Quantum symmetry groups of noncommutative spheres

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Abstract

We show that the noncommutative spheres of Connes and Landi are quantum homogeneous spaces for certain compact quantum groups. We give a general construction of homogeneous spaces which support noncommutative spin geometries.

1 Introduction

Noncommutative geometry [6] has established itself as a theory which goes beyond the realm of differentiable manifolds and deals in a unified fashion with many singular geometric spaces, too. A fundamental feature of NCG is that it fully incorporates all compact, boundaryless spin manifolds under the heading of "noncommutative spin geometries": see [7] and [17, Chap. 11].

Outstanding examples of singular geometric spaces are the noncommutative tori [5, 9, 25], orbit spaces of discrete group actions, and leaf spaces of foliations. Recently, a new class of examples has appeared, the "noncommutative spheres" of Connes and Landi [10], from a purely cohomological construction.

The Moyal-like nature of the twisted products introduced in [10] suggests that the underlying noncommutative spaces of these spin geometries may be obtained, as C^*-algebras, by the general deformation construction of Rieffel [27]. The question arises as to whether these are in fact noncommutative homogeneous spaces, that is, subalgebras of invariants of certain Hopf algebras which may be regarded as "quantized symmetry groups". This question is more delicate than it might seem, because it must be answered at the C^*-algebra level: these "symmetry groups" must be found in the category of "compact quantum groups" in the sense of Woronowicz [37] or perhaps in the wider category of "locally compact quantum groups" [20]. As it happens, the compact noncommutative spaces which we discuss below have compact (quantum) symmetry groups, so we shall restrict ourselves here to Woronowicz' version.

In Sections 2 and 3 we review the construction of noncommutative spheres and Rieffel's C^*-deformation theory. Section 4 treats compact quantum groups built by such deformations. In Section 5, we explain how both constructions mesh to yield the desired quantum homogeneous spaces. In the final section, we briefly discuss noncommutative spin geometries on these homogeneous spaces.

2 Quantized 4-spheres

The construction of noncommutative spin geometries by Connes and Landi proceeds in two stages. First, the data $(\mathcal{A}, \mathcal{H}, D, C, \chi)$ of an even real spectral triple [6, 17] are sought as possible solutions to a system of equations for the Chern character in cyclic homology:

$$\operatorname{ch}_k(p) \equiv \langle (p - \tfrac{1}{2})\, dp^{2k} \rangle = 0 \quad \text{for} \quad k = 0, 1, \dots, m-1, \tag{1a}$$

$$\pi_D(\operatorname{ch}_m(p)) = \chi, \tag{1b}$$

where $p = p^2 = p^*$ is an orthogonal projector in a matrix algebra $M_r(\mathcal{A})$, $\langle\cdot\rangle$ denotes the conditional expectation (or partial trace) onto $\mathcal{A}$, χ is the grading operator on the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, and $\pi_D(a_0\, da_1 \cdots da_n) := a_0\, [D, a_1] \cdots [D, a_n]$ represents elements of the universal graded differential algebra over $\mathcal{A}$ as operators on $\mathcal{H}$.

These equations impose restrictions, first of all, on the algebra $\mathcal{A}$ itself. In dimension two, i.e., when $m = 1$ and $r = 2$, only commutative solutions are found; in fact, Connes showed by an elementary argument [8] – see also [17, Sect. 11.A] and [22] – that (1a) alone forces $\mathcal{A}$ to be a commutative algebra whose Gelfand spectrum is a closed subset of the 2-sphere $\mathbb{S}^2$. This equation also makes $\operatorname{ch}_1(p)$ a Hochschild 2-cycle, whose associated volume form is the standard volume form on the sphere, so the Gelfand spectrum must be the whole $\mathbb{S}^2$, and thus $\mathcal{A} \simeq C^\infty(\mathbb{S}^2)$ on the basis of (1) alone!

Even in commutative cases such as this, where D may be taken as the Dirac operator given by some metric and spin structure on the spectrum of $\mathcal{A}$, the final condition (1b) does not determine the metric, but only its volume form; thus the cohomological conditions (1) allow for volume-preserving variations of the metric, as befits a theory which aspires to incorporate gravity.

In dimension four, with $m = 2$ and $r = 4$, there is also a commutative solution given in [8], namely the smooth function algebra $C^\infty(\mathbb{S}^4)$. Later, Connes and Landi [10] found a family of noncommutative solutions, parametrized by a complex number of modulus one $\lambda = e^{2\pi i\theta}$: these are the algebras $C^\infty(\mathbb{S}^4_\theta)$ (together with their corresponding Dirac operators), which may be called "smooth function algebras for noncommutative 4-spheres $\mathbb{S}^4_\theta$", in the standard parlance of quantum group theorists. Their representations are uniformly bounded and in each case a C^*-norm is quickly found, allowing to complete them to "continuous function algebras", denoted $C(\mathbb{S}^4_\theta)$.

This procedure extends directly to higher dimensions, yielding noncommutative spheres in any even dimension greater than 2 from the corresponding "instanton algebras" (so called because the finite projective modules $p\mathcal{A}^r$ may be regarded as vector bundles over $\mathcal{A}$). Starting from the odd Chern character in cyclic homology, one can also search for odd-dimensional noncommutative spaces with this method (in the odd case, $\mathcal{H}$ is ungraded and χ in (1b) is replaced by 1).

A striking feature of this construction is that these noncommutative manifolds are parametrized by numbers of modulus one, in contrast to the *real* numbers $q \neq \pm 1$ which label the well-known 2-spheres $\mathbb{S}^2_{qc}$ of Podleś [24], which were originally constructed as homogeneous spaces of the compact quantum groups $\mathrm{SU}_q(2)$. By combining features of both constructions, Dąbrowski, Landi and Masuda [12] built a family of quantized 4-spheres $\mathbb{S}^4_q$; on computing the Chern characters of the instantons, they found that (1a) is violated, inasmuch as $\operatorname{ch}_1(p) = (1 - q^2)$ times a nonvanishing term.

In any case, it is clear that the Connes–Landi spheres $\mathbb{S}^4_\theta$ lie outside the realm of q-spheres of the Podleś type. Indeed, several other variants on the $\mathbb{S}^4_q$ spheres have since appeared [2, 3, 31], which, however, do not incorporate the $\mathbb{S}^4_\theta$ family [11]. Of particular note is the construction by Hong and

Szymański [18] of a large family of quantized n-spheres $\mathbb{S}^n_q$, for $n \geqslant 2$ and $q > 0$, by deforming $C(\mathbb{S}^n)$ to Cuntz–Krieger C^*-algebras based on certain directed graphs; but again, the $\mathbb{S}^4_\theta$ family is not included. Therefore, it behooves us to ask whether that family may be realized as "quantum homogeneous spaces".

3 Deformations of homogeneous spaces

The second stage of the Connes–Landi construction is the provision of spin geometries on the spheres $\mathbb{S}^4_\theta$. This is accomplished by a deformation of the commutative spectral triple $(C^\infty(\mathbb{S}^4), \mathcal{H}, D\!\!\!/)$, where $D\!\!\!/$ denotes a Dirac operator on the Hilbert space $\mathcal{H}$ of square-integrable spinors over $\mathbb{S}^4$. In the deformation, $D\!\!\!/$ is kept fixed, so that all spectral data, including the classical dimension (four!) of the geometry are unchanged: only the algebra and its representation on $\mathcal{H}$ are modified.

One declares a kind of Moyal product on $C^\infty(\mathbb{S}^4)$ by the following recipe: first, note that there is an isometric action of the 2-torus $\mathbb{T}^2$ on $\mathbb{S}^4$, allowing us to decompose any smooth function on $\mathbb{S}^4$ as a series $f = \sum_r f_r$ indexed by $r \in \mathbb{Z}^2$, where f_r lies in the r^{th} spectral subspace:

$$(e^{2\pi i\phi_1}, e^{2\pi i\phi_2}) \cdot f_r = e^{2\pi i(r_1\phi_1 + r_2\phi_2)} f_r.$$

The series converges rapidly in the Fréchet topology of $C^\infty(\mathbb{S}^4)$. By introducing the following star-product of homogeneous elements:

$$f_r \times g_s := e^{2\pi i\theta r_1 s_2} f_r g_s, \tag{2}$$

Connes and Landi constructed a representation of $C(\mathbb{S}^4_\theta)$ on the spinor space $\mathcal{H}$ (having bounded commutators with $D\!\!\!/$); in essence, the representation is explicit only on the smooth subalgebra, which is just the vector space $C^\infty(\mathbb{S}^4)$ with the commutative product replaced by the star-product (2).

More generally, if M is a compact Riemannian manifold admitting a Lie group of isometries of rank $l \geqslant 2$, so that M carries an isometric action of the torus $\mathbb{T}^l$, one can decompose $C^\infty(M)$ into spectral subspaces indexed by $\mathbb{Z}^l$. The Moyal product of two homogeneous functions f_r and g_s is then given by

$$f_r \times g_s := \rho(r, s)\, f_r g_s, \tag{3}$$

where $\rho\colon \mathbb{Z}^l \times \mathbb{Z}^l \to \mathbb{T}$ is a 2-cocycle on the additive group $\mathbb{Z}^l$. The cocycle relation

$$\rho(r, s + t)\rho(s, t) = \rho(r, s)\rho(r + s, t)$$

guarantees associativity of the new product. For instance [17, 34], one may take

$$\rho(r, s) := \exp\bigl\{-2\pi i \textstyle\sum_{j<k} r_j\theta_{jk}s_k\bigr\},$$

where $\theta = [\theta_{jk}]$ is a real $l \times l$ matrix. Complex conjugation of functions remains an involution for the new product provided that the matrix θ is *skewsymmetric*.

The relation (3) is easily recognized as the product rule for the twisted group C^*-algebra $C^*(\mathbb{Z}^l, \rho)$. We may replace ρ by its skewsymmetrized version

$$\sigma(r, s) := \exp\bigl\{-\pi i \textstyle\sum_{j,k=1}^{l} r_j\theta_{jk}s_k\bigr\}, \tag{4}$$

because ρ and σ are cohomologous [26], and we obtain $C^*(\mathbb{Z}^l, \sigma) = C(\mathbb{T}^l_\theta)$, which is precisely the C^*-algebra of the noncommutative l-torus with parameter matrix θ.

The relation (3) is clearly, then, a discretized version of the usual Moyal product, due to the periodicity of the $\mathbb{T}^l$-action. Recall that the standard Moyal product on the phase space $\mathbb{R}^{2m}$ may be expressed either by the familiar series in powers of $\hbar$ whose first nontrivial term gives the Poisson bracket, or alternatively in the integral form [16]:

$$(f \times_J g)(x) := (2\pi\hbar)^{-n} \iint f(x+s)g(x+t)\, e^{is\cdot Jt/\hbar}\, ds\, dt,$$

where J is the skewsymmetric matrix giving the standard symplectic structure on $\mathbb{R}^{2m}$ (and the dot is the usual scalar product on $\mathbb{R}^{2m}$). This may be interpreted as an oscillatory integral for suitable classes of functions and distributions on $\mathbb{R}^{2m}$, and yields the familiar series as an *asymptotic* expansion in powers of $\hbar$ [14, 35]. It is, therefore, a better starting point than that series for a C^*-algebraic theory of deformations. Indeed, this was the form of the Moyal product used by Rieffel in his general deformation theory [27]. He found, in fact, an improvement over the board by rewriting it as

$$(f \times_J g)(x) := \iint f(x+Js)g(x+t)\, e^{2\pi i s\cdot t}\, ds\, dt.$$

He then generalized this to

$$a \times_J b := \iint_{V\times V} \alpha_{Js}(a)\alpha_t(b)\, e^{2\pi i s\cdot t}\, ds\, dt, \tag{5}$$

where a, b belong to a C^*-algebra A, $\alpha\colon V \longrightarrow \mathrm{Aut}(A)$ is a (strongly continuous) action of a vector group $V \simeq \mathbb{R}^l$ on A, and J is a skewsymmetric real $l \times l$ matrix. The oscillatory integral (5) makes sense, *a priori*, only for elements a, b of the smooth subalgebra A^∞ of A (under the action α), which is a Fréchet pre-C^*-algebra.

This problem of good definition is overcome [27] by introducing a suitable C^*-norm on A^∞ for which the $\times_J$ product is continuous, and then completing it in this norm to obtain the deformed C^*-algebra A_J. The construction is functorial in that morphisms of A restrict to A^∞ and then extend uniquely to morphisms of A_J. In more detail: if $(A, \alpha(V))$ and $(B, \beta(V))$ are two C^*-algebras carrying actions of V, and if $\phi\colon A \longrightarrow B$ is a $*$-homomorphism intertwining the actions α and β, then $\phi(A^\infty) \subseteq B^\infty$ and the restriction of ϕ to A^∞ extends uniquely to a $*$-homomorphism $\phi_J\colon A_J \longrightarrow B_J$. Moreover, if the original map ϕ is injective, then ϕ_J is injective, too; and ϕ_J is surjective whenever ϕ is surjective.

In particular, when $B = A$ and $\beta = \alpha$, each α_x intertwines α with itself since V is an abelian group, and this gives an action $\alpha_J\colon V \longrightarrow \mathrm{Aut}(A_J)$ whose restriction to A^∞ coincides with α. Then (A_J, α_J) can be deformed in turn, using a new skewsymmetric matrix K, say; and the result turns out to be isomorphic to A_{J+K}. By taking $K = -J$, we see that the change $A \mapsto A_J$ is reversible. It is therefore unsurprising, but still a deep and important result, that the smooth subalgebra remains unchanged during this mutation: $(A_J)^\infty = A^\infty$ as vector spaces, although they have different multiplications [27, Thm. 7.1].

The case of particular interest to us occurs when the action α of V is periodic, so that $\alpha_x = \mathrm{id}_A$ for $x \in L$, a cocompact lattice in V; in which case, α is effectively an action of the compact abelian group $H = V/L$. Then A^∞ decomposes into spectral subspaces labelled by elements of L (or characters of H) and one can check [27, Prop. 2.21] that if $\alpha_s(a_p) = e^{2\pi i p\cdot s} a_p$ and $\alpha_t(b_q) = e^{2\pi i q\cdot t} b_q$ with $p, q \in L$, then

$$a_p \times_J b_q = e^{-2\pi i p\cdot Jq} a_p b_q.$$

On comparing this with (3) (with the cocycle ρ replaced there by σ), we see that it suffices to take $A := C(\mathbb{T}^l)$ and $J := \frac{1}{2}\theta$ in order to obtain any noncommutative torus $C(\mathbb{T}^l_\theta) \simeq A_J$ by this algorithm.

In fine, the isospectral deformation procedure of [10], based on the star-product (2), is, as far as the algebra is concerned, a special case of Rieffel's C^*-deformation theory. (This same point is made by Sitarz in a recent announcement [32].)

Moreover, if the isometric action of $\mathbb{T}^l$ on M alluded to above is free on some orbit, so that M contains an embedded l-torus, then the restriction map $\pi\colon C(M) \to C(\mathbb{T}^l)$ induces a surjective $*$-homomorphism $\pi_{2J}\colon C(M)_{2J} \to C(\mathbb{T}^l_\theta)$, so that the noncommutative torus appears as a quotient of the deformed $C(M)$. In particular, if θ is irrational, then the noncommutative sphere $C(\mathbb{S}^4_\theta)$ is not a type I C^*-algebra.

4 Compact quantum groups from deformations

It stands to reason, then, that this C^*-deformation process should yield compact quantum groups when applied to the C^*-algebra $C(G)$ of continuous functions on a compact Lie group. This proves to be the case, by a further construction of Rieffel. There are two issues to address here: first, which vector group actions on $C(G)$ are admissible and useful, and second, how to deal with the coproduct, counit and antipode which define the Hopf algebra structure of $C(G)$ (or rather, of its dense subalgebra of representative functions).

The solution to the second problem could not be simpler: the coalgebra structure and antipode can be left completely untouched, and only the algebra structure need be deformed! The matter is not quite trivial, as one must ensure that the coproduct is still an algebra homomorphism for the new product. This possibility was pointed out by Dubois-Violette [13], who noticed that Woronowicz' matrix corepresentations for $C(\mathrm{SU}_q(N))$ and similar bialgebras could be seen as different star-products on the same coalgebra.

Now suppose that H is a closed connected abelian subgroup of G (usually we may take H to be a maximal torus, but it is not really necessary that it be maximal); following Rieffel [28], we consider the action of $H \times H$ on G given by $(h,k) \cdot x := hxk^{-1}$, and the corresponding action on $C(G)$:

$$[(h,k) \cdot f](x) := f(h^{-1}xk). \tag{6}$$

We may regard this as a periodic action of the Lie algebra $\mathfrak{h} \oplus \mathfrak{h}$, with the following notation. Choose and fix a basis for the vector space $\mathfrak{h} \simeq \mathbb{R}^l$, so that the exponential mapping from $\mathfrak{h}$ onto H may be expressed as a homomorphism $e\colon \mathbb{R}^l \to H$ whose kernel is the integer lattice $\mathbb{Z}^l$; by taking $\lambda := e(1,1,\dots,1)$ we may write $\lambda^s := e(s)$ for $s \in \mathbb{R}^l$ with a multiindex notation; the action of $V := \mathfrak{h} \oplus \mathfrak{h}$ on $C(G)$ is then written as

$$[\alpha(s,t)f](x) := f(\lambda^{-s}x\lambda^t). \tag{7}$$

(In the sequel, we shall refer to this as an action of $\mathfrak{h} \oplus \mathfrak{h}$ or of $H \times H$, interchangeably.) If J is now any skewsymmetric matrix in $M_{2l}(\mathbb{R})$, then (5) now defines a Moyal product on $C^\infty(G)$, and the procedure of Section 3 extends this to a C^*-algebra $C(G)_J$, which yields a quantization of G as a noncommutative space.

The remaining difficulty is that an arbitrary choice of J will not mesh well with the coalgebra structure of $C^\infty(G)$, so we shall not always get a quantization of the *group* structure of G. For that, one may follow the approach of Drinfeld by first equipping G with a compatible Poisson bracket (i.e., the product map $G \times G \to G$ must be a Poisson map). By a well-known procedure [4] this can be done at the infinitesimal level by equipping its Lie algebra $\mathfrak{g}$ with a cocycle $\phi\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$ whose dual defines a Lie bracket on $\mathfrak{g}^*$. For instance, one may take $\phi(X) = \mathrm{ad}_X(r)$, where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a skewsymmetric solution of the classical Yang–Baxter equation $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$. If $r = \sum_k X_k \otimes Y_k$, then since $[r_{12}, r_{13}] = \sum_{jk} [X_j, X_k] \otimes Y_j \otimes Y_k$ and similarly for the other terms, this equation is satisfied when $r \in \mathfrak{h} \otimes \mathfrak{h}$ for an *abelian* Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Although there are other solutions (see [21] for an exhaustive treatment of Poisson Lie group structures on *simple* compact Lie groups and the several algebraic quantizations of the Hopf algebra of representative functions), we shall focus on the case $r \in \mathfrak{h} \otimes \mathfrak{h}$. On using our previous identification of $\mathfrak{h}$ with $\mathbb{R}^l$, we can write r as a skewsymmetric $l \times l$ matrix Q. The corresponding Poisson structure on G is given by the bivector field W, where $W_x := \lambda_x(r) - \rho_x(r)$ is the difference of the left and right translates of r from $\Lambda^2\mathfrak{g}$ to $\Lambda^2 T_x G$. Therefore [29], at the infinitesimal level we should take

$$J := \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}$$

as the $2l \times 2l$ matrix of deformation parameters for the action of $\mathfrak{h} \oplus \mathfrak{h}$.

We can now write the twisted product on $C^\infty(G)$ as

$$(f \times_J g)(x) := \int_{\mathfrak{h}^4} f(\lambda^{-Qs} x \lambda^{-Qt}) g(\lambda^{-u} x \lambda^{v})\, e^{2\pi i(s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv. \tag{8}$$

The coproduct Δ, the counit ε and the antipode S, which are defined on the Hopf algebra of representative functions of G by

$$\Delta f(x, y) := f(xy), \qquad \varepsilon(f) := f(1), \qquad S f(x) := f(x^{-1}), \tag{9}$$

whereby Δ and ε are algebra homomorphisms and S is an antiisomorphism, obviously extend to algebra maps of $C(G)$ with the same properties. It is shown in [28,36] that they also satisfy the same algebraic relations for the twisted product. The formulas (9) make sense for $f \in C^\infty(G)$ or even $f \in C(G)$, although the usual requirement $\Delta(C^\infty(G)) \subseteq C^\infty(G) \otimes C^\infty(G)$ holds only if the algebraic tensor product is replaced by the completed tensor product, which we denote by $C^\infty(G) \mathbin{\widehat{\otimes}} C^\infty(G)$ and identify with $C^\infty(G \times G)$. We can make a formal check of these homomorphism properties for smooth functions:

$$\begin{aligned}
(\Delta f \times_J \Delta g)(x, y) &= \int_{\mathfrak{h}^8} f(\lambda^{-Qs} x \lambda^{-Qt - Qs'} y \lambda^{-Qt'}) g(\lambda^{-u} x \lambda^{v - u'} y \lambda^{v'})\, e^{2\pi i(s\cdot u + t\cdot v + s'\cdot u' + t'\cdot v')}\, ds \cdots dv' \\
&= \int_{\mathfrak{h}^8} f(\lambda^{-Qs} x \lambda^{-Qt''} y \lambda^{-Qt'}) g(\lambda^{-u} x \lambda^{-u''} y \lambda^{v'})\, e^{2\pi i(s\cdot u + t''\cdot v + s'\cdot u'' + t'\cdot v')}\, ds \cdots dv' \\
&= \int_{\mathfrak{h}^6} f(\lambda^{-Qs} x \lambda^{-Qt''} y \lambda^{-Qt'}) g(\lambda^{-u} x \lambda^{-u''} y \lambda^{v'})\, e^{2\pi i(s\cdot u + t'\cdot v')}\, \delta(t'')\, \delta(u'')\, ds \cdots dv' \\
&= \int_{\mathfrak{h}^4} f(\lambda^{-Qs} xy \lambda^{-Qt'}) g(\lambda^{-u} xy \lambda^{v'})\, e^{2\pi i(s\cdot u + t'\cdot v')}\, ds\, dt'\, du\, dv' \\
&= (f \times_J g)(xy) = \Delta(f \times_J g)(x, y).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
(f \times_J g)(1) &= \int_{\mathfrak{h}^4} f(\lambda^{-Q(s+t)}) g(\lambda^{v-u})\, e^{2\pi i(s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv \\
&= \int_{\mathfrak{h}^4} f(\lambda^{-Q(s')}) g(\lambda^{v'})\, e^{2\pi i(s'\cdot u + t\cdot v')}\, ds'\, dt\, du\, dv' \\
&= \int_{\mathfrak{h}^2} f(\lambda^{-Q(s')}) g(\lambda^{v'})\, \delta(s')\, \delta(v')\, ds'\, dv' = f(1)\, g(1),
\end{aligned}
$$

so $\varepsilon(f \times_J g) = \varepsilon(f)\varepsilon(g)$. Next, if Q is invertible, then

$$
\begin{aligned}
(Sf \times_J Sg)(x) &= \int_{\mathfrak{h}^4} f(\lambda^{Qt} x^{-1} \lambda^{Qs}) g(\lambda^{-v} x^{-1} \lambda^{u})\, e^{2\pi i(s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv \\
&= (\det Q)^{-2} \int_{\mathfrak{h}^4} f(\lambda^{-t'} x^{-1} \lambda^{s'}) g(\lambda^{-v} x^{-1} \lambda^{-u})\, e^{-2\pi i(Q^{-1}t'\cdot v + Q^{-1}s'\cdot u)}\, ds'\, dt'\, du\, dv \\
&= \int_{\mathfrak{h}^4} f(\lambda^{-t'} x^{-1} \lambda^{s'}) g(\lambda^{-Qv'} x^{-1} \lambda^{-Qu'})\, e^{2\pi i(t'\cdot v' + s'\cdot u')}\, ds'\, dt'\, du'\, dv' \\
&= (g \times_J f)(x^{-1}) = S(g \times_J f)(x),
\end{aligned}
$$

using the skewsymmetry of Q in the third step; on the other hand, if $Q = 0$, then $f \times_J g = fg$ and the calculation reduces to $(Sf \times_J Sg)(x) = f(x^{-1})g(x^{-1}) = S(g \times_J f)(x)$; since we may integrate separately over the nullspace of Q and its orthogonal complement, we conclude that $Sf \times_J Sg = S(g \times_J f)$ in all cases.

The defining property of the antipode may also be checked in this manner: indeed, if $m(f \otimes g) := f \times_J g$ for $f, g \in C^\infty(G)$, similar formal calculations quickly establish that

$$
m(\mathrm{id} \otimes S)(\Delta f) = \varepsilon(f)\, 1 = m(S \otimes \mathrm{id})(\Delta f) \tag{10}
$$

whenever $f \in C^\infty(G)$. However, it should be pointed out that the previous calculations in fact involve oscillatory integrals of functions of $s, t, u, v \in \mathbb{R}^l$ which have neither compact support nor fast decrease; but with some additional careful analysis, it is shown in [27] that they remain valid for smooth functions which have all derivatives bounded on $\mathbb{R}^l$, as is always the case when $f, g \in C^\infty(G)$.

In summary, the product (8) on $C^\infty(G)$ is fully compatible with its original coalgebra structure and antipode. The functoriality of the A_J construction then lifts Δ and S as algebra (anti)homomorphisms to the C^*-level. (Some bookkeeping is necessary because the source and target algebras carry different actions of $\mathfrak{h} \oplus \mathfrak{h}$ in each case.) With $A = C(G)$ and $J = Q \oplus (-Q)$ as before, we then obtain a continuous $*$-homomorphism $\Delta_J \colon A_J \to A_J \otimes A_J$ (with the minimal C^*-tensor product) and a continuous $*$-antihomomorphism $S_J \colon A_J \to A_J$. The counit ε on $C^\infty(G)$ also extends to a character of A_J.

Note, however, that the twisted product on $C^\infty(G)$ generally does not extend to a continuous linear map from $A_J \otimes A_J$ to A_J. (For one thing, m is not an algebra homomorphism unless G is abelian.) Thus, the relation (10) is not helpful at the C^*-level. This is an old problem, and for unital C^*-algebras there is a well-known solution, described in the fundamental paper of Woronowicz [37].

Given a *unital* C^*-algebra A and a unital $*$-homomorphism $\Delta\colon A \to A \otimes A$ which is coassociative, define linear maps W, W' on the algebraic tensor product of A with itself by

$$W(a \otimes b) := (\Delta a)(1 \otimes b) \quad \text{and} \quad W'(a \otimes b) := (a \otimes 1)(\Delta b).$$

(These are the Kac–Takesaki or "fundamental unitary" operators.) Woronowicz' postulate is that the maps W, W' have dense range. Then (A, Δ) is called a *compact quantum group*. The counit and antipode are automatically defined on a dense $*$-subalgebra, and A has a unique state (the "Haar state") which is both left and right invariant [37]. For $A = C(G)$, these maps are

$$W(f \otimes g)(x, y) := f(xy)g(y), \qquad W'(f \otimes g) := f(x)g(xy),$$

which have dense range in $C(G \times G)$. After deformation, these become

$$W(f \otimes g) := (\Delta f) \times_J (1 \otimes g), \qquad W'(f \otimes g) := (f \otimes 1) \times_J (\Delta g),$$

for $f, g \in C^\infty(G)$, and these extend to invertible maps on $C^\infty(G \times G)$. Concretely, for $h \in C^\infty(G \times G)$,

$$\begin{aligned} Wh(x, y) &= \int_{\mathfrak{h}^4} h(x\lambda^{-Qs} y\lambda^{-Qt}, \lambda^{-u} y\lambda^{v})\, e^{2\pi i(s\cdot u+t\cdot v)}\, ds\, dt\, du\, dv, \\ W^{-1}h(x, y) &= \int_{\mathfrak{h}^4} h(x\lambda^{Qt} y^{-1}\lambda^{-Qs}, \lambda^{-u} y\lambda^{v})\, e^{2\pi i(s\cdot u+t\cdot v)}\, ds\, dt\, du\, dv, \end{aligned}$$

as may be verified directly. It follows that W and likewise W' have dense range in $C(G \times G)$.

5 Noncommutative spheres as homogeneous spaces

The standard 4-sphere is a homogeneous space of the 5-dimensional rotation group, namely: $\mathbb{S}^4 \approx \mathrm{SO}(5)/\mathrm{SO}(4)$. Note that SO(5) is a compact simple Lie group of rank two. More generally, we may consider homogeneous spaces of the form $M = G/K$, where G is a compact Lie group (which need not be semisimple) and K is a closed subgroup. Let H be a closed abelian subgroup of K; then we can deform both $C(G)$ and $C(K)$ by the *same* action (7) of $H \times H$. Note in passing that a maximal torus in $\mathrm{SO}(2l)$ is carried onto a maximal torus of $\mathrm{SO}(2l+1)$ by the standard inclusion $\mathrm{SO}(2l) \subset \mathrm{SO}(2l+1)$, so that even-dimensional spheres $\mathbb{S}^{2l} = \mathrm{SO}(2l+1)/\mathrm{SO}(2l)$ fall under this heading.

The left action of G on G/K yields a $*$-homomorphism $\rho\colon C(G/K) \to C(G) \otimes C(G/K)$ by $\rho f(x, yK) := f(xyK)$. Restricted to smooth functions, this can be viewed as a left coaction of $C^\infty(G)$ on $C^\infty(G/K)$. Let $C(G)^K$ denote the subalgebra of $C(G)$ consisting of right-invariant functions under the action of K, so $f \in C(G)^K$ if $f(xw) = f(x)$ whenever $w \in K$, $x \in G$; and let $C^\infty(G)^K := C(G)^K \cap C^\infty(G)$. There is an obvious $*$-isomorphism $\zeta\colon C(G)^K \to C(G/K)$ given by $\zeta f(xK) := f(x)$, and $\zeta(C^\infty(G)^K) = C^\infty(G/K)$. The coproduct Δ of $C^\infty(G)$ maps $C^\infty(G)^K$ into $C^\infty(G) \mathbin{\widehat{\otimes}} C^\infty(G)^K$, the space of smooth functions h on $G \times G$ for which $h(x, yw) \equiv h(x, y)$ when $w \in K$. Moreover, if $f \in C^\infty(G)^K$, then

$$[\rho\zeta f](x, yK) = \zeta f(xyK) = f(xy) = \Delta f(x, y) = [(\mathrm{id} \otimes \zeta)\Delta f](x, yK),$$

so ζ intertwines the coactions ρ and Δ. In short, the algebra $C^\infty(G/K)$, together with its isomorphism onto $C^\infty(G)^K$, is an embedded homogeneous space in the Hopf algebra $C^\infty(G)$.

Now we come to the main point. Since $H \subseteq K$, the left-right action (6) of $H \times H$ on both G and K induces a left action of H on G/K, since the right action of H is absorbed in the right K-cosets. If we deform $C(G)$ and $C(K)$ via the $H\times H$ action along the direction $J = Q \oplus (-Q)$, the corresponding effect on $C(G/K)$ should be a deformation under an H-action along the direction Q. And so it proves.

To see that, we first notice that for $f, g \in C^\infty(G)^K$, (5) yields

$$\begin{aligned}(f \times_J g)(x) &= \int_{\mathfrak{h}^4} f(\lambda^{-Qs} x \lambda^{-Qt}) g(\lambda^{-u} x \lambda^{v})\, e^{2\pi i (s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv \\ &= \int_{\mathfrak{h}^4} f(\lambda^{-Qs} x) g(\lambda^{-u} x)\, e^{2\pi i (s\cdot u + t\cdot v)}\, ds\, dt\, du\, dv \\ &= \int_{\mathfrak{h}^2} f(\lambda^{-Qs} x) g(\lambda^{-u} x)\, e^{2\pi i s\cdot u}\, ds\, du,\end{aligned}$$

or

$$f \times_J g = \int_{\mathfrak{h}^2} \gamma_{Qs}(f) \gamma_u(g)\, e^{2\pi i s\cdot u}\, ds\, du, \tag{11}$$

where $(\gamma_t f)(x) := f(\lambda^t x)$ for $f \in C(G)^K$. The action of $\mathfrak{h}$ on $C(G/K)$ may be defined as $(\beta_t h)(xK) := h(\lambda^t xK)$, so that ζ intertwines the actions β and γ of H. Then (11) becomes simply

$$\zeta f \times_Q \zeta g = \zeta(f \times_J g) \quad \text{for all} \quad f, g \in C^\infty(G/K).$$

Finally, we can lift this isomorphism to the C^*-level, using the functoriality of C^*-deformations. First, since $\zeta\colon C(G/K) \to C(G)^K$ is a $*$-isomorphism intertwining β and γ, its restriction to $C^\infty(G/K)$ extends to a $*$-isomorphism of $C(G/K)_Q$ to $C(G)^K_Q$, where the latter comes from the action of γ on $C^\infty(G)^K$. Of course, γ can be regarded as an action of $H\times H$ where the second factor acts trivially; since elements of $C(G)^K$ are right-invariant under H, γ is just the restriction of the action α to $C(G)^K$. This means that the inclusion $C(G)^K \hookrightarrow C(G)$ is equivariant for the actions γ and α, and so its restriction to $C^\infty(G)^K$ extends to a $*$-homomorphism from $C(G)^K_Q$ to $C(G)_J$; by Proposition 5.8 of [27], this is still injective. In summary,

$$C(G/K) \simeq C(G)^K \hookrightarrow C(G) \quad \text{leads to} \quad C(G/K)_Q \simeq C(G)^K_Q \hookrightarrow C(G)_J.$$

If the subgroup H is not a maximal torus in either K or G, the space of smooth elements for the action of $H\times H$ will be strictly larger than $C^\infty(G)$ (for instance, if the action is trivial, all continuous functions are smooth in this sense); however, as clarified in Sect. 1 of [28], we may continue to use $C^\infty(G)$ instead, because it will be dense in the Fréchet topology of the space of all smooth elements, and therefore will remain dense in the deformed C^*-algebra $C(G)_J$. The same applies, *mutatis mutandis*, to $C^\infty(G/K)$ and $C(G/K)_Q$.

We have thus proved the following result.

Theorem 1. *The deformed C^*-algebra $C(G/K)_Q$ is an embedded homogeneous space for the compact quantum group $C(G)_J$.* □

Example 1. The even-dimensional noncommutative spheres $\mathbb{S}^{2l}_\theta$ of Connes and Landi come directly from this framework, for $l \geqslant 2$. Just take $G = \mathrm{SO}(2l+1)$, $K = \mathrm{SO}(2l)$ and let $H \simeq \mathbb{T}^l$ be a maximal torus for K; then let $Q = \frac{1}{2}\theta$, where θ is a skewsymmetric $l \times l$ matrix.

The odd-dimensional spheres $\mathbb{S}^{2l+1} = \mathrm{SO}(2l+2)/\mathrm{SO}(2l+1)$ have somewhat different deformations, since the l-dimensional maximal torus of $\mathrm{SO}(2l+1)$ is not maximal in $\mathrm{SO}(2l+2)$, so the twisted product reduces to the ordinary commutative product along some directions.

Example 2. Our construction yields several new examples of homogeneous spaces. For instance, if T is a maximal torus of G, the flag manifold G/T may be deformed in any direction $Q = -Q^t$ in $M_l(\mathbb{R})$ provided $l = \dim T \geqslant 2$. In particular, it yields a family of 6-dimensional quantized manifolds $C(\mathrm{SU}(3)/\mathbb{T}^2)_Q$. It would be of interest to classify these up to isomorphism or Morita equivalence.

At the algebraic level, there are other deformations of flag manifolds [21] which go beyond those considered here, in that more general solutions of the classical Yang–Baxter equation are used for the deformation directions. These could yield further examples of quantum homogeneous spaces.

6 Homogeneous noncommutative spin geometries

These new homogeneous spaces give rise to spectral triples, by the isospectral deformation procedure of [10]. We may start from the manifold G/K with, say, the normalized G-invariant metric. Suppose that G/K also has a homogeneous spin structure (if not, a homogeneous spin^c structure will do). Let D be the corresponding Dirac operator, let $X_1, \dots, X_l$ be the chosen basis of $\mathfrak{h}$, and let p_j be the selfadjoint operator representing X_j on the spinor space $\mathcal{H}$, for $j = 1, \dots, l$. Since the action of $\mathfrak{h}$ integrates to a representation of H on spinors, the operators p_j have integer or half-odd-integer spectra, and for each $r \in \mathbb{Z}^l$, there is a unitary operator $\sigma(p, r) := \exp\{-2\pi i \sum_{j,k} p_j Q_{jk} r_k\}$, using the notation of (4); its inverse is $\sigma(r, p)$. These operators commute with each other and also with D, although not with the representation of $C^\infty(G/K)$ on $\mathcal{H}$. Any bounded operator T in the common smooth domain of the transformations $T \mapsto \sigma(p, r) T \sigma(r, p)$ has a decomposition $T = \sum_{r\in\mathbb{Z}^l} T_r$, where $\sigma(p, r)\, T_s = T_s\, \sigma(p + s, r)$ for $r, s \in \mathbb{Z}^l$; define

$$L(T) := \sum_{r\in\mathbb{Z}^l} T_r\, \sigma(p, r).$$

The cocycle property of σ immediately gives $L(f)L(g) = L(f \times_Q g)$, so that L yields a representation of $(C^\infty(G/K), \times_Q)$ on $\mathcal{H}$, while $[D, L(f)] = \sum_r [D, f_r]\, \sigma(p, r) = L([D, f])$ is a bounded operator for all $f \in C^\infty(G/K)$. The charge conjugation operator C on spinors [17, Chap. 9] commutes with all $\sigma(p, r)$ and therefore $C p_j C^{-1} = -p_j$ for each j. It follows that $R(T) := C L(T)^* C^{-1}$ is given by

$$R(T) = \sum_{r\in\mathbb{Z}^l} \sigma(r, p)\, C T_r^* C^{-1} = \sum_{r\in\mathbb{Z}^l} C T_r^* C^{-1}\, \sigma(r, p).$$

Since $C f^* C^{-1} = f$ for f in the commutative algebra $C^\infty(G/K)$, this reduces to the relation $R(f) = \sum_{r\in\mathbb{Z}^l} f_r\, \sigma(r, p)$, and therefore $R(f)R(g) = R(f \times_{-Q} g)$. (Our use of the skewsymmetrized cocycle σ obviates the need to twist the conjugation as in [10].) It is easy to see – compare [16] –

that R gives an antirepresentation of $(C^\infty(G/K), \times_Q)$ on $\mathcal{H}$, which commutes with L because

$$\begin{aligned} L(f)R(g) &= \sum_{r,s} f_r\, \sigma(p,r)\, g_s\, \sigma(s,p) = \sum_{r,s} f_r g_s\, \sigma(p+s,r)\, \sigma(s,p) \\ &= \sum_{r,s} g_s f_r\, \sigma(s,p+r)\, \sigma(p,r) = \sum_{r,s} g_s\, \sigma(s,p)\, f_r\, \sigma(p,r) = R(g)L(f). \end{aligned}$$

This verifies the reality property of the spin geometry. It is readily checked that

$$[[D, L(f)], R(g)] = \sum_{r,s\in\mathbb{Z}^l} \sigma(p,r)\, [[D, f_r], g_s]\, \sigma(s,p) = 0,$$

so the first-order property of the spin geometry holds, too.

Such a spin geometry $(L(C^\infty(G/K)), \mathcal{H}, D, C, \chi)$ has maximal symmetry; we may indeed refer to the quantum group $C(G)_J$ as its "noncommutative symmetry group". They provide examples of spectral triples with noncommutative symmetries as discussed, for instance, in [23]. However, only the invariance of D under the abelian subgroup H is actually used, so we are free to build other spin geometries by deforming the commutative ones obtained from any H-invariant metric on G/K.

More elaborate examples of deformed geometries can also be built, starting from commutative spin geometries wherein the spin connection is replaced by a Clifford superconnection (as in [15], for instance), provided the latter is also H-invariant.

Finally, we consider whether the noncommutative homogeneous spaces constructed here may play the same role as noncommutative tori in quantum field theory. Recall that Seiberg and Witten [30] and Konechny and Schwarz [19] have extensively explored noncommutative gauge theories based on tori. In general, the divergent ultraviolet behaviour for field theories based on noncommutative tori [34] is no better than in the commutative case. This divergence holds also for field theories obtained by second-quantizing the spin geometries constructed here.

Without going into the detailed analysis, the matter may be summed up as follows. The action of a G-invariant Dirac operator over G/K decomposes into matrix actions on finite-dimensional subspaces of smooth spinors, for which explicit formulas are available [1, 33]. The sign operator $F := D|D|^{-1}$ preserves these subspaces, which are permuted by the representation of the algebra $(C^\infty(G/K), \times_Q)$. For any unitary u in this algebra, we can decompose the operator $[F, u]$ as in [34] or [17, Sect. 13.A] and estimate its Schatten class, which measures the degree of ultraviolet divergence of the theory. The norms $\|[F, u]\|_p$ turn out to be independent of the cocycle σ defining the product, provided $\sigma(r, r+s) = \sigma(r, s)$; in view of (4), this is immediate from the skewsymmetry of the parameter matrix Q. Therefore, the overall UV behaviour remains the same as in the commutative case when $Q = 0$: our deformations never soften the ultraviolet divergence.

The ubiquity of the Moyal product in noncommutative field theory is already familiar. While the present work cannot pretend to explain its pervasiveness, we have at any rate shown that noncommutative geometries with a high degree of symmetry are easy to deform along (at least two) commuting directions, leading always to Moyal products with a few parameters; thus the emphasis on noncommutative tori is by no means misplaced. Whether this is in the nature of things remains to be seen.

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