^2\right) - \frac{t^{2/3}}{K^2}dz^2 \tag{3.80}$$

In this case $a = b$ but $c \neq b$, so it is Petrov D, where K and F are functions of time.

Chapter 4

Anisotropic and homogeneous universe with Petrov D symmetry

4.1 Relevant solutions

We can start with the computation of the left part of the field equations (2.7), which would be the geometric part, as the metric is already known, the non-zero Christoffel symbols (2.38) are

$$\begin{gathered}\Gamma^0{}_{00} = \frac{\dot{F}}{2F}, \quad \Gamma^0{}_{11} = \Gamma^0{}_{22} = \frac{1}{6Ft^{1/3}}\left(3t\dot{K} + 2K\right), \\ \Gamma^0{}_{33} = \frac{1}{3Ft^{1/3}K^3}\left(-3t\dot{K} + K\right), \quad \Gamma^1{}_{01} = \Gamma^2{}_{02} = \frac{1}{6tK}\left(3t\dot{K} + 2K\right), \\ \Gamma^3{}_{30} = \frac{1}{3tK}\left(-3t\dot{K} + K\right).\end{gathered} \tag{4.1}$$

We can calculate the Einstein tensor G (2.5) with the help of (2.35), (2.36) and (2.37), obtaining

$$G_{00} = \frac{4K^2 - 9t^2\dot{K}^2}{12t^2K^2}, \tag{4.2}$$

$$G_{11} = G_{22} = \frac{3Ft^2(2K\ddot{K} - 5\dot{K}^2) - 3tK(\dot{F}t - 2F)\dot{K} + 4K^2(\dot{F}t + F)}{12t^{4/3}F^2K}, \tag{4.3}$$

$$G_{33} = -\frac{3Ft^2(4K\ddot{K} - \dot{K}^2) - 6\dot{K}Kt(\dot{F}t - 2F) - 4K^2(\dot{F}t + F)}{12t^{4/3}K^4F^2}. \tag{4.4}$$

In the case of the right-hand side of the equations, the covariant components of the SET (2.16) are $T_{00} = F\rho$, $T_{11} = T_{22} = t^{2/3}KP$ and $T_{33} = t^{2/3}P/K^2$. Then it follows that $T_{11} = K^3T_{33}$ and using the EFE $G_{\mu\nu} = T_{\mu\nu}$, where $\kappa = 1$, we obtained the relationship

$G_{11} - K^3 G_{33} = 0$ and by the expressions for the Einstein tensor previously calculated we have

$$\dot{K}K(2F - \dot{F}t) - 2Ft(\dot{K}^2 - K\ddot{K}) = 0. \tag{4.5}$$

The solution for this last differential equation is

$$K = K_0 e^{C_1 \int \frac{F^{1/2}}{t} dt}. \tag{4.6}$$

We will consider without loss of generality that $K_0 = 1$. While by the equation of state $P = w\mu$ and the EFE we can deduce that $G_{11} = t^{2/3} K w G_{00}/F$, using the results (4.2) and (4.3) together with (4.6) we have

$$F(w-1)(9FC_1^2 - 4) + 4\dot{F}t = 0. \tag{4.7}$$

These equations have two solutions for two cases of w. In the case of a stiff matter fluid where $w = 1$, F is a constant. If we integrate (4.6) the solutions for stiff matter are

$$F = 1, \quad K_{\pm} = t^{\pm 2/3\sqrt{1-3\beta}}, \tag{4.8}$$

where F was chosen to be equal to 1 and then $C_1 \equiv \pm 2/3\sqrt{1-3\beta}$ for β a positive constant. While for $w \neq 1$ the solution is of the form

$$F = \frac{4}{9C_1^2 + 4t^{1-w}C_2} = \frac{1}{1+\alpha t^{1-w}}, \quad K_{\pm} = \left(\frac{\alpha t^{1-w}}{(\sqrt{1+\alpha t^{1-w}} + 1)^2} \right)^{\pm \frac{2}{3(1-w)}}, \tag{4.9}$$

where α is a constant. The scale factors must satisfy that $\alpha > 0$ and $\beta > 0$ for $0 \leq \mu$, in addition $0 \leq \beta \leq 1/3$ so that there are no complex terms. Once we obtain the solutions for these function, the pressure and energy can be obtained by replacing F and K in the EFE, solving either for the pressure or energy, and then obtaining the other one with $P = w\mu$. Regardless of the sign chosen for K, we have

$$\mu = \frac{\alpha}{3t^{w+1}}, \quad P = \frac{w\alpha}{3t^{w+1}} \tag{4.10}$$

in the case that $w \neq 1$, while for $w = 1$ we have to change $\alpha/3$ for β.

The relations (4.10) describe a mixture of fluids through a linear sum of each term. As an example, if we consider a dark energy fluid $P = -\Lambda/8\pi G$, due to the conservation of the SET ${T^{\nu}}_{\mu;\nu} = 0$, the component that contains the density satisfies the following differential equation

$$\dot{\mu}t + \mu - \frac{\Lambda}{8\pi G} = 0. \tag{4.11}$$

The solution to this last differential equation is

$$\mu = \frac{\Lambda}{8\pi G} + \frac{D}{3t}, \tag{4.12}$$

where D is a constant, this relationship would be the linear sum of (4.10) for a fluid with $w = -1$ and $\alpha = 3\Lambda/8\pi G$ plus another one with $w = 0$ and $\alpha = D$, which is a mixture of dark energy and dust. The dust fluid is an implicit solution of the equations since its pressure is zero. To decouple this dust fluid or any other fluid, their respective parameters must be set equal to zero ($\alpha = 0$).

4.1.1 Kasner solution

If α or β are zero and there are no extra sources of matter, we have the following solution

$$ds_{\pm}^2 = dt^2 - t^{2/3 \pm 2/3}(dx^2 + dy^2) - t^{2/3 \mp 4/3} dz^2. \tag{4.13}$$

A particular case is to see if this cosmology can satisfy the criteria of Kasner exponents. The Kasner metric [47] is an anisotropic vacuum solution with the form

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2, \tag{4.14}$$

where p_1, p_2 and p_3 are the Kasner exponents that satisfy the following relations

$$\sum_{i=1}^{d-1} p_i = 1, \tag{4.15}$$

$$\sum_{i=1}^{d-1} p_i^2 = 1, \tag{4.16}$$

where in our case, the dimension is $d = 4$. If we take the negative sign in (4.13), we obtain $p_i = (0, 0, 1)$ which satisfies the previous relations, so it is a Kasner solution. The positive sign $p_i = (2/3, 2/3, -1/3)$ also satisfies the same relations, so this is also a Kasner solution. Note that both solutions are degenerate by having two equal exponents, which come from the fact that we have a Petrov type D symmetry.

These line elements are the simplest particular analytical solutions to the EFE for Bianchi type I symmetry ($d = 1$ in (3.71)) to a universe without matter, anisotropic and homogeneous. Except for the solution K_- where $p_i = (0, 0, 1)$, we will always see exponents with negative signs. It can be proved that there will always be a positive exponent, one non-positive exponent, and one non-negative exponent.

The universe with K_- is diffeomorphic (isomorphism between differentiable manifolds) to that of Minkowski by a coordinate transformation, so it is flat but anisotropic. The K_+ universe where $p_i = (2/3, 2/3, -1/3)$ is not flat since the Riemann tensor is not zero, here the universe contracts in the z direction (as $t^{-1/3}$), and expands in the other two (as $t^{2/3}$). Because the volume is $\sqrt{-g} = t$, there is a singularity at $t \to 0$, from then on the universe expands.

Finally, it is worth mentioning that at $t \to \infty$, the Kasner universes approximate asymptotically to the Einstein-de Sitter model, which is the FLRW model with $k = 0$ dominated by matter without a cosmological constant. This model has been of great interest in theories such as inflation to study observations about the flatness of the universe and its expansion (if fine-tuning with the critical density is considered). However, observations on the universe's accelerated expansion [13] have displaced this scenario to other more complete models such as the ΛCDM universe.

4.1.2 Dark energy solution

In this case $\alpha = 3\Lambda/8\pi G$ and $w = -1$, then the cosmological solution is of the form [5]

$$\begin{aligned} ds_\pm^2 = d\eta^2 - (\sinh(\sqrt{\alpha}\eta))^{2/3} \left(4\frac{\cosh(\sqrt{\alpha}\eta)-1}{\cosh(\sqrt{\alpha}\eta)+1}\right)^{\pm 1/3} \alpha^{\frac{-1\mp 1}{3}} \\ \times \left(dx^2 + dy^2 + \left(4\frac{\cosh(\sqrt{\alpha}\eta)-1}{\alpha(\cosh(\sqrt{\alpha}\eta)+1)}\right)^{\mp 1} dz^2\right), \end{aligned} \tag{4.17}$$

where we used a coordinate transformation for the temporal part $t = \sinh(\eta\sqrt{\alpha})/\sqrt{\alpha}$.

A result that we will study later with the mixture of three fluids is the limit $t \to 0$, or $\eta \to 0$, where in such case we obtain

$$ds_\pm^2 \approx d\eta^2 - \eta^{2/3\pm 2/3}\left(dx^2 + dy^2\right) - \eta^{2/3\mp 4/3} dz^2, \tag{4.18}$$

which is the result without fluids (4.13) from the previous section with a change of variable. In the case that $t \to \infty$ we have that

$$ds_\pm^2 \approx d\eta^2 - e^{2/3\sqrt{\alpha}\eta}\left(dx_\pm'^2 + dy_\pm'^2 + dz_\pm'^2\right) \tag{4.19}$$

In this last metric, we used the change of variables

$$x'_{\pm} = 2^{-1/3 \pm 1/3} \alpha^{-1/6 \mp 1/6} x,$$
$$y'_{\pm} = 2^{-1/3 \pm 1/3} \alpha^{-1/6 \mp 1/6} y,$$
$$z'_{\pm} = 2^{\mp 2/3 - 1/3} \alpha^{-1/6 \pm 1/3} z.$$

The solution (4.19) has the form of de Sitter space-time, a geometrically flat and matter-free solution dominated by dark energy with a positive cosmological constant. The de Sitter solution is important because it is an asymptotic solution to many models for a late universe dominated by dark energy with an accelerated expansion. The work in [82] was the first to demonstrate that all Bianchi models with a positive cosmological constant, except for type IX universes, present an asymptotic behavior of de Sitter. This study has also expanded to various inflationary scenarios as a possible solution to these universes and to study quantum gravity in anti-de Sitter models.

4.1.3 Zeldovich solution

The last single fluid we will take care of is the stiff matter fluid, which dominates in the early universe. We have that $w = 1$ and then by the results of (4.8)

$$ds^2_{\pm} = dt^2 - t^{2/3(1 \pm \sqrt{1-3\beta})} \left(dx^2 + dy^2\right) - t^{2/3(1 \mp 2\sqrt{1-3\beta})} dz^2. \tag{4.20}$$

In the case that $\beta = 0$ we have the Kasner solution (4.13). The stiff matter fluid dominates the early universe. We estimate the time in which this scenario with EoS $P = \mu$ could have existed by considering the density of matter in the transition [86] and the left-hand side of the equation (4.10) $t(\beta) = \sqrt{\beta/(\kappa c^2 \mu)}$. If we remember that $0 \leq \beta \leq 1/3$ and if $\mu_Z \sim 10^{52}\text{gr/cm}^3$ the maximum time for the transition is $t(1/3) \sim \times 10^{-23}\text{s}$ while the minimum time is the Plank time $t_{plank} \sim 10^{-44}\text{s}$ which also sets a bound for the coupling constant of $M_{min} \sim 10^{-40}$ [5]. These values will depend on the model. The one used as an example was proposed by Zeldovich for an EoS of neutral matter [86]. However, other models can be taken into consideration, such as the case of the dynamics of a Bose-Einstein condensate mentioned in Chapter 3.

4.2 The Hubble parameter and the deceleration parameter

For a Bianchi I universe where $d = 1$ in (3.71), we have to consider the average Hubble parameter

$$H \equiv \frac{1}{3}\left(H_x + H_y + H_z\right), \tag{4.21}$$

where $H_x = \dot{a}/a$, $H_y = \dot{b}/b$ and $H_z = \dot{c}/c$ are the Hubble parameters in the directions x, y and z respectively. Here, the average scale factor is defined as the positive cubic root of the volume $R \equiv (abc)^{1/3}$ so that the Hubble parameter is

$$H = \frac{1}{(abc)^{1/3}}\frac{d}{dt}\left(abc\right)^{1/3}. \tag{4.22}$$

If we use the change of variable dt to ddt, we obtained the general case (3.71)

$$H = \frac{1}{(abc)^{1/3}d}\frac{d}{dt}\left(abc\right)^{1/3}. \tag{4.23}$$

The definition of the time-dependent deceleration parameter is then

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = -\left(1 + \frac{\dot{H}}{dH^2}\right). \tag{4.24}$$

When calculating these parameters, we must take into account that for linear fluids, there are two different types of solutions, in the case that $w \neq 1$, by (4.8) we obtain [6]

$$H = \frac{\sqrt{1+\alpha t^{1-w}}}{3t}, \quad q = \frac{\alpha t(1+3w) + 4t^w}{2\left(t^w + \alpha t\right)}, \tag{4.25}$$

while for $w = 1$ by (4.9) we have that

$$H = \frac{1}{2t}, \quad q = 2. \tag{4.26}$$

These last solutions correspond to a fluid of stiff matter, whereas $t \to 0$ we have that $H \to \infty$, while the universe decelerates steadily. The FLRW universe has the same H and q because an isotropization of the parameters occurred when we considered the average of the scale factor in (4.21).

In the case of dark energy ($w = -1$), when $t \to 0$ we have that $H \to \infty$ and $q \to 2$, while when $t \to \infty$ we have that $H \to \sqrt{\alpha}/3$ and $q \to -1$. There is a transition from deceleration to an acceleration of the universe given by q at some point, in this case, when $t = \sqrt{2/\alpha}$. The existence of this transition is in agreement with observations of Supernovae Type Ia from the Supernova Cosmology Project and the High-z SN Search [69, 70], which has been corroborated with kinematic methods based on the Sunyaev-Zel'dovich effect and data from *X-ray surface brightness* for galaxy clusters [53], suggesting a relatively recent time of transition.

4.3 Kretschmann Invariant and Singularities

It is now convenient to analyze the possible singularities of our model. It must be taken into account that a singularity, such as the one presented in the Schwarzschild space-time in $r = 2M$, can be removed by making a suitable change of coordinates systems, like the Lemaître or Kruskal–Szekeres coordinates. However, suppose there is a singularity that any change of variables cannot remove, in that case, we can say that it is a truly geometric singularity and at that point, the space-time will not be well defined [24].

One way to search for this type of non-removable singularities is by determining divergences of curvature given by some scalar described in terms of the Riemann tensor since it is an indicator of space-time curvature. The Kretschmann invariant, generally described as the square of the Riemann tensor, will allow us to find the possible singularities of the model. The Kretschmann invariant has the following form

$$\mathcal{K} = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}. \tag{4.27}$$

For the Schwarzschild metric $K \sim m^2/r^6$, so there is a truly geometric singularity in $r = 0$.

Since the Kasner universe (4.14) is an approximation at $t \to 0$ to the solution with the Petrov symmetry D studied here, we have to see what the Kreschmann invariant says about it. To simplify the expression for $\mathcal{K}$, the Kasner exponents can be put in terms of a variable called the Lifshitz Khalatnikov parameter $u \in \mathbb{R}$ as follows

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}. \tag{4.28}$$

Now, since the Ricci tensor (2.36) vanishes in vacuum, by (3.43), the singularity can also be determined by the Weyl tensor, where it takes the form

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} = \frac{16(1+u)^2u^2}{(1+u+u^2)^3\, t^4} \geq 0. \tag{4.29}$$

The case when it is zero occurs for $u = -\infty, 0, 1, \infty$, which is the same as one of the exponents being equal to 1, so it is also diffeomorphic to Mikowski space-time; for the rest of the cases, there is a truly geometric singularity at $t \to 0$ [49]. We can conclude that the negative case of (4.13) does not present a singularity, unlike the positive case with a singularity that goes as t^4.

For non-empty space and taking into consideration the expressions for the energy density and pressure of (4.10), we have

$$\mathcal{K}_{\pm} = \frac{32t^{2w}(1 \pm \sqrt{1+\alpha t^{1-w}}) + t^2(5+6w+9w^2)\alpha^2 + 16\alpha t^{w+1}}{27t^{2w+4}}. \tag{4.30}$$

If we take the positive sign for dark energy, the singularity at $t \to 0$ is $\mathcal{K}_+ \to 64/(27t^4)$, while the negative case gives $\mathcal{K}_- \to 4\alpha^2/9$ and there is no singularity. Because of (3.43), we expect these limits since in the early universe the model with dark energy reduces to the particular case of Kasner (4.13). Meanwhile, in the case of $t \to \infty$, $\mathcal{K}_\pm$ does not present a singularity.

In the case of stiff matter, since $w = 1$ and $\alpha = 3\beta$ we have

$$\mathcal{K} = \frac{4(45\beta^2 + 12\beta \pm 8\sqrt{3\beta+1} + 8)}{27t^4} \tag{4.31}$$

which is a singularity with the same form of $\mathcal{K}_+$ with dark energy ($\sim t^{-4}$), but that falls more slowly towards the singularity if we consider $0 < \beta \leq 1/3$.

Chapter 5

The stability of cosmological solutions with a scalar field

5.1 Jacobi stability and the theory of Kosambi-Cartan-Chern

A scalar field coupled with the Einstein-Hilbert action possesses non-linear terms in the system's dynamical evolution. The non-linearity might cause exponential deviations in the geodesics lines and difficulties in the model's predictive power for late times due to the inevitable uncertainty in the initial conditions of the Universe. Therefore, it is imperative to determine the stability of the dynamical system. The approach we use is the Jacobi Stability, which can be regarded as the resistance or robustness of the system to exponential deviations of geodesics in a manifold from small perturbations [15].

Exponential deviations from initial conditions are present in many dynamical systems in physics. So, it will be relevant to answer briefly the origin of such behavior. Consider a system of first-order ordinary differential equations, where

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \tag{5.1}$$

and $\mathbf{y}(x_1, x_2, ..., x_i)$ depend on the phase space coordinates $x_1, x_2, ..., x_i$. For a small deviation $\delta\mathbf{y}$ in phase space, a first-order Taylor expansion gives

$$\delta\mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \delta\mathbf{y}) - \mathbf{f}(\mathbf{y}) \approx D\mathbf{f} \cdot \delta\mathbf{y}, \tag{5.2}$$

here $(D\mathbf{f})_{ij} = \partial f_i/\partial x^j$ is the Jacobian matrix [40]. It is straightforward to see that in the one-dimensional case,

$$\frac{d\delta y}{dt} = f'(x)\delta y, \tag{5.3}$$

there is a local exponential grow due to $\delta y \propto e^{f'(x)t}$. The sign of the exponent in the constant of proportionality, set by $f'(x)$ evaluated at a specific point, will give the stability of the solution: if $f'(x) < 0$, the solution is stable, and if $f'(x) > 0$ the solution is unstable. This criteria is enough since the exponential growth of the path dominates in the linear approximation.

The generalization to the n-dimensional case is made by determining the sign of the eigenvalues given by the Jacobian matrix evaluated at specific points, called fixed points. A fixed point on a ordinary differential equation (5.1) is a solution $\bar{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{f}(\bar{\mathbf{y}}) = 0$. If the initial condition is $\mathbf{y}(t_0) = \bar{\mathbf{y}}$, so for all t the path of the system in unchanged $\mathbf{y}(t) = \bar{\mathbf{y}}$. Therefore, for a deviation in the path $\mathbf{y} = \bar{\mathbf{y}} + \delta\mathbf{y}$, the one-dimensional Jacobian matrix is just $f(\mathbf{y})|_{\bar{\mathbf{y}}}$; for a two-dimensional phase space and considering a system of linear differential equations

$$\dot{x}_1 = f(x_1, x_2), \quad \dot{x}_2 = g(x_1, x_1), \tag{5.4}$$

the perturbations in the path set by the fixed points satisfy the following two systems of equations in matrix form

$$\frac{d}{dt}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix} = \begin{pmatrix}\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2}\\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2}\end{pmatrix}\Bigg|_{x_1=\bar{x}_1, x_2=\bar{x}_2}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix}, \tag{5.5}$$

so the Jacobian matrix comprises of partial derivatives of the functions evaluated at these fixed points. The next theorem is of help in determining the stability of these dynamical systems.

Theorem 3 *A fixed point in a linear homogeneous system is said to be stable if and only if all the real parts of the eigenvalues λ_i from the Jacobian matrix with $i \in \{1, 2, 3, .., n\}$ evaluated at the fixed point are strictly negative, otherwise is unstable if at least one is strictly positive.*

The stability is determined by obtaining the sign of the eigenvalues from the matrix in (5.5); for higher dimensions, the generalization of (5.5) follows the same form and theorem for the criteria of stability.

So far, this development is part of the Liapunov linear stability, a very well-known and used tool to study multitudes of dynamical systems. However, in the case of EFE coupled with a scalar field, there will be second-order ordinary differential equations. A powerful tool for computing the Jacobi stability in this case is the KCC, initially developed by the authors with the same name.

Following the formalism presented in [15,28], the idea is to use a set of second-order differentials equations, equivalent to Euler-Lagranges Equations and assigns two types of connections: a Bernwald connection and a non-linear connection. We perturb the trajectories in these equations by a small parameter to obtain the Jacobi Equation, which possesses the second invariant of five in KKC, such invariant is the deviation curvature tensor that gives the stability of the system.

The KKC theory is a geometrical approach based on the path deviations from nearby trajectories in a manifold. Let $\mathcal{M}$ be an n-dimensional smooth manifold with local coordinates (x^i) and $\mathcal{TM}$ its tangle bundle with a set of $2n{+}1$ induced coordinates (x^i, y^i) where

$$x = (x^1, x^2, \ldots, x^i), \tag{5.6}$$

$$y = (y^1, y^2, \ldots, y^i), \tag{5.7}$$

for $i \in \{1, 2, \ldots, n\}$ and $y^i = \dot{x}^i$. The second-order ordinary differential equation on $\mathcal{M}$ can be represented by a vector field on the tangent bundle $\mathcal{TM}$ called semispray S of the form

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial x^i}, \tag{5.8}$$

here $G(x, y)$ are some local coefficients of the semispray defined on domains of (x, y). For a curve $c(t) = (x^i(t))$ and its tangent lift $c'(t) = (x^i(t), dx^i(t)/dt) \in \mathcal{TM}$, $c(t)$ is a geodesic of S if and only if

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x^i, \frac{dx^i}{dt}\right) = 0, \tag{5.9}$$

thus the curve of (5.9) represents a second-order differential equation on $\mathcal{M}$ equivalent to a dynamical system, such as the one given by the Euler-Lagrange equations [16]. We can construct a geometrical structure by introducing a nonlinear connection on $\mathcal{M}$, called horizontal distribution N, using the following theorem [60].

Theorem 4 *Let S be a semispray with coefficients $G^i(x, y)$, then*

$$N^i_j \equiv \frac{\partial G^i}{\partial y^j} \tag{5.10}$$

are the coefficients of a nonlinear connection of N on $\mathcal{TM}$.

Now if the path deviates infinitesimally from its trajectory $\tilde{x}^i(t) = x^i(t) + \epsilon\xi^i(t)$ where ϵ is a small parameter and $\xi^t(t)$ goes along $x(t)$, because of (5.9), the perturbed dynamical system acquires the following form

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j\frac{d\xi^i}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0. \tag{5.11}$$

The KKC-covariant differential acting on a vector field v^i is defined as

$$\frac{Dv^i}{dt} = \frac{dv^i}{dt} + N^i_j v^j. \tag{5.12}$$

In the case of y^i and with the help of (5.9)

$$\frac{Dy^i}{dt} = N^i_j y^j - 2G^i. \tag{5.13}$$

Taking this into account, the equation (5.11) can finally be written as the Jacobi equation

$$\frac{D^2\xi^i}{dt^2} = P^i_j\xi^j, \tag{5.14}$$

P^i_j is the deviation curvature tensor and second invariant of the KKC theory

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l\frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j, \tag{5.15}$$

where the Bernwald connections G^i_{jl} are defined as [1]

$$G^i_{jl} \equiv \frac{\partial N^i_j}{\partial y^l}. \tag{5.16}$$

The work in [15] showed the equivalence between the linear stability of (5.4) and the Jacobi stability by comparing the signs of the eigenvalues of the Jacobi matrix and the deviation curvature tensors evaluated at fixed points. Hence, similarly to how the eigenvalues of the Jacobian matrix give the stability of linear systems near stationary points, the deviation curvature tensor will encode information about the robustness of the trajectories.

Theorem 5 *The trajectories of a second-order ordinary differential equation given by (5.9) are Jacobi stable if and only if all the real parts of the eigenvalues of the deviation curvature tensor P^i_j are strictly negative, otherwise, is Jacobi unstable if at least one is strictly positive.*

The last theorem will help establish the stability of the scalar field along its dynamical evolution. For the sign of the eigenvalues, the characteristic equations of P^i_j with polynomial form have negative real parts solutions if the Hurwitz determinants are strictly positive. This is the Routh-Hurwitz criteria for stability. So in the case of a two-dimensional dynamical system, the characteristic equation of the deviation curvature tensor is

$$\lambda^2 - (P^1_1 + P^2_2)\lambda + (P^1_1 P^2_2 - P^1_2 P^2_1) = 0. \tag{5.17}$$

From this last polynomial, the relevant Hurwitz determinants and the conditions for stability are

$$H_1 = -(P^1_1 + P^2_2) > 0, \tag{5.18}$$

$$H_2 = \begin{vmatrix} -(P^1_1 + P^2_2) & 0 \\ 1 & (P^1_1 P^2_2 - P^1_2 P^2_1) \end{vmatrix} > 0. \tag{5.19}$$

In a more precise way, the components of the deviation curvature tensor need to satisfy

$$P^1_1 + P^2_2 < 0, \quad P^1_1 P^2_2 - P^1_2 P^2_1 > 0, \tag{5.20}$$

so that the system is Jacobi stable. These inequalities (5.20) are sufficient since our system of equations is a two-dimensional dynamical system. The following section applies the KKC theory to cosmological models with scalar fields to find the stability.

5.2 Jacobi stability for scalar field cosmology: the flat FLRW universe

The procedure is as follows: 1) find a pair of coupled differential equations that describe the dynamical system entirely, 2) identify the phase space variables (y^i, x^i), 3) write down the equation $\ddot{x}^i + 2G^i(x^i, \dot{x}^i) = 0$ to recognize the function $G^i(x^i, \dot{x}^i)$, 4) calculate the P^i_j and 5) determine the stability with the inequalities of (5.20).

An introductory example is the flat FLRW with a scalar field, which was first developed in [28]. Referring back to the Friedmann equations (2.42) and (2.43), the scalar field is coupled with the fluids through (2.51) and (2.52), so if we eliminate the thermodynamics variables

$$3\left(\frac{\dot{a}}{a}\right)^2 = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{5.21}$$

$$2\left(\frac{\ddot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{\dot{a}}\right) = -\frac{\dot{\phi}^2}{2} + V(\phi). \tag{5.22}$$

We still have three variables, considering $H = \dot{a}/a$ a third one, so an extra equation is necessary to reduce the system to the two-dimensional case. The conservation of the SET gives this last equation, $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$. Now, it is possible to eliminate H

$$\ddot{\phi} + \frac{a}{2}\left(\dot{\phi}^2 - V(\phi)\right) = 0, \tag{5.23}$$

$$\ddot{\phi} + \dot{\phi}\sqrt{\frac{3}{2}\dot{\phi}^2 + 3V(\phi)} + V'(\phi) = 0. \tag{5.24}$$

Clearly, the phase-space variables have to be $x^1 = a, x^2 = \phi, y^1 = \dot{a}$ and $y^2 = \dot{\phi}$. By comparing to (5.9) the local coefficients are

$$G^1(a,\phi,\dot{a},\dot{\phi}) = \frac{a}{6}\left(\dot{\phi}^2 - V(\phi)\right), \tag{5.25}$$

$$G^2(a,\phi,\dot{a},\dot{\phi}) = \frac{\dot{\phi}}{2}\sqrt{\frac{3}{2}\dot{\phi}^2 + 3V(\phi)} + \frac{V'(\phi)}{2}. \tag{5.26}$$

$$\tag{5.27}$$

To compute P^i_j from (5.15), we have to known first the nonlinear connection N^i_j (5.10) and the Bernwald connection G^i_{jl} (5.16), after some relatively easy but tedious calculations

$$P^1_1 = \frac{1}{3}\left(-\dot{\phi}^2 + V(\phi)\right), \quad P^1_2 = \frac{1}{3}\dot{a}\dot{\phi} - \frac{a\dot{\phi}V(\phi)}{\sqrt{12V(\phi) + 6\dot{\phi}^2}}, \quad P^2_1 = 0, \tag{5.28}$$

$$P^2_2 = -V''(\phi) - \frac{\sqrt{6}\dot{\phi}V'(\phi)}{\sqrt{2V(\phi) + \dot{\phi}^2}} + \frac{9V(\phi)^2}{2(2V(\phi) + \dot{\phi}^2)} - \frac{3V(\phi)}{2}. \tag{5.29}$$

Finally, the system is stable if

$$P_1^1 + P_2^2 =$$
$$\frac{1}{3}\left(-\dot{\phi}^2 + V(\phi)\right) - V''(\phi) - \frac{\sqrt{6}\dot{\phi}V'(\phi)}{\sqrt{2V(\phi)+\dot{\phi}^2}} + \frac{9V(\phi)^2}{2(2V(\phi)+\dot{\phi}^2)} - \frac{3V(\phi)}{2} < 0, \quad (5.30)$$
$$P_1^1P_2^2 - P_2^1P_1^2 = P_1^1P_2^2 =$$
$$\frac{1}{3}\left(-\dot{\phi}^2 + V(\phi)\right)\left(-V''(\phi) - \frac{\sqrt{6}\dot{\phi}V'(\phi)}{\sqrt{2V(\phi)+\dot{\phi}^2}} + \frac{9V(\phi)^2}{2(2V(\phi)+\dot{\phi}^2)} - \frac{3V(\phi)}{2}\right) > 0. \quad (5.31)$$

This is the general expression for any potential. From this point, it will be necessary to either establish the potential $V(\phi)$ or assume the existence of some equation of state that can give its evolution, where in the case of this work, it will be the latter.

5.3 Jacobi stability in a Petrov type D with a scalar field

Finally, the procedure explained in the last section must be applied to the solution (3.80) with a scalar field to find the Jacobi Stability; this was first obtained in [8]. The lagrangian of the scalar field is given by

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \quad (5.32)$$

Because of homogeneity, the only term that survives here is the kinetic term together with the potential. The SET with the scalar field is given by (2.50) and has the form

$$T_\mu^\nu = \frac{\dot{\phi}}{F}\delta_0^\nu\delta_\mu^0 - \delta_\mu^\nu\mathcal{L}_\phi. \quad (5.33)$$

Now, with the help of the EFE and the expressions derived in the section (4.1), the next two equations follow

$$-\frac{F-1}{3Ft^2} = \frac{\dot{\phi}^2 + 2VF}{2F}, \quad (5.34)$$
$$\frac{F^2 - \dot{F}t - F}{3F^2t^2} = \frac{-\dot{\phi}^2 + 2VF}{2F}. \quad (5.35)$$

To eliminate the explicit dependence of the time variable, a third equation is necessary, which comes from the condition of conservation of the SET

$$\dot{\mu}t + \mu + P = 0. \tag{5.36}$$

or in terms of the scalar field

$$\ddot{\phi} + \dot{\phi}\left(\frac{1}{t} - \frac{1}{2}\frac{\dot{F}}{F}\right) + F\partial_\phi V = 0. \tag{5.37}$$

These equations specified the complete dynamical evolution of the scalar field, so by combining (5.34) with (5.35)

$$\frac{\dot{F}}{3tF^2} + 2V = 0, \tag{5.38}$$

and then with (5.37), it is easy to show that one of the second-order differential equations is

$$\ddot{\phi} + F\partial V_\phi - \frac{\dot{\phi}\dot{F}}{2F} - \frac{6F^2V\dot{\phi}}{\dot{F}} = 0, \tag{5.39}$$

and the other one comes from the time derivative of (5.38) and the original equation to remove the leftovers t's

$$\ddot{F} - \frac{2\dot{F}^2}{F} + 6VF^2 - \frac{\partial_\phi V\dot{\phi}\dot{F}}{V} = 0. \tag{5.40}$$

The new notation for the variables will be $x^1 = F, y^1 = \dot{F}, x^2 = \phi$, $y^2 = \dot{\phi}$ and $V' = \partial_{x^2}V$, now by comparison with (5.9), the local coefficients are

$$G^1 = -\frac{V'y^2y^1}{2V} + 3(x^1)^2V - \frac{(y^1)^2}{x^1}, \quad G^2 = \frac{x^1V'}{2} - \frac{y^2y^1}{4x^1} - \frac{3(x^1)^2y^2V}{y^1}. \tag{5.41}$$

Therefore, with the help of the last expressions, the nonlinear connections (5.10) are

$$N_1^1 = -\frac{V'y^2}{2V} - \frac{2y^1}{x^1}, \quad N_2^1 = -\frac{V'y^1}{2V}, \quad N_1^2 = -\frac{y^2}{4x^1} + \frac{3\left(x^1\right)^2Vy^2}{\left(y^1\right)^2}, \tag{5.42}$$

$$N_2^2 = -\frac{y^1}{4x^1} - \frac{3\left(x^1\right)^2V}{y^1}, \tag{5.43}$$

and the Bernwald connections (5.10) are

$$G^1_{11} = -\frac{2}{x^1}, \quad G^1_{21} = G^1_{12} = -\frac{V'}{2V}, \quad G^2_{22} = G^1_{22} = 0, \tag{5.44}$$

$$G^2_{11} = -\frac{6(x^1)^2 y^2 V}{(y^1)^3}, \quad G^2_{21} = G^2_{12} = -\frac{1}{4x^1} + \frac{3\,(x^1)^2\, V}{(y^1)^2}. \tag{5.45}$$

At last, the components of the deviation curvature tensor (5.15) are

$$P^1_1 = -\frac{y^1 y^2 V'}{8x^1 V} + \frac{x^1 V'^2}{2V} - \frac{9\,(x^1)^2\, y^2 V'}{2y^1} - \frac{(y^2)^2\, V''}{2V} + \frac{3\,(y^2)^2\, V'^2}{4V^2}, \tag{5.46}$$

$$P^2_2 = -x^1 V'' + \frac{9\,(x^1)^2\, y^2 V'}{2y^1} - \frac{y^1 y^2 V'}{8x^1 V} + 3x^1 V - \frac{9\,(x^1)^4\, V^2}{(y^1)^2} - \frac{3\,(y^1)^2}{16\,(x^1)^2}, \tag{5.47}$$

$$P^1_2 = \frac{y^1 y^2 V''}{2V} - \frac{3y^1 y^2 V'^2}{4V^2} - \frac{3\,(x^1)^2\, V'}{2} + \frac{(y^1)^2\, V'}{8x^1 V}, \tag{5.48}$$

$$P^2_1 = -\frac{3V'}{4} + \frac{3y^1 y^2}{16\,(x^1)^2} - \frac{9\,(x^1 y^2)^2\, V'}{2\,(y^1)^2} + \frac{45\,(x^1)^4\, y^2 V^2}{(y^1)^3} - \frac{3\,(x^1)^3\, V V'}{(y^1)^2} + \frac{(y^2)^2\, V'}{8x^1 V}. \tag{5.49}$$

The first derivative of the potential is given by the equation (5.37). Even though the first derivative does not depend on x^2, the second derivative with respect to this variable is still not zero because the potential was already evaluated at a particular solution. So by deriving (5.37) with respect to time and using the fact that $\dot{V}' = V'\dot{\phi}$, the second derivative is

$$V'' = -\frac{\dot{F}^2}{F^3} + \frac{1}{F^2}\left(\frac{3\ddot{\phi}\dot{F}}{2\dot{\phi}} + \frac{\ddot{F}}{2} + \frac{\dot{F}}{t}\right) + \frac{1}{F}\left(\frac{1}{t^2} - \frac{\dddot{\phi}}{\dot{\phi}} + \frac{\ddot{\phi}}{\dot{\phi}t}\right) \tag{5.50}$$

The deviation curvature tensor does not depend explicitly on x^2, so an expression for the scalar field is unnecessary. However, the time derivative y^2 is required, which can be obtained by eliminating V from (5.34) and (5.35)

$$\dot{\phi}^2 + \frac{2F^2 - 2F - \dot{F}t}{3Ft^2} = 0. \tag{5.51}$$

The only thing left is to specify the scale factors and the potentials from the fluids present in the universe, which will be done later in the next chapter with a proper discussion about the system's stability.

Chapter 6

Analysis of the Petrov D universe with dark energy, dust and stiff matter

6.1 Solutions for the scale factors and the scalar field

We now present the solutions for the anisotropic and homogenous Petrov type D universe with the three mixture of fluids. The general form of $F(t)$ in terms of the total energy density can be written as

$$F(t) = \frac{4}{9C_1^2 + 12t^2\mu_T}, \tag{6.1}$$

where μ_T is the total energy of the universe and C_1 is the same constant in $K(t)$ from (4.6). We know that the following EoS governs each fluid

$$\mu_w = \frac{C_w}{t^{1+w}}, \quad P_w = \frac{wC_w}{t^{1+w}}, \tag{6.2}$$

with $C_w > 0$, so the total pressure and the total energy density are

$$P_T = -\Lambda + \frac{Z}{t^2}, \tag{6.3}$$

$$\mu_T = \frac{Z}{t^2} + \frac{D}{t} + \Lambda. \tag{6.4}$$

Using (6.1), the scale factor $F(t)$ has the form

$$F(t) = \frac{1}{3\Lambda t^2 + 3Dt + 3Z + 1} \tag{6.5}$$

where $C_1 = \pm 2/3$. The scale factor $K(t)$ can be obtained by integrating the exponent in (4.6)

$$K_{\pm}(t) = K_0 \left(\frac{2\sqrt{1+3Z}\sqrt{3\Lambda t^2 + 3Dt + 3Z + 1} - 3Dt - 6Z - 2}{2\sqrt{1+3Z}\sqrt{3\Lambda t^2 + 3Dt + 3Z + 1} + 3Dt + 6Z + 2} \right)^{\pm \frac{1}{3\sqrt{1+3Z}}}. \tag{6.6}$$

A few comments follow about this last result. The first one is that if the stiff matter fluid is decoupled ($Z = 0$), $K(t)$ will take the expected form of a universe filled with a mixture of an isobaric fluid (such as dark energy) and a dust fluid found in [8], where $K_0 = 1$. If dust and dark energy fluids are zero, the scale factor $K(t)$ is also zero, which is a problem in the z-direction because the growth is always infinite. Therefore, reducing the scale factor to a single fluid of stiff matter is not possible. Even though it blows up at $t = 0$, an analysis is still possible in the proximity of $t \approx 0$ when the stiff matter is the most dominant. In the limit $t \to 0$, the scale factors are

$$F \approx \frac{1}{3Z+1}, \quad K_{\pm}(t) \approx K_{0\pm} \left(\frac{3t^2(-3D^2 + 4\Lambda + 12\Lambda Z)}{16(3Z+1)^2} \right)^{\pm 1/(3\sqrt{3Z+1})}, \tag{6.7}$$

so F approximates a constant that only depends on the stiff matter parameter, similar to the model with just this fluid. A way of understanding this result better is by substituting the time variable with $\eta = \frac{t}{\sqrt{3Z+1}}$ and choosing

$$K_{0\pm} = \left(\frac{3(-3D^2 + 4\Lambda + 12\Lambda Z)}{16(3Z+1)^2} \right)^{\mp 1/(3\sqrt{3Z+1})}, \tag{6.8}$$

then the line element takes the next form

$$ds^2 \approx d\eta^2 - t^{2/3 \pm 2/(3\sqrt{3Z+1})}(dx^2 + dy^2) - t^{2/3 \mp 4/(3\sqrt{3Z+1})} dz^2. \tag{6.9}$$

This last result is very similar to the universes of Kasner (4.13). If we use the criteria of Kasner exponents

$$p_{1\pm} = p_{2\pm} = \frac{1}{3} \pm \frac{1}{3\sqrt{3Z+1}}, \tag{6.10}$$

$$p_{3\pm} = \frac{1}{3} \mp \frac{2}{3\sqrt{3Z+1}}, \tag{6.11}$$

the first condition (4.15) is satisfied but for the second condition (4.16)

$$\sum_{i=1}^{d-1} p_i^2 = \frac{Z+1}{3Z+1}, \tag{6.12}$$

which means that there is a tendency to the behavior of Kasner models for very small values of Z. Also, we notice that the time coordinate η reduces to the original coordinate t, so we obtain the exact form of (4.13) at first order with only a difference in some constants. Of course, because the universe is non-empty, the model can never be reduced to the Kasner behavior, a vacuum solution.

Similar to the single-fluid model with stiff matter presented in (4.20), the positive value K_+ is an anisotropic space-time of Petrov D which contracts in the z direction and expands in the x and y directions, the negative value K_- is an anisotropic flat space-time.

Something to ask is what happened to the restrictions of the Z parameter (the β parameter in section 4.1.3). The solution presented (6.6) is using $C_1 = \pm\frac{2}{3}$, but the condition $F = 1$ requires that $C_1 = \pm 2/3\sqrt{1-3\beta}$ and this limits the fluid presence throughout the history of the universe for a distinct value to the one in (6.6).

In the case of the late universe when $t \to \infty$ and choosing

$$K_{0\pm} = \left(\frac{2\sqrt{3\Lambda + 9\Lambda Z} - 3D}{2\sqrt{3\Lambda + 9\Lambda Z} + 3D}\right)^{\mp 1/(3\sqrt{1+3Z})}, \tag{6.13}$$

the scale factors are approximated as

$$K_\pm \approx 1, \tag{6.14}$$

$$F \approx \frac{1}{3\Lambda t^2}. \tag{6.15}$$

To understand this last result better, consider a change of the temporal variable $t = e^{\sqrt{2\Lambda}\eta}$

$$ds^2 = d\eta^2 - e^{2/3\sqrt{3\Lambda}\eta}(dx^2 + dy^2 + dz^2). \tag{6.16}$$

So when $t \to \infty$ this solution has the same behavior of a single-fluid model of dark energy for the same Petrov D solution (4.19) and a FLWR with the same mixture of fluids (2.45). This solution is not a surprise because it is a de Sitter space-time, which is expected for this kind of model as an asymptotic solution.

To get the form of the scalar field, if $F(t)$ in (5.51) is replaced by (6.5)

$$\dot{\phi}^2 = \frac{Dt + 2Z}{t^2(3\Lambda t^2 + 3Dt + 3Z + 1)} \tag{6.17}$$

and taking the positive value of the square root and integrating with respect to time

$$\begin{aligned} \phi(t) = -\frac{2D}{\sqrt{3\Lambda}} L_1 L_2 \Bigg[& F\left(L_1\sqrt{Dt+2Z}, L_2\right) - \Pi\left(L_1\sqrt{Dt+2Z}, \frac{1}{2ZL_1^2}, L_2\right) \\ & + F\left(\sqrt{2Z}L_1, L_2\right) - \Pi\left(\sqrt{2Z}L_1, \frac{1}{2ZL_1^2}, L_2\right)\Bigg] \end{aligned} \tag{6.18}$$

where $F(z, k)$ is the incomplete elliptic integral of the first kind and $\Pi(z, \nu, k)$ the incomplete elliptic integral of the third kind respectively. Here we have that

$$L_1 \equiv \sqrt{\frac{6\Lambda}{12\Lambda Z - D\sqrt{9D^2 - (36Z+12)\Lambda} - 3D^2}}, \tag{6.19}$$

$$L_2 \equiv \sqrt{\frac{12\Lambda Z - D\sqrt{9D^2 - (36Z+12)\Lambda} - 3D^2}{12\Lambda Z + D\sqrt{9D^2 - (36Z+12)\Lambda} - 3D^2}}. \tag{6.20}$$

An easier way of studying the same behavior (6.5) and (6.6) is by taking the next new set of variables $\eta = t/\zeta$ and $x_i' = \zeta x_i$ with the change of constants $D' = D/\zeta$ and $\Lambda' = \Lambda$ where $\zeta = \sqrt{3Z+1}$. If we use this substitution, the scale factors remain the same, but the scalar field takes the following form

$$\phi(t) = -\frac{2}{\sqrt{3}}\sqrt{L_1^2 - 1}\left(F\left(\sqrt{L_2 t + 1}, \frac{L_1}{\sqrt{2}}\right) - K\left(\frac{L_1}{\sqrt{2}}\right)\right), \tag{6.21}$$

and in this case

$$L_1 = \sqrt{\frac{3\,D + \sqrt{9\,D^2 - 12\,(3\,Z+1)\,\Lambda}}{\sqrt{9\,D^2 - 12\,(3\,Z+1)\,\Lambda}}}, \tag{6.22}$$

$$L_2 = \frac{6\Lambda}{3\,D + \sqrt{9\,D^2 - 12\,(3\,Z+1)\,\Lambda}}. \tag{6.23}$$

Now in order to obtain the scalar potential $V(\phi)$ consider $V = 1/2(\mu - P)$

$$V = \frac{D}{2t} + \Lambda \tag{6.24}$$

and using the result of (6.21) then

$$V(\phi) = \Lambda - \frac{DL_2}{2}\mathrm{nc}\left(\frac{\sqrt{3}}{2\sqrt{L_1^2-1}}(\phi-\phi_0), \frac{L_1}{\sqrt{2}}\right)^2 \tag{6.25}$$

where $\phi_0 = \frac{2}{\sqrt{3}}\sqrt{L_1^2-1}K(L_1/\sqrt{2})$ and $\mathrm{nc}(x,n)$ is the elliptic function of Jacobi. It is interesting to see how the EoS given by the scalar field (2.53) without the change of variable (6.5) is

$$w_\phi = \frac{-\Lambda t^2 + Z}{\Lambda t^2 + Dt + Z}. \tag{6.26}$$

In the early universe when $t \to 0$, we have $w_\phi \approx 1 + \mathcal{O}(t)$, so the EoS is approximately of stiff matter; while in the late universe, when $t \to \infty$, $w_\phi \approx -1 + \mathcal{O}(1/t)$, which is dark energy. The model also has a radiation time around

$$t_R = \frac{\sqrt{D^2 + 32\Lambda Z} - D}{8\Lambda} \tag{6.27}$$

which happens after the stiff matter era if all parameters are non-zero. In the case of the change of variable suggested, the model only has negative values or zero $w_\phi = -\Lambda t/(\Lambda t + D)$, so the earliest fluid to appear is of dust.

6.2 Krestchman Invariant

The general form of this invariant (4.27) with the help of $K(t)$ in terms of $F(t)$ given by (4.6) is

$$\mathcal{K} = \frac{\pm 32F^{7/2} + 36F^4 - 24F^3 + 20F^2 + 24F\dot{F}t(1-F) + 9\dot{F}^2t^2}{27F^4t^4}. \tag{6.28}$$

Now with the use of $F(t)$ from (6.5) we have

$$\begin{aligned}\mathcal{K} = \frac{1}{27t^4}\Big[&\pm\sqrt{3At^2 + 3Bt + 3Z + 1} + 72A^2t^4 + 36ABt^3 \\ &+ ((-72Z + 48)\Lambda + 45B^2)t^2 + (144Z + 48)Bt + 180Z^2 + 48Z + 32\Big].\end{aligned} \tag{6.29}$$

The approximation to $t \to 0$ reveals a clear singularity

$$\mathcal{K} = \frac{4\left(45Z^2 + 12Z \pm 8\sqrt{3Z+1} + 8\right)}{27t^4} \tag{6.30}$$

with the same behavior $\sim t^{-4}$ of dark energy and the stiff fluid discussed before, and at first order, it has the same form as this last fluid. A change of the variables will not alter the singularity because the Krestchmann is independent of the coordinate system, so this kind of result is expected to behave similarly to the one obtained in [8].

6.3 Hubble and deceleration parameters

According to the expression for the average Hubble parameter presented in section 4.2, in this case, $H(t)$ is

$$H(t) = \frac{\sqrt{3\Lambda t^2 + 3Dt + 3Z + 1}}{3t} \tag{6.31}$$

In the early universe, when $t \to 0$, the Hubble parameter approximates $H \to \infty$. In the late universe, when $t \to \infty$, the Hubble tends to a constant value in terms of the dark energy parameter $H \to \sqrt{\Lambda/3}$.

In the case of the deceleration parameter

$$q(t) = -\frac{6\,\Lambda t^2 - 3Dt - 12Z - 4}{6\Lambda t^2 + 6Dt + 6Z + 2} \tag{6.32}$$

When $t \to 0$ the universe is decelerating $q \to 2$ and when $t \to \infty$ is accelerating $q \to -1$ with a time of transition given by

$$t_0 = \frac{3D + \sqrt{288\Lambda Z + 9D^2 + 96\Lambda}}{12\Lambda} \tag{6.33}$$

As already discussed, this transition has been observed by data of Supornavae Type Ia in [69, 70] and corroborated by other works, including kinematic methods of galaxy clusters in [53].

6.4 Jacobi Stability

To establish the criteria of stability (5.20), we need the following values in terms of the components of the deviation curvature tensor

$$
\begin{aligned}
P_1^1 + P_2^2 = & -\frac{(3Z+1)\left(144\Lambda^4 t^6 + 576\Lambda^3 D t^5\right)}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
& -\frac{12\Lambda^2(3Z+1)\left(69D^2 + 8\Lambda(3Z+1)\right)t^4}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
& -\frac{36\Lambda D(3Z+1)\left(11D^2 + 12\Lambda(3Z+1)\right)t^3}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
& -\frac{(3Z+1)\left(27D^4 + 336\Lambda D^2(3Z+1) + 64\Lambda^2(3Z+1)^2\right)t^2}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
& -\frac{4D(3Z+1)^2\left(9D^2 + 60\Lambda Z + 20\Lambda\right)t}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
& -\frac{12D^2(3Z+1)^3}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2},
\end{aligned}
\tag{6.34}
$$

and

$$
\begin{aligned}
P_1^1 P_2^2 - P_2^1 P_1^2 = & \frac{3D(3Z+1)^2\left(9D\Lambda^2 t^4 + 18D^2\Lambda t^3\right)}{16t^4(2\Lambda t + D)^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2} \\
& + \frac{3D(3Z+1)^3\left(18\Lambda D t^2 + 8\Lambda(3Z+1)t\right)}{t^4(2\Lambda t + D)^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2} \\
& + \frac{3D^2(3Z+1)^4}{16t^4(2\Lambda t + D)^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2}.
\end{aligned}
\tag{6.35}
$$

By using the equations (5.46)-(5.49), we can conclude that the model is stable for all the possible values of the parameters and for the entire time of the universe.

The stability in this case is not very common, at least in the models studied in the literature with minimally coupled scalar fields. In fact, in [28], an exponential potential $V = V_0 e^{\lambda\phi}$ in FLRW is Jacobi unstable due to the exponential growth of the deviation vector ξ^i near the origin and regardless the value to the λ parameter. The Higgs potential $V(\phi) = V_0 + \frac{1}{2}M^2\phi^2 + \frac{\lambda}{4}\phi^4$ where M is the Higgs boson's mass has regions of stability and instability for different values of λ, due the more complex oscillatory behavior of the field throughout the universe evolution. A scalar field with a power-law potential $V = V_0\phi^\alpha$ and a Tachyon scalar field with these types of potentials are also Jacobi unstable for all time intervals independent of the value of the parameter α [28].

In the flat FLRW universes, there might be the case of a model that has stable and unstable regions but becomes completely unstable for particular parameter values, such as the FLRW universe with a Higgs field. We can also have stable models that become unstable much time later than the current age of the universe, such as in [8] with a scalar field equivalent to dust and dark energy. Therefore, a stability analysis is an excellent tool for discriminating cosmological models that can be tested observationally.

Chapter 7

Summary

In this work, we found and studied the solutions for an anisotropic and homogeneous cosmological model of Petrov type D symmetry and a scalar field equivalent to a mixture of three fluids: dark energy, stiff matter, and dust. The stiff matter fluid is especially relevant in the early universe because the stiffness properties arise in systems with high density and pressure. We observed how this fluid dominates in the early FLRW universe with the same mixture of fluids due to $\mu_Z \propto 1/a^6$, even before the radiation time $\mu_r \propto 1/a^4$. In the late universe when $t \to \infty$ the FLRW tends to the de Sitter solution with $a_0 \propto e^{\beta_\Lambda H_0 t}$ where β_Λ depends on Λ, which means that the accelerated expansion in the late universes is dominated by dark energy.

We studied the Petrov classification with the NP formalism and set a null tetrad that satisfied the orthonormalization relations and can give us the Weyl scalars. We found how, by rotating the coordinates systems, we can set all the Weyl scalars to zero except Ψ_2 if any two of the spatial scale factors (a, b or c) are equal so that we obtain a Petrov type D symmetry solution.

We obtained the analytical solutions of the scale factors $F(t)$ and $K(t)$ for the Petrov type D symmetry studied here. These solutions can be reduced to previous results of simpler models. In the case of empty space, we obtained two Kasner solutions K_- and K_+ with exponents $p_i = (0, 0, 1)$ and $p_i = (2/3, 2/3, -1/3)$ respectively. The solution K_- is a flat universe that expands in the z direction and is diffeomorphic to the Minkowski space-time. The K_+ is a curve space-time which expands in the x and y direction but contracts in the z. When we have the mixture of three fluids, the stiff fluid dominates the early dynamics of the universe with similar behavior to the Kasner universes K_- and K_+, depending on the sign chosen. In the late universe when $t \to \infty$, we observe again the de Sitter solution dominated by Λ, which is expected as [82] proved that all Bianchi models with positive cosmological constants, except for the type IX models, approach asymptotically the de Sitter solution.

The solutions for the scalar field and the potential were also obtained. We found that with a new set of coordinates, a simpler scalar field and potential can give the same universe in terms of $K(t)$ and $F(t)$. In this model, the universe's matter content changes fundamentally without affecting the behavior of the metric. So for instance, when $t \to 0$, we observed that the universe with the equation of state of the scalar field, $P_\phi = w_\phi \mu_\phi$, is governed predominantly by dust instead of stiff matter.

The dynamical system with a scalar field in our cosmological scenario is Jacobi stable at all times. We proved this by using the criteria of stability given by the KKC theory developed in [15, 28]. The Jacobi stability permits us to conclude that this Petrov type D universe with a scalar representing dark energy, stiff matter, and dust is viable for predictions and observations about the evolution and composition of the universe.

Bibliography

[1] M. Abate et al. A characterization of the Chern and Berwald connections. *Houston J. Math*, 22(4):701–717, 1996.

[2] J.M. Aguirregabiria, A. Feinstein, and J. Ibáñez. Exponential-potential scalar field universes I: Bianchi type I models. *Physical Review D*, 48(10):4662, 1993.

[3] Z. Ahsan. *The Potential of Fields in Einstein's Theory of Gravitation.* Springer, 2019.

[4] P. K. Aluri et al. Is the observable universe consistent with the cosmological principle? *Classical and Quantum Gravity*, 40(9):094001, 2023.

[5] R. Alvarado. Cosmological exact solutions set of a perfect fluid in an anisotropic space-time in Petrov type D. *Advanced Studies in Theoretical Physics*, 10(6):267–295, 2016.

[6] R. Alvarado. The Hubble constant and the deceleration parameter in anisotropic cosmological spaces of Petrov type D. *Advanced Studies in Theoretical Physics*, 10(8):421–31, 2016.

[7] R. Alvarado. Exact cosmological solution of a scalar field of type +cosh in a anisotropic space-time of Petrov type D. *Advanced Studies in Theoretical Physics*, 12(3):121–128, 2018.

[8] R. Alvarado. Exact cosmological solutions of isobaric scalar fields in space-times: anisotropic of the type of Petrov D and isotropic homogeneous. *Advanced Studies in Theoretical Physics*, 12(7):319–333, 2018.

[9] R. Alvarado. Cosmologic solution of scalar and spinorial interactive fields with the dark energy pattern and a magnetic primordial not perturbed field in an anisotropic space-time of Petrov D. *Advanced Studies in Theoretical Physics*, 13(6):253–261, 2019.

[10] R. Alvarado. Cosmological exact solutions of Petrov type D. A mixture of three fluids: Quintessence, dust and radiation. *Advanced Studies in Theoretical Physics*, 14(7):327–334, 2020.

[11] R. Alvarado. Cosmological exact solution of Petrov type D of a nonlinear mixture of fluids of dark energy, dust, Zeldovich and a non-disrupted primordial magnetic field. *Advanced Studies in Theoretical Physics*, 16(4):265–272, 2022.

[12] R. Alvarado, A. Angulo, and M. Vargas. Cosmological exact solutions of Petrov type D. A mixture of two fluids: dark energy and radiation. *Revista de Matemática: Teoría y Aplicaciones*, 29(2):225–238, 2022.

[13] P. Astier and R. Pain. Observational evidence of the accelerated expansion of the universe. *Comptes Rendus Physique*, 13(6-7):521–538, 2012.

[14] J.D. Barrow and P. Saich. Scalar-field cosmologies. *Classical and Quantum Gravity*, 10(2):279, 1993.

[15] C.G. Boehmer, T. Harko, and S.V. Sabau. Jacobi stability analysis of dynamical systems—applications in gravitation and cosmology. *Advances in Theoretical and Mathematical Physics*, 16(4):1145–1196, 2012.

[16] I. Bucataru, O. Constantinescu, and M. F. Dahl. Anisotropic cosmologies containing isotropic background radiation. *Physical Review D*, 64(8):083502, 2001.

[17] S. Carneiro and G.A.M. Marugán. Anisotropic cosmologies containing isotropic background radiation. *Physical Review D*, 64(8):083502, 2001.

[18] B.E. Carvajal-Gámez, J. López-Bonilla, and R. López-Vázquez. Matrix approach to Petrov classification. *Prespacetime Journal*, 6(3):151–155, 2015.

[19] S. Chandrasekhar. *The mathematical theory of black holes*, volume 69. Oxford university press, 1998.

[20] R. Chaubey and R. Raushan. Qualitative study of Bianchi type-I, III and Kantowski–Sachs cosmological models with scalar field. *International Journal of Geometric Methods in Modern Physics*, 13(10):1650123, 2016.

[21] P.H. Chavanis. Growth of perturbations in an expanding universe with Bose-Einstein condensate dark matter. *Astronomy & Astrophysics*, 537:A127, 2012.

[22] P.H. Chavanis. Cosmology with a stiff matter era. *Physical Review D*, 92(10):103004, 2015.

[23] P.H. Chavanis. Partially relativistic self-gravitating Bose-Einstein condensates with a stiff equation of state. *The European Physical Journal Plus*, 130:1–38, 2015.

[24] I. Ciufolini and J. A. Wheeler. *Gravitation and inertia*. Princeton university press, 1995.

[25] R.G. Clowes et al. A structure in the early universe at $z \sim 1.3$ that exceeds the homogeneity scale of the RW concordance cosmology. *Monthly Notices of the Royal Astronomical Society*, 429(4):2910–2916, 2013.

[26] S. Coleman. Fate of the false vacuum: Semiclassical theory. *Physical Review D*, 15(10):2929–2936, 1977.

[27] Plank Collaboration. Planck 2018 results-VI. Cosmological parameters. *Astronomy & Astrophysics*, 641:A6, 2020.

[28] B. Dănilă et al. Jacobi stability analysis of scalar field models with minimal coupling to gravity in a cosmological background. *Advances in High Energy Physics*, 2016:7521464, 2016.

[29] R. D'Inverno. *Introducing Einstein's Relativity*. Clarendon Press, 1992.

[30] G. Efstathiou and S. Gratton. The evidence for a spatially flat universe. *Monthly Notices of the Royal Astronomical Society: Letters*, 496(1):L91–L95, 2020.

[31] C.R. Fadragas, G. Leon, and E.N. Saridakis. Dynamical analysis of anisotropic scalar-field cosmologies for a wide range of potentials. *Classical and Quantum Gravity*, 31(7):075018, 2014.

[32] A. Feinstein and J. Ibáñez. Exact anisotropic scalar field cosmologies. *Classical and Quantum Gravity*, 10(1):93, 1993.

[33] E.G.M. Ferreira. Ultra-light dark matter. *The Astronomy and Astrophysics Review*, 29(7):1–186, 2021.

[34] Y. Gouttenoire, G. Servant, and P. Simakachorn. Kination cosmology from scalar fields and gravitational-wave signatures. *arXiv, 2022.2111.01150 [astro-ph.CO]*, 2022.

[35] J.B. Griffiths. *Colliding plane waves in general relativity*. Courier Dover Publications, 2016.

[36] A. H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23(2):347–356, 1981.

[37] A. H. Guth. Inflation and eternal inflation. *Physics Reports*, 333:555–574, 2000.

[38] T. Harko. Scalar field cosmologies in a Bianchi type I universe. *Acta Physica Hungarica New Series Heavy Ion Physics*, 3(1):115–129, 1996.

[39] T. Harko. Evolution of cosmological perturbations in Bose–Einstein condensate dark matter. *Monthly Notices of the Royal Astronomical Society*, 413(4):3095–3104, 2011.

[40] M. D. Hartl. *Dynamics of spinning compact binaries in general relativity*. PhD thesis, California Institute of Technology, 2003.

[41] P. Helbig. The flatness problem and the age of the Universe. *Monthly Notices of the Royal Astronomical Society*, 495(4):3571–3575, 2020.

[42] S. Hossenfelder. Screams for explanation: finetuning and naturalness in the foundations of physics. *Synthese*, 198(Suppl 16):3727–3745, 2021.

[43] D. Hutsemékers et al. Mapping extreme-scale alignments of quasar polarization vectors. *Astronomy & Astrophysics*, 441(3):915–930, 2005.

[44] F. Jegerlehner. The hierarchy problem and the cosmological constant problem in the Standard Model. *arXiv preprint arXiv:1503.00809*, 2015.

[45] A. Joyce et al. Beyond the cosmological standard model. *Physics Reports*, 568:1–98, 2015.

[46] T.M. Kalotas and C.J. Eliezer. Petrov classification: An elementary approach. *American Journal of Physics*, 51(1):24–28, 1983.

[47] E. Kasner. Geometrical theorems on Einstein's cosmological equations. *American Journal of Mathematics*, 43(4):217–221, 1921.

[48] H.C. Kim and M. Minamitsuji. Scalar field in the anisotropic universe. *Physical Review D*, 81(8):083517, 2010.

[49] N. Kwidzinski. *Quantum Fate of Singularities in Anisotropic Cosmological Models.* PhD thesis, Universität zu Köln, 2020.

[50] B.S. Lakhal and A. Guezmir. The Horizon Problem. *Journal of Physics: Conference Series*, 1269(1):012017, 2019.

[51] J. Lankinen and I. Vilja. Gravitational particle creation in a stiff matter dominated universe. *Journal of Cosmology and Astroparticle Physics*, 2017(08):025, 2017.

[52] A.R. Liddle and R.J. Scherrer. Classification of scalar field potentials with cosmological scaling solutions. *Physical Review D*, 59(2):023509, 1998.

[53] J.A.S. Lima, R.F.L Holanda, and J.V. Cunha. Are galaxy clusters suggesting an accelerating universe? *arXiv preprint arXiv:1009.2736*, 2010.

[54] A. D. Linde. A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems. *Physics Letters B*, 108(6):389–393, 1982.

[55] A. M. Lopez, R. G. Clowes, and G. M. Williger. A giant arc on the sky. *Monthly Notices of the Royal Astronomical Society*, 516(2):1557–1572, 2022.

[56] J. Magana and T. Matos. A brief review of the scalar field dark matter model. *Journal of Physics: Conference Series*, 378(1):012012, 2012.

[57] D. McMahon and P. M. Alsing. *Relativity demystified.* McGraw Hill Professional, 2005.

[58] K. Migkas et al. Probing cosmic isotropy with a new X-ray galaxy cluster sample through the LX–T scaling relation. *Astronomy & Astrophysics*, 636:A15, 2020.

[59] K. Migkas et al. Cosmological implications of the anisotropy of ten galaxy cluster scaling relations. *Astronomy & Astrophysics*, 649:A151, 2021.

[60] R. Miron. Dynamical systems of lagrangian and hamiltonian mechanical systems. *Advance Studies in Pure Mathematics*, 48:309–340, 2007.

[61] E. Mitsou and J. Yoo. Tetrad formalism for exact cosmological observables. *arXiv preprint arXiv:1908.10757*, 2019.

[62] A. Paliathanasis et al. Dynamical analysis in scalar field cosmology. *Physical Review D*, 91(12):123535, 2015.

[63] Z. Perjés. Introduction to spinors and Petrov types in general relativity. *Acta Physica Academiae Scientiarum Hungaricae*, 41(3):173–185, 1976.

[64] J. Plebanski and A. Krasinski. *An Introduction to General Relativity and Cosmology*. Cambridge University Press, 2006.

[65] C. Quigg. Spontaneous symmetry breaking as a basis of particle mass. *Reports on Progress in Physics*, 70(7):1019, 2007.

[66] V.U.M. Rao and Y.V.S.S. Sanyasiraju. Exact Bianchi-type VIII and IX models in the presence of zero-mass scalar fields. *Astrophysics and Space Science*, 187(1):113–117, 1992.

[67] B. Ratra and P. J. E. Peebles. Cosmological consequences of a rolling homogeneous scalar field. *Physycal Review D*, 37:3406–3427, 1988.

[68] D.R.K. Reddy. Bianchi type-V inflationary universe in general relativity. *International Journal of Theoretical Physics*, 48(7):2036–2040, 2009.

[69] A. G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *The astronomical journal*, 116(3):1009, 1998.

[70] A. G. Riess et al. Type Ia supernova discoveries at z > 1 from the Hubble Space Telescope: Evidence for past deceleration and constraints on dark energy evolution. *The Astrophysical Journal*, 607(2):665, 2004.

[71] A. G. Riess et al. Cosmic distances calibrated to 1% precision with gaia EDR3 parallaxes and Hubble Space Telescope photometry of 75 Milky Way cepheids confirm tension with ΛCDM. *The Astrophysical Journal Letters*, 908(1):L6, 2021.

[72] B. Saha. Spinor fields in Bianchi type-I universe. *Physics of Particles and Nuclei*, 37(1):S13–S44, 2006.

[73] L. Shamir. Multipole alignment in the large-scale distribution of spin direction of spiral galaxies. *arXiv preprint arXiv:2004.02963*, 2020.

[74] C.P. Singh and M. Srivastava. Minimally coupled scalar field cosmology in anisotropic cosmological model. *Pramana*, 88(2):1–10, 2017.

[75] P. J. Steinhardt. A quintessential introduction to dark energy. *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 361(1812):2497–2513, 2003.

[76] P. J. Steinhardt, L. Wang, and I. Zlatev. Cosmological tracking solutions. *Physical Review D*, 59(12):123504, 1999.

[77] S. Tsujikawa. Introductory review of cosmic inflation. *arXiv preprint arXiv:hep-ph/0304257*, 2003.

[78] L. A. Ureña-López. Scalar fields in Cosmology: dark matter and inflation. 761(1):012076, 2016.

[79] H. Velten and T.R.P Carams. To conserve, or not to conserve: A review of non-conservative theories of gravity. *Universe*, 7(2):38, 2021.

[80] H.E.S. Velten, R.F. Vom Marttens, and W. Zimdahl. Aspects of the cosmological "coincidence problem". *The European Physical Journal C*, 74:1–8, 2014.

[81] F.J. Vieira. Conceptual problems in cosmology. *arXiv preprint arXiv:1110.5634*, 2011.

[82] R. M. Wald. Asymptotic behavior of homogeneous cosmological models in the presence of a positive cosmological constant. *Physical Review D*, 28(8):2118, 1983.

[83] S. Weinberg. *Cosmology*. OUP Oxford, 2008.

[84] S. Westmoreland. Energy conditions and scalar field cosmology. Master's thesis, Kansas State U., 2013.

[85] J. K. Yadav, J.S. Bagla, and N. Khandai. Fractal dimension as a measure of the scale of homogeneity. *Monthly Notices of the Royal Astronomical Society*, 405(3):2009–2015, 2010.

[86] Y. B. ZelDovich. Equation of state of neutral matter and fluctuations. *Soviet Physics—JETP*, 29(6), 1969.

[87] Y. B. Zeldovich. A hypothesis, unifying the structure and the entropy of the Universe. *Monthly Notices of the Royal Astronomical Society*, 160(1):1P–3P, 1972.