# Quantum electrodynamics in external fields from the spin representation

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**Abstract**

Systematic use of the infinite-dimensional spin representation simplifies and rigorizes several questions in quantum field theory. This representation permutes "Gaussian" elements in the fermion Fock space, and is necessarily projective: we compute its cocycle at the group level, and obtain Schwinger terms and anomalies from infinitesimal versions of this cocycle. Quantization, in this framework, depends on the choice of the "right" complex structure on the space of solutions of the Dirac equation. We show how the spin representation allows one to compute exactly the *S*-matrix for fermions in an external field; the cocycle yields a causality condition needed to determine the phase.

## 1 Introduction

It should have been clear since Shale and Stinespring's seminal paper [1] that the description of fermions coupled to external classical fields, such as gauge fields, reduces to a problem in representation theory of the infinite dimensional orthogonal group. That paper was couched in general theoretical terms and no explicit calculation was made. The spin (and pin) representation came to the fore in Quantum Field Theory again as the cornerstone of the remarkable books by Pressley and Segal [2], dealing with loop groups – not unrelated to the subjects of string theory, Kac–Moody algebras and integrable systems – and Mickelsson [3], dealing with current algebras. However, the spin representation is not calculated in all generality in these books.

We give the pin representation of the infinite-dimensional orthogonal group *à la* Pressley and Segal, in full detail, and we derive from it, among other things, the scattering matrix in closed form and the Feynman rules. For simplicity, we consider only fermions coupled to external electromagnetic fields in Minkowski space (the external field problem in QED), although the realm of applicability of the spin representation is much wider. We hope to convince the reader that ours is a very economical approach to linear Quantum Field Theory, and that there is nothing in this branch of quantum electrodynamics that cannot be traced back to the representation. Applications of the pin representation to nonlinear field theories will be examined in a separate paper. We have taken pains to give a rigorous treatment; except in the last Section, smeared field operators are used throughout.

Section 2 reviews the algebraic theory of infinite dimensional vector spaces with a symmetric form, dealing with the respective complex structures and polarizations. Section 3 brings in the fermion Fock space and canonical anticommutation relations. The main tool is a series expansion of the general "Gaussian" element of this space, whose coefficients are Pfaffians of skewsymmetric operators. This will be needed in the following Section 4, where we construct the pin representation for the orthogonal group; actually, the group acting on Fock space is an extension by $\mathrm{U}(1)$.

The pin representation immediately proves its worth in yielding the quantization prescription by means of its infinitesimal version. We show how the cocycle of the pin representation gives the general anomaly and the anomalous commutators (Schwinger terms) for linear fermion fields. All this is dealt with in Section 5. Our formulas concerning the anomaly appear to be new. In Section 6 we rewrite the representation in terms of field operators on Fock space.

In Section 7, after a discussion of the space of solutions of the Dirac equation in the framework of Section 2, an all-important step is taken, to wit, the choice of complex structure, which is determined by the nature of the vacuum in quantum electrodynamics. It turns out – somewhat mysteriously – that the correct complex structure is closely related to the phase of the Dirac operator, which plays a prominent role in Connes's noncommutative differential geometry [4]. We then complete a careful translation between the language of group representation theory and that of quantum electrodynamics. After treating charge sectors, we develop the exact expression of the $S$-matrix for charged fermions in an external field, and we reexpress the Schwinger terms and the general anomaly formula directly in terms of the scattering operator for the Dirac equation.

In Section 8, we derive in a completely rigorous manner the Feynman rules for electrodynamics in external fields, within the validity conditions of the classical perturbation expansion. Section 9 briefly deals with vacuum polarization.

Throughout, units are so taken that $c = 1$ and $\hbar = 1$.

## 2 The infinite-dimensional orthogonal group: algebraic aspects

The treatment of the symmetries of the fermion field which we develop here starts from the observation that the space of solutions of the Dirac equation with an external potential is a *real* vector space with a distinguished symmetric form.

### 2.1 Orthogonal complex structures

We start with a real vector space $V$ and a symmetric bilinear form $d$, given *a priori*. We lose nothing by supposing $V$ to be complete in the metric induced by $d$, so we take $(V, d)$ to be a real Hilbert space, either infinite-dimensional or of finite even dimension.

An orthogonal *complex structure* $J$ is a real-linear operator on $V$ satisfying:

$$J^2 = -1, \qquad \text{and} \qquad d(Ju, Jv) = d(u, v) \quad \text{for} \quad u, v \in V.$$

Now, regarding $V$ as a complex vector space via the rule $(\alpha + i\beta)v := \alpha v + \beta Jv$ for $\alpha, \beta$ real, the hermitian form

$$\langle u \mid v \rangle_J := d(u, v) + i d(Ju, v)$$

makes $(V, d, J)$ a complex Hilbert space.

The orthogonal group $\mathrm{O}(V)$ is $\{\, g \in \mathrm{GL}_{\mathbb{R}}(V) : d(gu, gv) = d(u, v), \text{ for all } u, v \in V \,\}$. Note that $g$ is orthogonal iff $g^t g = 1$ where the transpose is with respect to $d$.

The set $\mathcal{J}(V)$ of orthogonal complex structures on $V$ can be seen as a subset of the orthogonal Lie algebra $\mathfrak{o}(V) = \{\, X \colon V \to V \text{ real-linear} : d(\cdot, X\cdot) + d(X\cdot, \cdot) = 0 \,\}$ of $\mathrm{O}(V)$, preserved by the adjoint action $J' \mapsto gJ'g^{-1}$ (with $J' \in \mathcal{J}(V)$) of the orthogonal group.

We select a particular complex structure called $J$ and decompose elements of $\mathrm{O}(V)$ as $g = p_g + q_g$ where $p_g$, $q_g$ are its linear and antilinear parts: $p_g := \frac{1}{2}(g - JgJ)$, $q_g := \frac{1}{2}(g + JgJ)$. We find that $p_{g^{-1}} = \frac{1}{2}(g^{-1} - Jg^{-1}J) = \frac{1}{2}(g^t - Jg^tJ) = p_g^t$, whereas $q_{g^{-1}} = \frac{1}{2}(g^{-1} + Jg^{-1}J) = \frac{1}{2}(g^t + Jg^tJ) = q_g^t$. Linear and antilinear parts of $gg^{-1} = g^{-1}g = 1$ yield, for $g \in \mathrm{O}(V)$:

$$p_g p_g^t + q_g q_g^t = p_g^t p_g + q_g^t q_g = 1, \quad p_g q_g^t = -q_g p_g^t, \quad p_g^t q_g = -q_g^t p_g. \tag{2.1}$$

The complexification $V_{\mathbb{C}} = V \oplus iV$ is a complex Hilbert space under the positive-definite hermitian form:

$$\langle\!\langle w_1 \mid w_2 \rangle\!\rangle := 2d(w_1^*, w_2).$$

Writing $P_J := \frac{1}{2}(1 - iJ)$, $W_0 := P_J V_{\mathbb{C}} = P_J V$ is a Hilbert subspace of $V_{\mathbb{C}}$, satisfying $V_{\mathbb{C}} = W_0 \oplus W_0^*$, and

$$\langle\!\langle P_J u \mid P_J v \rangle\!\rangle = \langle u \mid v \rangle, \qquad \langle\!\langle P_{-J} u \mid P_{-J} v \rangle\!\rangle = \langle v \mid u \rangle \quad \text{for} \quad u, v \in V.$$

Moreover, $W_0$ is isotropic for $d$, i.e., $d(u - iJu, v - iJv) = 0$ for $u, v \in V$.

A (complex) *polarization* for $d$ is any complex subspace $W \leqslant V_{\mathbb{C}}$ which is isotropic for $d$ and satisfies $W \cap W^* = 0$, $W \oplus W^* = V_{\mathbb{C}}$. If $w \in W$, then $w = u - iv$ for unique elements $u, v \in V$, and $J_W \colon u \mapsto v$ is real-linear; thus $W = \{\, u - iJ_W u : u \in V \,\}$. Now $\Re\, d(w_1, w_2) = 0$ for $w_1, w_2 \in W$ shows that $J_W$ is orthogonal, and $\Im\, d(w_1, w_2) = 0$ gives $J_W^2 = -1$. Conversely, if $J' \in \mathcal{J}(V)$, then $W' := \{\, u - iJ'u : u \in V \,\}$ is a polarization for $d$. The correspondence $W \mapsto J_W$ intertwines the adjoint action of $\mathrm{O}(V)$ on $\mathcal{J}(V)$ and its action $W \mapsto gW$ on the set of polarizations.

If $W_1$, $W_2$ are two polarizations for $d$, we can find a unitary transformation $U \colon W_1 \to W_2$; if $C$ denotes complex conjugation in $V_{\mathbb{C}}$, then $g = U \oplus CUC$ is a unitary operator on $V_{\mathbb{C}}$ commuting with $C$, so $g \in \mathrm{O}(V)$ and $gW_1 = W_2$. Thus $\mathrm{O}(V)$ acts transitively on polarizations, and hence also on $\mathcal{J}(V)$. The isotropy subgroup of $J$ in $\mathcal{J}(V)$ is $U_J(V)$, the unitary group of the Hilbert space $(V, d, J)$.

## 2.2 The restricted orthogonal group

We define the *restricted orthogonal group* $\mathrm{O}'(V)$ to be the subgroup of $\mathrm{O}(V)$ consisting of those $g$ for which $[J, g]$ is a Hilbert–Schmidt operator, or equivalently, for which $q_g$ is Hilbert–Schmidt. Similarly, we restrict the set of complex structures by introducing $\mathcal{J}'(V) := \{\, J' \in \mathcal{J}(V) : J - J' \text{ is Hilbert–Schmidt} \,\}$. Also, we call "restricted polarizations" those $W$ for which $J - J_W$ is Hilbert–Schmidt; these form the orbit of $W_0$ under $\mathrm{O}'(V)$. Since $U_J(V) = \{\, g \in \mathrm{O}(V) : [J, g] = 0 \,\}$, it is again the isotropy subgroup of $J$ or $W_0$ under the respective actions of $\mathrm{O}'(V)$.

In the finite-dimensional case, we thus have $\mathcal{J}'(V) \approx \mathrm{O}(2n)/\mathrm{U}(n)$, which has two connected components, one of which is $\mathrm{SO}(2n)/\mathrm{U}(n)$. It can be shown that $J_W$ is in the same connected component as $J$ iff $\dim(W \cap W_0^*)$ is even. In the infinite-dimensional case, the same results hold true [5]: $\mathcal{J}'(V)$ has two connected components, and the component of any $J_W \in \mathcal{J}'(V)$ is determined by the parity of $\dim(W \cap W_0^*)$. Likewise, $\mathrm{O}'(V)$ has two components: $g$ lies in the neutral component iff $\dim(gW_0 \cap W_0^*)$ is even. We shall denote by $\mathrm{SO}'(V)$ the component of the identity of $\mathrm{O}'(V)$.

To see that $\dim(W \cap W_0^*)$ is finite, we note that $B := \frac{1}{2}(1 - J_W J)$ is a Fredholm operator on the complex Hilbert space $V_{\mathbb{C}}$, since $B^\dagger B - 1 = BB^\dagger - 1 = \frac{1}{4}(J_W - J)^2$ is traceclass. Moreover,

$$\begin{aligned} \ker B = \ker B^\dagger &= \{\, z \in V_{\mathbb{C}} : Jz = -J_W z \,\} \\ &= \{\, z \in V_{\mathbb{C}} : \tfrac{1}{2}(1 \mp iJ)z = \tfrac{1}{2}(1 \pm iJ_W)z \,\} \\ &= (W \cap W_0^*) \oplus (W^* \cap W_0). \end{aligned} \tag{2.2}$$

In particular, $B$ has index: $\dim(\ker B) - \dim(\ker B^\dagger) = 0$. Since $W^* \cap W_0 = C(W \cap W_0^*)$, it follows that $\dim(W \cap W_0^*) = \frac{1}{2}\dim(\ker B)$, which is finite. Note that $B = gp_g^t$, from which $\dim_{\mathbb{C}}(\ker p_g) = \dim_{\mathbb{C}}(\ker p_g^t) = \frac{1}{2}\dim(\ker B)$; so $p_g$ is a Fredholm operator of index zero.

Note also that $W \cap W_0^* = \{0\}$ iff $\frac{1}{2}(1 - J_W J)$ is invertible iff $\|J_W - J\| < 2$. For such $W$, we can write $w = z_1 + z_2^*$ with $z_j = u_j - iJu_j \in W_0$ for $j = 1, 2$. This defines a real-linear operator $T_W$ on $V$ by $T_W(u_1) := u_2$. By examining $iw = iz_1 - (iz_2)^*$, we find that $T_W J = -JT_W$, i.e., $T_W$ is antilinear. Moreover,

$$0 = \tfrac{1}{2}\Re\, d(w, w') = \tfrac{1}{2}\Re\, (d(z_1, z_2'^*) + d(z_1', z_2^*)) = d(u_1, T_W u_1') + d(T_W u_1, u_1'),$$

so that $T_W$ is *skewsymmetric*. Since $w = z_1 + z_2^* = (1 + T_W)u_1 - iJ(1 - T_W)u_1 = (1 + T_W)z_1$, we find that $J_W$ and $T_W$ are related by a pair of Cayley transformations:

$$J_W = J(1 - T_W)(1 + T_W)^{-1}, \qquad T_W = (J - J_W)(J + J_W)^{-1},$$

using that $\ker(J_W + J) = \{0\}$ whenever $W \cap W_0^* = \{0\}$ by (2.2); hence $T_W$ is a Hilbert–Schmidt operator. In synthesis, $T_W \in \mathrm{Sk}(V)$, where $\mathrm{Sk}(V)$ denotes the vector space of *antilinear skewsymmetric Hilbert–Schmidt operators* on $V$.

When $p_g$ is invertible, we can define $T_g := q_g p_g^{-1}$. We have $T_g \in \mathrm{Sk}(V)$, since it equals $T_{gW_0}$, as is readily checked; and from (2.1) we see that $p_g^t(1 - T_g^2)p_g = 1$. This shows that we may regard the pair of operators $(p_g, T_g)$ as a parametrization of $\mathrm{SO}'_*(V) := \{\, g \in \mathrm{SO}'(V) : p_g^{-1} \text{exists} \,\}$. While this is not a subgroup of $\mathrm{SO}'(V)$, it is an open neighbourhood of the identity in the topology of $\mathrm{SO}'(V)$ induced by the norm $g \mapsto \|g\|_{\mathrm{op}} + \|[J, g]\|_{\mathrm{HS}}$. Any element of $\mathrm{O}'(V)$ not in $\mathrm{SO}'_*(V)$ satisfies $\dim_{\mathbb{C}}(\ker p_g) = n > 0$ and one can find $r \in \mathrm{O}'(V)$ which is a product of $n$ reflections fixing $(\ker p_g^t)^\perp$, for which $rg \in \mathrm{SO}'_*(V)$. We shall therefore devote most attention to the case $g \in \mathrm{SO}'_*(V)$.

A few formulas for $p_g$ and $T_g$ will be very useful later. Let us abbreviate $\widehat{T}_g := T_{g^{-1}}$. From the antilinear part of the equation $1 = gg^{-1} = (1 + T_g)p_g(1 + \widehat{T}_g)p_g^{-1}$ we obtain $0 = T_g p_g + p_g \widehat{T}_g$, which yields:

$$\widehat{T}_g = -p_g^{-1} T_g p_g.$$

If we write $p_g^{-t} := (p_g^t)^{-1} = (p_g^{-1})^t$ for $g \in \mathrm{SO}'_*(V)$, we then find that

$$p_g + q_g \widehat{T}_g = p_g + q_g(q_g^t p_g^{-t}) = (p_g p_g^t + q_g q_g^t) p_g^{-t} = p_g^{-t}.$$

So, whenever $g^{-1}, h, gh \in \mathrm{SO}'_*(V)$, we get

$$\begin{aligned} T_{gh} &= q_{gh} p_{gh}^{-1} = (q_g + p_g T_h)(p_g + q_g T_h)^{-1} = (q_g + p_g T_h)(1 - \widehat{T}_g T_h)^{-1} p_g^{-1} \\ &= q_g p_g^{-1} + (q_g + p_g T_h - q_g(1 - \widehat{T}_g T_h))(1 - \widehat{T}_g T_h)^{-1} p_g^{-1} \\ &= T_g + p_g^{-t} T_h (1 - \widehat{T}_g T_h)^{-1} p_g^{-1}. \end{aligned} \tag{2.3}$$

A few words on traces and determinants are in order too. If $A$ is a traceclass linear operator, we will denote its *complex trace* $\sum_k \langle e_k \mid A e_k \rangle_J$ by $\mathrm{Tr}_{\mathbb{C}} A$; where $\{e_k\}_{k \in \mathbb{N}}$ is any orthonormal basis on $(V, d, J)$. If $S, T$ are antilinear Hilbert–Schmidt operators on $V$, then $\mathrm{Tr}_{\mathbb{C}}([S, T])$ need not vanish; indeed, $\mathrm{Tr}_{\mathbb{C}}([S, T]) = \sum_k \langle S e_k \mid T e_k \rangle - \langle T e_k \mid S e_k \rangle$ is a purely imaginary number. (The complex trace does, of course, vanish on commutants of linear operators.) Likewise, we will use $\det_{\mathbb{C}}$ to denote a *complex determinant*: if $A$ is a linear operator on $V$ and $V$ is finite-dimensional, we define $\det_{\mathbb{C}} A$ to be the determinant of the matrix with entries $\langle e_i \mid A e_j \rangle$. When $V$ is infinite-dimensional, $\det_{\mathbb{C}} A$ still makes sense provided $A - 1$ is traceclass, and the following [5] may be adopted as a definition: $\det_{\mathbb{C}}(\exp N) := \exp(\mathrm{Tr}_{\mathbb{C}} N)$ for traceclass $N$.

# 3 Operators and "Gaussians" on the fermion Fock space

## 3.1 Gaussian elements

We first recall briefly the construction of the fermion Fock space. We fix a complex structure $J \in \mathcal{J}(V)$ and regard $V$ as the complex Hilbert space $(V, d, J)$. Its *exterior algebra* is $\Lambda(V) := \bigoplus_{n=0}^{\infty} V^{\wedge n}$, where $V^{\wedge n}$ is the complex vector space algebraically generated by the alternating products $v_1 \wedge v_2 \wedge \cdots \wedge v_n$, with $V^{\wedge 0} = \mathbb{C}$ by convention. The inner product on $\Lambda(V)$ is given by $\langle u_1 \wedge \cdots \wedge u_m \mid v_1 \wedge \cdots \wedge v_n \rangle := \delta_{mn} \det(\langle u_k \mid v_l \rangle)$.

The vacuum $\Omega$ is a fixed unit vector in $V^{\wedge 0}$. The antisymmetric Fock space $\mathcal{F}_J(V)$ is the Hilbert-space completion of $\Lambda(V)$; most of the time we shall write only $\mathcal{F}(V)$ instead of $\mathcal{F}_J(V)$. An orthonormal basis is given by the elements $\varepsilon_K := e_{k_1} \wedge \cdots \wedge e_{k_r}$, where $\{e_n\}$ denotes a fixed orthonormal basis for $(V, d, J)$, and $K = \{k_1, k_2, \dots, k_r\}$ is any finite set of positive integers written in increasing order: $k_1 < \cdots < k_r$ (with $\varepsilon_K = \Omega$ if $K$ is void).

The fermion Fock space splits as $\mathcal{F}(V) = \mathcal{F}_0 \oplus \mathcal{F}_1$, where $\mathcal{F}_0$ is the completion of the even part $\bigoplus_{k=0}^{\infty} V^{\wedge(2k)}$ of the exterior algebra, and $\mathcal{F}_1$ is the completion of $\bigoplus_{k=0}^{\infty} V^{\wedge(2k+1)}$.

If $T \in \mathrm{Sk}(V)$, the series

$$H_T := \sum_{i,j=1}^{\infty} \langle e_i \mid T e_j \rangle e_i \wedge e_j \tag{3.1}$$

converges in $\mathcal{F}_0$ since $\sum_{i,j=1}^{\infty} |\langle e_i \mid T e_j \rangle|^2 = \sum_{j=1}^{\infty} \langle T e_j \mid T e_j \rangle < \infty$, the sum being independent of the basis chosen.

We call *Gaussians* the following "quadratic exponential" elements of $\mathcal{F}_0$:

$$f_T := \exp^{\wedge}(\tfrac{1}{2} H_T) := \sum_{m=0}^{\infty} \frac{1}{2^m\, m!} H_T^{\wedge m}. \tag{3.2}$$

If $T_m \in \mathrm{Sk}(V)$ is determined by $T_m(e_{2m-1}) = -e_{2m}$, $T_m(e_{2m}) = e_{2m-1}$, $T_m(e_j) = 0$ for other $j$, then $H_{T_m} = 2e_{2m-1} \wedge e_{2m}$, and $f_{T_m} = \Omega + e_{2m-1} \wedge e_{2m}$. Furthermore, if $T = T_1 + \cdots + T_m$, then $H_T = 2\sum_{k=1}^m e_{2m-1} \wedge e_{2m}$, and $f_T = \Omega + e_1 \wedge \cdots \wedge e_{2m}$ + (lower order terms). It follows that the linear span of $\{ f_T : T \in \mathrm{Sk}(V) \}$ is dense in $\mathcal{F}_0$.

We can expand the Gaussian $f_T$ in the orthonormal basis $\{\varepsilon_K\}$. First, note that

$$\frac{1}{2^m\, m!} H_T^{\wedge m} = \sum_{|K|=2m} \frac{1}{2^m\, m!} (\pm)_K \langle e_{k_1} \mid T e_{k_2}\rangle \cdots \langle e_{k_{2m-1}} \mid T e_{k_{2m}}\rangle e_{k_1} \wedge \cdots \wedge e_{k_{2m}} \tag{3.3}$$

where $(\pm)_K$ is the sign of the permutation putting $K = \{k_1, \dots, k_{2m}\}$ in increasing order, and $T_K$ denotes the skewsymmetric $2m \times 2m$ matrix with entries $\langle e_k \mid T e_{k'}\rangle$ for $k, k' \in K$. Recall [6] that the *Pfaffian* of a skewsymmetric $2m \times 2m$ matrix $A$ is given by

$$\mathrm{Pf}\, A := \frac{1}{2^m\, m!} \sum_{\sigma \in S_{2m}} \pm_\sigma\, a_{\sigma(1)\sigma(2)}\, a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2m-1)\sigma(2m)},$$

and satisfies the crucial property $(\mathrm{Pf}\, A)^2 = \det A$. If $A$ is a skewsymmetric $(2m+1) \times (2m+1)$ matrix, then $\det A = 0$, so one defines $\mathrm{Pf}\, A := 0$. Thus we can rewrite (3.3) as

$$\frac{1}{2^m\, m!} H_T^{\wedge m} = \sum_{|K|=2m} \mathrm{Pf}(T_K)\, \varepsilon_K.$$

By convention, we take $\mathrm{Pf}(T_K) := 1$ when $K$ is the void set. We thus arrive at the expansion for (3.2):

$$f_T = \sum_{K \text{ finite}} \mathrm{Pf}(T_K)\, \varepsilon_K, \tag{3.4}$$

where only the even subsets $K \subset \mathbb{N}$ contribute to the sum.

It is shown in [2] that $\sum_K \mathrm{Pf}(S_K)\,\mathrm{Pf}(T_K)$ is a square root of $\det(1 - TS)$ whenever $S$, $T$ are skewsymmetric real matrices. For complex matrices, the corresponding formula is:

$$\left(\sum_K \mathrm{Pf}(S_K)^*\, \mathrm{Pf}(T_K)\right)^2 = \det_{\mathbb{C}}(1 - TS).$$

We may summarize the foregoing discussion as follows (we suppress the subscript '$\mathbb{C}$' on fractional powers of complex determinants).

**Lemma 1.** *Let $S, T \in \mathrm{Sk}(V)$ where $\dim V = 2n$ is finite. Then if $\det^{1/2}(1 - TS)$ denotes the square root of $\det_{\mathbb{C}}(1 - TS)$ which equals $1$ when $S = 0$ or $T = 0$, the following expansion is valid:*

$$\det{}^{1/2}(1 - TS) = \sum_K \mathrm{Pf}(S_K)^*\, \mathrm{Pf}(T_K), \tag{3.5}$$

*where the sum ranges over the principal submatrices of even dimension.*

When $V$ is infinite dimensional and $T \in \mathrm{Sk}(V)$, we obtain

$$\sum_{K \text{ finite}} |\operatorname{Pf}(T_K)|^2 = \det{}^{1/2}(1 - T^2) = \det{}^{1/2}(1 + T^t T),$$

using (3.5) for finite-rank $T$ and a limiting argument. The series $\sum_{K \text{ finite}} \operatorname{Pf}(S_K)^* \operatorname{Pf}(T_K)$ converges for $S, T \in \mathrm{Sk}(V)$ by the Schwarz inequality, so (3.5) holds for all $S, T \in \mathrm{Sk}(V)$.

The inner product of two fermionic Gaussians follows at once from the expansions (3.4) and (3.5):

$$\langle f_S \mid f_T \rangle = \det{}^{1/2}(1 - TS), \quad \text{for} \quad S, T \in \mathrm{Sk}(V). \tag{3.6}$$

### 3.2 Representing the field algebra

The basic object in quantization is the field algebra over the space $(V, d)$, which is defined prior to any choice of the complex structure. The field algebra over the space $(V, d)$ is just the complexified Clifford algebra $\mathfrak{A}(V_{\mathbb{C}}) := \mathrm{Cl}(V, d) \otimes \mathbb{C}$, complete with respect to the natural (inductive limit) $C^*$-norm [7]. There is a linear map $B \colon V_{\mathbb{C}} \to \mathfrak{A}(V_{\mathbb{C}})$ – the "fermion field" – satisfying $B(w^*) = B(w)^\dagger$ and

$$\{B(v), B(v')\} = 2\, d(v, v') \quad \text{for all} \quad v, v' \in V. \tag{3.7}$$

Any $C^*$-algebra generated by a set of operators $\{\, B'(w) : w \in V_{\mathbb{C}} \,\}$ satisfying the same rules is isomorphic [5] to $\mathfrak{A}(V_{\mathbb{C}})$ via $B(w) \mapsto B'(w)$.

One can construct a faithful irreducible representation $\pi_J$ of $\mathfrak{A}(V_{\mathbb{C}})$ by the GNS construction with respect to the "Fock state" $\omega_J$ determined by $\omega_J(B(u)B(v)) := \langle u \mid v \rangle_J$; see [5] for details. It turns out that this is equivalent to the standard representation of the canonical anticommutation relations (CAR) on the fermion Fock space $\mathcal{F}_J(V)$. The annihilation and creation operators for the free fermion field may be defined as real-linear operators on $\mathcal{F}(V)$:

$$a_J(v) := \pi_J B(P_{-J} v), \qquad a_J^\dagger(v) := \pi_J B(P_J v). \tag{3.8}$$

Note that the vacuum sector is associated to the polarization: $\pi_J B(w)\Omega = 0$ for $w \in W_0^*$. Since $P_{\pm J} = \frac{1}{2}(1 \mp iJ)$, it follows that $a_J(Jv) = -i\, a_J(v)$ and $a_J^\dagger(Jv) = i\, a_J^\dagger(v)$ for $v \in V$; and $\pi_J B(v) = a_J(v) + a_J^\dagger(v)$. Since $V^{\wedge 1} \subset \mathcal{F}(V)$ is the complex Hilbert space $(V, d, J)$, we obtain $iv = Jv$ in $V^{\wedge 1}$; thus $a_J^\dagger(v)\Omega = \frac{1}{2}v - \frac{i}{2}Jv = v$, $a_J(v)\Omega = \frac{1}{2}v + \frac{i}{2}Jv = 0$ in $V^{\wedge 1}$. More generally, $a_J^\dagger(v_1) a_J^\dagger(v_2) \dots a_J^\dagger(v_k)\Omega = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in V^{\wedge k}$. It is straightforward to check that $\{\pi_J B(v), \pi_J B(v')\} = 2\, d(v, v')$ on vectors of the form $v_1 \wedge \cdots \wedge v_k$. The canonical anticommutation relations $\{a_J(v), a_J(v')\} = 0$, $\{a_J(v), a_J^\dagger(v')\} = \langle v \mid v' \rangle$ follow directly from (3.8).

In summary, to each complex structure, in principle there corresponds a different identification of the Clifford algebra with a CAR algebra. Since $\pi_J$ is faithful, as long as we consider a single complex structure – thus a single Fock space representation – we will no longer need to distinguish between $B(v)$ and $\pi_J B(v)$ in the notation. This amounts to regarding $\mathfrak{A}(V_{\mathbb{C}})$ as the algebra of field operators on $\mathcal{F}_J(V)$.

If $U \in U_J(V)$, one defines the unitary operator $\Gamma_J(U)$ on $\mathcal{F}_J(V)$ by the usual rule $\Gamma_J(U) := U \wedge U \wedge \cdots \wedge U$ with $\Gamma_J(U)\Omega := \Omega$. Using $B(v) = a_J(v) + a_J^\dagger(v)$, we get the intertwining property:

$$\Gamma_J(U) B(v) \Gamma_J(U)^{-1} = B(Uv). \tag{3.9}$$

Observables of the one-particle theory are given by elements of the Lie algebra $\mathfrak{o}'(V)$ of the restricted orthogonal group consisting of skewsymmetric real-linear operators $X \in \mathrm{End}_{\mathbb{R}}(V)$. We can quantize at the present stage the Lie algebra elements commuting with $J$. If $X \in \mathfrak{o}'(V)$ and $[X, J] = 0$, then it is immediately seen that $-JX$ is selfadjoint on $(V, d, J)$. If $A$ is a selfadjoint operator on $(V, d, J)$, then the quantized operator or *current* $d\Gamma_J(A)$ on $F(V)$ defined as $d\Gamma_J(A) := \frac{d}{dt}\big|_{t=0}\Gamma_J(\exp(itA))$ is explicitly given by

$$d\Gamma_J(A)(v_1 \wedge v_2 \wedge \cdots \wedge v_n) := \sum_{k=1}^{n} v_1 \wedge \cdots \wedge v_{k-1} \wedge Av_k \wedge v_{k+1} \wedge \cdots \wedge v_n,$$

whenever $v_1, v_2, \ldots, v_n \in \mathrm{Dom}(A)$, and $d\Gamma_J(A)\Omega := 0$. Such vectors span a dense subspace of $\mathcal{F}(V)$ which is invariant under the one-parameter group $\Gamma_J(\exp(itA))$, and hence is a core for $d\Gamma_J(A)$ [8]. In the terminology of [2], $\Gamma$ is a *positive-energy* representation of $U_J(V)$. We remark that $(\mathcal{F}_J(V), \pi_J B, \Omega, \Gamma)$ is a *full quantization* of $(V, d)$ in the sense of Segal [9]. That is to say:

(a) $\mathcal{F}_J(V)$ is a separable Hilbert space;

(b) $\pi_J B(V)$ is a system of selfadjoint operators on this Hilbert space, satisfying (3.7);

(c) $\Omega$ is a unit vector in $\mathcal{F}_J(V)$ which is a cyclic vector for $\pi_J B(V)$; and

(d) $\Gamma$ is a unitary representation of $U_J(V)$ intertwining $\pi_J B(V)$, for which $\Omega$ is stationary, such that $d\Gamma(A)$ is positive on $\mathcal{F}_J(V)$ whenever $A$ is positive on the Hilbert space $(V, d, J)$.

# 4 The pin and spin representations

## 4.1 Vacuum sectors and the Shale–Stinespring criterion

Given such a full quantization of $(V, d)$, then for any $g \in \mathrm{O}(V)$ it follows from (3.7) that $v \mapsto \pi_J B(gv)$ defines another full quantization acting on the same Hilbert space. Indeed, $w \mapsto B(gw)$ (for $w \in V_{\mathbb{C}}$) extends to a $*$-automorphism of the CAR algebra $\mathfrak{A}(V_{\mathbb{C}})$. We then ask whether these two quantizations are unitarily equivalent, i.e., whether this $*$-automorphism is unitarily implementable on $\mathcal{F}_J(V)$. For a given $g \in \mathrm{O}(V)$, we seek a unitary operator $\mu(g)$ on $\mathcal{F}(V)$ so that, extending (3.9):

$$\mu(g)B(v) = B(gv)\mu(g), \quad \text{for all} \quad v \in V. \tag{4.1}$$

The complex structure $J$ is transformed to $gJg^{-1}$; the creation and annihiliation operators undergo a *Bogoliubov transformation*:

$$a_{gJg^{-1}}(gv) = a_J(p_g v) + a_J^\dagger(q_g v), \qquad a^\dagger_{gJg^{-1}}(gv) = a_J(q_g v) + a_J^\dagger(p_g v). \tag{4.2}$$

Were $\mu(g)$ to exist, it would follow that

$$\mu(g)a_J(v) = a_{gJg^{-1}}(gv)\mu(g), \qquad \mu(g)a_J^\dagger(v) = a^\dagger_{gJg^{-1}}(gv)\mu(g).$$

Thus the *out-vacuum* $\mu(g)\Omega$ is annihilated by $a_{gJg^{-1}}(gv)$, for all $v \in V$. Since $\{\, a_J(v) : v \in V \,\}$ vanishes only on scalar multiples of $\Omega$, the kernel of $\{\, a_{gJg^{-1}}(gv) : v \in V \,\}$ is one-dimensional also.

Indeed, since $\{ a_{gJg^{-1}}(gv) : v \in V \} = \{ B(z) : z \in gW_0^* \}$, this kernel, if it is nonzero, must be the vacuum sector for the polarization $gW_0^*$.

We first of all observe that when $g \in \mathrm{SO}'_*(V)$, this vacuum sector is generated by a Gaussian element:

$$a_{gJg^{-1}}(gv) f_{T_g} = 0 \quad \text{for all} \quad v \in V \quad \text{if} \quad g \in \mathrm{SO}'_*(V).$$

Indeed, if we replace $v$ by $p_g^{-1} v$ and use (4.2), we need only show $(a_J(v) + a_J^\dagger(T_g v)) f_{T_g} = 0$ for all $v \in V$; we can moreover take $v = e_1$, the first element in an orthonormal basis for $V$. From (3.4), it is not hard to establish that

$$a_J(e_1) f_T + a_J^\dagger(Te_1) f_T = 0 \quad \text{for any} \quad T \in \mathrm{Sk}(V), \tag{4.3}$$

on expanding the Pfaffians with a little linear algebra.

From this we see that $a_{gJg^{-1}}(gv)\Psi = 0$ for all $v$ implies that $\Psi = c_g f_{T_g}$, for some constant $c_g$.

If $g \in \mathrm{O}'(V)$ with $\dim(\ker p_g) = \dim(\ker p_g^t) = n > 0$, one proceeds as follows [10]. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for the subspace $\ker p_g^t$. Let $r_k$ be the reflection of $(V, d)$ for which $r_k(e_k + Je_k) = -(e_k + Je_k)$, $r_k(v) = v$ if $d(v, e_k + Je_k) = 0$, and let $r = r_1 \dots r_n$. Then $rg \in \mathrm{SO}'_*(V)$, and one can check that

$$(a_J(p_g v) + a_J^\dagger(q_g v))[e_1 \wedge \cdots \wedge e_n \wedge f_{T_{rg}}] = 0.$$

Using (3.4) and recalling that $\mathrm{Pf}(T_K) = 0$ unless $K$ is even, we observe that out-vacuum sector $\mathbb{C}(e_1 \wedge \cdots \wedge e_n \wedge f_{T_{rg}})$ lies in $\mathcal{F}_0(V)$ or $\mathcal{F}_1(V)$ according as $n$ is even or odd.

We turn now to the necessary conditions for the existence of the unitary operators $\mu(g)$. Since the representation $\pi_J$ of the CAR algebra is irreducible, (4.1) shows that $\mu(g)$ is determined by $\mu(g)\Omega$; and this vector lies in the one-dimensional subspace spanned by $e_1 \wedge \cdots \wedge e_n \wedge f_{T_{rg}}$ if $g \in \mathrm{O}'(V)$. Shale and Stinespring [1] have shown that a necessary condition for a vacuum vector for $B(gW_0^*)$ to exist inside $\mathcal{F}_J(V)$ is that $[J, g]$ be Hilbert–Schmidt. The proof of the Shale–Stinespring theorem in the present treatment amounts to showing that such a vector must necessarily be a multiple of $e_1 \wedge \cdots \wedge e_n \wedge f_{T_{rg}}$. In order that this expression be convergent, $n$ must be finite and $f_{T_{rg}}$ must lie in $\mathcal{F}(V)$; by (3.6), the operator $1 - T_{rg}^2$ must have a determinant, i.e., $T_{rg}$ must be Hilbert–Schmidt. Both conditions are fulfilled iff $g \in \mathrm{O}'(V)$, since the finiteness of $n$ amounts to $p_g$ being Fredholm. From now on, we shall assume that $g \in \mathrm{O}'(V)$.

Thus $\mu(g)\Omega = c_g e_1 \wedge \cdots \wedge e_n \wedge f_{T_{rg}}$. Unitarity demands that

$$1 = \langle \mu(g)\Omega \mid \mu(g)\Omega \rangle = |c_g|^2 \langle f_{T_{rg}} \mid f_{T_{rg}} \rangle = |c_g|^2 \det{}^{1/2}(1 - T_{rg}^2).$$

Thus $|c_g| = \det^{-1/4}(1 - T_{rg}^2)$. We now *fix the phase* by taking $c_g > 0$, i.e.,

$$c_g := \det{}^{-1/4}(1 - T_{rg}^2).$$

(Although there is some arbitrariness in the choice of $r$, it happens that $T_{rg}$ vanishes on $\ker p_g^t$ and one finds that $c_g = \det^{1/4}(p_g^t p_g|_{(\ker p_g)^\perp})$, independently of $r$.)

### 4.2 Construction of the pin and spin representations

We are now ready to write down the pin representation of $\mathrm{O}'(V)$, along the lines of Pressley and Segal [2]. Their treatment is explicit only for a finite-dimensional $V$, and can be checked to coincide with the more usual Clifford-algebra construction of the pin representation [11, 12] for the double cover $\mathrm{Pin}(2m)$ of $\mathrm{O}(2m)$. The technique is to permute Gaussian elements, paying due regard to exceptional cases. We present it here in a somewhat different form from [2], in order to make clear that the infinite-dimensional case poses no extra difficulty. An excellent treatment from the Clifford-algebra viewpoint was given by Carey and Palmer [13, 14]. For supersymmetric generalizations, consult the sketch in [15] and the detailed treatment of a finite dimensional case in [16].

The first step is to note that there is a local action of $\mathrm{SO}'(V)$ on $\mathrm{Sk}(V)$, given by:

$$g \cdot S := (q_g + p_g S)(p_g + q_g S)^{-1}.$$

(If $p_g$ is invertible, so is $p_g + q_g S$ for $S$ small enough in the Hilbert–Schmidt norm.) Since the relations (2.1) yield

$$(p_g + q_g S)^t (q_g + p_g S) = -(q_g + p_g S)^t (p_g + q_g S)$$

whenever $S^t = -S$, it follows that $g \cdot S$ is also skew, and it is clearly antilinear and Hilbert–Schmidt; thus $g \cdot S \in \mathrm{Sk}(V)$. It is readily checked that $gh \cdot S = g \cdot (h \cdot S)$ whenever $h \cdot S$ and $gh \cdot S$ are defined; the group action is "local" since they may be undefined in particular cases. From (2.3) with $T_h = S$, we obtain the useful alternative form:

$$g \cdot S = T_g + p_g^{-t} S (1 - \widehat{T}_g S)^{-1} p_g^{-1}. \tag{4.4}$$

For a given $g \in \mathrm{SO}'_*(V)$, other than the identity, those $S$ for which $p_g + q_g S$ is invertible form an open neighbourhood of zero in $\mathrm{Sk}(V)$, so $\{ f_S : g \cdot S \text{ exists} \}$ spans a dense subset of $\mathcal{F}_0(V)$. Thus we may define a unitary operator on $\mathcal{F}_0(V)$ by the prescription:

$$\mu(g) f_S := c_g \phi_g(S) f_{g \cdot S}, \tag{4.5}$$

where $\phi_g(S) \in \mathbb{C}$ is to be chosen to make $\mu(g)$ unitary. A suitable choice is:

$$\phi_g(S) := \det{}^{1/2}(1 - S\widehat{T}_g). \tag{4.6}$$

We have $\phi_g(0) = 1$ and $c_g^2 \langle \phi_g(S) f_{g\cdot S} \mid \phi_g(T) f_{g\cdot T} \rangle = \langle f_S \mid f_T \rangle$, using (3.6).

From the definition (4.6) and the formula (4.4), it follows that

$$\begin{aligned}
\phi_{gh}(S) &= \det{}^{1/2}(1 - S(\widehat{T}_h + p_h^{-1} \widehat{T}_g (1 - T_h \widehat{T}_g)^{-1} p_h^{-t})) \\
&= \det{}^{1/2}(p_h^{-t}(1 - S\widehat{T}_h) p_h^t - p_h^{-t} S p_h^{-1} \widehat{T}_g (1 - T_h \widehat{T}_g)^{-1})) \\
&= \det{}^{-1/2}(1 - T_h \widehat{T}_g)\ \det{}^{1/2}((1 - S\widehat{T}_h) p_h^t (1 - T_h \widehat{T}_g) p_h^{-t} - S p_h^{-1} \widehat{T}_g p_h^{-t}) \\
&= \det{}^{-1/2}(1 - T_h \widehat{T}_g)\ \phi_h(S)\ \det{}^{1/2}(1 - T_h \widehat{T}_g - p_h^{-t}(1 - S\widehat{T}_h)^{-1} S p_h^{-1} \widehat{T}_g) \\
&= \det{}^{-1/2}(1 - T_h \widehat{T}_g)\ \phi_h(S)\ \phi_g(h \cdot S),
\end{aligned}$$

whenever $g, h, gh \in \mathrm{SO}'_*(V)$, provided $h \cdot S$ and $gh \cdot S$ both exist. The set of such $S$ is a neighbourhood of zero in $\mathrm{Sk}(V)$, so that the corresponding Gaussians $f_S$ are total in $\mathcal{F}_0(V)$; we thus arrive at

$$\mu(g)\mu(h) = c(g, h)\mu(gh), \tag{4.7}$$

where the *cocycle* $c(g,h)$ is given by

$$c(g,h) := c_g c_h c_{gh}^{-1} \det{}^{1/2}(1 - T_h \widehat{T}_g) = \exp(i \arg \det{}^{1/2}(1 - T_h \widehat{T}_g)). \tag{4.8}$$

Thus the $g \mapsto \mu(g)$ is a *projective* representation of the restricted orthogonal group. Actually, its definition is incomplete, since in (4.5) we have defined $\mu(g)$ only for $g \in \mathrm{SO}'_*(V)$. To finish the job, we must give the formulae for $\mu(g)\Psi$ and the respective cocycles, when $g = rh$ with $h \in \mathrm{SO}'_*(V)$, $r$ is a product of reflections in $\mathrm{O}'(V)$, and $\Psi \in \mathcal{F}_0(V)$ or $\mathcal{F}_1(V)$. This involves a considerable amount of bookkeeping; we shall summarize the results.

If $v \in V$ is a unit vector, i.e., $d(v,v) = 1$, we define $r_v \in \mathrm{O}'(V)$ by

$$r_v(u) := 2d(v,u)v - u. \tag{4.9}$$

Notice that $-r_v$ is the reflection across the hyperplane orthogonal to $v$. It is an improper orthogonal transformation, since $\ker p_{r_v}$ is one-dimensional. We define simply:

$$\mu(r_v) := B(v).$$

If $u, v$ are unit vectors in $V$ with $\langle u \mid v\rangle \neq 0$, then $s = r_u r_v \in \mathrm{SO}'_*(V)$. Then

$$\mu(r_u)\mu(r_v)\Omega = c(r_u, r_v)\mu(r_u r_v)\Omega = \langle u \mid v\rangle\Omega + u \wedge v,$$

with

$$c(r_u, r_v) = \exp(i \arg\langle u \mid v\rangle).$$

One then checks that $\mu(r_u)\mu(r_v)\Psi = c(r_u, r_v)\mu(r_u r_v)\Psi$ for all $\Psi \in \mathcal{F}_0(V)$.

We complete now the definition of $\mu(g)$ for $g \in \mathrm{SO}'_*(V)$ by defining it on $\mathcal{F}_1(V)$. In order to achieve $\mu(g)B(v) = B(gv)\mu(g)$ on $\mathcal{F}_0(V)$, we must set

$$\mu(g)(B(v)f_S) := B(gv)\mu(g)f_S, \tag{4.10}$$

for $S \in \mathrm{Sk}(V)$ such that $g \cdot S$ is defined. We are also free to define

$$\mu(gr_u) := \mu(g)\mu(r_u) \quad \text{for} \quad g \in \mathrm{SO}'_*(V),$$

and consequently $c(g, r_u) := 1$, so that (4.10) yields $\mu(gr_u)f_S = B(gu)\mu(g)f_S$. It can be verified that these partial definitions are consistent. Now from (3.7) we obtain $B(u)B(v) = B(r_u v)B(u)$, and one must then check that $\mu(gr_u)B(v) = B(gr_u v)\mu(gr_u)$; thus (4.1) holds for elements of the form $g = hr_u$ with $h \in \mathrm{SO}'_*(V)$. Since $hr_u h^{-1} = r_{hu}$, we can equivalently say that it holds for elements of the form $r_v h$, $h \in \mathrm{SO}'_*(V)$.

The general case now follows easily. We can always write $g = rh$ where $p_h$ is invertible and $r = r_{e_1} \dots r_{e_n}$ is a product of $n$ elements of the form (4.9), with $\{e_1, \dots, e_n\}$ an orthonormal basis of $\ker p_g^t$; we then define

$$\mu(g) = B(e_1) \dots B(e_n)\mu(h).$$

In particular, $\mu(g)\Omega = e_1 \wedge \cdots \wedge e_n \wedge f_{T_h}$, as expected.

The cocycle (4.7) is extended to all of $\mathrm{O}'(V)$ as follows. We define $c(g, r_u) := 1$ if $g \in \mathrm{SO}'_*(V)$ and $r_u$ is of the form (4.9). We set $c(gr_u, r_v) := c(g, r_u r_v)c(r_u, r_v)$ if $d(u,v) \neq 0$; otherwise we are free to take $c(g, r_u r_v)$, $c(r_u, r_v)$, $c(gr_u, r_v)$ all equal to 1. In general, we take $\mu(g, r_{e_1} \cdots r_{e_k}) = 1$ if

$g \in \mathrm{SO}'(V)$ and the $e_i$ are an orthonormal set of vectors in $V$. The cocycle equation $c(g,h)c(gh,k) = c(g,hk)c(h,k)$ then determines the remaining values of $c(g,h)$, in such a way that (4.7) remains valid for all $g, h \in \mathrm{O}'(V)$.

We have now obtained the full "pin representation" of $\mathrm{O}'(V)$ on $\mathcal{F}(V)$. By construction, it is an irreducible projective representation. Its restriction to $\mathrm{SO}'(V)$ has two orthogonal irreducible subspaces, namely $\mathcal{F}_0(V)$ and $\mathcal{F}_1(V)$; this is the "spin representation" of $\mathrm{SO}'(V)$.

# 5 Fermionic anomalies

## 5.1 The infinitesimal spin representation and the quantization procedure

The spin representation allows us to quantize all elements of the Lie algebra of the restricted orthogonal group $\mathrm{O}'(V)$. If $X \in \mathfrak{o}'(V)$ we write $C_X := \frac{1}{2}(X - JXJ)$, $A_X := \frac{1}{2}(X + JXJ)$ for the linear and antilinear parts of $X$. If $t \mapsto \exp tX$ is a one-parameter group with values in $\mathrm{SO}'(V)$, then $p_{\exp tX}$ is invertible for small enough $t$, and

$$\frac{d}{dt}\bigg|_{t=0} p_{\exp tX} = C_X, \qquad \frac{d}{dt}\bigg|_{t=0} T_{\exp tX} = A_X.$$

In particular, $A_X$ is Hilbert–Schmidt. Thus the antilinear part of $\mathfrak{o}'(v)$ is just $\mathrm{Sk}(V)$.

We define the *infinitesimal spin representation* $\dot{\mu}(X)$ of $X \in \mathfrak{o}'(V)$ by:

$$\dot{\mu}(X)\Psi := \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)}\mu(\exp tX)\Psi \tag{5.1}$$

for $\Psi \in \mathcal{F}(V)$, where $\theta_X(t)$ is such that $t \mapsto e^{i\theta_X(t)}\mu(\exp tX)$ is a homomorphism.

Writing $g(t) := \exp tX$ for small $t$, we obtain

$$\dot{\mu}(X)\Omega = \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)} c_{g(t)} f_{T_{g(t)}} = i\theta'_X(0)\Omega + H_{A_X}.$$

Recall that if $A_X = 0$, the quantized counterpart of $X$ is given by $d\Gamma(-JX)$, for which $d\Gamma(-JX)\Omega = 0$. The vacuum expectation value of $-i\dot{\mu}(X)$ is $-i\langle\Omega \mid \dot{\mu}(X)\Omega\rangle = \theta'_X(0)$. Since we may choose $\theta'_X(0)$ arbitrarily, we set $\theta'_X(0) = 0$ for all $X \in \mathfrak{o}'(V)$. Hence the quantization rule $X \mapsto -i\dot{\mu}(X)$ is uniquely specified by (5.1) together with the condition of vanishing vacuum expectation values.

We shall write $dG(X) := -i\dot{\mu}(X)$, for $X \in \mathfrak{o}'(V)$, to denote our quantization rule. As we have just remarked, $dG(X) = d\Gamma(-JX)$ whenever the latter makes sense: this is already clear from the fact that the spin representation generalizes the intertwining property (3.9).

To be more precise about the domains of the (unbounded) operators $\dot{\mu}(X)$ on $\mathcal{F}(V)$, we recall that for any $S \in \mathrm{Sk}(V)$, both $p_{\exp tX}$ and $(p_{\exp tX} + q_{\exp tX}S)$ will be invertible for small $t$, and so:

$$\begin{aligned}
\dot{\mu}(X) f_S &= \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)} c_{g(t)} \phi_{g(t)}(S) f_{g(t)\cdot S} \\
&= \frac{d}{dt}\bigg|_{t=0} \phi_{g(t)}(S) f_S + \sum_K \frac{d}{dt}\bigg|_{t=0} \mathrm{Pf}((g(t)\cdot S)_K)\, \varepsilon_K \\
&= \tfrac{1}{2}\,\mathrm{Tr}_{\mathbb{C}}[SA_X]\, f_S + \tfrac{1}{2} H_{\psi(X,S)} \wedge f_S,
\end{aligned} \tag{5.2}$$

where $\psi(X,S) = [C_X, S] + A_X - SA_XS$. This shows that $f_S \in \mathrm{Dom}(\dot\mu(X))$ – and in fact a similar calculation shows that the application $X \mapsto \dot\mu(X) f_S$ is differentiable.

In like manner, the vectors $B(v)f_S$ lie in the domain of $\dot\mu(X)$; indeed,

$$\begin{aligned}\dot\mu(X)(B(v)f_S) &= \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)} c_{g(t)}\phi_{g(t)}(S)B(g(t)v)f_{g(t)\cdot S}\\ &= B(v)\dot\mu(X)f_S + \frac{d}{dt}\bigg|_{t=0} B((\exp tX)v)f_S\\ &= B(v)\dot\mu(X)f_S + B(Xv)f_S \end{aligned}\tag{5.3}$$

by real-linearity of $B$. In the same way we get

$$\dot\mu(X)B(v)(B(u)f_S) = B(v)\dot\mu(X)(B(u)f_S) + B(Xv)(B(u)f_S). \tag{5.4}$$

Combining (5.3) and (5.4), we arrive at the fundamental commutation relations:

$$[\dot\mu(X), B(v)] = B(Xv) \tag{5.5}$$

as an operator-valued equation valid on the dense domain in $\mathcal{F}(V)$ generated by all $f_S$ and $B(u)f_S$. Indeed, (5.5) is the smeared expression for the formal commutation relations between field operators and currents; this justifies the name "currents" for the quantized observables.

### 5.2 Schwinger terms and cyclic cohomology

The extended orthogonal group $\widetilde{\mathrm{O}'}(V)$ is the one-dimensional central extension of $\mathrm{O}'(V)$ by $\mathrm{U}(1)$ whose elements are pairs $(g,\lambda)$, where $g \in \mathrm{O}'(V)$, $\lambda \in \mathrm{U}(1)$, with group law

$$(g,\lambda)\cdot(h,\lambda') = (gh, \lambda\lambda' c(g,h)),$$

so that $(g,\lambda) \mapsto \lambda\mu(g)$ is a *linear* unitary representation of the extended group. Its Lie algebra $\widetilde{\mathfrak{o}'}(V)$ is a one-dimensional central extension of $\mathfrak{o}'(V)$ by $i\mathbb{R}$, with commutator

$$[(X,ir),(Y,is)] := ([X,Y], \alpha(X,Y)),$$

where

$$\alpha(X,Y) = \frac{d^2}{dt\,ds}\bigg|_{t=s=0} c(\exp sX, \exp tY) - \frac{d^2}{dt\,ds}\bigg|_{t=s=0} c(\exp tY, \exp sX). \tag{5.6}$$

The Lie algebra cocycle $\alpha$ has the physical meaning of an anomalous commutator or *Schwinger term*. Indeed, if $X, Y \in \mathfrak{o}'(V)$, the Campbell–Baker–Hausdorff formula gives:

$$\alpha(X,Y) = [\dot\mu(X), \dot\mu(Y)] - \dot\mu([X,Y]).$$

**Proposition 2.** *If $X, Y \in \mathfrak{o}'(V)$, then*

$$\alpha(X,Y) = -\tfrac12 \operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]). \tag{5.7}$$

*Proof.* The linear and antilinear parts of $[X,Y] = [C_X + A_X, C_Y + A_Y]$ are given by

$$C_{[X,Y]} = [C_X, C_Y] + [A_X, A_Y], \qquad A_{[X,Y]} = [A_X, C_Y] + [C_X, A_Y].$$

The commutator $[\dot\mu(X), \dot\mu(Y)]$ may be computed from the quantization formula (6.4) by substituting (6.1) – see below. Using the CAR, one finds that

$$\begin{aligned}
[a^\dagger A_X a^\dagger, a^\dagger C_Y a] &= a^\dagger [A_X, C_Y] a^\dagger, \\
[a^\dagger C_X a, a A_Y a] &= a [C_X, A_Y] a, \\
[a^\dagger C_X a, a^\dagger C_Y a] &= a^\dagger [C_X, C_Y] a, \\
[a^\dagger A_X a^\dagger, a A_Y a] + [a A_X a, a^\dagger A_Y a^\dagger] &= -4\, a^\dagger [A_X, A_Y] a + 2 \operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]),
\end{aligned}$$

It then follows that $[\dot\mu(X), \dot\mu(Y)] = \dot\mu([X,Y]) - \frac{1}{2} \operatorname{Tr}_{\mathbb{C}}([A_X, A_Y])$. □

One can obtain the same result directly from (5.6).

The formula (5.7) yields the Schwinger term directly from the obstruction to linearity of the pin representation. When $V$ is finite-dimensional, the following reformulation is possible: since the linear commutant $[C_X, C_Y]$ is traceless, (5.7) reduces to $\alpha(X,Y) = -\frac{1}{2} \operatorname{Tr}_{\mathbb{C}}[C_{[X,Y]}]$, which is a trivial cocycle; thus the pin representation appears as a *linear* representation of a double covering group $\mathrm{Pin}(2m)$ of $\mathrm{O}(2m)$. In the infinite-dimensional case, this is no longer true, since $[C_X, C_Y]$ is in general not traceclass.

The Lie algebra cocycle $\alpha$ turns out to be also a cocycle for the *cyclic cohomology* of Connes [4, 12]; this has been pointed out by Araki [5]. We start from the observation that

$$\alpha(X,Y) = \tfrac{i}{4} \operatorname{Tr}(J[J,X][J,Y]) \quad \text{for} \quad X, Y \in \mathfrak{o}'(V).$$

Here Tr is the usual trace over the real Hilbert space $(V, d)$. Notice that

$$\operatorname{Tr}(J[J,Y][J,X]) = \operatorname{Tr}([J,X]J[J,Y]) = -\operatorname{Tr}(J[J,X][J,Y]);$$

antisymmetrization of the right-hand side yields $\frac{i}{2} \operatorname{Tr}(J[A_X, A_Y]) = -\frac{1}{2} \operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]) = \alpha(X,Y)$. Now a cyclic 1-cochain is simply an antisymmetric bilinear form $\omega$; and the cyclic coboundary operator $b$, defined by

$$b\omega(X,Y,Z) := \omega(XY, Z) - \omega(X, YZ) + \omega(ZX, Y),$$

yields $b\alpha(X,Y,Z) = (i/4) \sum_{\text{cyclic}} \operatorname{Tr}(J[J,XY][J,Z]) = 0$, so the fermionic Schwinger term $\alpha$ is a cyclic 1-cocycle. Higher-order cyclic cocycles also appear, somewhat mysteriously, in current algebras [17], in a manner closely related to the present approach [18].

## 5.3 Anomalies

The group $\mathrm{O}'(V)$ acts on $\widetilde{\mathfrak{o}'}(V)$ by the adjoint action of the central extension:

$$\widetilde{\operatorname{Ad}}(g) : (X, ir) \mapsto (\operatorname{Ad}(g)X, ir + \gamma(g,X)),$$

where the *anomaly* $\gamma(g,X) \in i\mathbb{R}$ depends linearly on $X$. Since

$$\widetilde{\operatorname{Ad}}(g)[(X,ir),(Y,is)] = [\widetilde{\operatorname{Ad}}(g)(X,ir), \widetilde{\operatorname{Ad}}(g)(Y,is)],$$

we obtain

$$\gamma(g,[X,Y]) = \alpha(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) - \alpha(X,Y),$$

for $X, Y \in \mathfrak{o}'(V)$. Therefore, *the anomaly is determined by the Schwinger terms*. Moreover, for $g \in \mathrm{O}'(V)$, $X \in \mathfrak{o}'(V)$, there holds:

$$\gamma(g,X) = \mu(g)\dot{\mu}(X)\mu(g)^{-1} - \dot{\mu}(\mathrm{Ad}(g)X). \tag{5.8}$$

Indeed, from (5.1) we obtain

$$\begin{aligned}\mu(g)\dot{\mu}(X)\mu(g)^{-1} &= \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)} c(g, \exp tX) c(g\exp tX, g^{-1})\mu(g\exp tXg^{-1}) \\ &= \frac{d}{dt}\bigg|_{t=0} c(g,\exp tX)c(g\exp tX, g^{-1}) + \dot{\mu}(\mathrm{Ad}(g)X)\end{aligned} \tag{5.9}$$

(where we have used $\dot{\theta}_X(0) = \dot{\theta}_{\mathrm{Ad}(g)X}(0) = 0$).

Thus the right hand side of (5.8) is an (imaginary) scalar; call it $\gamma'(g,X)$. That $\gamma'(g,[X,Y]) = \gamma(g,[X,Y])$ in general is clear from:

$$\begin{aligned}\gamma'(g,[X,Y]) &= \mu(g)\dot{\mu}([X,Y])\mu(g)^{-1} - \dot{\mu}([\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]) \\ &= \mu(g)[\dot{\mu}(X),\dot{\mu}(Y)]\mu(g)^{-1} - \alpha(X,Y) \\ &\qquad - [\dot{\mu}(\mathrm{Ad}(g)X), \dot{\mu}(\mathrm{Ad}(g)Y)] + \alpha(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) \\ &= [\dot{\mu}(\mathrm{Ad}(g)X) + \gamma'(g,X), \dot{\mu}(\mathrm{Ad}(g)Y) + \gamma'(g,Y)] \\ &\qquad - [\dot{\mu}(\mathrm{Ad}(g)X), \dot{\mu}(\mathrm{Ad}(g)Y)] + \gamma(g,[X,Y]).\end{aligned}$$

It is now straightforward to compute the fermionic anomaly.

**Proposition 3.** *For $g \in \mathrm{SO}'_*(V)$, $X \in \mathfrak{o}'(V)$, there holds*

$$\gamma(g,X) = -\tfrac{1}{2}\,\mathrm{Tr}_{\mathbb{C}}\big((1-\widehat{T}_g^2)^{-1}([A_X, \widehat{T}_g] - \widehat{T}_g[C_X, \widehat{T}_g])\big). \tag{5.10}$$

*Proof.* Let us write $h := \exp tX$. From (5.9), we observe that $\gamma(g,X)$ is given by the formula $\gamma(g,X) = \frac{d}{dt}\big|_{t=0} c(g,h)c(gh,g^{-1})$. The right-hand side equals

$$\begin{aligned}&\frac{d}{dt}\bigg|_{t=0} \exp(i\arg(\det{}^{1/2}(1 - T_h\widehat{T}_g) + \det{}^{1/2}(1 - \widehat{T}_g\widehat{T}_{gh}))) \\ &= i\Im\left(\frac{d}{dt}\bigg|_{t=0} \det{}^{1/2}(1 - T_h\widehat{T}_g) + \det{}^{-1/2}(1-\widehat{T}_g^2)\frac{d}{dt}\bigg|_{t=0}\det{}^{1/2}(1-\widehat{T}_g\widehat{T}_{gh})\right) \\ &= \tfrac{i}{2}\Im\,\mathrm{Tr}_{\mathbb{C}}\left(\frac{d}{dt}\bigg|_{t=0}(1 - T_h\widehat{T}_g) + (1-\widehat{T}_g^2)^{-1}\frac{d}{dt}\bigg|_{t=0}(1-\widehat{T}_g\widehat{T}_{gh})\right) \\ &= -\tfrac{i}{2}\Im\,\mathrm{Tr}_{\mathbb{C}}\left(A_X\widehat{T}_g + (1-\widehat{T}_g^2)^{-1}\widehat{T}_g\frac{d}{dt}\bigg|_{t=0}\widehat{T}_{gh}\right) \\ &= -\tfrac{i}{2}\Im\,\mathrm{Tr}_{\mathbb{C}}\left(A_X\widehat{T}_g + (1-\widehat{T}_g^2)^{-1}\widehat{T}_g\frac{d}{dt}\bigg|_{t=0}(\widehat{T}_h + p_h^{-1}\widehat{T}_g(1 - T_h\widehat{T}_g)^{-1}p_h^{-t})\right) \\ &= -\tfrac{1}{2}\,\mathrm{Tr}_{\mathbb{C}}\Big((1-\widehat{T}_g^2)^{-1}([A_X,\widehat{T}_g] - \widehat{T}_g[C_X,\widehat{T}_g])\Big).\end{aligned}$$

Notice that the linear operator $[A_X, \widehat{T}_g] - \widehat{T}_g[C_X, \widehat{T}_g] = [A_X - \widehat{T}_g C_X, \widehat{T}_g]$, as a commutator of two antilinear operators, has purely imaginary trace. ◻

Formula (5.10) seems to be new: appraisal of its strengths and weaknesses is in order. Within its range of validity it is completely general. This makes it very useful in conformal field theory. For instance, the anomalous conservation laws for the energy-momentum tensor and other observables, obtained in [19] by a direct procedure, can be computed with less labour from (5.10) by the well known trick of embedding the Virasoro group in an infinite dimensional orthogonal group. The appearance of the *commuting part* of $X$ in (5.10) also deserves a comment: whereas observables that are linear (in the sense of commuting with the complex structure) have nonanomalous commutators for their corresponding quantum currents, they still suffer in general from anomalous transformation laws. On the other hand, (5.10) is not applicable to the anomalies of the charge and chiral charge conservation laws, which are partly "topological" in nature: see our treatment in subsection 7.3.

# 6 The spin representation in terms of field operators

## 6.1 Currents

We reexpress the quantization prescription of subsection V.1 in the more congenial Fock space language. Given orthonormal bases $\{e_j\}$, $\{f_k\}$ of $(V, d, J)$, let us introduce the quadratic expressions:

$$\begin{aligned} a^\dagger T a^\dagger &:= \sum_{j,k} a_J^\dagger(f_k)\langle f_k \mid Te_j\rangle a_J^\dagger(e_j), \\ aTa &:= \sum_{j,k} a_J(e_j)\langle Te_j \mid f_k\rangle a_J(f_k), \\ a^\dagger Ca &:= \sum_{j,k} a_J^\dagger(f_k)\langle f_k \mid Ce_j\rangle a_J(e_j). \end{aligned} \tag{6.1}$$

Formulas (6.1) are independent of the orthonormal bases used if and only if $T$ is antilinear and skew and $C$ is linear as operators in $\mathrm{End}_{\mathbb{R}}(V)$. Also,

$$a^\dagger T a^\dagger\, \Omega = \sum_{j,k} \langle e_k \mid Te_j\rangle\, e_k \wedge e_j = H_T, \tag{6.2}$$

which lies in $\mathcal{F}_0(V)$ if and only if $T$ is Hilbert–Schmidt; and more generally, $a^\dagger T a^\dagger \Psi = H_T \wedge \Psi$ for $\Psi \in \mathcal{F}(V)$. Thus the series for $a^\dagger T a^\dagger$ is meaningful and defines a bounded operator on $\mathcal{F}(V)$ iff $T \in \mathrm{Sk}(V)$.

One easily sees that $aTa$ is the adjoint of $a^\dagger T a^\dagger$. If $T, S \in \mathrm{Sk}(V)$, then we find that:

$$(aTa) f_S = H_{STS} \wedge f_S - \mathrm{Tr}_{\mathbb{C}}[ST]\, f_S.$$

If $C$ is a skewadjoint linear operator on $V$, using (4.3) we find that

$$(a^\dagger Ca) f_S = \tfrac{1}{2} H_{[C,S]} \wedge f_S.$$

Now from (5.2) we obtain

$$2\dot{\mu}(X) f_S = H_{A_X} \wedge f_S + H_{[C_X,S]} \wedge f_S + \mathrm{Tr}_{\mathbb{C}}[SA_X]\, f_S - H_{SA_XS} \wedge f_S,$$

so we arrive at

$$\dot{\mu}(X) f_S = \tfrac{1}{2}(a^\dagger A_X a^\dagger + 2a^\dagger C_X a - aA_X a) f_S. \tag{6.3}$$

Moreover, since $B(v) = a_J^\dagger(v) + a_J(v)$, it is readily checked from the CAR that

$$\tfrac{1}{2}[a^\dagger A_X a^\dagger + 2a^\dagger C_X a - aA_X a, B(v)] = B(A_X v + C_X v) = B(Xv),$$

so (5.3) shows that (6.3) holds with $f_S$ replaced by $B(v) f_S$. We get, finally, for the current associated to $X$:

$$dG(X) = -\tfrac{i}{2}(a^\dagger A_X a^\dagger + 2a^\dagger C_X a - aA_X a), \qquad (6.4)$$

as an unbounded operator on the domain spanned by all $f_S$ and $B(v) f_S$.

### 6.2 Factorization of the scattering matrix

We interpret orthogonal transformations on $V$ as classical scattering transformations $S_{\mathrm{cl}}$. By "classical" we mean the operator living in the one-particle space, to distinguish it from the quantum scattering operator in Fock space. But for a phase factor, $\mu(S_{\mathrm{cl}})$ is precisely the $S$-matrix for a fermion system.

We seek to factorize the operator $\mu(g)$ in a convenient manner. Let us *define*, for $g \in \mathrm{SO}'_*(V)$, the operators $S_1, S_2, S_3$:

$$S_1(g) = \exp(\tfrac{1}{2}a^\dagger T_g a^\dagger), \quad S_2(g) = {:}\exp(a^\dagger(p_g^{-t} - 1)a){:}\,, \quad S_3(g) = \exp(\tfrac{1}{2}a\widehat{T}_g a).$$

We compute the effect of these operators on Gaussians in a few steps.

**Lemma 4.** *$S_1$ is a bounded operator on $\mathcal{F}(V)$, with $S_1 f_R = f_{T_g+R}$ for any $R \in \mathrm{Sk}(V)$.*

*Proof.* If $\Psi \in V^{\wedge k}$ for any finite $k$, from (6.2) we see that $(a^\dagger T_g a^\dagger)^m \Psi = H_{T_g}^{\wedge m} \wedge \Psi$; moreover, we get the norm estimate $\|H_{T_g}^{\wedge m} \wedge \Psi\| \leqslant \|H_{T_g}\|^m\, \|\Psi\| = \|T_g\|_{\mathrm{HS}}^m \|\Psi\|$, with $\|T_g\|_{\mathrm{HS}}$ denoting the Hilbert–Schmidt norm of $T_g$. Hence the series $\exp(\tfrac{1}{2}a^\dagger T_g a^\dagger) := \sum_{m=0}^\infty (2^m m!)^{-1}(a^\dagger T_g a^\dagger)^m$ converges in the bounded operator norm on $\mathcal{F}_0(V)$, with the estimate $\|\exp(\tfrac{1}{2}a^\dagger T_g a^\dagger)\| \leqslant \exp(\tfrac{1}{2}\|T_g\|_{\mathrm{HS}})$.

It is now immediate that $S_1 \Psi = f_{T_g} \wedge \Psi$ for any $\Psi \in \mathcal{F}(V)$. In particular, for $\Psi = f_R$, this gives $S_1 f_R = f_{T_g} \wedge f_R = f_{T_g+R}$. □

**Lemma 5.** *If $R \in \mathrm{Sk}(V)$ and $(1 - R\widehat{T}_g)$ is invertible, then*

$$S_3 f_R = \det{}^{1/2}(1 - R\widehat{T}_g) f_{R(1-\widehat{T}_g R)^{-1}}.$$

*Proof.* This is straightforward, from $\langle f_S \mid S_3 f_R\rangle = \langle S_3^\dagger f_S \mid f_R\rangle = \langle S_1(g^{-1}) f_S \mid f_R\rangle$. □

**Lemma 6.** *If $R \in \mathrm{Sk}(V)$, then $S_2 f_R = f_{p_g^{-t} R p_g^{-1}}$.*

*Proof.* Firstly, if $A$ is any bounded linear operator on $(V, d, J)$, then $ARA^t \in \mathrm{Sk}(V)$ and

$$f_{ARA^t} = \sum_{K \text{ finite}} \mathrm{Pf}(R_K)\, Ae_{k_1} \wedge \cdots \wedge Ae_{k_{2m}}; \qquad K = \{k_1, \dots, k_{2m}\}. \qquad (6.5)$$

This is obtained from (3.3) on noting that $H_{ARA^t} = \sum_{i,j} \langle e_i \mid Re_j\rangle Ae_i \wedge Ae_j$, which in turn follows from the definition (3.1).

Secondly, we must check that, if $K_m = \{1, \dots, 2m\}$, then

$$:\!\exp(a^\dagger C a)\!: \varepsilon_{K_m} = (1 + C)e_1 \wedge \cdots \wedge (1 + C)e_{2m}. \tag{6.6}$$

For then, if $K = \{k_1, \dots, k_{2m}\}$, we obtain $:\!\exp(a^\dagger C a)\!: \varepsilon_K = (1 + C)e_{k_1} \wedge \cdots \wedge (1 + C)e_{k_{2m}}$ by a change of orthonormal basis. Taking $C = p_g^{-t} - 1$, (6.5) gives the result.

To verify (6.6), note that the left hand side is a finite series, since the terms $a(e_{l_j})$ with $l_j > 2m$ give no contribution. Thus

$$\begin{aligned}
&:\!\exp(a^\dagger C a)\!: \varepsilon_{K_m} \\
&= \sum_{n=0}^{2m} \frac{1}{n!} \sum_{\substack{k_1 \dots k_n \\ l_1 \dots l_n}} a^\dagger(e_{k_1}) \dots a^\dagger(e_{k_n}) \prod_{j=1}^{n} \langle e_{k_j} \mid C e_{l_j} \rangle a(e_{l_n}) \dots a(e_{l_1}) \varepsilon_{K_m} \\
&= \sum_{L \subset K_m} \frac{1}{|L!|} \eta_L \, a^\dagger(C e_{l_1}) \dots a^\dagger(C e_{l_n}) \varepsilon_{K_m \setminus L} = \sum_{L \subset K_m} f_1 \wedge \cdots \wedge f_{2m},
\end{aligned}$$

where $\eta_L$ is the sign of the permutation $K_m \mapsto (L, K_m \setminus L)$ and $f_j = C e_j$ if $j \in L$, $f_j = e_j$ otherwise. But the latter sum is just an expansion of the right hand side of (6.6). □

It is worth noting that since

$$f_{p_g^{-t} R p_g^{-1}} = \sum_K \operatorname{Pf}((p_g^{-t} R p_g^{-1})_K) \varepsilon_K = \sum_K \det_{\mathbb{C}}((p_g^{-1})_K) \operatorname{Pf}(R_K) \varepsilon_K$$

and $|\det_{\mathbb{C}}((p_g^{-1})_K)| = \det^{1/2}((p_g^{-t})_K (p_g^{-1})_K) \leqslant \det^{1/2}(p_g^{-t} p_g^{-1}) = \det^{-1/2}(1 - T_g^2)$, then $S_2$ extends to $\mathcal{F}_0(V)$ as a bounded operator with norm at most $c_g^2$.

Now we see that, on applying $S_1$, $S_2$, $S_3$ in turn to $f_R$, the index of the Gaussian transforms as $R \mapsto T_g + p_g^{-t} R (1 - \widehat{T}_g R)^{-1} p_g^{-1} = g \cdot R$ by (4.4), and so by (4.5):

$$c_g S_1(g) S_2(g) S_3(g) \, f_R = c_g \det^{1/2}(1 - R\widehat{T}_g) f_{g \cdot R} = \mu(g) f_R, \tag{6.7}$$

whenever $p_g^{-1}$ and $g \cdot R$ exist. Thus $\mu(g) = c_g S_1 S_2 S_3$ holds on $\mathcal{F}_0(V)$ whenever $g \in \mathrm{SO}'_*(V)$.

Finally, if $p_g$ is not invertible, the general factorization is obtained from

$$\mu(g) = c_{rg} \, B(e_1) \cdots B(e_n) \, S_1(rg) S_2(rg) S_3(rg) \tag{6.8}$$

where $n = \dim(\ker p_g)$.

# 7 Quantization of the Dirac equation

## 7.1 The choice of complex structures

We examine only the case of a charged field. Thus we think of $V$ as the space of complex solutions of the free Dirac equation:

$$i \frac{\partial}{\partial t} \psi = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m) \psi =: H\psi,$$

where $\boldsymbol{p} = -i\,\partial/\partial\boldsymbol{x}$, regarded as a *real* vector space with the symmetric form:

$$d(\psi_1, \psi_2) = \frac{1}{2}\Big(\int \psi_1^* \psi_2 \, d^3x + \int \psi_2^* \psi_1 \, d^3x\Big).$$

For definiteness, we shall adopt the Dirac representation of the $\boldsymbol{\alpha}$ and $\beta$ matrices:

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}; \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator $H$ is selfadjoint with domain $\mathrm{Dom}(H) \subset \mathcal{H} := \mathbb{C}^4 \otimes L^2(\mathbb{R}^3)$. Define the two-spinor functions:

$$u_s(\boldsymbol{k}) := \begin{pmatrix} \sqrt{\omega(\boldsymbol{k}) + m}\, \chi_s \\ \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{k}}{\sqrt{\omega(\boldsymbol{k}) + m}}\, \chi_s \end{pmatrix}; \qquad v_s(\boldsymbol{k}) := \begin{pmatrix} \dfrac{\boldsymbol{\sigma} \cdot \boldsymbol{k}}{\sqrt{\omega(\boldsymbol{k}) + m}}\, \chi_s \\ \sqrt{\omega(\boldsymbol{k}) + m}\, \chi_s \end{pmatrix};$$

where $s = \uparrow$ or $\downarrow$, as the case may be, $\chi_\uparrow = \binom{1}{0}$ and $\chi_\downarrow = \binom{0}{1}$. Denote by $(-,-)$ the ordinary hermitian product on $\mathbb{C}^2$. Then one checks that:

$$\begin{gathered} \big(u_s(\boldsymbol{k}), u_{s'}(\boldsymbol{k})\big) = \big(v_s(\boldsymbol{k}), v_{s'}(\boldsymbol{k})\big) = 2\omega(\boldsymbol{k})\delta_{ss'}, \\ \big(u_s(\boldsymbol{k}), v_{s'}(-\boldsymbol{k})\big) = 0. \end{gathered}$$

We consider also the projectors:

$$P_\pm = \tfrac{1}{2}\big(1 \pm (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\omega^{-1}\big),$$

corresponding respectively to the positive and negative parts of the spectrum of $H$. Then $P_+ + P_- = 1$ and $HP_\pm = \pm\omega P_\pm$; we may write $V_\pm = P_\pm V$. We note also the relations:

$$\begin{aligned} P_+(\boldsymbol{k})u_s(\boldsymbol{k}) &= u_s(\boldsymbol{k}), & P_+(\boldsymbol{k})v_s(-\boldsymbol{k}) &= 0, \\ P_-(\boldsymbol{k})u_s(\boldsymbol{k}) &= 0, & P_-(\boldsymbol{k})v_s(-\boldsymbol{k}) &= v_s(-\boldsymbol{k}), \end{aligned} \tag{7.1}$$

where we have denoted by $P_\pm(\boldsymbol{k})$ the projectors on the Fourier transformed space of $\mathcal{H}$, which are multiplication operators. Moreover,

$$\begin{aligned} u_\uparrow(\boldsymbol{k})u_\uparrow^\dagger(\boldsymbol{k}) + u_\downarrow(\boldsymbol{k})u_\downarrow^\dagger(\boldsymbol{k}) &= 2\omega(\boldsymbol{k})P_+(\boldsymbol{k}), \\ v_\uparrow(-\boldsymbol{k})v_\uparrow^\dagger(-\boldsymbol{k}) + v_\downarrow(-\boldsymbol{k})v_\downarrow^\dagger(-\boldsymbol{k}) &= 2\omega(\boldsymbol{k})P_-(\boldsymbol{k}). \end{aligned} \tag{7.2}$$

Besides $\mathcal{H}$, we shall consider the Hilbert space $\widetilde{\mathcal{H}}$, which is $V$ endowed with the new complex Hilbert space structure given by $d(-,-) + i d(J-,-)$, with $J := i(P_+ - P_-)$. In other words, we preserve the real part $d$ of the inner product and we introduce a nonlocal imaginary part through the nonlocal complex structure. In this way, complex multiplication is dynamically built into the space of solutions, in such a manner to make possible a direct interpretation of "negative energy" solutions as antiparticles.

Now define the 2-spinor functions:

$$b(\boldsymbol{k}) := \begin{pmatrix} \big(u_\uparrow(\boldsymbol{k}), \mathcal{F}\psi(\boldsymbol{k})\big) \\ \big(u_\downarrow(\boldsymbol{k}), \mathcal{F}\psi(\boldsymbol{k})\big) \end{pmatrix}, \qquad d(\boldsymbol{k}) := \begin{pmatrix} \big(\mathcal{F}^{-1}\psi(\boldsymbol{k}), v_\uparrow(\boldsymbol{k})\big) \\ \big(\mathcal{F}^{-1}\psi(\boldsymbol{k}), v_\downarrow(\boldsymbol{k})\big) \end{pmatrix}.$$

In view of (7.1) and (7.2), this transformation is inverted by:

$$\psi(\boldsymbol{x}) = (2\pi)^{-3/2} \sum_{s=\uparrow,\downarrow} \int \left(b_s(\boldsymbol{k}) u_s(\boldsymbol{k}) e^{i\boldsymbol{kx}} + d_s^\dagger(\boldsymbol{k}) v_s(\boldsymbol{k}) e^{-i\boldsymbol{kx}}\right) d\mu(k),$$

where $d\mu(k) := d^3k/\omega(\boldsymbol{k})$. It is seen now that the map to momentum space $\widetilde{\mathcal{H}} \to \mathcal{H}_m^{\frac12,+} \oplus \mathcal{H}_m^{\frac12,+}$ where $\mathcal{H}_m^{\frac12,+} = \mathbb{C}^2 \otimes L^2(H_m^+, d\mu)$, given by $\psi \mapsto \binom{b}{d}$, is an isometry such that $J\psi \mapsto i\binom{b}{d}$. Here we make contact with Weinberg's quantization method [20], based on Wigner's theory of the unitary irreducible representations of the Poincaré group. It will follow from our treatment in the next subsection that the fermion field has the gauge transformation properties required in the Weinberg construction.

### 7.2 Quantization and the charge operator

We now prove that the *standard* construction of fermion Fock space of Section III, applied to $\widetilde{\mathcal{H}}$ or $\mathcal{H}_m^{\frac12,+} \oplus \mathcal{H}_m^{\frac12,+}$, gives the charged fermion field. In order to fully grasp what is involved here, we need some further reflection on the relation between the two Hilbert space structures for the space of solutions of the Dirac equation. We can define the charge operator on the one-particle space for the Dirac equation as the generator of gauge transformations: $Q\psi := i\psi$. It is an infinitesimally orthogonal operator:

$$d(\psi_1, Q\psi_2) + d(Q\psi_1, \psi_2) = 0.$$

By definition, $(V, d, Q) \equiv \mathcal{H}$ and $(V, d, J) \equiv \widetilde{\mathcal{H}}$. One can pass from the "natural" complex structure $Q$ to $J$ by means of an orthogonal transformation of $V$:

$$g_0 Q g_0 = J, \quad \text{with} \quad g_0 := \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} = g_0^{-1}, \tag{7.3}$$

in the $V_+ \oplus V_-$ splitting. Thus the Dirac equation on $\mathcal{H}$:

$$i\frac{\partial}{\partial t}\psi = H\psi$$

becomes:

$$J\frac{\partial}{\partial t}\psi = -iJH\psi,$$

the transformed operator $\widetilde{H} := -iJH$ being selfadjoint on $\widetilde{\mathcal{H}}$. Obviously $-iH$ and $-J\widetilde{H}$ are the same element of the orthogonal Lie algebra (in the generalized sense, since these are unbounded operators). The crucial difference is that $\widetilde{H}$ is bounded below, in fact positive:

$$\widetilde{H}P_\pm = \omega P_\pm.$$

The orthogonal transformation $g_0$ has changed the spectrum! The idea is apparently due to Bongaarts, although its implementation in [21] is rather murky; one should look also at [22] and [23]. As $g_0$ does not fulfil the Shale–Stinespring criterion, $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\widetilde{\mathcal{H}})$ are not equivalent upon quantization. We must choose $\widetilde{\mathcal{H}}$ for the *standard* construction of fermion Fock space – which allows the straightforward particle interpretation – to yield the charged fermion field. This is a crucial point in our argument, because otherwise we would not be able to use the spin representation in QED.

In view of (7.3), we can apply the Bogoliubov transformation philosophy to relate creation and annihilation operators defined with respect to each complex structure. Let us abbreviate $\psi_+ := P_+\psi$, $\psi_- := P_-\psi$. Then $p_{g_0}\psi = \psi_+$, $q_{g_0}\psi = \psi_-^*$. We get on $\mathcal{F}(\mathcal{H})$, from (4.2):

$$a_Q(g_0\psi) = a_J(\psi_+) + a_J^\dagger(\psi_-^*); \qquad a_Q^\dagger(g_0\psi) = a_J^\dagger(\psi_+) + a_J(\psi_-^*),$$

so that

$$a_Q(\psi) = a_J(\psi_+) + a_J^\dagger(\psi_-); \qquad a_Q^\dagger(\psi) = a_J^\dagger(\psi_+) + a_J(\psi_-)$$

and similar converse equations in $\mathcal{F}(\widetilde{\mathcal{H}})$.

Note also the simple formula:

$$\langle \psi \mid \phi \rangle_J = \langle\!\langle \psi_+ \mid \phi_+ \rangle\!\rangle + \langle\!\langle \phi_- \mid \psi_- \rangle\!\rangle, \tag{7.4}$$

where $\langle\!\langle - \mid - \rangle\!\rangle$ will now denote the "natural" inner product $\langle - \mid - \rangle_Q$ of $\mathcal{H}$.

Now we prove – for operators on $V$ preserving both structures – that our quantization method gives the same result as the usual procedure of first performing the Fock quantization with respect to the "natural" complex structure and then amending the result with a "normal ordering" recipe; the tricky subtraction of infinities is avoided.

**Proposition 7.** *If $X$ commutes with both $J$ and $Q$, then $dG(X) = {:}d\Gamma_Q(-iX){:}$.*

*Proof.* From (6.4) we obtain $dG(X) = d\Gamma_J(-JX) = -i\, a_J^\dagger X a_J$ since $XJ = JX$. Let us choose orthonormal bases $\{\varphi_k\}$ and $\{\psi_k\}$ for $V_+$ and $V_-$. We write $b^{(\dagger)}$ rather than $a^{(\dagger)}$ on $V_+$, $d^{(\dagger)}$ rather than $a^{(\dagger)}$ on $V_-$, as is customary. If $Y = -JX$, then $Y$ is selfadjoint on $\widetilde{\mathcal{H}}$, and so

$$\begin{aligned} -i\, a_J^\dagger X a_J &= \sum_{j,k} b_J^\dagger(\varphi_k)\langle \varphi_k \mid Y\varphi_j\rangle_J b_J(\varphi_j) + d_J^\dagger(\psi_k)\langle \psi_k \mid Y\psi_j\rangle_J d_J(\psi_j) \\ &= \sum_{j,k} b_J^\dagger(\varphi_k)\langle\!\langle \varphi_k \mid -iX\varphi_j\rangle\!\rangle b_J(\varphi_j) - d_J^\dagger(\psi_k)\langle\!\langle \psi_j \mid -iX\psi_k\rangle\!\rangle d_J(\psi_j) \\ &= {:}d\Gamma_Q(-iX){:}. \end{aligned}$$

□

In particular,

$$\mathbb{Q} := dG(Q) = b^\dagger b - d^\dagger d$$

in units of electronic charge. This current has integer eigenvalues and we call charge sectors the eigenspaces $\mathcal{F}_k(V)$, for $k \in \mathbb{Z}$. We remark that the 1-particle charge conjugation operator, which is antilinear in $\mathcal{H}$, is linear in $\widetilde{\mathcal{H}}$ [22, 23].

The particle interpretation of the quantum field and the possibility of a direct *translation into physics of our results on the spin representation* hinge on our choice of $J$ as complex structure, one that allows us to dry up the Dirac sea – which is what the physical vacuum looks like using the "natural" (and wrong) complex structure. However, as local conservation of the charge in interactions is a basic physical principle, we cannot dispense entirely with $Q$ in the quantization process; the interplay of both complex structures is characteristic of the theory of charged fields. This is reflected in the fact that the invariance group of the theory is not $\mathrm{O}'(V)$, but its subgroup $\mathrm{U}'_Q(V)$ of (restricted) unitary operators on $H$, which has a very different topological structure: whereas $\mathrm{O}'(V)$ has two connected pieces, we will soon see that the group $\mathrm{U}'_Q(V)$ has an infinite number of connected pieces, naturally indexed by $\mathbb{Z}$.

## 7.3 Charge sectors

Now we are prepared to translate the group-representation machinery into the usual language for QED. We relabel the vacuum vector $\Omega$ as $|0\rangle$, and write $|0_{\mathrm{out}}\rangle := \mu(g)\,|0\rangle$ for the out vacuum. From (3.2) and (6.2), it follows at once that

$$\exp(\tfrac{1}{2}a^\dagger T a^\dagger)\,|0\rangle = f_T$$

for $T \in \mathrm{Sk}(V)$. Thus $|0_{\mathrm{out}}\rangle$ is proportional to $\exp(\frac{1}{2}a^\dagger T_g a^\dagger)\,|0\rangle$ whenever $g \in \mathrm{SO}'_*(V)$.

If the classical scattering operator $g = S_{\mathrm{cl}}$ lies in $\mathrm{O}'(V)$ but with $\dim(\ker p_g) = n > 0$, we again write $g = r_{e_1} \dots r_{e_n} h$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $\ker p_g^t$; the out-vacuum can thus be written as

$$|0_{\mathrm{out}}\rangle \propto e_1 \wedge \cdots \wedge e_n \wedge f_{T_{rg}} = a^\dagger(e_1) \dots a^\dagger(e_n) \exp(\tfrac{1}{2}a^\dagger T_{rg} a^\dagger)\,|0\rangle. \tag{7.5}$$

In this subsection and most of what follows, we shall always assume that a particle-antiparticle fermion field can be built over the one-particle space $V$, and that $J = i(P_+ - P_-)$ always, where $P_\pm = 1 - P_\mp$ denote orthogonal projectors on infinite dimensional subspaces, the outstanding example being the space of solutions of a Dirac equation. This is to say, we propose to deal with charged fermion fields; even so, all we have to say in the next subsection is also valid for neutral fields.

We write then $g, p_g, q_g, T_g, \widehat{T}_g$ in matricial form with respect to the decomposition $V = V_+ \oplus V_-$ with the proviso that $T_g$ exists if and only if $p_g$ is invertible:

$$g = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}, \quad \text{thus} \quad p_g = \begin{pmatrix} S_{++} & 0 \\ 0 & S_{--} \end{pmatrix}, \quad q_g = \begin{pmatrix} 0 & S_{+-} \\ S_{-+} & 0 \end{pmatrix}.$$

The fact that $g \in \mathrm{U}'_Q(V)$ means precisely (as remarked at the end of the previous section) that the $S_{\pm\pm}$ are *complex-linear* operators acting between the complex spaces $V_+, V_-$. We leave to the care of the reader to rewrite (2.1) in terms of the $S$'s. It is immediate that

$$T_g = \begin{pmatrix} 0 & S_{+-}S_{--}^{-1} \\ S_{-+}S_{++}^{-1} & 0 \end{pmatrix}, \quad \widehat{T}_g = \begin{pmatrix} 0 & -S_{++}^{-1}S_{+-} \\ -S_{--}^{-1}S_{-+} & 0 \end{pmatrix}.$$

We see that $p_g^{-1}$ exists if and only if $S_{++}$ and $S_{--}$ are invertible, and that $g \in \mathrm{O}'(V)$ iff $S_{+-}$ and $S_{-+}$ are Hilbert–Schmidt (actually, since $S_{+-}(g^{-1}) = (S_{-+}(g))^\dagger$, it suffices that $S_{+-}$ be Hilbert–Schmidt).

From the fact, remarked in Section II, that $\operatorname{ind} p_g = 0$, we get $\operatorname{ind} S_{++} = -\operatorname{ind} S_{--}$. One checks that $g(\ker S_{\pm\pm}) = \ker S^\dagger_{\mp\mp}$, and so $\operatorname{ind} S_{\pm\pm} = \dim\ker S^\dagger_{\mp\mp} - \dim\ker\ S^\dagger_{\pm\pm}$.

Now, the $f_{T_g}$ are all charge zero states, since $\gamma(g, Q) = 0$ for all $g \in \mathrm{SO}'_*(V) \cap \mathrm{U}'_Q(V)$.

For out-vacua of the form (7.5), we can always choose orthonormal bases $\{\varphi_1, \dots, \varphi_l\}$ and $\{\psi_1, \dots, \psi_m\}$ for $\ker S^\dagger_{++}$ and $\ker S^\dagger_{--}$ respectively, so that $l + m = n$. Clearly, the expectation value of the charge in the out-vacuum $|0_{\mathrm{out}}\rangle = \mu(g)\,|0\rangle$ is

$$\langle 0_{\mathrm{out}} \mid \mathbb{Q} \mid 0_{\mathrm{out}}\rangle = l - m = \dim\ker S^\dagger_{++} - \dim\ker S^\dagger_{--} = \operatorname{ind} S_{--} = -\operatorname{ind} S_{++},$$

and then:

$$\mu(g)\mathcal{F}_k(V) = \mathcal{F}_{k+\operatorname{ind} S_{--}}(V),$$

which can be rewritten as

$$\mu(g)\mathbb{Q}\mu(g)^{-1} = \mathbb{Q} + \operatorname{ind} S_{++}. \tag{7.6}$$

Thus the group $\mathrm{U}'_Q(V)$ has infinitely many connected components, indexed by the Fredholm index of $S_{++}$ or $S_{--}$; which components interchange the charged vacua. Note that (7.6) is an *anomalous* identity, since $Q$ commutes both with $g$ and $J$. However, it has been shown by Carey and O'Brien [24] in QED and then for quite general gauge fields by Matsui [25] that under reasonable circumstances the scattering matrix belongs to the component of the identity $\mathrm{U}'_{Q,0}(V)$ of the group; thus vacuum polarization in this sense does not occur in the external field problem – where consequently only pair creation arises.

A similar treatment is possible for the chiral charge anomaly. Let us consider, for simplicity, a theory of massless fermions in 1+1 spacetime dimensions. Then again $Q_5 := i\gamma_5$ is an infinitesimally orthogonal operator, commuting with $J$, which takes in momentum space the following form, with respect to the $V = V_+ \oplus V_-$ splitting:

$$Q_5 = \begin{pmatrix} i\varepsilon(k) & 0 \\ 0 & i\varepsilon(-k) \end{pmatrix}.$$

The support in momentum space of the elements of an orthonormal basis for $p^t_g$ must now lie either in the right or the left half axis. With an obvious notation, the chiral current is

$$\mathbb{Q}_5 := dG(Q_5) = b^\dagger_R b_R - b^\dagger_L b_L + d^\dagger_L d_L - d^\dagger_R d_R.$$

Again $\mathbb{Q}_5$ has integer eigenvalues and (for scattering operators $g$ such that $[g, Q_5] = 0$) a formula of the type (7.6) intervenes; only now the index of the scattering operator is directly related to the Chern number of the gauge field [25] and it is generally nonzero for elements of $\mathrm{U}'_{Q,0}(V)$. This is why the chiral charge anomaly is local in the usual parlance [26]. Index formulae for currents associated with special gauge transformations are given in [27]. Further consideration of these questions would take us too far afield.

### 7.4 The scattering matrix for a charged fermion field

For simplicity, we shall assume for the rest of this subsection that $p_g$ is invertible. We need the formulas:

$$\mathrm{Tr}_{\mathbb{C}} \begin{pmatrix} A_{++} & 0 \\ 0 & A_{--} \end{pmatrix} = \mathrm{Tr}(A_{++}) + \mathrm{Tr}(A^\dagger_{--}), \tag{7.7}$$

and by exponentiating:

$$\det_{\mathbb{C}} \begin{pmatrix} A_{++} & 0 \\ 0 & A_{--} \end{pmatrix} = \det(A_{++})\ \det(A^\dagger_{--}), \tag{7.8}$$

which come from (7.4). For instance, using (7.7) and since $X^\dagger_{+-} = -X_{-+}$, the Schwinger terms (5.7) reduce to:

$$\begin{aligned}[dG(X), dG(Y)] = -\alpha(X,Y) &= \tfrac{1}{2}\,\mathrm{Tr}(X_{+-}Y_{-+} - Y_{+-}X_{-+}) + \tfrac{1}{2}\,\mathrm{Tr}(Y_{-+}X_{+-} - X_{-+}Y_{+-}) \\ &= \mathrm{Tr}(X_{+-}Y_{-+} - Y_{+-}X_{-+}) = 2i\,\Im\,\mathrm{Tr}(X_{+-}Y_{-+}).\end{aligned}$$

The fermionic anomaly $\gamma(g, X)$ can also be recomputed in the QED language. From (5.10) and (7.7) one gets:

$$\gamma(g,X) = \operatorname{Tr}\big((1 - S_{++}^{-1}S_{+-}S_{--}^{-1}S_{-+})^{-1}(X_{+-}S_{--}^{-1}S_{-+} - S_{++}^{-1}S_{+-}X_{-+} \\ - S_{++}^{-1}S_{+-}S_{--}^{-1}S_{-+}X_{++} + S_{++}^{-1}S_{+-}X_{--}S_{--}^{-1}S_{-+})\big).$$

For the absolute value of the vacuum persistence amplitude we obtain, since $T_g$ is skewsymmetric:

$$\begin{aligned} |\langle 0_{\rm in} \mid 0_{\rm out}\rangle| &= \det{}^{-1/4}(1 - T_g^2) = \det{}^{-1/2}(1 + (S_{+-}S_{--}^{-1})^\dagger S_{+-}S_{--}^{-1}) \\ &= \det{}^{-1/2}((S_{--}^\dagger)^{-1}(S_{--}^\dagger S_{--} + S_{+-}^\dagger S_{+-})S_{--}^{-1}) \\ &= \det{}^{-1/2}((S_{--}^\dagger)^{-1}S_{--}^{-1}) = \det{}^{1/2}(S_{--}S_{--}^\dagger), \end{aligned}$$

using $S_{--}^\dagger S_{--} + S_{+-}^\dagger S_{+-} = 1$ (on $V_-$). On the other hand,

$$|\langle 0_{\rm in} \mid 0_{\rm out}\rangle| = \det{}^{1/4}(p_g p_g^t) = \det{}^{1/4}(S_{++}S_{++}^\dagger)\ \det{}^{1/4}(S_{--}S_{--}^\dagger),$$

and hence both factors on the right hand side are equal. We thus arrive at

$$\begin{aligned} |\langle 0_{\rm in} \mid 0_{\rm out}\rangle| &= \det{}^{1/2}(S_{--}S_{--}^\dagger) = \det{}^{1/2}(S_{++}S_{++}^\dagger) \\ &= \det{}^{1/2}(1 - S_{+-}S_{+-}^\dagger) = \det{}^{1/2}(1 - S_{-+}S_{-+}^\dagger). \end{aligned}$$

We are finally ready for the computation of the full $S$-matrix. We start from the factorization (6.7). We choose orthonormal bases $\{\varphi_k\}$ and $\{\psi_k\}$ for $V_+$ and $V_-$ and regard their union as an orthonormal basis for $V$. We distinguish the particle and antiparticle sectors by setting $b^\dagger(\varphi_k) := a_J^\dagger(\varphi_k)$, $d^\dagger(\psi_k) := a_J^\dagger(\psi_k), \dots$ Then we find:

$$\begin{aligned} \tfrac12 a^\dagger T_g a^\dagger &= \frac12 \sum_{j,k} a_J^\dagger(\varphi_k)\langle \varphi_k \mid T_g\psi_j\rangle a_J^\dagger(\psi_j) + a_J^\dagger(\psi_j)\langle \psi_j \mid T_g\varphi_k\rangle a_J^\dagger(\varphi_k) \\ &= \frac12 \sum_{j,k} b^\dagger(\varphi_k)\langle\!\langle \varphi_k \mid T_g\psi_j\rangle\!\rangle d^\dagger(\psi_j) + d^\dagger(\psi_k)\langle\!\langle T_g\varphi_j \mid \psi_k\rangle\!\rangle b^\dagger(\varphi_j) \\ &= \frac12 \sum_{j,k} b^\dagger(\varphi_k)\langle\!\langle \varphi_k \mid S_{+-}S_{--}^{-1}\psi_j\rangle\!\rangle d^\dagger(\psi_j) + d^\dagger(\psi_j)\langle\!\langle S_{-+}S_{++}^{-1}\varphi_k \mid \psi_j\rangle\!\rangle b^\dagger(\varphi_k) \\ &= \sum_{j,k} b^\dagger(\varphi_k)\langle\!\langle \varphi_k \mid S_{+-}S_{--}^{-1}\psi_j\rangle\!\rangle d^\dagger(\psi_j) =: b^\dagger S_{+-}S_{--}^{-1}d^\dagger, \end{aligned} \tag{7.9}$$

using the CAR $\{b^\dagger(\varphi_j), d^\dagger(\psi_k)\} = 0$, the relation $(S_{-+}S_{++}^{-1})^\dagger = -(S_{+-}S_{--}^{-1})$, and (7.4). In like manner, we obtain

$$\begin{aligned} \tfrac12 a\widehat{T}_g a &= \frac12 \sum_{j,k} a_J(\varphi_k)\langle \widehat{T}_g\varphi_k \mid \psi_j\rangle a_J(\psi_j) + a_J(\psi_j)\langle \widehat{T}_g\psi_j \mid \varphi_k\rangle a_J(\varphi_k) \\ &= \frac12 \sum_{j,k} b(\varphi_k)\langle\!\langle \psi_j \mid \widehat{T}_g\varphi_k\rangle\!\rangle d(\psi_j) + d(\psi_j)\langle\!\langle \widehat{T}_g\psi_j \mid \varphi_k\rangle\!\rangle b(\varphi_k) \\ &= -\frac12 \sum_{j,k} b(\varphi_k)\langle\!\langle \psi_j \mid S_{--}^{-1}S_{-+}\varphi_k\rangle\!\rangle d(\psi_j) + d(\psi_j)\langle\!\langle S_{++}^{-1}S_{+-}\psi_j \mid \varphi_k\rangle\!\rangle b(\varphi_k) \\ &= \sum_{j,k} d(\psi_j)\langle\!\langle \psi_j \mid S_{--}^{-1}S_{-+}\varphi_k\rangle\!\rangle b(\varphi_k) =: dS_{--}^{-1}S_{-+}b. \end{aligned} \tag{7.10}$$

The Wick-ordered product $:\!\exp(a^\dagger(p_g^{-t}-1)a)\!:$ can be written as $S_{2b}S_{2d}$ by separating the $b^{(\dagger)}$ and $d^{(\dagger)}$ terms. Since $\langle\!\langle \varphi_k \mid (p_g^{-t}-1)\varphi_l \rangle\!\rangle = \langle\!\langle \varphi_k \mid ((S_{++}^\dagger)^{-1}-1)\varphi_l \rangle\!\rangle$, it follows that $S_{2b} = {:\!\exp(b^\dagger((S_{++}^\dagger)^{-1}-1)b)\!:}$, and for $S_{2d}$ we get

$$
\begin{aligned}
S_{2d} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1\dots k_n\\ l_1\dots l_n}} d^\dagger(\psi_{k_1})\cdots d^\dagger(\psi_{k_n}) \prod_{j=1}^{n} \langle\!\langle \psi_{l_j} \mid (1-S_{--}^{-1})\psi_{k_j} \rangle\!\rangle d(\psi_{l_n})\cdots d(\psi_{l_1}) \\
&= {:\!\exp(d(1-S_{--}^{-1})d^\dagger)\!:}\,.
\end{aligned}
\tag{7.11}
$$

Putting the equations (7.9)–(7.11) together, we arrive at the exact $S$-matrix for the charged fermion field:

$$
\begin{aligned}
\boldsymbol{S} = e^{i\theta}\mu(g) &= \langle 0_{\text{in}} \mid 0_{\text{out}} \rangle \, \exp(b^\dagger S_{+-}S_{--}^{-1}d^\dagger) \\
&\quad \times {:\!\exp(b^\dagger((S_{++}^\dagger)^{-1}-1)b - d(S_{--}^{-1}-1)d^\dagger)\!:}\, \exp(dS_{--}^{-1}S_{-+}b).
\end{aligned}
\tag{7.12}
$$

Whenever $S_{++}$ and $S_{--}$ are not invertible, this formula must be modified in accordance with (6.8).

It is instructive to compare the form (7.12) of the fermionic $S$-matrix with the bosonic $S$-matrix expression obtained in a parallel way from the metaplectic representation [28]. We remark that the exact quantum $S$-matrix for the external field problem has direct application in laser physics and heavy ion collisions.

Let $\varphi \in V_+$ and $\psi \in V_-$. The following commutation rules (and their adjoints) are very useful for calculations involving the $S$-matrix:

$$
\begin{aligned}
&[e^{b^\dagger A d^\dagger}, b(\varphi)] = -d^\dagger(A^\dagger\varphi)e^{b^\dagger A d^\dagger}, \qquad && e^{b^\dagger A d^\dagger}, d(\psi)] = b^\dagger(A\psi)e^{b^\dagger A d^\dagger}, \\
&{:\!e^{b^\dagger(A-1)b}\!:}\, b^\dagger(\varphi) = b^\dagger(A\varphi)\, {:\!e^{b^\dagger(A-1)b}\!:}\,, && {:\!e^{d(1-A)d^\dagger}\!:}\, d^\dagger(\psi) = d^\dagger(A^\dagger\psi)\, {:\!e^{d(1-A)d^\dagger}\!:}\,.
\end{aligned}
\tag{7.13}
$$

## 8 The Feynman rules for electrodynamics of external fields

We next derive the Feynman rules for quantum electrodynamics of external fields from the exact $S$-matrix. The Dirac equation in an external electromagnetic field is

$$
i\frac{\partial}{\partial t}\psi = (H+V)\psi,
$$

where the external field $V$ is given by

$$
V\psi = e(-\boldsymbol{\alpha}\cdot\boldsymbol{A} + A_0)\psi = e\gamma^0 A\!\!\!/\psi,
$$

with the usual notation $A\!\!\!/ := \gamma^\mu A_\mu$. The classical scattering matrix corresponding to this problem is given by a Dyson expansion:

$$
S_{\text{cl}} = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_{n-1}} \widetilde{V}(t_1)\cdots\widetilde{V}(t_n)\, dt_n\cdots dt_2\, dt_1 =: \sum_{n=0}^{\infty} S_{:n}.
$$

Here $\widetilde{V}(t) := e^{iHt}V(t)e^{-iHt}$. We write out $S_{:n}$ as an integral kernel in momentum space:

$$S_{:n}(\boldsymbol{k},\boldsymbol{k}') := \frac{(-i)^n}{(2\pi)^{3n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int \cdots \int e^{iH(\boldsymbol{k})t_1} V(t_1, \boldsymbol{k}-\boldsymbol{k}_1)$$
$$\times\, \theta(t_1-t_2) e^{-iH(\boldsymbol{k}_1)(t_1-t_2)} V(t_2, \boldsymbol{k}_1-\boldsymbol{k}_2) \ldots \theta(t_1-t_n) e^{-iH(\boldsymbol{k}_{n-1})(t_{n-1}-t_n)}$$
$$\times\, V(t_n, \boldsymbol{k}_{n-1}-\boldsymbol{k}') e^{-iH(\boldsymbol{k}')t_n}\, d^3k_1 \cdots d^3k_{n-1}\, dt_n \cdots dt_1,$$

where $\theta$ denotes the Heaviside function.

We shall not dwell on the question of the conditions on $V$ such that $S_{\mathrm{cl}}$ is implementable. An apparently more ambitious endeavour would be to try to implement the interacting time evolution operator $U(t,t')$ with $U(\infty,-\infty) = S_{\mathrm{cl}}$. It can be shown, however, that $U(t,t')$ can be implemented only for electric fields; hence the implementability of time evolution has no covariant meaning and the particle interpretation has meaning only in the realm of scattering theory.

To derive the Feynman rules, the first step is to rewrite $S_{:n}$ in covariant form. We follow the treatment of [29]. From the formula:

$$\theta(t)e^{-iH(\boldsymbol{k})t} = \frac{i}{2\pi}\int dk^0 S_{\mathrm{ret}}(k)\gamma^0 e^{-ik^0 t},$$

where

$$S_{\mathrm{ret}}(k) := \frac{\not{k}+m}{k^2-m^2+ik^0 0},$$

we obtain:

$$(S_{\pm\pm})_{:n}(\boldsymbol{k},\boldsymbol{k}') = -i(2\pi)^{-2n+1} e^n P_\pm(\boldsymbol{k})\gamma^0 \biggl(\int \cdots \int \not{A}(k-k_1) S_{\mathrm{ret}}(k_1)$$
$$\times\, \not{A}(k_1-k_2) \cdots S_{\mathrm{ret}}(k_{n-1})\not{A}(k_{n-1}-k')\, d^4k_1 \cdots d^4k_{n-1}\biggr) P_\pm(\boldsymbol{k}')$$

(where $k^0 = \omega(\boldsymbol{k})$ and $k'^0 = \omega(\boldsymbol{k}')$ are understood).

It is well known and physically obvious – as beautifully discussed in the classic paper [30] – that the $n$-pair amplitudes are Slater determinants of the one-pair amplitudes. It is enough thus to derive the one-pair amplitudes. There are four of them, which are not altogether independent; their expressions may be computed from (7.12) and the rules (7.13) of commutation of the creation and annihilation operators with the quadratic exponentials.

(a) For electron scattering from initial state $\varphi_i$ to final state $\varphi_f$:

$$S_{fi} := \langle b^\dagger(\varphi_f)0_{\mathrm{in}} \mid \boldsymbol{S} b^\dagger(\varphi_i)0_{\mathrm{in}}\rangle = \langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}}\rangle\, \langle\!\langle S_{++}^{-1}\varphi_f \mid \varphi_i\rangle\!\rangle;$$

(b) For positron scattering from initial state $\psi_i$ to final state $\psi_f$:

$$S_{fi} := \langle d^\dagger(\psi_f)0_{\mathrm{in}} \mid \boldsymbol{S} d^\dagger(\psi_i)0_{\mathrm{in}}\rangle = \langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}}\rangle\, \langle\!\langle \psi_i \mid S_{--}^{-1}\psi_f\rangle\!\rangle;$$

(c) For creation of an electron-positron pair, in respective states $\varphi, \psi$:

$$S_{fi} := \langle b^\dagger(\varphi)d^\dagger(\psi)0_{\mathrm{in}} \mid \boldsymbol{S} 0_{\mathrm{in}}\rangle = \langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}}\rangle\, \langle\!\langle \varphi \mid S_{+-}S_{--}^{-1}\psi\rangle\!\rangle;$$

(d) For annihilation of an electron-positron pair, in respective states $\varphi, \psi$:

$$S_{fi} := \langle 0_{\text{in}} \mid \boldsymbol{S} b^\dagger(\varphi) d^\dagger(\psi) 0_{\text{in}} \rangle = \langle 0_{\text{in}} \mid 0_{\text{out}} \rangle \, \langle\!\langle \psi \mid S_{--}^{-1} S_{-+} \varphi \rangle\!\rangle.$$

Note that if $V$ is time-independent, by a well-known result of scattering theory, one has $[S_{\text{cl}}, H] = 0$; thus $[S_{\text{cl}}, J] = 0$ and there cannot be creation or annihilation of pairs. In such a context, the quantized and the one-particle theory are essentially equivalent.

We need to compute an expansion for $S_{\pm\pm}^{-1}$ from the expansion of $S_{\text{cl}}$, in order to proceed. From the identity $1 + \sum_{n\geqslant 1} (S^{-1})_{:n} = S^{-1} = (1 + \sum_{n\geqslant 1} S_{:n})^{-1}$, we obtain

$$(S^{-1})_{:n} = -\big(S_{:n} + S_{:n-1}(S^{-1})_{:1} + S_{:n-2}(S^{-1})_{:2} + \cdots + S_{:1}(S^{-1})_{:n-1}\big), \tag{8.1}$$

where juxtaposition means convolution of kernels: $ST(\boldsymbol{k}, \boldsymbol{k}') := \int S(\boldsymbol{k}, \boldsymbol{k}_1)\, T(\boldsymbol{k}_1, \boldsymbol{k}')\, d^3k_1$. For instance:

$$\begin{aligned}
(S_{++}^{-1})_{:1}(\boldsymbol{k}, \boldsymbol{k}') &= -(S_{++})_{:1}(\boldsymbol{k}, \boldsymbol{k}') = \frac{i}{2\pi} e\, P_+(\boldsymbol{k}) \gamma^0 A\!\!\!/(k - k') P_+(\boldsymbol{k}'); \\
(S_{++}^{-1})_{:2}(\boldsymbol{k}, \boldsymbol{k}') &= -(S_{++})_{:2}(\boldsymbol{k}, \boldsymbol{k}') + \int (S_{++})_{:1}(\boldsymbol{k}, \boldsymbol{k}_1)(S_{++})_{:1}(\boldsymbol{k}_1, \boldsymbol{k}')\, d^3k_1 \\
&= i(2\pi)^{-3} e^2\, P_+(\boldsymbol{k}) \gamma^0 \bigg( \int A\!\!\!/(k - k_1) S_{\text{ret}}(k_1) A\!\!\!/(k_1 - k')\, d^4k_1 \bigg) P_+(\boldsymbol{k}') \\
&\quad - (2\pi)^{-2} e^2\, P_+(\boldsymbol{k}) \gamma^0 \bigg( \int A\!\!\!/(k - k_1) P_+(\boldsymbol{k}_1) \gamma^0 A\!\!\!/(k_1 - k') \delta(k_1^0 - \omega(\boldsymbol{k}_1))\, d^4k_1 \bigg) P_+(\boldsymbol{k}') \\
&= i(2\pi)^{-3} e^2\, P_+(\boldsymbol{k}) \gamma^0 \bigg( \int A\!\!\!/(k - k_1) \gamma^0 S_F(k_1)^\dagger \gamma^0 A\!\!\!/(k_1 - k')\, d^4k_1 \bigg) P_+(\boldsymbol{k}').
\end{aligned}$$

Here we have used:

$$\big[S_{\text{ret}}(k) + 2\pi i P_+(\boldsymbol{k}) \gamma^0 \delta(k^0 - \omega(\boldsymbol{k}))\big]^\dagger = \gamma^0 S_F(k) \gamma^0, \tag{8.2}$$

where $S_F$ is the Feynman propagator,

$$S_F(k) := \frac{k\!\!\!/ + m}{k^2 - m^2 + i0}.$$

Taking adjoints and using $A\!\!\!/(k)^\dagger = \gamma^0 A\!\!\!/(k) \gamma^0$, we get:

$$\begin{aligned}
((S_{++}^{-1})^\dagger)_{:1}(\boldsymbol{k}, \boldsymbol{k}') &= -\frac{i}{2\pi} e\, P_+(\boldsymbol{k}) \gamma^0 A\!\!\!/(k - k') P_+(\boldsymbol{k}'), \\
((S_{++}^{-1})^\dagger)_{:2}(\boldsymbol{k}, \boldsymbol{k}') &= -i(2\pi)^{-3} e^2\, P_+(\boldsymbol{k}) \gamma^0 \bigg( \int A\!\!\!/(k - k_1) S_F(k_1) A\!\!\!/(k_1 - k')\, d^4k_1 \bigg) P_+(\boldsymbol{k}').
\end{aligned}$$

Proceeding recursively according to (8.1), in the same way we obtain:

$$\begin{aligned}
((S_{++}^{-1})^\dagger)_{:n}(\boldsymbol{k}, \boldsymbol{k}') = -i(2\pi)^{-2n+1} e^n\, P_+(\boldsymbol{k}) \gamma^0 \bigg( \int A\!\!\!/(k - k_1) S_F(k_1) A\!\!\!/(k_1 - k_2) \cdots \\
S_F(k_{n-1}) A\!\!\!/(k_{n-1} - k')\, d^4k_1 \cdots d^4k_{n-1} \bigg) P_+(\boldsymbol{k}').
\end{aligned}$$

The argument for the scattering of a positron is entirely analogous.

Similarly, for pair creation we must compute:

$$((S_{--}^{-1})^{\dagger} S_{+-}^{\dagger})_{:n} = ((S_{--}^{-1})^{\dagger})_{:0} (S_{+-}^{\dagger})_{:n} + \cdots + ((S_{--}^{-1})^{\dagger})_{:n-1} (S_{+-}^{\dagger})_{:1}.$$

Note that

$$\begin{aligned}(S_{+-}^{\dagger})_{:n}(\boldsymbol{k}, \boldsymbol{k}') = i(2\pi)^{-2n+1} e^n \, P_-(\boldsymbol{k}) \gamma^0 \int d^4 k_1 \cdots \int d^4 k_{n-1} \\ \times \not{A}(k - k_1) S_{\mathrm{adv}}(k_1) \cdots S_{\mathrm{adv}}(k_{n-1}) \not{A}(k_{n-1} - k') \, P_+(\boldsymbol{k}'),\end{aligned}$$

where $S_{\mathrm{adv}}(k) := (\not{k} + m)/(-k^2 + m^2 + ik^0 0)$.

By using as needed (8.2) under the form:

$$S_{\mathrm{adv}}(k) - 2\pi i P_+(\boldsymbol{k}) \gamma^0 \delta(k^0 - \omega(\boldsymbol{k})) = S_F(k),$$

we can finally reexpress the whole expansion for this process in terms of the Feynman propagators, obtaining:

$$\begin{aligned}(S_{+-} S_{--}^{-1})_{:n}(\boldsymbol{k}, \boldsymbol{k}') = -i(2\pi)^{-2n+1} e^n \, P_+(\boldsymbol{k}) \gamma^0 \Big( \int \not{A}(k - k_1) S_F(k_1) \not{A}(k_1 - k_2) \cdots \\ S_F(k_{n-1}) \not{A}(k_{n-1} - k') \, d^4 k_1 \cdots \, d^4 k_{n-1} \Big) P_-(\boldsymbol{k}').\end{aligned}$$

We leave as an exercise for the reader the treatment of pair annihilation.

Note that the phase factor of $\langle 0_{\mathrm{in}} \,|\, 0_{\mathrm{out}} \rangle$ has no bearing on the probabilities for electron scattering, pair production, and the like.

## 9 On virtual vacuum polarization

One can compute the polarization of the vacuum, following [31], as the vacuum expectation value of the current: $\langle 0_{\mathrm{in}} \mid j^{\mu}(x) 0_{\mathrm{in}} \rangle$, where the current is defined as the functional derivative

$$j^{\mu}(x) := i \boldsymbol{S}^{\dagger} \frac{\delta \boldsymbol{S}}{\delta A_{\mu}(x)} \, .$$

Thus, one must study the functional dependence of the phase factor $e^{i\theta[A]}$ on the vector potential. The information we need is encoded in the vacuum persistence amplitude. Recall that the effective action $W$ is defined by $\langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}} \rangle =: e^{iW}$. Introduce $\gamma(k) = 2m^2/k^2$ and

$$\begin{aligned}G(k) := \frac{\alpha}{3} \int k^{-2} (1 + \gamma(k))(1 - 2\gamma(k))^{1/2} \theta(1 - 2\gamma(k)) \, d^4 k, \\ G^{\mu\nu}(k) := (k^{\mu} k^{\nu} k^2 - g^{\mu\nu} k^4) \, G(k),\end{aligned}$$

where $\alpha = e^2/4\pi$, the fine structure constant. It is easy to see that $G(x)$ and $G^{\mu\nu}(x)$ have no support outside the light cone. Perturbatively, to first order of approximation we get:

$$\begin{aligned}|\langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}} \rangle| \simeq \exp(-\tfrac{1}{2} \operatorname{Tr}(S_{+-} S_{+-}^{\dagger})) \simeq 1 - \tfrac{1}{2} \operatorname{Tr}(S_{+-} S_{+-}^{\dagger}) \\ = 1 - \int G^{\mu\nu}(k) A_{\mu}(k) A_{\nu}^{*}(k) \, d^4 k = 1 - \int G(k) |j(k)|^2 \, d^4 k,\end{aligned}$$

after a routine calculation, where the Maxwell equations have been used to conjure up the sources of the classical field. This gives the imaginary part of $W$.

The real part of the effective action is found by means of a dispersion relation. We gloss here the very detailed treatment in [29]. The dispersion relation can be written nonperturbatively in the form:

$$\frac{\delta}{\delta A_\nu(y)} \boldsymbol{S}^\dagger[A] \frac{\delta \boldsymbol{S}[A]}{\delta A_\mu(x)} = 0$$

for $A$ arbitrary and $y^0 > x^0$.

Adding the real and imaginary components of the effective action, we finally arrive at:

$$W[j] = \frac{\alpha}{3\pi} \int d^4k\, H(k) |j(k)|^2,$$

where

$$H(k) = \int_{4m^2}^{\infty} d\lambda \, \frac{(1+\gamma(\lambda))(1-2\gamma(\lambda))^{-1/2}}{\lambda(\lambda - k^2 - i0)} .$$

The latter is precisely the renormalized expression from which one can immediately compute, for instance, the Uehling correction to the Coulomb potential [32].

Now, what is the meaning of the dispersion relation? It is the functional-differential form of the "causality condition" [29, 31]:

$$\boldsymbol{S}[A_1 + A_2] = \boldsymbol{S}[A_2]\, \boldsymbol{S}[A_1]$$

when $A_2$ is to the future of $A_1$. This is nothing but our cocycle condition (4.8) for orthogonal transformations parametrized by the gauge potential:

$$e^{i\theta[A_1+A_2]} = e^{i\theta[A_1]} e^{i\theta[A_2]} \exp\bigl(i \arg\det(1 + S_{--}^{-1}[A_2] S_{-+}[A_2] S_{+-}[A_1] S_{--}^{-1}[A_1])\bigr).$$

In the last formula the equation (7.8) has been used.


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