The metaplectic representation and boson fields

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Abstract

We explicitly the infinite-dimensional metaplectic representation and show how its use simplifies and rigorizes several questions in bosonic Quantum Field Theory. The representation permutes Gaussian elements in the boson Fock space, and is necessarily projective. We compute its cocycle at the group level, and obtain Schwinger terms and anomalies from different versions of the cocycle; for instance, the Virasoro anomalous terms are obtained in this manner. We show how the choice of a complex structure on the space of solutions of a wave equation is related to the covariant Feynman propagator methods. We then show how the metaplectic representation allows one to compute exactly the S-matrix for bosons in an external field from the classical scattering operator.

1 Introduction

The main purpose of this paper is to give a detailed, rigorous account of the metaplectic representation of the infinite-dimensional symplectic group and its applications in Quantum Field Theory. A companion paper [1] does the same for the pin representation of the infinite-dimensional orthogonal group. It has been known for a long time that linear field theory (e.g., for bosons or fermions in an external field) can be mathematically described entirely by the above mentioned representation theory. For the boson case at least, this was recognized as far back as the work of I. E. Segal in the sixties. However, it seems to us that the advantages of explicitly working with the metaplectic or pin representations, when available, have not been adequately recognized in the physics literature. This makes textbooks treatments of scattering theory, such as that of Reed and Simon [2] for bosons, appear more complicated than the subject really warrants.

We contend that the construction of the algebras of field operators is straightforward in the group-theoretical context. It accordingly makes sense to clear up the rubble at this level before trying to tackle specific problems. The emphasis of these articles is on the explicit calculation of the representations. Once this has been achieved, the parameters of the representation are reinterpreted in physical terms and the answer to pertinent physical questions becomes surprisingly simple.

Existence of the metaplectic and the infinite-dimensional pin representations is proved in the basic papers of Shale [3] and Shale and Stinespring [4], respectively. However, explicit presentations have been rather late in coming. Our approach tends to unify the treatment of ordinary quantum-mechanical systems of bosons and fermions and systems with infinitely many degrees of freedom. But there are significant differences between the finite and the infinite-dimensional cases, related to the existence of a nonsplit extension by the circle group of the (restricted) symplectic and orthogonal groups. The corresponding metaplectic cocycle was first exhibited, to the best of our knowledge, by G. Segal [5]. We wish to point out that the pin representation is the cornerstone of the book *Loop Groups* [6] by A. Pressley and G. Segal; but it is not computed there in all generality. The only paper purporting to do something equivalent is the masterly survey by Araki [7]. Our methods are rather different, even so.

There are several things in the present review that we think are new: we extend the "real-variable" treatment of Gaussian integrals by Robinson and Rawnsley [8] to the infinite dimensional case; the treatment of the derived metaplectic representation in infinite dimensions, in relation to quantization; the derivation of the anomaly in Section 7 from the nonequivariance of the adjoint action, in particular formula (7.8); the derivation of the S-matrix from the representation. The connoisseur will find here and there little "technological" improvements. But the article has a pedagogical bent. We hope to revive the conceptually appealing and uncomplicated, but mathematically rigorous, approach to quantum fields by I. E. Segal [9], fusing it with methods patented by G. Segal. We trust that our paper may provide a bridge easy to cross for physicists and mathematicians familiar with Classical Mechanics and Lie group representations, wishing to get acquainted on their own terms with the basics of Quantum Field Theory.

In Section 2 the basic algebraic facts concerning infinite-dimensional symplectic vector spaces, complex structures and polarizations are laid out. We introduce here an important computational tool, which is a parametrization of the symplectic group [8] that deserves to be better known. The subgroup of symplectic transformations whose antilinear part is Hilbert–Schmidt is introduced and its action on the infinite-dimensional analogue of the Cartan–Siegel disk is discussed.

In Section 3, the Fock–Bargmann–Segal construction of Fock space is performed; we develop the important Gaussian integrals in infinite-dimensional spaces. From this a simple proof of Shale's theorem is given. Section 4 treats general Weyl systems, their derived systems (boson fields) and the Wick theorem for bosons.

Section 5 is the heart of the paper. We compute the metaplectic representation in the Fock–Bargmann–Segal space, with its cocycle. In Section 6 the metaplectic procedure is examined from the general standpoint of quantization. Here we obtain the derived metaplectic representation, and we show how this formalism, applied in the finite-dimensional case, reproduces the standard coherent-state approach to ordinary Quantum Mechanics, together with Berezin's "covariant quantization" scheme.

Section 7 deals with Schwinger terms and anomalies in linear field theories. A complete treatment is given and the relation with Quillen's and Connes' cyclic cohomology is discussed.

Section 8 treats the Virasoro group and Lie algebra within the framework of bosonic field quantization. The Hilbert transform of functions on the circle provides the appropriate complex structure on the tangent space of the loop group. The results of Section 7 are then employed to derive the anomaly in the Virasoro group and algebra from the metaplectic cocycle.

In Section 9, the quantization of the space of solutions of a Klein–Gordon equation is performed, using the metaplectic representation. We show how one relates, for free fields in a given class of spacetimes, the approach based on complex structures and the Fock–Segal–Bargmann spaces to the covariant methods based on the Feynman propagator.

In Section 10, we compute exactly the S-matrix for bosons in an external field, in the standard Fock-space language. We point out the relation between the metaplectic cocycle and the phase of the vacuum persistence amplitude. In Section 11, we show how the formalism may be adapted for charged fields; charge conservation follows from the vanishing of the corresponding Schwinger term. In both cases, the group representations yield the respective S-matrices as quantizations of the classical scattering operator.

An Appendix deals with the question of the appropriate choice of complex structures suitable for quantization; *a fortiori* it is concerned with classical Hamiltonian systems in infinite-dimensional spaces.

Although we are not concerned with them here, it must be said that there exist noteworthy applications of the theory developed in this paper to problems in filtering theory, digital signal processing and optics [10].

Throughout the paper, units are taken so that $c = 1$ and $\hbar = 1$.

2 Symplectic vector spaces

The classical manifold underlying the boson fields is just a symplectic vector space, i.e., a *real vector space* V with a *symplectic form* s (i.e., a nondegenerate antisymmetric bilinear form) on V. If V is finite-dimensional, its dimension must be even, but we shall mainly be concerned with the infinite-dimensional case. The primary examples of symplectic vector spaces are spaces of solutions of dynamical equations, such as the Klein–Gordon equation.

2.1 Complex structures

In order to quantize, a real symplectic space is not enough; we need a complex (Hilbert) space. We must then choose a *complex structure* J, i.e., a real-linear operator on V which satisfies

$$J^2 = -1, \tag{2.1}$$

and moreover:

$$s(Ju, Jv) = s(u, v), \quad \text{for } u, v \in V, \tag{2.2a}$$

$$s(v, Jv) > 0, \quad \text{for } 0 \neq v \in V. \tag{2.2b}$$

The condition (2.2a) is that the complex structure be also symplectic; if so, we shall say that J is *compatible* with the given symplectic form s. The positivity condition (2.2b) is equivalent to demanding that the symmetric bilinear form

$$d(u, v) \equiv d_J(u, v) := s(u, Jv)$$

be positive definite on V. This allows to regard V as a *complex* vector space under the rule

$$(\alpha + i\beta)v := \alpha v + \beta Jv \quad \text{for } \alpha, \beta \text{ real}, \tag{2.3}$$

and in that case the *hermitian* form

$$\langle u \mid v \rangle := s(u, Jv) + is(u, v) = d(u, v) + id(Ju, v) \tag{2.4}$$

is a positive definite scalar product on V.

Complex structures satisfying (2.2b) need not always exist. A sufficient condition is that V be a real Hilbert space under some given positive definite symmetric form d_0, with respect to which s is continuous (on such a V, s need only be nondegenerate in the weak sense, i.e., $s(u, v) = 0$ for all $v \in V$ if and only if $u = 0$). This is proved in the Appendix. Now s appears as the imaginary part of a scalar product (2.4). The Hilbert space structure determined by (2.4) is complete if and only if s is nondegenerate in the strong sense, i.e., the bounded real-linear operator B on (V, d_0) determined by $d_0(Bu, v) = s(u, v)$ is bijective.

One should regard positive compatible complex structures on (V, s) as a device for the dense embedding of the real symplectic space V into a complex Hilbert space $\mathcal{H}$, such that $\Im\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$, restricted to V, equals s; then $\Re\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$ gives d_J. It has recently been shown by Kay and Wald [11] that, given d_0 (a positive definite symmetric form on V) such that

$$|s(u, v)|^2 \leqslant d_0(u, u) d_0(v, v), \tag{2.5}$$

a suitable Hilbert space exists, and V may be densely embedded in it provided the inequality (2.5) is sharp. Moreover, any two such embeddings are unitarily equivalent. We shall show in later sections how such embeddings may be constructed in practice.

We say V is *hilbertizable* when a suitable J can be found, and we shall assume this to be the case. It is known that hilbertizability is a minimum condition for the existence of a free boson field on V [12]. To simplify the discussion, then, we shall henceforth assume that V is complete for the scalar product (2.4), and is thus the underlying real space of a complex Hilbert space.

The real part d of the scalar product (whose imaginary part is s) is not unique, since it depends on J. However, as shown below, the induced metric topology on V does not depend on the chosen complex structure. We shall denote this Hilbert space by V also, or by (V, s, J) whenever precision demands it. We shall further assume that (V, s, J) is separable.

Note, in particular, that $\langle u \mid Jv \rangle = i\langle u \mid v \rangle$ but $\langle Ju \mid v \rangle = -i\langle u \mid v \rangle$.

► We write $A \in \mathrm{End}_{\mathbb{R}}(V)$ if A is a real-linear endomorphism on V. A^t will denote its transpose with respect to d, i.e., $d(u, A^t v) := d(Au, v)$. We let $\mathrm{GL}_{\mathbb{R}}(V)$ denote the group of invertible endomorphisms, and write $\mathrm{Sp}(V, s)$, or simply $\mathrm{Sp}(V)$, for the symplectic group

$$\mathrm{Sp}(V) := \{\, g \in \mathrm{GL}_{\mathbb{R}}(V) : s(gu, gv) = s(u, v) \text{ for all } u, v \in V \,\}.$$

Any $A \in \mathrm{End}_{\mathbb{R}}(V)$ which commutes with J is also complex-linear on V regarded as a complex space via (2.3); we shall simply say that A is *linear*. If B is a real-linear operator on V such that $BJ = -JB$, we shall call B *antilinear*.

In terms of the scalar product (2.4), the hermitian conjugate of a real-linear operator coincides with its transpose, since $s(u, A^t v) = s(Au, v)$ and hence $\langle u \mid A^t v \rangle = \langle Au \mid v \rangle$ if A is linear; whereas $s(u, B^t v) = -s(Bu, v) = s(v, Bu)$ and hence $\langle u \mid B^t v \rangle = \langle v \mid Bu \rangle$ if B is antilinear. We shall also write $A^{-t} := (A^{-1})^t = (A^t)^{-1}$ for $A \in \mathrm{End}_{\mathbb{R}}(V)$.

2.2 The algebra of symplectic transformations

An invertible real-linear operator g is symplectic iff $Jg = g^{-t}J$ iff $g^{-t} = -JgJ$, since $g \in \mathrm{Sp}(V)$ iff $d(Jgu, w) = d(Ju, g^{-1}w)$ for all $u, v \in V$. Thus also $g \in \mathrm{Sp}(V)$ iff $g^{-t} \in \mathrm{Sp}(V)$ iff $g^t \in \mathrm{Sp}(V)$.

We may decompose any real-linear operator g on V into linear and antilinear parts by

$$p_g := \tfrac{1}{2}(g - JgJ), \qquad q_g := \tfrac{1}{2}(g + JgJ). \tag{2.6}$$

Note that $g \in \mathrm{Sp}$ if and only if $p_g = \frac{1}{2}(g + g^{-t})$, $q_g = \frac{1}{2}(g - g^{-t})$. We shall write simply p, q whenever a fixed g is understood. If $g \in \mathrm{Sp}$, then p is invertible, since

$$p^t p = \tfrac{1}{4}(g^t + g^{-1})(g + g^{-t}) = \tfrac{1}{4}(g^t g + g^{-1}g^{-t} + 2) \geqslant \tfrac{1}{2}$$

and similarly $pp^t \geqslant \frac{1}{2}$ (we shall soon see that in fact $p^t p \geqslant 1$, $pp^t \geqslant 1$).

We define $T_g := q_g p_g^{-1}$ for $g \in \mathrm{Sp}(V)$. It will be convenient to abbreviate $\widehat{T}_g := T_{g^{-1}}$. We can parametrize $g \in \mathrm{Sp}(V)$ by the pair (p, q), or alternatively by the pair (p, T). We summarize the algebraic properties of these parameters as follows.

Proposition 2.1. *If $g \in \mathrm{Sp}(V)$, then g may be expressed in a unique manner as $g = (1 + T)p$, where T is antilinear and symmetric, and $1 - T^2$ is positive definite; p is linear and satisfies $p^t(1 - T^2)p = 1$. Conversely, given a pair (p, T) of real-linear operators on V satisfying these conditions, the operator $g := (1 + T)p$ belongs to* $\mathrm{Sp}(V)$. *Moreover, for $g \in \mathrm{Sp}(V)$ these relations hold:*

$$p_{g^{-1}} = p_g^t; \qquad \widehat{T}_g := T_{g^{-1}} = -p_g^{-1} T_g p_g.$$

Proof. If $g \in \mathrm{Sp}$, p_g is invertible and $g = (1 + T_g)p_g$ follows from the definitions of T_g and p_g. It is immediate that $p_{g^{-1}} = \frac{1}{2}(g^{-1} - Jg^{-1}J) = \frac{1}{2}(-Jg^tJ + g^t) = p_g^t$. The antilinear part of the equation $1 = gg^{-1} = (1 + T_g)p_g(1 + \widehat{T}_g)p_g^t$ then yields $0 = T_g p_g + p_g \widehat{T}_g$, giving $\widehat{T}_g = -p_g^{-1} T_g p_g$.

Now we get $g = p_g + T_g p_g = p_g(1 - \widehat{T}_g)$; replacing g by g^{-1} gives $g^{-1} = p_g^t(1 - T_g)$, from which we conclude that $p_g^t(1 - T_g^2)p_g = g^{-1}g = 1$.

It is clear that p_g is linear and T_g is antilinear, and $1 - T_g^2 = (p_g^{-1})^t p_g^{-1}$ is positive definite. To see that T_g is symmetric, we must show that $s(u, T_g v) + s(T_g u, v) = 0$ for all $u, v \in V$. This follows from

$$\begin{aligned} s(u, (1 - T_g)v) &= s(u, p_g^{-t} g^{-1} v) = s(p_g^{-1} u, g^{-1} v) \\ &= s(g p_g^{-1} u, v) = s((1 + T_g)u, v). \end{aligned}$$

We remark that, since T_g is antilinear, its symmetry may alternatively be expressed as

$$d(u, T_g v) = d(T_g u, v), \quad \text{or} \quad \langle u \mid T_g v \rangle = \langle v \mid T_g u \rangle.$$

The uniqueness of the decomposition $g = (1 + T_g)p_g$ is clear, for p_g must be the linear part of g and $T_g p_g$ the antilinear part.

Conversely, given (p, T) satisfying the stated conditions, write $g := (1 + T)p$. Then g is invertible, and

$$g^{-t} = (g^{-1})^t = (p^t(1 - T))^t = (1 - T)p = -J(1 + T)J(-JpJ) = -JgJ,$$

so $g \in \mathrm{Sp}(V)$. □

The corresponding algebraic properties of the pairs (p_g, q_g) are obtained by noting that $q_{g^{-1}} = \frac{1}{2}(g^{-1} + Jg^{-1}J) = -\frac{1}{2}(Jg^tJ + g^t) = -q_g^t$. Taking linear and antilinear parts of the equalities $gg^{-1} = 1$, $g^{-1}g = 1$, we find that

$$p_g p_g^t - q_g q_g^t = p_g^t p_g - q_g^t q_g = 1, \quad p_g q_g^t = q_g p_g^t, \quad p_g^t q_g = q_g^t p_g. \tag{2.7}$$

Suppose $g, h \in \mathrm{Sp}(V)$; then $(1 + T_{gh})p_{gh} = (1 + T_g)p_g(1 + T_h)p_h$ and the uniqueness of the decomposition leads to

$$p_{gh} := p_g(1 - \widehat{T}_g T_h)p_h, \tag{2.8a}$$

$$T_{gh} := p_g(T_h - \widehat{T}_g)(1 - \widehat{T}_g T_h)^{-1} p_g^{-1} \tag{2.8b}$$

$$= (p_g T_h + q_g)(q_g T_h + p_g)^{-1}. \tag{2.8c}$$

Another expression for T_{gh} is also quite useful. From the identity $p_g^t(1 - T_g^2)p_g = 1$ we obtain $p_g(1 - \widehat{T}_g^2)p_g^t = 1$ on substituting g^{-1} for g; thus $p_g = p_g^{-t} + p_g\widehat{T}_g^2$. This yields $p_g(T_h - \widehat{T}_g) = p_g^{-t}T_h - p_g\widehat{T}_g(1 - \widehat{T}_g T_h)$; then, using (2.8b) and $T_g = -p_g\widehat{T}_g p_g^{-1}$, we arrive at

$$T_{gh} = T_g + p_g^{-t} T_h (1 - \widehat{T}_g T_h)^{-1} p_g^{-1}. \tag{2.9}$$

► We now define $\mathcal{D}(V) := \{\, X \in \mathrm{End}_{\mathbb{R}} V : XJ = -JX,\ X^t = X,\ 1 - X^2 > 0 \,\}$, which we may call the *open Cartan–Siegel disk* of V. We have shown that if $g \in \mathrm{Sp}$, then $T_g \in \mathcal{D}(V)$. Conversely, if $T \in \mathcal{D}(V)$, we may take $p := (1 - T^2)^{-1/2}$ and thereby $h_T := (1 + T)(1 - T^2)^{-1/2} \in \mathrm{Sp}(V)$ whose T-part is the given $T \in \mathcal{D}(V)$. In view of (2.8c), we see that $\mathrm{Sp}(V)$ acts transitively on $\mathcal{D}(V)$ by fractional linear transformations.

The isotropy subgroup of $0 \in \mathcal{D}(V)$ under this action consists of those $g \in \mathrm{Sp}(V)$ for which $T_g = 0$, i.e., the *complex-linear* subgroup $\mathrm{U}_J(V) := \{\, g \in \mathrm{Sp}(V) : gJ = Jg \,\}$. Since $g \in \mathrm{U}_J(V)$ iff $g = p_g$ iff $g^t g = 1$, we see that $\mathrm{U}_J(V) = \mathrm{Sp}(V) \cap \mathrm{O}(V, d)$ is the *unitary group* for the Hilbert space (V, s, J).

The set $\Sigma(V)$ of positive compatible complex structures on V, i.e., those real-linear operators J' satisfying (2.1) and (2.2b), also forms a homogeneous space for the group $\mathrm{Sp}(V)$. Indeed, any compatible complex structure belongs to the Lie algebra

$$\mathfrak{sp}(V) = \{\, X \in \mathrm{End}_{\mathbb{R}}(V) : s(\cdot, X\cdot) + s(X\cdot, \cdot) = 0 \,\}$$

of $\mathrm{Sp}(V)$. The *adjoint action* $J' \mapsto gJ'g^{-1}$ clearly preserves $\Sigma(V)$, and we shall shortly establish that $\mathrm{Sp}(V)$ acts transitively on $\Sigma(V)$.

2.3 Polarizations

We now consider the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$. We shall identify real-linear operators $A \in \mathrm{End}_{\mathbb{R}} V$ with their natural amplifications to complex-linear operators on $V_{\mathbb{C}}$ by the rule $A(u + iv) := Au + iAv$. The complex Hilbert space $V_{\mathbb{C}}$ carries a natural conjugation $(u + iv)^* := u - iv$.

A complex subspace $W \leqslant V_{\mathbb{C}}$ is *isotropic* with respect to (the complex amplification of) s if $s(z, w) = 0$ for all $z, w \in W$. A *polarization* for s is a maximal isotropic subspace. A polarization W is *complex* if $W \cap W^* = \{0\}$. Notice that a complex polarization satisfies $W \cap V = W \cap iV = \{0\}$.

If W is a complex polarization, we can write any $w \in W$ as $w = u - iv$ for unique elements $u, v \in V$. The maps $w \mapsto u$, $w \mapsto v$ are real-linear and one-to-one; they have continuous inverses since the scalar product (2.4) extends to $V_\mathbb{C}$ so that $\langle w \mid w\rangle = \langle u \mid u\rangle + \langle v \mid v\rangle$. The composite map $u \mapsto w \mapsto v$ is thus an invertible real-linear operator J_W on V. We can thus write

$$W = \{\, w = u - iJ_W u : u \in V \,\}. \tag{2.10}$$

Since W is a polarization, from $\Re\, s(w_1, w_2) = 0$ we get $s(J_W u_1, J_W u_2) = s(u_1, u_2)$, so J_W is symplectic. Also, $\Im\, s(w_1, w_2) = 0$ implies $s(J_W u_1, u_2) = -s(u_1, J_W u_2)$, so $J_W^2 = -1$.

The complex polarization W is called *positive* if $s(u, J_W u) > 0$ for nonzero u. Alternatively, we may notice that $s(u, J_W u) = \frac{i}{2} s(w^*, w)$, so that W is a positive polarization if and only if the sesquilinear form r on $V_\mathbb{C}$ given by $r(w_1, w_2) := 2is(w_1^*, w_2)$ is positive definite on the subspace W. (It is then negative definite on the complementary subspace W^*.) Notice that a symplectic space with a positive polarization carries a reflection operator satisfying an Osterwalder–Schrader type positivity condition [13]; in this case the reflection is conjugation followed by multiplication by i.

Let W_0 denote the polarization $\{\, u - iJu : u \in V \,\}$ for the initially chosen J. Let $P_+ := \frac{1}{2}(1 - iJ)$, $P_- := \frac{1}{2}(1 + iJ)$ denote the projectors on $V_\mathbb{C}$ with ranges W_0, W_0^* respectively. Then if $u, v \in V$, we note that $r(P_+ u, P_+ v) = \langle u \mid v\rangle$. Thus (W_0, r) is a complex Hilbert space and $v \mapsto P_+ v$ is a unitary map from (V, s, J) to W_0. Analogous projectors may be defined for any positive polarization.

Given the positive polarization W_0, we can define a (positive definite) scalar product on $V_\mathbb{C}$ by

$$\langle\!\langle w_1 \mid w_2 \rangle\!\rangle := 2s(w_1^*, Jw_2). \tag{2.11}$$

Notice that J acts as multiplication by i on W_0 and by $(-i)$ on W_0^*, so that P_+, P_- are the orthogonal projectors on W_0 and W_0^* with respect to this Hilbert space structure on $V_\mathbb{C}$; and there holds

$$\langle\!\langle P_+ u \mid P_+ v \rangle\!\rangle = \langle u \mid v\rangle, \qquad \langle\!\langle P_- u \mid P_- v \rangle\!\rangle = \langle v \mid u\rangle \tag{2.12}$$

for $u, v \in V$. Conversely, if J' is a complex structure satisfying (2.2b), then $W := \{\, u - iJ'u : u \in V \,\}$ is a positive polarization.

We can decompose $w \in W$ uniquely as $w = z_1 + z_2^*$ with $z_1, z_2 \in W_0$. Now $W \cap W_0^* = \{0\}$, since $r(\cdot, \cdot)$ is positive definite on W and negative definite on W_0^*. Thus $w \mapsto z_1$ is a one-to-one complex-linear map which has a continuous inverse since $\langle\!\langle w \mid w \rangle\!\rangle = \langle\!\langle z_1 \mid z_1 \rangle\!\rangle + \langle\!\langle z_2^* \mid z_2^* \rangle\!\rangle$. Let T_W denote the composite mapping

$$u_1 \mapsto u_1 - iJu_1 = z_1 \mapsto w \mapsto z_2^* = u_2 + iJu_2 \mapsto u_2, \tag{2.13}$$

which is a real-linear operator on V. Since the map $z_1 \mapsto w \mapsto z_2^*$ is complex-linear, so that $Ju_1 + iu_1 = iz_1$ maps to $iz_2^* = -Ju_2 + iu_2$, we find that $T_W J = -JT_W$. Moreover, T_W is symmetric; indeed, if $u_1, u_1' \in V$, then

$$0 = \tfrac{i}{2} \Im\, s(w, w') = \tfrac{i}{2} \Im\, (s(z_1, z_2'^*) - s(z_1', z_2^*)) = d(u_1, T_W u_1') - d(T_W u_1, u_1'). \tag{2.14}$$

We may also observe that

$$\begin{aligned} w = z_1 + z_2^* &= (1 + T_W)u_1 - iJ(1 - T_W)u_1 \\ &= (1 + T_W)(u_1 - iJu_1) = (1 + T_W)z_1, \end{aligned} \tag{2.15}$$

so $(1 + T_W)$ is invertible, and J and J_W are related by a *Cayley transformation*:

$$J_W = J(1 - T_W)(1 + T_W)^{-1}. \tag{2.16}$$

Furthermore, the calculation

$$\begin{aligned}\langle v \mid (1 - T_W^2)v\rangle &= d(v + T_W v, v - T_W v) = s((1 + T_W)v, J(1 - T_W)v) \\ &= s((1 + T_W)v, J_W(1 + T_W)v) > 0,\end{aligned}$$

shows that $1 - T_W^2$ is positive definite.

We summarize this discussion with the following result.

Proposition 2.2. *The correspondences $W \leftrightarrow J_W \leftrightarrow T_W$ are bijections between the set of positive polarizations for s, the set $\Sigma(V)$ of positive compatible complex structures on V, and the Cartan–Siegel disk $\mathcal{D}(V)$. The symplectic group $\mathrm{Sp}(V)$ acts transitively on these spaces, and these correspondences are equivariant for the group actions.*

Proof. The map $W \mapsto T_W$ is inverted by $T \mapsto (1 + T)W_0$, in view of (2.16). If $g \in \mathrm{Sp}(V)$, then $p_g W_0 = W_0$ since p_g commutes with $(1 - iJ)$, so $(1 + T_g)W_0 = (1 + T_g)p_g W_0 = gW_0$: left translation by (the complex amplifications of) elements of $\mathrm{Sp}(V)$ permute the positive polarizations. Also,

$$J(1 - T_g)(1 + T_g)^{-1} = (1 + T_g)J(1 + T_g)^{-1} = (1 + T_g)p_g J p_g^{-1}(1 + T_g)^{-1} = gJg^{-1},$$

so the actions $W \mapsto gW$, $J_W \mapsto gJ_W g^{-1}$ and $T_W \mapsto p_g(T_W - \widehat{T}_g)(1 - \widehat{T}_g T_W)^{-1}p_g^{-1}$ are equivariant under the given correspondences.

If J' is any positive compatible complex structure, take $W' := (1 - iJ')V, T := T_{W'} \in \mathcal{D}(V)$; then $h' := (1 + T)(1 - T^2)^{-1/2} \in \mathrm{Sp}(V)$ satisfies $h'^2 = (1 + T)(1 - T)^{-1} = -J'J$. Since T is symmetric (with respect to d) and $1 - T^2 > 0$, we find that $(1 + T)$ and h' are positive definite real-linear operators on (V, d), so $(-J'J)$ is also positive definite and h' is its positive square root. [We might also consider h' as a real-linear operator on (V, d'), where $d'(u, v) := s(u, J'v)$; by (2.16), the roles of J' and J may be reversed with T replaced by $(-T)$. Since $(-T)$ is symmetric with respect to d' and $1 - T^2 > 0$ on (V, d'), h' is also positive definite on (V, d').]

By (2.16), we see that $h'Jh'^{-1} = Jh'^{-2} = J(1 - T)(1 + T)^{-1} = J'$. We have shown that $\mathrm{Sp}(V)$ acts transitively on $\Sigma(V)$, with

$$(-J'J)^{1/2}J(-JJ')^{1/2} = J'. \tag{2.17}$$

It is now clear why the topologies on V induced by different d_J are equivalent. We have also shown that $J' \mapsto (-J'J)^{1/2}$ is a global section of the principal fibre bundle $\mathrm{Sp}(V) \to \Sigma(V)$. Moreover, the inverse of the correspondence $\mathcal{D}(V) \to \Sigma(V) : T \mapsto J(1 - T)(1 + T)^{-1}$ is given by inverting the Cayley transformation (2.16):

$$J' \mapsto T = (J - J')(J + J')^{-1}. \qquad \square \tag{2.18}$$

2.4 The restricted symplectic group

The *restricted symplectic group* consists of those symplectic transformations whose T-part is a Hilbert–Schmidt operator, on the real Hilbert space (V, d). Letting $\mathrm{HS} \equiv \mathrm{HS}(V)$ denote the class of Hilbert–Schmidt operators, we take note that

$$T_g \in \mathrm{HS} \iff q_g \in \mathrm{HS} \iff [J, g] \in \mathrm{HS} \iff J - gJg^{-1} \in \mathrm{HS}.$$

Thus we define

$$\begin{aligned} \mathrm{Sp}'(V) &:= \{\, g \in \mathrm{Sp}(V) : T_g \in \mathrm{HS}(V) \,\}, \\ \Sigma'(V) &:= \{\, J' \in \Sigma(V) : [J, J'] \in \mathrm{HS}(V) \,\}, \\ \mathcal{D}'(V) &:= \mathcal{D}(V) \cap \mathrm{HS}(V). \end{aligned}$$

Then $\mathrm{Sp}'(V)$ is a subgroup of $\mathrm{Sp}(V)$, and $\Sigma'(V)$, $\mathcal{D}'(V)$ are respectively the orbits of $J \in \Sigma(V)$ and $0 \in T(V)$ under the action of $\mathrm{Sp}'(V)$. The isotropy subgroup $\mathrm{U}_J(V)$ is contained in $\mathrm{Sp}'(V)$.

The homogeneous spaces $\Sigma'(V)$ and $\mathcal{D}'(V)$ are Kähler manifolds based on the Hilbert space of antilinear Hilbert–Schmidt operators on V [14]. Note that the global sections $J' \mapsto (-J'J)^{1/2}$, $T \mapsto (1+T)(1-T^2)^{-1/2}$ have values in $\mathrm{Sp}'(V)$.

The set of all positive polarizations may be identified with $\Sigma(V)$ or $\mathcal{D}(V)$ under the correspondences of Proposition 2.2. Now $\Sigma(V)$ is partitioned into equivalence classes, where J_1 and J_2 are equivalent if and only if $J_1 - J_2$ is Hilbert–Schmidt. Likewise, two polarizations W_1 and W_2 are equivalent if and only if $W_2 = gW_1$ for some $g \in \mathrm{Sp}'(V)$. We may then call "restricted polarizations" those W for which $J_W - J$ or T_W is Hilbert–Schmidt: these form the orbit under $\mathrm{Sp}'(V)$ of the reference polarization W_0.

Since the action of $\mathrm{Sp}'(V)$ on $\mathcal{D}'(V)$ is transitive, we obtain a useful formula for this action by replacing T_h by a general $S \in \mathcal{D}'(V)$ in (2.9). Let us write $g \cdot S$ for the image of S under $g \in \mathrm{Sp}'(V)$. Then we can rewrite (2.9) as

$$g \cdot S = T_g + p_g^{-t} S (1 - \widehat{T}_g S)^{-1} p_g^{-1}. \tag{2.19}$$

► If (and only if) $g \in \mathrm{Sp}'(V)$, then $p_g p_g^t = (1 - T_g^2)^{-1}$ has a determinant, since the operator $p_g p_g^t - 1 = T_g^2 (1 - T_g^2)^{-1}$ is trace-class; and $\det(1 - T_g^2) = \det(p_g p_g^t)^{-1} = \det(p_g^t p_g)^{-1} = \det(1 - \widehat{T}_g^2)$. For the theory of infinite determinants, including the justification of such expected properties as $\det(AB) = \det(BA)$, we refer to [7, Appendix A].

The determinants we need to compute in the present context are *complex* determinants. Whenever $T \in \mathcal{D}'(V)$, $1 - T^2$ is a linear trace-class positive operator on the complex Hilbert space (V, s, J), whose determinant is

$$\det_{\mathbb{C}}(1 - T^2) = \prod_{k=1}^{\infty} (1 - \lambda_k^2),$$

where the λ_k^2 are the eigenvalues of T^2. (The subscript $\mathbb{C}$ emphasizes the nature of the complex determinant; we shall usually omit it if no ambiguity is likely.) The determinant of $(1 - T^2)$ as a real-linear operator on V is the square of this complex determinant. Indeed, we can find an orthonormal basis $\{e_1, e_2, \dots\}$ for (V, s, J) so that $T^2 e_k = \lambda_k^2 e_k$ for each k; and moreover, since T is antilinear and symmetric, we can select the vectors e_k so that

$$T e_k = \lambda_k J e_k, \qquad T J e_k = \lambda_k e_k. \tag{2.20}$$

The eigenvalues of T are $\{\pm\lambda_k\}$, since $T(e_k \pm J e_k) = \pm\lambda_k (e_k \pm J e_k)$.

An alternative formulation in terms of the polarization W_0 is often useful. The complex amplification of $(1 - T^2)$ on $V_{\mathbb{C}}$ is a trace-class operator, and its eigenvectors $\frac{1}{2}(e_k - iJe_k) \in W_0$, $\frac{1}{2}(e_k + iJe_k) \in W_0^*$ span a dense subspace of $V_{\mathbb{C}}$. One sees at a glance that

$$\det_{\mathbb{C}}(1 - T^2) = \prod_{k=1}^{\infty} (1 - \lambda_k^2) = \det(P_+ (1 - T^2) P_+).$$

► Let $\Pi'(V)$ denote the set of $A \in \mathrm{GL}(V)$ such that A is linear, $A + A^t$ is positive definite, and $1 - A$ is trace-class. For example, $1 - T^2 \in \Pi'(V)$ whenever $T \in \mathcal{D}'(V)$. Moreover, if $S, T \in \mathcal{D}'(V)$, then $1 - ST \in \Pi'(V)$.

The trace norm $\|A_1 - A_2\|_{\mathrm{tr}}$ defines a metric on $\Pi'(V)$. Now $\Pi'(V)$ is contractible, since $(1-t)1 + tA \in \Pi'(V)$ if $A \in \Pi'(V)$ and $0 \leqslant t \leqslant 1$, and $A \mapsto \det A$ is continuous on $\Pi'(V)$; so if m is a nonzero integer (positive or negative), we can define a unique continuous function $\det^{1/m}$ on $\Pi'(V)$ which satisfies

$$(\det{}^{1/m} A)^m = \det{}_{\mathbb{C}} A, \qquad \det{}^{1/m} 1 = 1.$$

In particular, $\det^{1/m} A > 0$ if A is positive definite. For $T \in \mathcal{D}'(V)$, it follows that

$$\det{}^{1/m}(1 - T^2) = \prod_{k=1}^{\infty} (1 - \lambda_k^2)^{1/m}.$$

For complex determinants, the following identity is generally valid [7]:

$$\det(\exp N) = \exp(\operatorname{Tr} N)$$

for trace-class operators N. The trace in this formula is a *complex trace*. We therefore define, for $A \in \mathrm{End}_{\mathbb{R}}(V)$,

$$\operatorname{Tr}_{\mathbb{C}}[A] := \operatorname{Tr}[P_+ A P_+], \tag{2.21}$$

where the trace on the right is that of the complex Hilbert space W_0. Now $\operatorname{Tr}_{\mathbb{C}}[A] = 0$ if A is antilinear, since $P_+ A = A P_-$ on $V_{\mathbb{C}}$, but $\operatorname{Tr}_{\mathbb{C}}$ need not vanish on commutators of antilinear operators; it does, of course, vanish on commutators of linear operators.

With these notations, then, derivatives of determinants obey the rule [7]:

$$\frac{d}{dt} \det{}_{\mathbb{C}} A(t) = \det{}_{\mathbb{C}} A(t) \ \operatorname{Tr}_{\mathbb{C}}\Big(A(t)^{-1} \frac{d}{dt} A(t) \Big), \tag{2.22}$$

whenever $t \mapsto A(t) \in \Pi'(V)$ is a differentiable map.

3 The Fock space of antiholomorphic functions

3.1 The Segal–Bargmann construction of Fock space

The Fock space which carries the representations of the canonical commutation relations can be constructed directly by applying creation operators to a vacuum state, or abstractly as a space of analytic functions [15] on the underlying symplectic linear manifold V. The equivalence of these constructions for systems with finitely many degrees of freedom is guaranteed by the Stone–von Neumann theorem. In a field-theoretic context a more detailed treatment is necessary.

It has been shown by I. E. Segal [16] that the presentation via "functions on V" extends without essential change to the infinite-dimensional case. An equivalent presentation of Fock space is as the completion of the symmetric algebra on a positive polarization of (V, s); see G. Segal [5], for instance. We prefer to work directly with the real manifold V, to keep the classical picture more clearly in view and to define the symplectic action without altering the base space. In this Section, we establish the connection between the "symmetric algebra" and "function space" viewpoints.

Fix a positive compatible complex structure J on V; throughout this section we shall regard V as a complex Hilbert space via (2.3), with scalar product (2.4).

The *symmetric algebra* $S(V)$ of V is defined as $S(V) := \bigoplus_{n=0}^{\infty} V^{\vee n}$, where $V^{\vee n}$ is the complex vector space algebraically generated by the symmetric products

$$v_1 \vee v_2 \vee \cdots \vee v_n := \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)},$$

with $V^{\vee 0} = \mathbb{C}$ by convention. The scalar product on V extends to a scalar product on $S(V)$ by declaring

$$\langle u_1 \vee \cdots \vee u_m \mid v_1 \vee \cdots \vee v_n \rangle := \delta_{mn} \operatorname{per}(\langle u_k \mid v_l \rangle) \equiv \delta_{mn} \sum_{\sigma \in S_n} \prod_{j=1}^{n} \langle u_j \mid v_{\sigma(j)} \rangle. \tag{3.1}$$

If $\{e_n\}$ is an orthonormal basis for the complex Hilbert space (V, s, J), an orthonormal family in $S(V)$ is given by the elements $\varepsilon_\alpha := (\alpha!)^{-1/2} e_1^{\vee \alpha_1} \vee \cdots \vee e_r^{\vee \alpha_r}$, where α is a sequence of nonnegative integers with finitely many nonzero entries, and $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_r!$ is a multifactorial. (The ε_α have norm 1 since the permanent of a square matrix of 1's with α_k rows is $\alpha_k!$.) This family is an orthonormal basis for the Hilbert-space completion of $S(V)$, which is the symmetric Fock space.

The antilinear function $u \mapsto \frac{1}{\sqrt{2}} \langle u \mid v \rangle$ on V will be denoted simply by v. An *antiholomorphic homogeneous polynomial* of degree n on V is a function on V of the form $L(u, u, \ldots, u)$, where L is a continuous function which is antilinear in each of its n arguments. A typical example is the function

$$u \mapsto 2^{-n/2} \langle u \mid v_1 \rangle \langle u \mid v_2 \rangle \cdots \langle u \mid v_n \rangle = \frac{1}{n!} \langle (u/\sqrt{2})^{\vee n} \mid v_1 \vee \cdots \vee v_n \rangle, \tag{3.2}$$

which we identify with $v_1 \vee \cdots \vee v_n \in V^{\vee n}$. By these identifications, we regard $S(V)$ as the space of antiholomorphic polynomials on V.

Remark. The scalar factor $1/\sqrt{2}$ could well be suppressed, but would reappear in a more awkward fashion in other formulas; its inclusion at this stage is tantamount to regarding elements of $S(V)$ as "functions of $u/\sqrt{2}$".

If V is finite-dimensional with orthonormal basis $\{e_1, \ldots, e_N\}$, the polynomials ε_α are orthonormal with respect to the Gaussian integral:

$$\begin{aligned}\int_V \varepsilon_\alpha^*(u) \varepsilon_\beta(u)\, e^{-\frac{1}{2}\langle u|u\rangle}\, du &:= \prod_{k=1}^{N} \frac{1}{2\pi} \int 2^{-(\alpha_k+\beta_k)/2} \frac{u_k^{\alpha_k} u_k^{*\beta_k}}{\sqrt{\alpha_k!\, \beta_k!}}\, e^{-\frac{1}{2}|u_k|^2}\, d\Re u_k\, d\Im u_k \\ &= \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

We normalize the Lebesgue measure du by a factor of $(2\pi)^{-N}$ so that the integral of 1 equals 1. Let the vectors $v_1, \ldots, v_n$ span the complex subspace V' of V; we may suppose that V' has orthonormal basis $\{e_1, \dots, e_M\}$ with $M \leqslant N$. Then the Gaussian integral of the polynomial $v_1 \vee \cdots \vee v_n$ is

$$\begin{aligned}
2^{-n/2} \int_V \prod_{j=1}^n \langle u \mid v_j \rangle \, e^{-\frac{1}{2}\langle u|u\rangle} \, du &= 2^{-n/2} \int_V \prod_{j=1}^n \sum_{k_j=1}^M \langle u \mid e_{k_j} \rangle \langle e_{k_j} \mid v_j \rangle \, e^{-\frac{1}{2}\langle u|u\rangle} \, du \\
&= 2^{-n/2} \int_{V'} \prod_{j=1}^n \langle u \mid v_j \rangle \, e^{-\frac{1}{2}\langle u|u\rangle} \, du. \qquad (3.3)
\end{aligned}$$

Thus the integral depends only on the linear span of $\{v_1, \dots, v_n\}$.

If $F := \sum_\alpha c_\alpha \varepsilon_\alpha$ is a finite sum of basic polynomials on V, then

$$\|F\|^2 := \int_V |F(u)|^2 \, e^{-\frac{1}{2}\langle u|u\rangle} \, du = \sum_\alpha |c_\alpha|^2.$$

By the Schwarz inequality,

$$|F(u)|^2 = \sum_{\alpha,\beta} c_\alpha^* c_\beta \prod_{j,k=1}^N \frac{(u_j/\sqrt{2})^{\alpha_j}}{\sqrt{\alpha_j!}} \frac{(u_k/\sqrt{2})^{\beta_k *}}{\sqrt{\beta_k!}} \leqslant \sum_\alpha |c_\alpha|^2 \prod_{k=1}^N e^{\frac{1}{2}|u_k|^2} = \|F\|^2 \, e^{\frac{1}{2}\langle u|u\rangle}.$$

Now take V to be infinite-dimensional. Then V does not support a Gaussian measure, but we may nevertheless extend the integral as follows. We say a function F on V is *antiholomorphic* if its restriction to any finite-dimensional subspace of V is antiholomorphic. (In particular, any homogeneous polynomial of the form $v_1 \vee \cdots \vee v_n$ is an antiholomorphic function.) For such a function F, write

$$\|F\|^2 := \sup_Y \int_Y |F(u)|^2 \, e^{-\frac{1}{2}\langle u|u\rangle} \, du, \qquad (3.4)$$

where Y ranges over finite-dimensional complex subspaces of V. The *Segal–Bargmann space* $\mathcal{B}(V)$ is the space of antiholomorphic functions F for which $\|F\|$ is finite. Now, on any finite-dimensional Y, the estimate

$$|F(u)|^2 \leqslant \|F\|^2 \, e^{\frac{1}{2}\langle u|u\rangle} \qquad (3.5)$$

holds for $u \in Y$; on account of the definition (3.4), this estimate holds for all $u \in V$.

For $F = v_1 \vee \cdots \vee v_n$, the supremum is attained on any finite-dimensional Y which contains $\{v_1, \dots, v_n\}$, in view of (3.3), and coincides with the norm determined by (3.1). The completion of $S(V)$ in this norm coincides with $\mathcal{B}(V)$. Indeed, in view of (3.5), a Cauchy sequence of antiholomorphic polynomials converges uniformly on finite-dimensional compact sets, so its pointwise limit is an antiholomorphic function on V. On the other hand, if F is an antiholomorphic function for which $\|F\|$ is finite, then F can be approximated by polynomials on the subspace Y_N with orthonormal basis $\{e_1, \dots, e_N\}$. Thus we can write $F(u) = \sum_\alpha c_\alpha \varepsilon_\alpha(u)$ for u in any Y_N, with

$$\sum_\alpha |c_\alpha|^2 = \sup_N \int_{Y_N} |F(u)|^2 \, e^{-\frac{1}{2}\langle u|u\rangle} \, du = \|F\|^2, \qquad (3.6)$$

so that F lies in the completion of $S(V)$. On account of (3.6), we shall use the notation

$$\int |F(u)|^2 \, e^{-\frac{1}{2}\langle u|u\rangle} \, du := \|F\|^2$$

to denote the right hand side of (3.4). Then $\mathcal{B}(V)$ consists of those entire antiholomorphic functions on V for which this "integral" is finite.

3.2 Operator kernels on the Segal–Bargmann space

If $v \in V$, write $E_v(u) := e^{\frac{1}{2}\langle u|v\rangle}$. Then E_v is antiholomorphic on V and

$$\|E_v\|^2 = \sup_N \int_{Y_N} e^{\frac{1}{2}(\langle u|v\rangle + \langle v|u\rangle - \langle u|u\rangle)}\, du = \sup_N \exp(\tfrac{1}{2}\langle P_N v \mid P_N v\rangle) = e^{\frac{1}{2}\langle v|v\rangle},$$

where P_N denotes the orthogonal projector on V with range Y_N. Hence $E_v \in \mathcal{B}(V)$. An analogous computation shows that

$$\langle E_v \mid E_w\rangle = e^{\frac{1}{2}\langle v|w\rangle}.$$

Now $(d^n/dt^n)E_{tv}(u) = 2^{-n}\langle u \,|\, v\rangle^n E_{tv}(u)$, so that $v^{\vee n} = 2^{n/2}(d^n/dt^n)\big|_{t=0}E_{tv}$ lies in the closed linear span of the E_v. The combinatorial formula

$$v_1 \vee v_2 \vee \cdots \vee v_n = \frac{1}{n!}\sum_{r=1}^{n}(-1)^{n-r} \sum_{1\leqslant k_1<\cdots<k_r\leqslant n} (v_{k_1} + \cdots + v_{k_r})^{\vee n}$$

then shows that this closed linear span contains all of $S(V)$. In other words, the set $\{\, E_v : v \in V \,\}$ spans a dense subspace of $\mathcal{B}(V)$. Indeed, a dense subspace of $\mathcal{B}(V)$ is spanned by $\{\, E_v : v \in V' \,\}$ whenever V' is a dense subspace of V, since the closure of this subspace of $\mathcal{B}(V)$ contains all $V^{\vee n}$.

We may think of E_v as the symmetric exponential $\exp^\vee(v/\sqrt{2})$, by which is meant the power series $\sum_{n=0}^\infty (2^{-n/2}v^{\vee n})/n!$. Indeed, this series does converge to the function E_v both in norm, as is easily checked, and uniformly on finite-dimensional compact subsets of V.

The functions E_v are "principal vectors" for $\mathcal{B}(V)$, since

$$\begin{aligned}
\langle E_v \mid \varepsilon_\alpha\rangle &= \lim_N \int_{Y_N} e^{\frac{1}{2}\langle v|u\rangle}\varepsilon_\alpha(u)\, e^{-\frac{1}{2}\langle u|u\rangle}\, du \\
&= \lim_N \prod_{k=1}^{N} \frac{2^{-\alpha_k/2}}{2\pi\sqrt{\alpha_k!}} \int e^{\frac{1}{2}v_k^* u_k} u_k^{*\,\alpha_k} e^{-\frac{1}{2}|u_k|^2}\, d\Re u_k\, d\Im u_k \\
&= \lim_N \prod_{k=1}^{N} \frac{v_k^{*\,\alpha_k}}{2^{\alpha_k/2}\sqrt{\alpha_k!}} = \varepsilon_\alpha(v),
\end{aligned}$$

and so $\langle E_v \mid F\rangle = F(v)$ for any $F \in \mathcal{B}(V)$. Hence $E_v(u) = \exp(\frac{1}{2}\langle u \mid v\rangle)$ is a *reproducing kernel* for $\mathcal{B}(V)$.

The existence of a reproducing kernel implies that an operator A on $\mathcal{B}(V)$ has a kernel

$$K_A(u, v) := \langle E_u \mid AE_v\rangle = AE_v(u), \tag{3.7}$$

which is antiholomorphic in u and holomorphic in v, provided that the principal vectors E_v lie in the domains of A and its hermitian conjugate $A^\dagger$. In particular, any bounded operator has a kernel. Thus

$$AF(u) = \int K_A(u, v)F(v)\, e^{-\frac{1}{2}\langle v|v\rangle}\, dv := \lim_Y \int_Y K_A(u, v)F(v)\, e^{-\frac{1}{2}\langle v|v\rangle}\, dv,$$

where Y ranges over finite-dimensional complex subspaces of V.

It often happens that an unbounded operator A on $\mathcal{B}(V)$ will contain $E_v \in \operatorname{Dom} A$ and also $E_w \in \operatorname{Dom} A^\dagger$, for v, w ranging over dense subspaces of V, in which case the kernel K_A is densely defined; we may then compute kernel compositions and adjoints with due regard to these domains.

We write $\Omega := E_0$ (the constant function 1 on V). Then $\langle \Omega \mid A\Omega\rangle = K_A(0, 0)$ for any operator A on $\mathcal{B}(V)$ whose domain contains Ω.

3.3 A Gaussian integral

In the sequel we shall need to evaluate Gaussian integrals over the Segal–Bargmann space. If $T \in \mathcal{D}'(V)$, we define the (unnormalized) Gaussian $f_T \in \mathcal{B}(V)$ by

$$f_T(u) := \exp(\tfrac{1}{4}\langle u \mid Tu\rangle). \tag{3.8}$$

As before, we can choose the orthonormal basis $\{e_k\}$ for V so that $Te_k = \lambda_k Je_k$, $TJe_k = \lambda_k e_k$ where $\{\pm\lambda_k\}$ is the square-summable sequence of eigenvalues of T. Thereby, we get

$$\begin{aligned}\langle u \mid Tu\rangle + \langle Tu \mid u\rangle &= \sum_{k=1}^{\infty} \langle u \mid e_k\rangle\langle e_k \mid Tu\rangle + \langle Tu \mid e_k\rangle\langle e_k \mid u\rangle \\ &= \sum_{k=1}^{\infty} \langle u \mid e_k\rangle\langle u \mid Te_k\rangle + \langle Te_k \mid u\rangle\langle e_k \mid u\rangle \\ &= \sum_{k=1}^{\infty} i\lambda_k u_k^{*2} - i\lambda_k u_k^2.\end{aligned}$$

Hence the norm of the Gaussian f_T is given by

$$\begin{aligned}\|f_T\|^2 &= \int \exp\tfrac{1}{4}\big\{\langle u \mid Tu\rangle + \langle Tu \mid u\rangle\big\}\, e^{-\frac{1}{2}\langle u|u\rangle}\, du \\ &= \lim_N \int_{Y_N} \exp\tfrac{1}{4}\big\{\langle u \mid Tu\rangle + \langle Tu \mid u\rangle\big\}\, e^{-\frac{1}{2}\langle u|u\rangle}\, du \\ &= \prod_{k=1}^{\infty} \frac{1}{2\pi}\int \exp\tfrac{1}{4}\{i\lambda_k(u_k^{*2} - u_k^2)\}\, e^{-\frac{1}{2}|u_k|^2}\, d\Re u_k\, d\Im u_k \\ &= \prod_{k=1}^{\infty}(1-\lambda_k^2)^{-1/2} = \det{}^{-1/2}(1-T^2).\end{aligned} \tag{3.9}$$

Although the formula (3.8) defines an antiholomorphic function for any antilinear and symmetric $T \in \mathrm{End}_{\mathbb{R}}(V)$, this function lies in $\mathcal{B}(V)$ if and only if T is both contractive (i.e., $1 - T^2$ is positive definite) and Hilbert–Schmidt. This is the essence of Shale's theorem [3] on the unitary implementability of the boson-field scattering operator. Indeed, when the out vacuum is given by a state vector in the Fock space, this vector turns out to be just a normalized Gaussian, i.e., a multiple of (3.8) for some $T \in \mathcal{D}'(V)$.

► Suppose $T \in \mathcal{D}'(V)$ and $v, w \in V$; we must evaluate the integral

$$\int \exp\tfrac{1}{4}\big\{\langle u \mid Tu\rangle + \langle Tu \mid u\rangle + 2\langle u \mid v\rangle + 2\langle w \mid u\rangle\big\}\, e^{-\frac{1}{2}\langle u|u\rangle}\, du. \tag{3.10}$$

Again using the basis of eigenvectors for T^2, this integral becomes

$$
\begin{aligned}
&\prod_{k=1}^{\infty} \frac{1}{2\pi} \int \exp \tfrac{1}{4}\{i\lambda_k(u_k^{*2} - u_k^2) + 2\langle e_k \mid v\rangle u_k^* + 2\langle w \mid e_k\rangle u_k\}\, e^{-\frac{1}{2}|u_k|^2}\, d\Re u_k\, d\Im u_k \\
&= \prod_{k=1}^{\infty} (1-\lambda_k^2)^{-1/2} \exp\bigl(\tfrac{1}{4}(1-\lambda_k^2)^{-1}\{i\lambda_k\langle w \mid e_k\rangle^2 + 2\langle w \mid e_k\rangle\langle e_k \mid v\rangle - i\lambda_k\langle e_k \mid v\rangle^2\}\bigr) \\
&= \prod_{k=1}^{\infty} (1-\lambda_k^2)^{-1/2} \exp \tfrac{1}{4}\{\langle w \mid e_k\rangle\langle e_k \mid T(1-T^2)^{-1}w\rangle + 2\langle w \mid e_k\rangle\langle e_k \mid (1-T^2)^{-1}v\rangle \\
&\qquad\qquad + \langle T(1-T^2)^{-1}v \mid e_k\rangle\langle e_k \mid v\rangle\} \\
&= \det{}^{-1/2}(1-T^2) \exp \tfrac{1}{4}\{\langle w \mid T(1-T^2)^{-1}w\rangle + 2\langle w \mid (1-T^2)^{-1}v\rangle + \langle T(1-T^2)^{-1}v \mid v\rangle\}
\end{aligned}
$$

on account of the relations:

$$
\begin{aligned}
i\lambda_k\langle w \mid e_k\rangle &= \langle w \mid \lambda_k J e_k\rangle = \langle w \mid T e_k\rangle = \langle e_k \mid T w\rangle, \\
-i\lambda_k\langle e_k \mid v\rangle &= \langle \lambda_k J e_k \mid v\rangle = \langle T e_k \mid v\rangle = \langle T v \mid e_k\rangle.
\end{aligned}
$$

► We may extend the environment of the previous computations as follows. First suppose that V is finite-dimensional and let $T, S \in \mathcal{D}'(V)$. Then $(1 - TS) \in \Pi'(V)$ has matrix elements $\langle e_j \mid e_k\rangle - \langle Se_k \mid Te_j\rangle$ with respect to a given orthonormal basis $\{e_1, \dots, e_N\}$ of V. Thus

$$
\det{}^{-1/2}(1 - TS) = \det{}^{-1/2}\bigl[\langle e_j \mid e_k\rangle - \langle Se_k \mid Te_j\rangle\bigr] \tag{3.11}
$$

is holomorphic in T and antiholomorphic in S as a function of two elements of the Kähler manifold $\mathcal{D}'(V)$. The function $f_T(u) f_S^*(u) = \exp \frac{1}{4}\{\langle u \mid Tu\rangle + \langle Su \mid u\rangle\}$ is likewise holomorphic in T and antiholomorphic in S. Therefore, the inner product of two Gaussians is given by

$$
\langle f_S \mid f_T\rangle = \int \exp \tfrac{1}{4}\bigl\{\langle u \mid Tu\rangle + \langle Su \mid u\rangle\bigr\}\, e^{-\frac{1}{2}\langle u|u\rangle}\, du = \det{}^{-1/2}(1 - TS), \tag{3.12}
$$

by analytic continuation from the diagonal $S = T$, where equality holds by (3.9).

If V is infinite-dimensional, then (3.12) holds at any rate for finite-rank elements $T, S \in \mathcal{D}'(V)$, on account of (3.3). Both sides of the equation are finite, by the Schwarz inequality and because $(1 - TS) \in \Pi'(V)$; by continuity of $\det^{-1/2}$ on $\Pi'(V)$, equality holds in (3.12) for all $S, T \in \mathcal{D}'(V)$.

Finally, the general Gaussian integral may be treated similarly. We state it as follows.

Proposition 3.1. *If $T, S \in \mathcal{D}'(V)$ and $v, w \in V$, then:*

$$
\begin{aligned}
&\int \exp \tfrac{1}{4}\bigl\{\langle u \mid Tu\rangle + \langle Su \mid u\rangle + 2\langle u \mid v\rangle + 2\langle w \mid u\rangle\bigr\}\, e^{-\frac{1}{2}\langle u|u\rangle}\, du \\
&= \det{}^{-1/2}(1 - TS) \exp \tfrac{1}{4}\bigl\{\langle w \mid T(1 - ST)^{-1}w\rangle + 2\langle w \mid (1 - TS)^{-1}v\rangle + \langle S(1 - TS)^{-1}v \mid v\rangle\bigr\}.
\end{aligned} \tag{3.13}
$$

Proof. If V is finite-dimensional, both sides of this equation are holomorphic in T and antiholomorphic in S. They coincide on the diagonal $S = T$ and by analytic continuation the equation is valid for all $T, S \in \mathcal{D}'(V)$. In the infinite-dimensional case, (3.13) is thus valid for S, T of finite rank, and by continuity it holds on all of $\mathcal{D}'(V)$. □

Remark. Several versions of this Gaussian integral calculation exist in the literature for the finite-dimensional case. The original treatment is due to Bargmann [15], for T, S scalars, and was extended by Itzykson [17] for general T, S. Itzykson's result is described in detail by Folland [18]. The integrands are expressed therein by complex symmetric matrices, since the Bargmann–Segal space is built over the polarization W_0 rather than over V itself. A treatment in the spirit of the "real" approach taken here, eschewing polarizations at this stage, is given by Robinson and Rawnsley [8], whose path we have followed. The main point of this subsection is that the integral formula (3.13) extends to the infinite dimensional case without further ado.

4 Weyl systems and free boson fields

4.1 Weyl systems

A *boson field* over the Hilbert space (V, s, J) may be thought of as a rule assigning creation and annihilation operators to elements of V (the "test function" space) in such a way that the canonical commutation relations are satisfied. Mathematically, the simplest approach is to start with the exponentiated version of the CCR. We define a *Weyl system* on the symplectic linear manifold (V, s) to be a strongly continuous map β to the group of unitary operators on some separable Hilbert space $\mathcal{K}$, which satisfies

$$\beta(v)\beta(w) = \beta(v + w) \exp[-\tfrac{i}{2}s(v, w)] \qquad \text{for all } v, w \in V. \tag{4.1}$$

In other words, β is a projective unitary representation of the additive group of V, whose cocycle is given by the symplectic form on V.

The existence question may be settled by taking $\mathcal{K} = \mathcal{B}(V)$, and defining

$$\beta(v)F(u) := \exp(\tfrac{1}{4}\langle 2u - v \mid v\rangle)F(u - v). \tag{4.2}$$

In particular,

$$\beta(v)E_w = \exp(-\tfrac{1}{4}\langle v \mid 2w + v\rangle)E_{w+v}.$$

It is immediate that $\langle \beta(v)E_u \mid \beta(v)E_w\rangle = \exp(\frac{1}{2}\langle u \mid w\rangle) = \langle E_u \mid E_w\rangle$. Since the E_w generate a dense subspace of $\mathcal{B}(V)$, the operators $\beta(v)$ are bounded and unitary on $\mathcal{B}(V)$. Notice also that $\beta(v)\Omega = \exp(-\frac{1}{4}\langle v \mid v\rangle)E_v$.

Moreover, β is irreducible. For if $\mathcal{K}_0$ is a closed subspace of $\mathcal{B}(V)$ invariant under all $\beta(v)$, let P denote the orthogonal projector on $\mathcal{B}(V)$ with range $\mathcal{K}_0$. If $\{ F_1, F_2, \dots \}$ is an orthonormal basis for $\mathcal{K}_0$, then so is $\{ \beta(-v)F_1, \beta(-v)F_2, \dots \}$ for any $v \in V$; it follows that

$$\begin{aligned} K_P(0,0) &= \sum_k \langle \Omega \mid \beta(-v)F_k\rangle\langle \beta(-v)F_k \mid \Omega\rangle = \sum_k \langle \beta(v)\Omega \mid F_k\rangle\langle F_k \mid \beta(v)\Omega\rangle \\ &= \sum_k e^{-\frac{1}{2}\langle v|v\rangle}\langle E_v \mid F_k\rangle\langle F_k \mid E_v\rangle = e^{-\frac{1}{2}\langle v|v\rangle}K_P(v, v) \end{aligned}$$

so that $K_P(v, v) = e^{\frac{1}{2}\langle v|v\rangle}K_P(0, 0)$. Now since $K_P(u, v)$ is antiholomorphic in u and holomorphic in v, we find by analytic continuation that $K_P(u, v) = e^{\frac{1}{2}\langle u|v\rangle}K_P(0, 0) = K_P(0, 0)E_v(u)$, and so $P = K_P(0, 0)1$ on $\mathcal{B}(V)$. Thus $K_P(0, 0) = 0$ or 1, corresponding to the cases $\mathcal{K}_0 = \{0\}$ or $\mathcal{B}(V)$ respectively, which establishes the irreducibility of $\beta(V)$.

► The particular Weyl system we have introduced on the Segal–Bargmann space $\mathcal{B}(V)$ is intertwined by the group of "one-particle" unitary operators on $\mathcal{B}(V)$. The group $\mathrm{U}_J(V)$ is the unitary group of the Hilbert space (V, s, J). For any element $U \in \mathrm{U}_J(V)$, the formula

$$\Gamma(U)F(v) := F(U^{-1}v)$$

defines a unitary operator $\Gamma(U)$ on $\mathcal{B}(V)$. Clearly, $\Gamma(U)\Omega = \Omega$ for all U. From (4.2) one sees that

$$\Gamma(U)\,\beta(v)\,\Gamma(U)^{-1} = \beta(Uv). \tag{4.3}$$

The representation Γ of the unitary group $\mathrm{U}_J(V)$ has a further *positivity property*: if A is a positive selfadjoint operator on V, its image $d\Gamma(A)$ under the derived representation of Γ is a positive selfadjoint operator on $\mathcal{B}(V)$. To see this, first let A be a selfadjoint (not necessarily bounded) operator on the complex Hilbert space (V, s, J). Then

$$\Gamma(\exp(itA))E_v(u) = E_v(\exp(-itA)u) = \exp\tfrac{1}{2}\langle u \mid \exp(itA)v\rangle = E_{\exp(itA)v}(u). \tag{4.4}$$

The infinitesimal generator of the one-parameter group $t \mapsto \Gamma(\exp(itA))$, which we shall denote by $d\Gamma(A)$, leaves invariant the subspace $\mathcal{D}_0 := \operatorname{span}\{\, E_v : v \in \operatorname{Dom} A \,\}$, which is dense in $\mathcal{B}(V)$ since $\operatorname{Dom} A$ is dense in V. Thus $\mathcal{D}_0$ is a core for $d\Gamma(A)$ [19, Prop. B.3]. For $v \in \operatorname{Dom} A$, we obtain

$$\begin{aligned}\langle E_w \mid d\Gamma(A)E_v\rangle &= -i\frac{d}{dt}\bigg|_{t=0} \langle E_w \mid \Gamma(\exp(itA))E_v\rangle \\ &= -i\frac{d}{dt}\bigg|_{t=0} \exp(\tfrac{1}{2}\langle w \mid \exp(itA)v\rangle) = \tfrac{1}{2}\langle w \mid Av\rangle.\end{aligned}$$

More generally, if $F = \sum_{j=1}^n c_j E_{v_j} \in \mathcal{D}_0$, then

$$\langle F \mid d\Gamma(A)F\rangle = \frac{1}{2}\sum_{j,k=1}^{n} c_j^* c_k \langle v_j \mid Av_k\rangle = \tfrac{1}{2}\langle w \mid Aw\rangle \tag{4.5}$$

where $w = \sum_{j=1}^n c_j v_j \in \operatorname{Dom} A$. Hence the restriction of $d\Gamma(A)$ to $\mathcal{D}_0$ is symmetric and positive, and its closure $d\Gamma(A)$ is a positive selfadjoint operator on $\mathcal{B}(V)$.

A particularly simple example occurs when $A = 1$, which generates an action of the circle group $\mathrm{U}(1)$ on $\mathcal{B}(V)$. We may regard $\mathrm{U}(1)$ as a subgroup of $\mathrm{U}_J(V)$, with $e^{i\phi}$ acting as the operator $\cos\phi\, 1 + \sin\phi\, J$. Thus

$$\Gamma(\cos\phi\, 1 + \sin\phi\, J)E_v(u) = E_{\exp(i\phi)v}(u),$$

and by repeated differentiation at $\phi = 0$, we obtain

$$d\Gamma(1)v^{\vee k} = kv^{\vee k} \qquad (k \geqslant 0).$$

We see that $d\Gamma(1) = N$, the number operator on $\mathcal{B}(V)$, which thus has nonnegative integer spectrum. [Had we chosen to define $\mathcal{B}(V)$ as the space of holomorphic rather than antiholomorphic functions, N would have *negative* spectrum.] We also see that $\Gamma(\cos\phi\, 1 + \sin\phi\, J)v^{\vee k} = e^{ik\phi}v^{\vee k}$. In the language of "loop groups", this says that the projective unitary representation β of V is intertwined with a representation of $\mathrm{U}(1)$ in such a way as to yield a "positive energy" projective representation of V, in the terminology of G. Segal [5, 6].

The term "positive energy" is more often used in connection with some given classical dynamics on V. Suppose $t \mapsto g(t)$ is a one-parameter group of symplectic transformations of (V, s), and suppose that a positive, compatible complex structure J can be found for which $g(t) \in \mathrm{U}_J(V)$ for all t (here we refer again to the Appendix). We say that the one-parameter group $t \mapsto g(t)$ has positive energy if its generator A, a selfadjoint operator on (V, s, J) – the energy operator – has nonnegative spectrum and does not have a 0 eigenvalue. It has been shown [12] that a Weyl system $\tilde\beta$ exists for which $t \mapsto \tilde\beta(g(t)\cdot)$ is implementable as a positive-energy unitary group on the representation space $\widetilde{\mathcal{K}}$ of $\tilde\beta$ only if $t \mapsto g(t)$ already has positive energy on the one-particle space (V, s, J).

We summarize the foregoing construction in the following definition.

Definition 4.1. A *full quantization* of a symplectic linear manifold (V, s) with a preferred positive compatible complex structure J consists of

(a) a separable Hilbert space $\mathcal{K}$;

(b) a strongly continuous map β from V to the group of unitary operators on $\mathcal{K}$ satisfying (4.1);

(c) a unit vector $\Omega \in \mathcal{K}$ such that $\operatorname{span}\{\, \beta(v)\Omega : v \in V \,\}$ is dense in $\mathcal{K}$;

(d) a unitary representation Γ of $\mathrm{U}_J(V)$ on $\mathcal{K}$ satisfying (4.3), for which Ω is stationary, such that $d\Gamma(A)$ is positive selfadjoint on $\mathcal{K}$ whenever A is positive selfadjoint on the Hilbert space (V, s, J).

The question of uniqueness is settled by the following theorem of I. E. Segal [16]: any two full quantizations are unitarily equivalent; moreover, a unitary equivalence can be constructed between two quantizations satisfying (a–c) and the apparently weaker condition (d′): that $\mathcal{K}$ supports a one-parameter unitary group $\Gamma(\exp(itA))$ intertwining $\beta(V)$ as in (4.3), for which Ω is stationary and $d\Gamma(A)$ is positive selfadjoint, where A is a positive selfadjoint operator without 0 eigenvalue on (V, s, J).

► A more algebraic approach to Weyl systems is to consider (4.1) to be the defining relation of an abstract C^*-algebra, the CCR algebra $\overline{\Delta}(V, s)$ on V, densely spanned by elements $\{\, \tilde\beta(v) : v \in V \,\}$, subject only to $\tilde\beta(v)^\dagger = \tilde\beta(-v)$ and to (4.1) with β replaced by $\tilde\beta$. Such a C^*-algebra may be defined as the C^*-inductive limit, over the set of finite dimensional symplectic subspaces (V', s) of (V, s), of the corresponding algebras $\overline{\Delta}(V', s)$, which are uniquely determined by the Schrödinger representations of (4.1) in each V': for details, we refer to [20]. The functional

$$\omega_J(\tilde\beta(v)) := \exp(-\tfrac14 \langle v \mid v\rangle)$$

extends by linearity and continuity to a faithful state of $\overline{\Delta}(V, s)$, since if $a = \sum_{k=1}^n \alpha_k \tilde\beta(v_k)$, then

$$\omega_J(a^\dagger a) = \sum_{j,k=1}^n \alpha_j^* \alpha_k \exp\bigl(-\tfrac14\langle v_j \mid v_j\rangle - \tfrac14\langle v_k \mid v_k\rangle + \tfrac12\langle v_j \mid v_k\rangle\bigr) > 0 \quad \text{unless} \quad a = 0.$$

Thus the Gelfand–Naĭmark–Segal construction [20] produces a faithful representation π_J of $\overline{\Delta}(V, s)$ on a Hilbert space $\mathcal{K}_J$, containing a cyclic vector Ω_J such that $\omega_J(\tilde\beta(v)) = \langle \Omega_J \mid \pi_J(\tilde\beta(v))\Omega_J\rangle$.

It follows that

$$\langle \pi_J(\tilde\beta(u))\,\Omega_J \mid \pi_J(\tilde\beta(v))\,\Omega_J\rangle = \omega_J(\tilde\beta(u)^\dagger\tilde\beta(v))$$
$$= \exp(-\tfrac14\langle u \mid u\rangle - \tfrac14\langle v \mid v\rangle + \tfrac12\langle u \mid v\rangle) = \langle \beta(u)\,\Omega \mid \beta(v)\,\Omega\rangle, \quad (4.6)$$

so $\pi_J(\tilde\beta(v))\Omega_J \mapsto \beta(v)\Omega$ extends to a unitary isomorphism from $\mathcal{K}_J$ to $\mathcal{B}(V)$, intertwining π_J and the representation $\tilde\beta(v) \mapsto \beta(v)$ of $\overline{\Delta}(V,s)$ on $\mathcal{B}(V)$. Also, if $U \in \mathrm{U}_J(V)$, then $\tilde\beta(v) \mapsto \tilde\beta(Uv)$ extends to an automorphism α_U of $\overline{\Delta}(V,s)$ leaving ω_J invariant, and thus is implemented on $\mathcal{K}_J$ by a unitary operator $\Gamma_J(U)$, i.e., $\pi_J(\alpha_U(a)) = \Gamma_J(U)\pi_J(a)\Gamma_J(U)^{-1}$, leading to the analogue of (4.3) for $\tilde\beta$ and Γ_J; and Ω_J is stationary for $\Gamma_J(\mathrm{U}_J(V))$. The argument of (4.4) and (4.5) may be repeated to show that $d\Gamma_J$ preserves positivity. Thus the GNS representation of $(\overline{\Delta}(V,s),\omega_J)$ is a full quantization, for which $\mathcal{B}(V)$ with the Weyl system (4.1) is an explicit presentation.

4.2 The derived representation of the Weyl system

The derived representation of the Weyl system is easily computed. We set

$$\dot\beta(v)F(u) := \frac{d}{dt}\bigg|_{t=0} \beta(tv)F(u)$$

and from (4.2) we get at once:

$$\dot\beta(v)F(u) = \tfrac12\langle u \mid v\rangle F(u) - D_vF(u), \tag{4.7}$$

where D_v is the directional derivative in the direction v. It is immediate that

$$[\dot\beta(v),\dot\beta(w)] = -is(v,w).$$

The domain of the operator $\dot\beta(v)$ is the space of F in $\mathcal{B}(V)$ for which the right hand side of (4.7) has finite norm (it is evidently antiholomorphic in u). An element $F \in \mathcal{B}(V)$ is a *smooth vector* for β if $t \mapsto \beta(tv)F$ is an infinitely differentiable function for any v or, equivalently, if $t \mapsto \beta(tv)F(u)$ is smooth, for any $u, v \in V$; for such F, the right hand side of (4.7) makes sense. It is readily seen that the principal vectors E_w are smooth vectors for $\dot\beta$ and that

$$\dot\beta(v)E_w(u) = \tfrac12(\langle u \mid v\rangle - \langle v \mid w\rangle)E_w(u).$$

Let us also write $\phi(v) := -i\dot\beta(v)$, so that

$$[\phi(v),\phi(w)] = is(v,w). \tag{4.8}$$

Then $\phi(v)$ is a symmetric operator with kernel $K_{\phi(v)}(u,w) = -\frac{i}{2}\langle u \mid v\rangle + \frac{i}{2}\langle v \mid w\rangle$. Since the one-parameter group $t \mapsto \beta(tv)$ leaves $\operatorname{span}\{\, E_w : w \in V \,\}$ invariant, the principal vectors generate a common domain of essential selfadjointness for all $\phi(v)$.

We define the complexified representation of $V_{\mathbb{C}}$,

$$\dot\beta(v_1 + iv_2)F(u) := \dot\beta(v_1)F(u) + i\dot\beta(v_2)F(u)$$
$$= \tfrac12\langle u \mid v_1 + Jv_2\rangle F(u) - D_{v_1 - Jv_2}F(u), \tag{4.9}$$

on the space of smooth vectors for β.

If W is a positive polarization of (V, s), we define the *vacuum sector* associated to W as the subspace of β-smooth vectors F verifying $\beta(w^*)F = 0$ for all $w \in W$. (We shall soon interpret the $\beta(w^*)$ as annihilation operators.) Writing $w^* = v + iJ_W v$ for $v \in V$, such an F satisfies the differential equation:

$$D_{v - JJ_W v} F = \tfrac{1}{2}\langle \cdot \mid v + JJ_W v\rangle F. \tag{4.10}$$

Since $T_W = (1 + JJ_W)(1 - JJ_W)^{-1}$ by (2.18), this equation may be rewritten as $D_v F = \frac{1}{2}\langle \cdot \mid T_W v\rangle F$. The vacuum sector associated to W is thus the 1-dimensional space of solutions of this equation, which are scalar multiples of the Gaussian labelled by T_W:

$$F(u) = C f_{T_W}(u) = C \exp(\tfrac{1}{4}\langle u \mid T_W u\rangle). \tag{4.11}$$

We have already seen that $f_{T_W} \in \mathcal{B}(V)$ if and only if $T_W \in \mathcal{D}'(V)$, or equivalently, if and only if $J_W \in \Sigma'(V)$. In view of (3.9), we may normalize (4.11) by defining

$$\Omega_W(u) := \det{}^{1/4}(1 - T_W^2) f_{T_W}(u) = \det{}^{1/4}(1 - T_W^2) \exp(\tfrac{1}{4}\langle u \mid T_W u\rangle). \tag{4.12}$$

In particular, if W_0 is the reference polarization for which $J_W = J$ and $T_W = 0$, we recover the vacuum vector Ω.

4.3 Creation and annihilation operators

The annihilation and creation operators for the boson field ϕ may now be defined as real-linear (unbounded) operators on $\mathcal{B}(V)$:

$$a(v) := \tfrac{1}{\sqrt{2}}[\phi(v) + i\phi(Jv)], \qquad a^\dagger(v) := \tfrac{1}{\sqrt{2}}[\phi(v) - i\phi(Jv)]. \tag{4.13}$$

Clearly $a(Jv) = -ia(v)$ and $a^\dagger(Jv) = ia^\dagger(v)$ since $v \mapsto \phi(v)$ is real-linear. Thus $a(v)$ is antilinear and $a^\dagger(v)$ is linear in v.

From (4.8), we directly obtain the *canonical commutation relations*:

$$[a(v), a(w)] = 0, \qquad [a(v), a^\dagger(w)] = \langle v \mid w\rangle. \tag{4.14}$$

On account of (4.9), we also get the explicit expressions

$$a(v) = i\sqrt{2}\, D_v, \qquad a^\dagger(v) = -\tfrac{i}{\sqrt{2}}\, v \tag{4.15}$$

as differentiation and multiplication operators on $\mathcal{B}(V)$. In particular, each $a(v)$ annihilates the vacuum Ω, as expected. Notice also that

$$a^\dagger(v_1)a^\dagger(v_2)\cdots a^\dagger(v_n)\,\Omega = (-i)^n v_1 \vee v_2 \vee \cdots \vee v_n$$

in $\mathcal{B}(V)$, on account of the convention (3.2).

From (4.13), (4.15) and the relation $\dot\beta(v) = i\phi(v)$, there follows:

$$\dot\beta(v)(v_1 \vee \cdots \vee v_k) := \frac{1}{\sqrt{2}}\, v \vee v_1 \vee \cdots \vee v_k - \frac{1}{\sqrt{2}} \sum_{j=1}^{k} \langle v \mid v_j\rangle\, v_1 \vee \cdots \vee \widehat{v_j} \vee \cdots \vee v_k. \tag{4.16}$$

The principal vectors are generated from the vacuum by

$$E_v = \exp\big(\tfrac{i}{\sqrt{2}} a^\dagger(v)\big)\Omega.$$

These are smooth vectors for all creation and annihilation operators. It is immediate that

$$\exp\big(\tfrac{i}{\sqrt{2}}\, a^\dagger(v)\big)E_w = E_{v+w}, \qquad \exp\big(-\tfrac{i}{\sqrt{2}}\, a(v)\big)E_w = e^{\frac{1}{2}\langle v|w\rangle} E_w. \tag{4.17}$$

The n-point functions for the derived representations are readily found from the Segal–Bargmann representation. We wish to compute

$$\langle \Omega \mid \phi(v_1)\cdots\phi(v_m)\,\Omega\rangle$$

for $v_1, \dots, v_m \in V$. This can be rewritten, using the Weyl relation (4.1), as

$$\begin{aligned}
&(-i)^m \frac{\partial^m}{\partial t_1\cdots\partial t_m}\bigg|_{t_1=\cdots=t_m=0} \langle \Omega \mid \beta(t_1 v_1)\cdots\beta(t_m v_m)\,\Omega\rangle \\
&= (-i)^m \frac{\partial^m}{\partial t_1\cdots\partial t_m}\bigg|_{t_1=\cdots=t_m=0} \exp\biggl(-\frac{i}{2}\sum_{i<j} t_i t_j\, s(v_i,v_j)\biggr)\langle \Omega \mid \beta(t_1 v_1 + \cdots + t_m v_m)\,\Omega\rangle \\
&= (-i)^m \frac{\partial^m}{\partial t_1\cdots\partial t_m}\bigg|_{t_1=\cdots=t_m=0} \exp\biggl(-\frac{i}{2}\sum_{i<j} t_i t_j\, s(v_i,v_j) - \frac{1}{4}\sum_{r,s=1}^{m}\langle t_r v_r \mid t_s v_s\rangle\biggr) \\
&= (-i)^m \frac{\partial^m}{\partial t_1\cdots\partial t_m}\bigg|_{t_1=\cdots=t_m=0} \exp\biggl(-\frac{1}{4}\sum_{k=0}^{m} t_k^2\langle v_k \mid v_k\rangle - \frac{1}{2}\sum_{i<j} t_i t_j\langle v_i \mid v_j\rangle\biggr),
\end{aligned}$$

which vanishes if m is odd. If $m = 2n$ is even, the term $\sum_k t_k^2\langle v_k \mid v_k\rangle$ contributes nothing to the mixed partial derivative at 0; and so

$$\begin{aligned}
\langle \Omega \mid \phi(v_1)\cdots\phi(v_{2n})\,\Omega\rangle &= (-1)^n \frac{\partial^{2n}}{\partial t_1\cdots\partial t_{2n}}\bigg|_{t_1=\cdots=t_{2n}=0} \exp\biggl(-\frac{1}{2}\sum_{i<j} t_i t_j\langle v_i \mid v_j\rangle\biggr) \\
&= \frac{1}{2^n}\sum_{I<J}\langle v_{i_1} \mid v_{j_1}\rangle\cdots\langle v_{i_n} \mid v_{j_n}\rangle,
\end{aligned}$$

where the last sum runs over the $(2n)!/2^n\, n!$ "pairings" (I, J) which are permutations of $\{1, \dots, 2n\}$ such that $i_r < j_r$ for $r = 1, \dots, n$.

► We take the opportunity to introduce a few quadratic expressions in the creation and annihilation operators that will prove useful in the sequel; as well as notations profusely used later.

If $B \in \mathrm{End}_{\mathbb{R}}(V)$ is an *antilinear* symmetric operator on V, let us write

$$aBa := \sum_{j,k} a(e_j)\,\langle Be_j \mid f_k\rangle\, a(f_k) \tag{4.18}$$

with respect to any pair of orthonormal bases $\{e_j\}$, $\{f_k\}$ for the Hilbert space (V, s, J), provided the series converges in some suitable sense (to be made precise later on). Note that the right hand side is actually independent of the chosen orthonormal bases.

Similarly, let us write

$$a^\dagger B a^\dagger := \sum_{j,k} a^\dagger(f_k)\,\langle f_k \mid Be_j\rangle\, a^\dagger(e_j) \tag{4.19}$$

If C is a *linear* operator on V, we also set

$$a^\dagger C a := \sum_{j,k} a^\dagger(f_k)\,\langle f_k \mid Ce_j\rangle\, a(e_j). \tag{4.20}$$

If B is a bounded operator, the series $(aBa)F$, $(a^\dagger B a^\dagger)F$ converge whenever F lies in $S(V)$, i.e., F is a finite sum of vectors of the form $a^\dagger(v_1)\cdots a^\dagger(v_m)\Omega$. However, in order that the principal vectors E_v belong to the domains of aBa and $a^\dagger B a^\dagger$, we need B to be Hilbert–Schmidt. Indeed, if T is antilinear, symmetric and Hilbert–Schmidt, let $\{e_k\}$ be an orthonormal basis of V so that (2.20) holds, and take $f_k = e_k$. Then

$$a^\dagger T a^\dagger = \sum_k i\lambda_k\, a^\dagger(e_k)a^\dagger(e_k), \qquad aTa = \sum_k (-i\lambda_k)\, a(e_k)a(e_k).$$

Thus

$$(aTa)E_v = \frac{i}{2}\sum_k \lambda_k \langle e_k \mid v\rangle^2 E_v = -\frac{1}{2}\sum_k \langle Te_k \mid v\rangle\langle e_k \mid v\rangle\, E_v = -\tfrac{1}{2}\langle Tv \mid v\rangle\, E_v. \tag{4.21}$$

We also get

$$(a^\dagger T a^\dagger)E_v(u) = \langle E_u \mid (a^\dagger T a^\dagger)E_v\rangle = \langle (aTa)E_u \mid E_v\rangle = -\tfrac{1}{2}\langle u \mid Tu\rangle\, E_v(u). \tag{4.22}$$

Moreover, on using (4.15), we obtain

$$(a^\dagger C a)E_v(u) = \sum_{j,k} \frac{1}{2}\langle u \mid f_k\rangle\langle f_k \mid Ce_j\rangle\langle e_j \mid v\rangle\, E_v(u) = \tfrac{1}{2}(Cv)\, E_v. \tag{4.23}$$

5 The metaplectic representation

5.1 Kernel operators for the metaplectic representation

If β is any Weyl system on the symplectic space (V, s), then $v \mapsto \beta(gv)$ is also a Weyl system acting on the same Hilbert space, since the relations (4.1) remain valid. The question of central importance is whether these two quantizations of (V, s) are unitarily equivalent.

For definiteness, let us take the full quantization β already constructed on the Segal–Bargmann space $\mathcal{B}(V)$. Notice that if $U \in \mathrm{U}_J(V)$, then the intertwining property (4.3) just says that unitary conjugation by $\Gamma(U)$ implements an equivalence between β and $\beta \circ U$. More generally, let us suppose that for some given $g \in \mathrm{Sp}(V)$, there is a unitary operator $\nu(g)$ on $\mathcal{B}(V)$ so that

$$\nu(g)\beta(v) = \beta(gv)\nu(g) \quad \text{for all} \quad v \in V. \tag{5.1}$$

Clearly $\nu(g)$ maps the smooth vectors for β to smooth vectors for $\beta \circ g$, so we may differentiate (5.1) to obtain

$$\nu(g)\dot\beta(v) = \dot\beta(gv)\nu(g) \tag{5.2}$$

for all $v \in V$, or in fact for $v \in V_{\mathbb{C}}$. Thus $\nu(g)$ must map the vacuum sector associated to the polarization W_0 to that associated to the polarization gW_0. Therefore, $\nu(g)$ can only be defined for g an element of the *restricted* symplectic group $\mathrm{Sp}'(V)$.

Let us then suppose that $g \in \mathrm{Sp}'(V)$. By unitarity of $\nu(g)$, we obtain

$$\nu(g)\Omega(u) = c_g f_{T_g}(u) = c_g \exp(\tfrac{1}{4}\langle u \mid T_g u\rangle), \tag{5.3}$$

where $|c_g| = \det^{1/4}(1 - T_g^2)$.

We may *fix the phase of c_g by choosing it to be positive*:

$$c_g := \det^{1/4}(1 - T_g^2). \tag{5.4}$$

If V is finite-dimensional, an arguably more appropriate choice of phase would be to take $c_g = \det^{-1/2} p_g^t$. Note that $\det^{1/4}(1 - T_g^2) = \det^{-1/4}(p_g p_g^t)$. However, this choice is ruled out in the infinite-dimensional case, since p_g^t will not have a determinant for most $g \in \mathrm{Sp}'(V)$. (When p_g is positive definite, so that $p_g = (1 - T_g^2)^{-1/2}$, both definitions coincide.) This loss of freedom in the infinite-dimensional case is what gives rise to the bosonic anomaly.

An advantage of working in the Segal–Bargmann space is that $\nu(g)$ may be computed explicitly as a kernel operator. Indeed:

$$\begin{aligned} K_{\nu(g)}(u,v) &= \nu(g)E_v(u) = e^{\langle v|v\rangle/4}\beta(gv)\nu(g)\Omega(u) \\ &= c_g \exp\tfrac{1}{4}\big\{\langle v \mid v\rangle + \langle 2u - gv \mid gv\rangle + \langle u - gv \mid T_g(u - gv)\rangle\big\} \\ &= c_g \exp\tfrac{1}{4}\big\{\langle v \mid v\rangle - \langle (1 + T_g)p_g v \mid (1 - T_g^2)p_g v\rangle + \langle u \mid T_g u\rangle + 2\langle u \mid (1 - T_g)gv\rangle\big\} \\ &= c_g \exp\tfrac{1}{4}\big\{\langle u \mid T_g u\rangle - \langle T_g p_g v \mid p_g^{-t} v\rangle + 2\langle u \mid p_g^{-t} v\rangle\big\} \\ &= c_g \exp\tfrac{1}{4}\big\{\langle u \mid T_g u\rangle + 2\langle p_g^{-1} u \mid v\rangle + \langle \widehat{T}_g v \mid v\rangle\big\}. \end{aligned} \tag{5.5}$$

This kernel, for the infinite-dimensional restricted symplectic group, was first derived, without the computation of c_g, by Vergne [21]. With our choice for the phase of c_g, $\nu|_{\mathrm{U}_J} = \Gamma$ holds.

With formula (5.5) in hand, it is straightforward to compute the kernel of $\nu(g)\nu(h)$ for $g, h \in \mathrm{Sp}'(V)$, using the Gaussian integral formula (3.13):

$$\begin{aligned} &\int K_{\nu(g)}(u,s)K_{\nu(h)}(s,v)\, e^{-\frac{1}{2}\langle s|s\rangle}\, ds \\ &\quad = c_g c_h \exp\tfrac{1}{4}\big\{\langle u \mid T_g u\rangle + \langle \widehat{T}_h v \mid v\rangle\big\} \\ &\qquad\quad \times \int \exp\tfrac{1}{4}\big\{\langle s \mid T_h s\rangle + \langle \widehat{T}_g s \mid s\rangle + 2\langle s \mid p_h^{-t} v\rangle + 2\langle p_g^{-1} u \mid s\rangle\big\}\, e^{-\frac{1}{2}\langle s|s\rangle}\, ds \\ &\quad = c_g c_h \det^{-1/2}(1 - T_h\widehat{T}_g) \exp\tfrac{1}{4}\big\{\langle u \mid T_g u\rangle + \langle \widehat{T}_h v \mid v\rangle + \langle \widehat{T}_g p_h^{-t} v \mid (1 - T_h\widehat{T}_g)^{-1} p_h^{-t} v\rangle \\ &\qquad\quad + 2\langle (1 - \widehat{T}_g T_h)^{-1} p_g^{-1} u \mid p_h^{-t} v\rangle + \langle (1 - \widehat{T}_g T_h)^{-1} p_g^{-1} u \mid T_h p_g^{-1} u\rangle\big\} \\ &\quad = c_g c_h \det^{-1/2}(1 - T_h\widehat{T}_g) \exp\tfrac{1}{4}\big\{\langle u \mid T_{gh} u\rangle + \langle \widehat{T}_{gh} v \mid v\rangle + 2\langle p_{gh}^{-1} u \mid v\rangle\big\}. \end{aligned} \tag{5.6}$$

The last equality follows on rearranging the exponents of the Gaussians by employing the formulas (2.8).

Thus we arrive at

$$\nu(g)\,\nu(h) = c(g,h)\,\nu(gh), \tag{5.7}$$

which says that ν is a *projective* representation of the restricted symplectic group $\mathrm{Sp}'(V)$ on $\mathcal{B}(V)$. This is the metaplectic representation.

The scalar $c(g,h)$ must be a phase factor, in order that each $\nu(g)$ be unitary. From the computation (5.6), we find directly that

$$\begin{aligned} c(g,h) &= c_g c_h c_{gh}^{-1} \det{}^{-1/2}(1 - T_h\widehat{T}_g) \\ &= \det{}^{1/4}(1 - T_g^2) \det{}^{1/4}(1 - T_h^2) \det{}^{-1/4}(1 - T_{gh}^2) \det{}^{-1/2}(1 - T_h\widehat{T}_g) \\ &= \exp\bigl(i \arg\det{}^{-1/2}(1 - T_h\widehat{T}_g)\bigr) \\ &= \exp\bigl(-i \arg\det{}^{-1/2}(p_g^{-1} p_{gh} p_h^{-1})\bigr). \end{aligned} \tag{5.8}$$

► The metaplectic representation is reducible. The Gaussians $\{ f_T : T \in \mathcal{D}'(V) \}$ generate a closed subspace $\mathcal{B}_0(V)$ of $\mathcal{B}(V)$ which we shall show to be invariant under $\nu(\mathrm{Sp}'(V))$.

First observe that $\mathcal{B}_0(V)$ is the closure of the even subalgebra of the symmetric algebra $S(V)$, and as such is a nontrivial closed subspace of $\mathcal{B}(V)$. Indeed, the quadratic function $H_T(u) := \frac{1}{2}\langle u \mid Tu\rangle$ equals $(d^2/dt^2)\big|_{t=0} f_T(tu)$, so that $H_T \in \mathcal{B}_0(V)$. On the other hand, if $\{e_1, e_2, \dots\}$ is the orthonormal basis of eigenvectors of T^2, we can write

$$H_T(u) = \frac{1}{2}\sum_k \langle u \mid e_k\rangle\langle u \mid Te_k\rangle = \sum_k \lambda_k (e_k \vee Je_k)(u) = \sum_k i\lambda_k (e_k \vee e_k)(u),$$

so that $f_T = \exp\bigl(\sum_{k=1}^\infty \frac{i}{2}\lambda_k\, e_k \vee e_k\bigr)$ lies in the closure of the subalgebra $S_{\mathrm{even}}(V)$ generated by the homogeneous polynomials of even degree. Since $\mathcal{B}_0(V)$ contains every $i(e_k \vee e_k)$ as particular cases of H_T, the closure of $S_{\mathrm{even}}(V)$ equals $\mathcal{B}_0(V)$.

From (3.13) it follows that

$$\begin{aligned} \nu(g)\, f_{T_h}(u) &= \int K_{\nu(g)}(u,v) f_{T_h}(v)\, e^{-\frac{1}{2}\langle v|v\rangle}\, dv \\ &= c_g \int \exp \tfrac{1}{4}\bigl\{\langle u \mid T_g u\rangle + \langle v \mid T_h v\rangle + 2\langle p_g^{-1}u \mid v\rangle + \langle \widehat{T}_g v \mid v\rangle\bigr\}\, e^{-\frac{1}{2}\langle v|v\rangle}\, dv \\ &= c_g \det{}^{-1/2}(1 - T_h\widehat{T}_g) \exp \tfrac{1}{4}\bigl\{\langle u \mid T_g u\rangle + \langle p_g^{-1}u \mid T_h(1 - \widehat{T}_g T_h)^{-1} p_g^{-1}u\rangle\bigr\} \\ &= \det{}^{1/4}(1 - T_g^2) \det{}^{-1/2}(1 - T_h\widehat{T}_g)\, f_{T_{gh}}(u) \end{aligned} \tag{5.9}$$

on using the expression (2.9) for T_{gh}. [Alternatively, since $f_{T_h} \propto \nu(h)\,\Omega$, the relation (5.7) implies that $\nu(g)\, f_{T_h} \propto f_{T_{gh}}$. The proportionality constant equals $(\nu(g)\, f_{T_h})(0) = c_h^{-1}(\nu(g)\,\nu(h)\,\Omega)(0) = c_h^{-1} c_{gh} c(g,h) = c_g \det^{-1/2}(1 - T_h\widehat{T}_g)$ from (5.8).] Since $\mathrm{Sp}'(V)$ acts transitively on $\mathcal{D}'(V)$, we see that $\nu(\mathrm{Sp}'(V))$ permutes the 1-dimensional subspaces generated by the Gaussians, and so leaves $\mathcal{B}_0(V)$ invariant and acts on it irreducibly.

The orthogonal complement of $\mathcal{B}_0(V)$ in $\mathcal{B}(V)$ is the closure $\mathcal{B}_1(V)$ of the subspace $S_{\mathrm{odd}}(V)$ of $S(V)$ generated by the odd-degree homogeneous polynomials. From (4.16), the operators $\dot\beta(v)$ exchange $S_{\mathrm{even}}(V)$ and $S_{\mathrm{odd}}(V)$. Notice also that $f_S \in \operatorname{Dom}\dot\beta(v)$ by (4.15), if $v \in V$ and $S \in \mathcal{D}'(V)$; and that $\mathcal{B}_1(V)$ is densely generated by $\{ \dot\beta(v) f_S : v \in V,\ S \in \mathcal{D}'(V) \}$. From (5.2) and (5.9), we see that $\nu(g)[\dot\beta(v) f_{T_h}] \propto \dot\beta(gv) f_{T_{gh}}$, and so ν acts irreducibly on the subspace $\mathcal{B}_1(V)$, too.

Furthermore, since $\nu(g) = \Gamma(g)$ for $g \in \mathrm{U}_J(V)$, as is clear from (5.5) in the case $T_g = 0$, and since the only stationary vectors for Γ are the constant functions in $\mathcal{B}(V)$, we see that $\mathcal{B}_0(V)$ contains nonzero stationary vectors for $\nu(\mathrm{U}_J(V))$ whereas $\mathcal{B}_1(V)$ does not. Hence the two subrepresentations of ν – on $\mathcal{B}_0(V)$ and on $\mathcal{B}_1(V)$ – are inequivalent.

In summary, the metaplectic representation, while not irreducible, is the direct sum of two irreducible (projective) subrepresentations, one of which is given explicitly by (5.9).

► The metaplectic representation may alternatively be defined in a more abstract way, as follows. We can define a complex line bundle on the Kähler manifold $\mathcal{D}'(V)$ [or $\Sigma'(V)$], whose total space is $E := \{\, \lambda f_S : S \in \mathcal{D}'(V) \,\} \subset \mathcal{B}_0(V)$, with the obvious projection $\eta\colon E \to \mathcal{D}'(V) : \lambda f_S \mapsto S$. E is a trivial line bundle, with $\lambda f_S \mapsto (S, \lambda)$ being an obvious trivialization. A family of holomorphic sections of this line bundle is given by

$$\psi_S(T) := \det{}^{-1/2}(1 - TS)\, f_T$$

for $S \in \mathcal{D}'(V)$. These sections generate a prehilbert space whose inner product is given by

$$\langle \psi_R \mid \psi_S \rangle := \det{}^{-1/2}(1 - SR). \tag{5.10}$$

From (5.9), we see that the action $S \mapsto g \cdot S$ of $\mathrm{Sp}'(V)$ on $\mathcal{D}'(V)$ given by (2.19) induces a linear mapping $\check{\nu}(g)\colon \psi_S \mapsto c_g \phi_g(S)\, \psi_{g\cdot S}$ on the sections, where we have written

$$\phi_g(S) := \det{}^{-1/2}(1 - S\widehat{T_g}).$$

It can then be checked that $\check{\nu}(g)$ preserves the inner product (5.10) and that $\check{\nu}(g)\, \check{\nu}(h)\, \psi_S = c(g,h)\check{\nu}(gh)\, \psi_S$, where $c(g,h)$ is the cocycle (5.8). The correspondence $\psi_S \mapsto f_S$ extends to a unitary equivalence of $\check{\nu}$ with the subrepresentation of ν on $\mathcal{B}_0(V)$.

The condition (5.7) amounts to saying that the group acting on E is not $\mathrm{Sp}'(V)$ but rather a 1-dimensional central extension of $\mathrm{Sp}'(V)$ by $\mathrm{U}(1)$, $c(g,h)$ being the cocycle of the extension [22].

Rather than give full details of these computations here, we refer the reader to the forthcoming [1], where this path is followed in constructing the spin representation. In the fermion case, the corresponding line bundle is *not* a trivial one.

5.2 The generalized metaplectic representation

One may well ask whether the explicit calculation of a kernel for the metaplectic representative of a restricted symplectic transformation can be of use for other symplectic transformations lying outside $\mathrm{Sp}'(V)$. Indeed, it has recently been shown by I. E. Segal and coworkers [23] that the kernel (5.5) can be used to implement many (non-restricted) symplectic transformations in a generalized sense.

Let us assume that there exists a positive selfadjoint operator B on (V, s, J) with a bounded inverse. Suppose moreover that e^{-tB} is trace-class for all $t > 0$. Then the description of the full quantization $(\mathcal{B}(V), \beta, \Omega, \Gamma)$ may be refined as follows.

We say that $v \in V$ is an *entire vector* for B if $v \in \mathrm{Dom}(e^{tB})$ for all t; denote the (dense) subspace of entire vectors by V_{ent}. The positive selfadjoint operator $d\Gamma(B)$ on $\mathcal{B}(V)$ is such that $e^{t\, d\Gamma(B)}$ is also trace-class for $t > 0$; denote the space of entire vectors for $d\Gamma(B)$ by $\mathcal{E}(V)$. This is a Fréchet space under the natural topology for which every $e^{t\, d\Gamma(B)}\colon \mathcal{E}(V) \to \mathcal{B}(V)$ is continuous; with this topology, any $e^{t\, d\Gamma(B)}$ is a continuous linear operator on $\mathcal{E}(V)$. The antidual (space of continuous antilinear forms) of $\mathcal{E}(V)$, denoted $\mathcal{E}^\times(V)$, can be represented as a space of antiholomorphic functions on V_{ent}, so that $\mathcal{E}(V) \subset \mathcal{B}(V) \subset \mathcal{E}^\times(V)$ with continuous dense inclusions. Also, the operators $e^{t\, d\Gamma(B)}$ act on $\mathcal{E}^\times(V)$ by transposition, i.e., $\langle F \mid e^{t\, d\Gamma(B)} G \rangle := \langle e^{t\, d\Gamma(B)} F \mid G \rangle$ for $F \in \mathcal{E}(V)$, $G \in \mathcal{E}^\times(V)$. (The sesquilinear pairing of $\mathcal{E}(V)$ and $\mathcal{E}^\times(V)$ extends the scalar product on $\mathcal{B}(V)$, so we use the same notation.) Moreover, the formula $e^{t\,d\Gamma(B)}G(u) = G(e^{-tB}u)$ holds, for $G \in \mathcal{E}^\times(V)$, $u \in V_{\mathrm{ent}}$, $t \in \mathbb{R}$. These properties of the boson Fock space are proved in [23].

The principal vectors $\{\, E_v : v \in V_{\mathrm{ent}} \,\}$ are thus entire vectors for all $e^{t\,d\Gamma(B)}$, and we may consider the kernels

$$K_T(u, v) := \langle E_u \mid T E_v \rangle,$$

whenever $T \colon \mathcal{E}(V) \to \mathcal{E}^\times(V)$ is a continuous linear operator. These kernels are defined for $u, v \in V_{\mathrm{ent}}$, and are antiholomorphic in u and holomorphic in v. The continuity of T is equivalent to the requirement that $e^{s\,d\Gamma(B)}Te^{-s\,d\Gamma(B)}$ be the restriction to $\mathcal{E}(V)$ of a bounded operator on $\mathcal{B}(V)$, for some $s > 0$, and it imposes on the kernel K_T the estimate

$$|K_T(u, v)| \leqslant C \exp \tfrac{1}{2}\big\{\langle e^{sB}u \mid e^{sB}u\rangle + \langle e^{sB}v \mid e^{sB}v\rangle\big\} \tag{5.11}$$

for some $s > 0$, $C > 0$. Furthermore, if K_T does satisfy such an estimate, it determines a unique continuous operator T.

If $g \in \mathrm{Sp}'(V)$, $\lambda(g) := c_g^{-1}\nu(g)$ is the bounded operator on $\mathcal{B}(V)$ with kernel

$$K_{\lambda(g)}(u, v) := \exp \tfrac{1}{4}\big\{\langle u \mid T_g u\rangle + 2\langle p_g^{-1}u \mid v\rangle + \langle \widehat{T}_g v \mid v\rangle\big\}. \tag{5.12}$$

Since $\lambda(g)$ is not normalized, it is no longer unitary, but the fundamental intertwining property (5.1) still holds. Thus we may follow [23] and call an element of $\mathrm{Sp}(V)$ "projectively implementable" if there exists a continuous linear operator $\lambda(g) \colon \mathcal{E}(V) \to \mathcal{E}^\times(V)$ such that

$$\lambda(g)\,\beta(v) = \beta(gv)\,\lambda(g) \qquad \text{for all} \quad v \in V_{\mathrm{ent}}, \tag{5.13}$$

and $\langle \Omega \mid \lambda(g)\Omega\rangle = 1$. (We waive the requirement of unitarity, so this *conventional* normalization should not be thought of as a vacuum persistence amplitude.) Now the conditions $1 - T_g^2 > 0$, $p_g p_g^t = (1 - T_g^2)^{-1}$ show that the kernel (5.12) satisfies the estimate (5.11) for all $s > 0$, and so represents a continuous operator from $\mathcal{E}(V)$ to $\mathcal{E}^\times(V)$. The intertwining property follows by an approximation argument, since we have already shown its validity for the subgroup $\mathrm{Sp}'(V)$: given $g \in \mathrm{Sp}(V)$, let P_n be finite-rank orthogonal projectors commuting with $1 - T_g^2$ so that $P_n \to 1$ strongly on V; write $T_{(n)} := P_n T_g$, $p_{(n)} := P_n p_g + (1 - P_n)(1 - T_g^2)^{1/2} p_g$; then $g_n := (1 + T_{(n)})p_{(n)} \in \mathrm{Sp}'(V)$ and $K_{\lambda(g_n)}(u, v) \to K_{\lambda(g)}(u, v)$ pointwise; from this one deduces that $\lambda(g_n) \to \lambda(g)$ as operators, and then (5.13) is immediate.

The family of operators $\{\, \lambda(g) : g \in \mathrm{Sp}(V) \,\}$ thereby determined may be called a "generalized metaplectic representation" of the full symplectic group $\mathrm{Sp}(V)$. Although, on account of its distributional nature, it needs careful handling, it opens the way to extending the validity of many of the results discussed here, in particular the S-matrices of Sections 10 and 11.

5.3 Bogoliubov transformations

The metaplectic representation intertwines with the boson field ϕ according to (5.2). Its effect on the creation and annihilation operators can be readily determined. Since the operators are dependent on the chosen polarization, we write

$$a_g(v) := \tfrac{1}{\sqrt{2}}[\phi(v) + i\phi(gJg^{-1}v)], \qquad a_g^\dagger(v) := \tfrac{1}{\sqrt{2}}[\phi(v) - i\phi(gJg^{-1}v)], \tag{5.14}$$

in accordance with (4.13), for any $g \in \mathrm{Sp}'(V)$. Since $gJv = (p_g + q_g)Jv = J(p_g - q_g)v$, we obtain the *Bogoliubov transformation*:

$$a_g(gv) = a(p_g v) + a^\dagger(q_g v), \qquad a_g^\dagger(gv) = a(q_g v) + a^\dagger(p_g v). \tag{5.15}$$

From (4.13), (5.2) and (5.14) we immediately get

$$\nu(g)\, a(v) = a_g(gv)\, \nu(g), \qquad \nu(g)\, a^\dagger(v) = a_g^\dagger(gv)\, \nu(g), \tag{5.16}$$

so that each $a_g(gv)$ annihilates the vacuum sector associated to the polarization gW_0, which consists of multiples of the "out-vacuum" vector $\nu(g)\,\Omega = c_g f_{T_g}$.

6 The metaplectic representation as a quantization procedure

6.1 The derived metaplectic representation

The Lie algebra $\mathfrak{sp}'(V)$ of the restricted symplectic group $\mathrm{Sp}'(V)$ consists of real-linear operators $X \in \mathrm{End}_{\mathbb{R}}(V)$; let us write

$$C_X := \tfrac{1}{2}(X - JXJ), \qquad A_X := \tfrac{1}{2}(X + JXJ)$$

to denote its linear and antilinear parts. Then C_X is skewsymmetric and A_X is symmetric. Moreover, since $T_g = q_g p_g^{-1}$ for $g \in \mathrm{Sp}(V)$, differentiation gives

$$\frac{d}{dt}\bigg|_{t=0} p_{\exp tX} = C_X, \qquad \frac{d}{dt}\bigg|_{t=0} T_{\exp tX} = A_X. \tag{6.1}$$

Thus A_X is Hilbert–Schmidt. The linear part C_X may well be unbounded, as an operator on the Hilbert space (V, s, J).

Elements X of $\mathfrak{sp}'(V)$ can be regarded as quadratic Hamiltonians H_X on (V, s), under the identification $H_X(u) := \frac{1}{2}s(u, Xu)$. Thus we ask whether the metaplectic representation of $\mathrm{Sp}'(V)$ can yield a quantization rule for quadratic functions at the infinitesimal level.

First of all, for a given $X \in \mathfrak{sp}'(V)$, the assignment $t \mapsto \nu(\exp tX)$ need not be a one-parameter group, since the representation ν is projective; however, we can always find a real-valued function θ_X so that $t \mapsto e^{i\theta_X(t)}\nu(\exp tX)$ is a homomorphism. The group law demands that

$$e^{i\theta_X(s+t)} = e^{i\theta_X(s)} e^{i\theta_X(t)} c(\exp sX, \exp tX); \tag{6.2}$$

differentiating with respect to s at $s = 0$ and solving the resulting equation for $\theta_X(t)$, we obtain

$$\theta_X(t) = \alpha t - i \int_0^t h(\tau)\, d\tau,$$

where

$$h(\tau) := \frac{d}{ds}\bigg|_{s=0} c(\exp sX, \exp \tau X) = \frac{1}{4}\mathrm{Tr}_{\mathbb{C}}[A_X, T_{\exp \tau X}] \tag{6.3}$$

is computed in Section 7, and $\alpha = \dot\theta_X(0)$ is an undetermined real constant.

The *derived representation* of ν may thus be defined, for $X \in \mathfrak{sp}'(V)$, by:

$$\dot\nu(X)F := \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)}\nu(\exp tX)F. \tag{6.4}$$

A formal computation of the kernel of $\dot\nu(X)$ gives, in view of (5.5),

$$K_{\dot\nu(X)}(u,v) = \dot\nu(X)E_v(u) = \big(i\alpha + \tfrac14\big\{\langle u \mid A_X u\rangle + 2\langle u \mid C_X v\rangle - \langle A_X v \mid v\rangle\big\}\big)\exp(\tfrac12\langle u \mid v\rangle). \tag{6.5}$$

We see that E_v is a smooth vector for $\dot\nu(X)$ if and only if $v \in \operatorname{Dom} C_X$. Thus $\dot\nu(X)$ has a dense subspace of smooth vectors – generated by such E_v – whenever C_X is densely defined.

Now, $\Omega = E_0$ is a smooth vector for $\dot\nu(X)$ in any case. Since the vacuum expectation value $\langle \Omega \mid -i\dot\nu(X)\Omega\rangle = -iK_{\dot\nu(X)}(0,0) = \alpha$ remains unspecified, we are free to choose it arbitrarily. We shall set $\alpha = 0$ for every $X \in \mathfrak{sp}'(V)$. Thus the quantization rule $X \mapsto -i\dot\nu(X)$ is uniquely specified by (6.4) together with the condition

$$\langle \Omega \mid \dot\nu(X)\Omega\rangle = 0 \tag{6.6}$$

of vanishing vacuum expectation values.

► The intertwining rule (5.2) is mirrored at the infinitesimal level. In fact, if $X \in \mathfrak{sp}'(V)$, and $v, w \in V$, then

$$\nu(\exp tX)\,\dot\beta(v)\,E_w = \dot\beta((\exp tX)v)\,\nu(\exp tX)\,E_w \quad \text{for} \quad t \in \mathbb{R},$$

and differentiation at $t = 0$ yields

$$\dot\nu(X)\,\dot\beta(v)\,E_w = \dot\beta(Xv)\,E_w + \dot\beta(v)\,\dot\nu(X)\,E_w \quad \text{for} \quad v, w \in \operatorname{Dom} C_X.$$

In other words,

$$[\dot\nu(X), \dot\beta(v)] = \dot\beta(Xv) \quad \text{for} \quad v \in \operatorname{Dom} C_X.$$

6.2 The Wick dequantization rule and its inverse

Given an operator A on $\mathcal{B}(V)$, we define its *Wick* or *covariant* symbol Q_A as the function on V given by

$$Q_A(v) := e^{-\frac12\langle v|v\rangle}K_A(v,v).$$

That is to say, Q_A is the expected value of A in the (normalized) state represented by E_v. This can be called a "dequantization" rule, because it associates a function to each sufficiently regular operator. Actually, the correspondence $A \mapsto Q_A$ is one-to-one under fairly general hypotheses. To see that, remark that a function $\widetilde{Q}(u,v)$ defined in $V \times V$, which is antiholomorphic in u and holomorphic in v, is determined by its restriction to the diagonal $Q(v) := \widetilde{Q}(v,v)$. If we now consider $\widetilde{Q}_A(u,v) := e^{-\frac12\langle u|v\rangle}K_A(u,v)$, one can clearly recover A from $\widetilde{Q}_A$ and hence from Q_A; thus there exists an inverse quantization rule.

Proposition 6.1. $Q_{\phi(v)} = \big[u \mapsto s(u,v) = d(Ju,v)\big]$.

Proof. Just observe that

$$e^{-\frac12\langle u|u\rangle}\langle E_u \mid \phi(v)E_u\rangle = -\tfrac{i}{2}e^{-\frac12\langle u|u\rangle}\big(\langle u \mid v\rangle - \langle v \mid u\rangle\big)\langle E_u \mid E_u\rangle = s(u,v). \qquad \square$$

From now on, we write $dG(X) := -i\dot{v}(X)$ for $X \in \mathfrak{sp}(V)$, remarking that $dG(X) = d\Gamma(-JX)$ whenever the latter makes sense. Its symbol is easily computed.

Proposition 6.2. *The covariant symbol of $dG(X)$ is given by*

$$Q_{dG(X)} = \bigl[u \mapsto \tfrac{1}{2}s(u, Xu)\bigr]. \tag{6.7}$$

Proof. Since A_X is (antilinear) selfadjoint and C_X is skewadjoint, we obtain

$$\begin{aligned} e^{-\frac{1}{2}\langle u|u\rangle}\langle E_u \mid -i\dot{v}(X)\, E_u\rangle &= -\tfrac{i}{4}\bigl\{\langle u \mid A_X u\rangle - \langle A_X u \mid u\rangle + 2\langle u \mid C_X u\rangle\bigr\} \\ &= \tfrac{1}{2}s(u, (A_X + C_X)u) = \tfrac{1}{2}s(u, Xu). \qquad \square \end{aligned} \tag{6.8}$$

What is of interest in the previous propositions is that the dequantization rule gives in both cases the classical Hamiltonian function associated with the Hamiltonian vector fields $u \mapsto v$ and $u \mapsto Xu$ (on identifying V with its tangent spaces by associating to each $v \in V$ the vector $\dot{v} \in T_uV$ at any point $u \in V$ for which

$$\dot{v}\, f(u) = \frac{d}{dt}\bigg|_{t=0} f(u + tv),$$

where f is any smooth function on V). More precisely, it is easy to check that

$$\begin{aligned} i(v)s &= -dQ_{\phi(v)} \qquad \text{for} \quad v \in V, \\ i(X)s &= -dQ_{dG(X)} \quad \text{for} \quad X \in \mathfrak{sp}(V); \end{aligned}$$

where $i(Y)s$ denotes the contraction of the vector field Y with the symplectic form s. In other words, the expectation of the quantum Hamiltonian $\phi(v)$ or $dG(X)$ is equal to the classical energy. This is a characteristic property of normal ordering.

► The quantization rule inverting (6.7) is found by comparing (6.8) with the formulas for the expressions (4.18) to (4.20) as quadratic forms on the principal vectors in $\mathcal{B}(V)$; polarizing (6.8) gives

$$\langle E_u \mid dG(X)E_v\rangle = -\tfrac{i}{4}\bigl\{\langle u \mid A_X u\rangle - \langle A_X v \mid v\rangle + 2\langle u \mid C_X v\rangle\bigr\}E_v(u),$$

and from (4.21) to (4.23) we obtain at once:

$$dG(X) = \tfrac{i}{2}(a^\dagger A_X a^\dagger - 2a^\dagger C_X a - aA_X a), \tag{6.9}$$

using the notations (4.18) to (4.20). In particular, the number operator appears as the Wick quantization of J:

$$N = d\Gamma(1) = dG(J) = -i\, a^\dagger J a = a^\dagger a. \tag{6.10}$$

The discussion so far remains in the infinite-dimensional context. It should be clear, however, that ordinary Quantum Mechanics is described by the theory, when $\dim V = 2n < \infty$. In that case, the space of motions for a spinless particle is identified to the space of initial conditions, i.e., ordinary phase space. In the latter context the Weyl–Moyal or "symmetric" quantization rule can be used and usually is preferred. The relations between the Wick rule, the "anti-Wick" or "contravariant" quantization rule and the Weyl–Moyal rule are discussed in [24], where the transformations between the corresponding symbols are described. An important property of the Weyl–Moyal rule is full covariance under linear symplectic transformations. In order to appreciate that, we must turn to the so-called metaplectic group.

6.3 The metaplectic group in Quantum Mechanics

As noted in Section 5, it is possible, when V is finite-dimensional, to take $c_g := \det^{-1/2} p_g^t$ rather than $c_g := \det^{1/4}(1 - T_g^2)$. With this choice, a glance at (5.8) is enough to verify that the redefined metaplectic representation $\tilde{\nu}$ fulfils

$$\tilde{\nu}(g)\, \tilde{\nu}(h) = \pm\tilde{\nu}(gh), \quad \text{for} \quad g, h \in \mathrm{Sp}(V).$$

In fact, $\tilde{\nu}$ is a faithful representation of a nonsplit $\mathbb{Z}_2$ extension of the symplectic group, called the metaplectic group.

That extension is of course invisible at the infinitesimal level. Then $d\widetilde{G}$ is a Lie algebra isomorphism between $\mathfrak{sp}(V)$ – or the set of quadratic Hamiltonians with the Poisson bracket as the Lie algebra operation – and $d\widetilde{G}(\mathfrak{sp}(V))$. [This is why there are no Schwinger terms in ordinary quantum mechanics: see next section for Schwinger terms in linear quantum field theory.]

Remark. Even in the ordinary brand of quantization, however, the extension by a circle $\mathrm{Mp}^c(V)$ of the symplectic group can be of some help. Not every symplectic manifold can be lifted to a metaplectic manifold, but it can be lifted to an Mp^c-manifold. This property has been used to refine and simplify geometric quantization techniques in [8].

Repeating the computation (6.9) – with $\theta_X(t) = 0$ since the new cocycle is ± 1 – gives

$$d\widetilde{G}(X) = dG(X) - \tfrac{i}{2}\,\mathrm{Tr}_{\mathbb{C}}[C_X]. \tag{6.11}$$

Notice that the last term is real. This gives Moyal quantization of (the quadratic Hamiltonian associated to) X. Comparing with (6.9), we get

$$d\widetilde{G}(X) = \tfrac{i}{2}(a^\dagger A_X a^\dagger - a^\dagger C_X a - a C_X a^\dagger - a A_X a),$$

with the definition $aC_Xa^\dagger := \sum_{j,k} a(e_j)\, \langle f_k \mid C_X e_j\rangle\, a^\dagger(f_k) = a^\dagger C_X a + \mathrm{Tr}_{\mathbb{C}}[C_X]$, using the CCR (4.14). This shows that Moyal quantization is halfway between Wick quantization and the "antinormal" rule.

For finite-dimensional V, it is readily seen that all irreducible Weyl systems yield full quantizations. Then Shale's theorem implies that all irreducible representations of the canonical commutation relations are equivalent, which is the main contention of the Stone–von Neumann theorem, usually considered the cornerstone of quantum mechanics. In order to make contact with the standard formulations, it will be enough to identify our Weyl systems with the standard system of coherent states.

We shall simplify the notation by assuming $V \simeq \mathbb{R}^2$. We shall also suppose that Darboux coordinates (q, p) have been chosen for s so that:

$$s\left(\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}\right) = q_1 p_2 - q_2 p_1,$$

and we shall take J conventionally of the form $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, all other choices being equivalent. Hence

$$d_J\left(\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}\right) = q_1 q_2 + p_1 p_2,$$

and (V, s, J) is identified to $\mathbb{C}$ by $\binom{q}{p} \leftrightarrow q + ip$. According to the above results, the function $\binom{q}{p} \mapsto aq + bp$ quantizes to $\phi\binom{-b}{a}$. Thus $Q = \phi\binom{0}{1}$, $P = \phi\binom{-1}{0}$, $2^{-1/2}(Q + iP) = a\binom{0}{1}$. Note that $[Q, P] = i$, as expected. We can rewrite the Weyl system β as a "symplectic exponential" or a "displacement operator":

$$\beta\begin{pmatrix} q \\ p \end{pmatrix} = e^{i(pQ - qP)} = e^{\alpha a^\dagger - \alpha^* a},$$

where $\alpha := (q + ip)/\sqrt{2}$ and $a := a\binom{0}{1}$. The theory of coherent states in quantum mechanics can be developed from here on as in [25] (which has slightly different conventions from what is natural in our context). The number operator $dG(J)$ is essentially the harmonic oscillator Hamiltonian and Ω is the harmonic oscillator ground state; this explains the privileged role of that system in ordinary quantum mechanics. In the Schrödinger representation, homogeneous components of the symmetric algebra correspond to the span of the Hermite functions of a given degree; on these subspaces, Γ acts cyclically.

Before we leave the subject of ordinary quantum mechanics, we point out that the metaplectic representation has been used for calculating geometrical (Aharonov–Anandan) phases in [26].

7 Bosonic anomalies

7.1 The extended symplectic Lie algebra

One may reformulate the discussion of derived representations in subsection 6.1 by passing to the extended symplectic group $\widetilde{\mathrm{Sp}}'(V)$ and the extended symplectic Lie algebra $\widetilde{\mathfrak{sp}}'(V)$. Here $\widetilde{\mathrm{Sp}}'(V)$ is the one-dimensional central extension of $\mathrm{Sp}'(V)$ by $\mathrm{U}(1)$ which is determined by the metaplectic representation; its elements can be written as (g, λ), where $g \in \mathrm{Sp}'(V)$, $\lambda \in \mathrm{U}(1)$, with group law

$$(g, \lambda) \cdot (h, \mu) = (gh, \lambda\mu c(g, h)), \tag{7.1}$$

so that $(g, \lambda) \mapsto \lambda\nu(g)$ is a (linear) unitary representation of the extended group. Its Lie algebra $\widetilde{\mathfrak{sp}}'(V)$ is a 1-dimensional central extension of $\mathfrak{sp}'(V)$ by $i\mathbb{R}$, with commutator

$$[(X, ir), (Y, is)] := ([X, Y], \alpha(X, Y)) \tag{7.2}$$

where

$$\alpha(X, Y) = \frac{d^2}{dt\, ds}\bigg|_{t=s=0} c(\exp sX, \exp tY) - \frac{d^2}{dt\, ds}\bigg|_{t=s=0} c(\exp tY, \exp sX),$$

obtained directly from (7.1) applied to the commutator $(g, \lambda)(h, \mu)(g, \lambda)^{-1}(h, \mu)^{-1}$ in the extended group, with $g = \exp sX$, $h = \exp tY$.

The Lie algebra cocycle α has the physical meaning of a *Schwinger term*. Indeed:

Proposition 7.1. *If $X, Y \in \mathfrak{sp}'(V)$, then*

$$\alpha(X, Y) = [\dot\nu(X), \dot\nu(Y)] - \dot\nu([X, Y]). \tag{7.3}$$

Proof. Because of the normal ordering (6.6), we obtain

$$\frac{d^2}{dt\, ds}\bigg|_{t=s=0} \nu(\exp sX)\, \nu(\exp tY) = \dot\nu(X)\, \dot\nu(Y),$$

and by the Campbell–Baker–Hausdorff formula, there holds

$$\nu(\exp sX)\, \nu(\exp tY) = c(\exp sX, \exp tY)\, \nu\big(\exp(sX + tY + \tfrac{1}{2}st[X,Y] + \text{higher order})\big).$$

Thus,

$$\begin{aligned} [\dot\nu(X), \dot\nu(Y)] &= \frac{d^2}{dt\, ds}\bigg|_{t=s=0} \nu(\exp sX)\, \nu(\exp tY) - \nu(\exp tY)\, \nu(\exp sX) \\ &= \frac{d^2}{dt\, ds}\bigg|_{t=s=0} \big(c(\exp sX, \exp tY) - c(\exp tY, \exp sX)\big) + \tfrac{1}{2}st[X,Y] - \tfrac{1}{2}st[Y,X] \\ &= \alpha(X,Y) + \dot\nu([X,Y]). \end{aligned}$$

□

It is not hard to compute explicitly the Schwinger terms in our framework.

Proposition 7.2. *If $X, Y \in \mathfrak{sp}'(V)$, then*

$$\alpha(X,Y) = \tfrac{1}{2}\operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]). \tag{7.4}$$

Proof. Note first that the linear and antilinear parts of $[X,Y] = [C_X + A_X, C_Y + A_Y]$ are given by $C_{[X,Y]} = [C_X, C_Y] + [A_X, A_Y]$, $A_{[X,Y]} = [A_X, C_Y] + [C_X, A_Y]$. The commutator $[\dot\nu(X), \dot\nu(Y)]$ may be computed from the quantization formula (6.9) by substituting equations (4.18) to (4.20); it is readily checked that

$$\begin{aligned} [a^\dagger A_X a^\dagger, a^\dagger C_Y a] &= a^\dagger [A_X, C_Y] a^\dagger, \\ [a^\dagger C_X a, a A_Y a] &= a[C_X, A_Y]a, \\ [a^\dagger C_X a, a^\dagger C_Y a] &= a^\dagger [C_X, C_Y] a, \\ [a^\dagger A_X a^\dagger, a A_Y a] + [a A_X a, a^\dagger A_Y a^\dagger] &= -4\, a^\dagger [A_X, A_Y] a - 2\operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]), \end{aligned}$$

using the canonical commutation relations. It then follows that

$$[\dot\nu(X), \dot\nu(Y)] = \dot\nu([X,Y]) + \tfrac{1}{2}\operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]).$$

□

It is also instructive to see how the Schwinger terms may be obtained directly from (7.2). Let us abbreviate $g := \exp sX$, $h := \exp tY$. We obtain

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} c(g,h) &= \frac{d}{ds}\bigg|_{s=0} \exp(i \arg \det{}^{-1/2}(1 - T_h \widehat{T}_g)) \\ &= c(1,h) \frac{d}{ds}\bigg|_{s=0} (i \arg \det{}^{-1/2}(1 - T_h \widehat{T}_g)) \\ &= i\Im\bigg(\frac{d}{ds}\bigg|_{s=0} \det{}^{-1/2}(1 - T_h \widehat{T}_g)\bigg) \\ &= -\frac{1}{4}\operatorname{Tr}_{\mathbb{C}}\bigg(\frac{d}{ds}\bigg|_{s=0}(1 - T_h \widehat{T}_g) - \frac{d}{ds}\bigg|_{s=0}(1 - \widehat{T}_g T_h)\bigg) = -\frac{1}{4}\operatorname{Tr}_{\mathbb{C}}([T_h, A_X]), \end{aligned}$$

which verifies (6.3). We then get

$$\frac{d^2}{dt\, ds}\bigg|_{t=s=0} c(g,h) = -\frac{1}{4}\frac{d}{dt}\bigg|_{t=0} \operatorname{Tr}_{\mathbb{C}}([T_h, A_X]) = \frac{1}{4}\operatorname{Tr}_{\mathbb{C}}([A_X, A_Y]).$$

In like manner, we find that $(d^2/dt\, ds)\big|_{t=s=0} c(h,g) = \frac{1}{4}\operatorname{Tr}_{\mathbb{C}}([A_Y, A_X])$. Subtracting these two derivatives then gives (7.4).

The formula (7.4) yields the Schwinger term directly from the obstruction to linearity of the metaplectic representation. When V is finite-dimensional, the following reformulation is possible: since the linear commutant $[C_X, C_Y]$ is traceless, (7.3) reduces to:

$$\alpha(X,Y) = \tfrac{1}{2}\operatorname{Tr}_{\mathbb{C}}[C_{[X,Y]}],$$

which is a trivial cocycle – compare equation (6.10). In the infinite-dimensional case, this is no longer true, since $[C_X, C_Y]$ is in general not trace-class. In other words, there is an obstruction to Moyal quantization at this level. (This does not mean that large classes of functions on V cannot be Moyal-quantized: we owe this remark to E. C. G. Sudarshan.)

► We end this subsection by checking directly that α is a 2-cocycle for the Lie algebra cohomology of $\mathfrak{sp}'(V)$ [22]. The coboundary operator for this cohomology is:

$$\delta\alpha(X,Y,Z) := \alpha([X,Y],Z) + \alpha([Y,Z],X) + \alpha([Z,X],Y) = \sum_{\text{cyclic}} \alpha([X,Y],Z),$$

where $\sum_{\text{cyclic}}$ denotes a sum over the three cyclic permutations of (X,Y,Z). The identity $\delta\alpha = 0$ can be checked from (7.4), the Jacobi identity and tracelessness of commutants of linear operators:

$$\begin{aligned}
2\delta\alpha(X,Y,Z) &= \operatorname{Tr}_{\mathbb{C}}\Big(\sum_{\text{cyclic}} [A_{[X,Y]}, A_Z]\Big) \\
&= \operatorname{Tr}_{\mathbb{C}}\Big(\sum_{\text{cyclic}} [[A_X, C_Y], A_Z] - [[A_Y, C_X], A_Z]\Big) \\
&= \operatorname{Tr}_{\mathbb{C}}\Big(\sum_{\text{cyclic}} [[A_X, C_Y], A_Z] + [[C_X, A_Z], A_Y]\Big) \\
&= \operatorname{Tr}_{\mathbb{C}}\Big(\sum_{\text{cyclic}} [[A_X, C_Y], A_Z] + [[C_Y, A_X], A_Z]\Big) = 0.
\end{aligned}$$

In summary, α acts as a generator for the cohomology space $H^2(\mathfrak{sp}', \mathbb{R}) = \mathbb{R}$.

7.2 The adjoint representation and the anomaly

The exponential map from $\widetilde{\mathfrak{sp}'}(V)$ to $\widetilde{\mathrm{Sp}'}(V)$ is given by $\exp t(X, ir) := (\exp tX, \exp(irt + i\theta_X(t)))$, in view of (6.2). Now the group $\mathrm{Sp}'(V)$ acts on $\widetilde{\mathfrak{sp}'}(V)$ by the adjoint action of the central extension; this action is of the form

$$\widetilde{\mathrm{Ad}}(g) : (X, ir) \longmapsto (\mathrm{Ad}(g)X, ir + \gamma(g,X)),$$

where the *anomaly* $\gamma(g,X) \in i\mathbb{R}$ depends linearly on X.

The term measuring the nonequivariance of the adjoint action has a direct physical meaning: look at equation (7.6), thinking of g as a classical scattering operator and suppose that it commutes with the observable X. Then the formula says that this classical symmetry will not be preserved at the quantum level in general. Also, see the remark at the end of next section, justifying the name chosen for γ.

Since
$$\widetilde{\mathrm{Ad}}(g)[(X,ir),(Y,is)] = [\widetilde{\mathrm{Ad}}(g)(X,ir),\ \widetilde{\mathrm{Ad}}(g)(Y,is)],$$
using (7.2), we obtain
$$\gamma(g,[X,Y]) = \alpha(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) - \alpha(X,Y), \tag{7.5}$$
for $X, Y \in \mathfrak{sp}'(V)$. We conclude that at least for $[\mathfrak{sp}'(V), \mathfrak{sp}'(V)]$, the anomaly is determined by the Schwinger terms. Moreover, the following relation holds.

Proposition 7.3. *If $g \in \mathrm{Sp}'(V)$, $X \in \mathfrak{sp}'(V)$, then*
$$\gamma(g,X) = \nu(g)\,\dot\nu(X)\,\nu(g)^{-1} - \dot\nu(\mathrm{Ad}(g)X). \tag{7.6}$$

Proof. From (6.4) we obtain
$$\begin{aligned}
\nu(g)\,\dot\nu(X)\,\nu(g)^{-1} &= \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)}\nu(g)\,\nu(\exp tX)\,\nu(g)^{-1} \\
&= \frac{d}{dt}\bigg|_{t=0} e^{i\theta_X(t)}c(g,\exp tX)c(g\exp tX, g^{-1})\,\nu(g\exp tXg^{-1}) \\
&= \frac{d}{dt}\bigg|_{t=0} e^{i\theta_{\mathrm{Ad}(g)X}(t)}c(g,\exp tX)c(g\exp tX, g^{-1})\,\nu(\exp t\,\mathrm{Ad}(g)X) \\
&= \frac{d}{dt}\bigg|_{t=0} c(g,\exp tX)c(g\exp tX, g^{-1}) + \dot\nu(\mathrm{Ad}(g)X),
\end{aligned} \tag{7.7}$$
where we have used $\dot\theta_X(0) = \dot\theta_{\mathrm{Ad}(g)X}(0) = 0$, from which it is clear that the right hand side of (7.6) is an (imaginary) scalar; call it $\gamma'(g,X)$. It suffices to show that $\gamma'(g,[X,Y]) = \gamma(g,[X,Y])$ in general. We now compute
$$\begin{aligned}
\gamma'(g,[X,Y]) &= \nu(g)\dot\nu([X,Y])\nu(g)^{-1} - \dot\nu([\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y]) \\
&= \nu(g)[\dot\nu(X),\dot\nu(Y)]\nu(g)^{-1} - \alpha(X,Y) \\
&\qquad - [\dot\nu(\mathrm{Ad}(g)X), \dot\nu(\mathrm{Ad}(g)Y)] + \alpha(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) \\
&= [\dot\nu(\mathrm{Ad}(g)X) + \gamma'(g,X),\ \dot\nu(\mathrm{Ad}(g)Y) + \gamma'(g,Y)] \\
&\qquad - [\dot\nu(\mathrm{Ad}(g)X),\ \dot\nu(\mathrm{Ad}(g)Y)] + \gamma(g,[X,Y]),
\end{aligned}$$
which reduces to $\gamma(g,[X,Y])$ since the $\gamma'(g,\cdot)$ are scalars. □

The methods of the previous subsection now allow us to compute the bosonic anomaly explicitly, in terms of the classical quantities.

Proposition 7.4. *For $g \in \mathrm{Sp}'(V)$, $X \in \mathfrak{sp}'(V)$, the bosonic anomaly is given by*
$$\gamma(g,X) = \tfrac12 \mathrm{Tr}_{\mathbb{C}}\big((1-\widehat{T}_g^2)^{-1}\big([A_X, \widehat{T}_g] - \widehat{T}_g[C_X, \widehat{T}_g]\big)\big). \tag{7.8}$$

Proof. From (7.7), we see that $\gamma(g,X)$ is indeed given by the formula
$$\gamma(g,X) = \frac{d}{dt}\bigg|_{t=0} c(g,\exp tX)\,c(g\exp tX, g^{-1}).$$

Writing $h := \exp tX$, the right hand side equals

$$
\begin{aligned}
&\frac{d}{dt}\bigg|_{t=0} \exp\bigl(i \arg(\det{}^{-1/2}(1 - T_h\widehat{T}_g) + \det{}^{-1/2}(1 - \widehat{T}_g\widehat{T}_{gh}))\bigr) \\
&= i \frac{d}{dt}\bigg|_{t=0} \arg(\det{}^{-1/2}(1 - T_h\widehat{T}_g) + \det{}^{-1/2}(1 - \widehat{T}_g\widehat{T}_{gh})) \\
&= i\Im\biggl(\frac{d}{dt}\bigg|_{t=0} \det{}^{-1/2}(1 - T_h\widehat{T}_g) + \det{}^{1/2}(1 - \widehat{T}_g^2)\frac{d}{dt}\bigg|_{t=0} \det{}^{-1/2}(1 - \widehat{T}_g\widehat{T}_{gh})\biggr) \\
&= -\frac{i}{2}\Im \operatorname{Tr}_{\mathbb{C}}\biggl(\frac{d}{dt}\bigg|_{t=0}(1 - T_h\widehat{T}_g) + (1 - \widehat{T}_g^2)^{-1}\frac{d}{dt}\bigg|_{t=0}(1 - \widehat{T}_g\widehat{T}_{gh})\biggr) \\
&= \frac{i}{2}\Im \operatorname{Tr}_{\mathbb{C}}\biggl(A_X\widehat{T}_g + (1 - \widehat{T}_g^2)^{-1}\widehat{T}_g\frac{d}{dt}\bigg|_{t=0}\widehat{T}_{gh}\biggr).
\end{aligned} \tag{7.9}
$$

Using (2.9), we find that

$$
\frac{d}{dt}\bigg|_{t=0}\widehat{T}_{gh} = \frac{d}{dt}\bigg|_{t=0}\Bigl(\widehat{T}_h + p_h^{-1}\widehat{T}_g(1 - T_h\widehat{T}_g)^{-1}p_h^{-t}\Bigr) = -A_X - [C_X, \widehat{T}_g] + \widehat{T}_g A_X \widehat{T}_g.
$$

Since the commutator has purely imaginary trace, on substituting this in (7.9) we arrive at (7.8). □

The appearance of the *commuting part* of X in (7.8) deserves a comment: whereas observables that are linear in the sense of commuting with the complex structure have non-anomalous commutators in the corresponding linear quantum field theory, they still suffer in general from anomalous transformation laws.

7.3 The Schwinger term as a cyclic cocycle

It turns out that the Lie algebra cocycle α is also a cocycle for the *cyclic cohomology* of Connes [27]; this provides a link with noncommutative geometry, which has already yielded an interesting approach to the classical action for the Standard Model [28]. We start from the observation that

$$
\alpha(X, Y) = -\frac{i}{8}\operatorname{Tr}(J[J, X][J, Y]), \tag{7.10}
$$

for $X, Y \in \mathfrak{sp}'(V)$. Here Tr denotes the usual trace over the polarization W_0; since $[J, X] = 2JA_X$, the commutators are Hilbert–Schmidt operators on W_0 – again we identify elements of $\operatorname{End}_{\mathbb{R}} V$ with their complex amplifications on $V_{\mathbb{C}}$ – and so the trace exists. One checks that

$$
\operatorname{Tr}(J[J, Y][J, X]) = \operatorname{Tr}([J, X]J[J, Y]) = -\operatorname{Tr}(J[J, X][J, Y]) \tag{7.11}
$$

since J and $[J, X]$ anticommute; on using (2.21), skewsymmetrization of the right hand side of (7.10) yields $-\frac{i}{4}\operatorname{Tr}(J[A_X, A_Y]) = \frac{1}{2}\operatorname{Tr}_{\mathbb{C}}([A_X, A_Y])$, as claimed.

► The cyclic cohomology theory is now defined as follows. Let $\mathcal{A}$ be an associative algebra. A Hochschild n-cochain over $\mathcal{A}$ is a complex $(n+1)$-linear form $\omega(X_0, X_1, \dots, X_n)$ defined for $X_0, X_1, \dots, X_n \in \mathcal{A}$; it is called *cyclic* if it satisfies:

$$
\omega(X_0, X_1, \dots, X_n) = (-1)^n \omega(X_1, \dots, X_n, X_0). \tag{7.12}
$$

The factor $(-1)^n$ is the sign of the cyclic permutation of the arguments. The Hochschild coboundary operator b is defined by

$$
\begin{aligned}
& b\omega(X_0, \dots, X_{n+1}) \\
& \quad := \sum_{j=0}^{n} (-1)^j \omega(X_0, \dots, X_j X_{j+1}, \dots, X_{n+1}) + (-1)^{n+1} \omega(X_{n+1} X_0, X_1, \dots, X_n);
\end{aligned} \tag{7.13}
$$

It is clear that if ω is a cyclic n-cocycle, then $b\omega$ is a cyclic $(n+1)$-cocycle; one checks that $b^2 = 0$. Thus the cyclic cochains over $\mathcal{A}$ form a complex $CC^\bullet(\mathcal{A})$. It is a subcomplex of the Hochschild complex obtained by dropping the cyclicity condition (7.12).

If α is a 0-cochain, $b\alpha(X, Y) = \alpha([X, Y])$; so a 0-cocycle is just a trace on $\mathcal{A}$. If β is a cyclic 1-cochain, then $b\beta(X, Y, Z) = \sum_{\text{cyclic}} \beta(XY, Z)$.

► Now take as $\mathcal{A}$ the algebra of bounded operators on V whose antilinear part is Hilbert–Schmidt. A cyclic n-cocycle is given by:

$$
\omega(X_0, X_1, \dots, X_n) := \operatorname{Tr}(J[J, X_0] \dots [J, X_n]). \tag{7.14}
$$

For *even* n, this is identically zero. For *odd* n, cyclicity is obvious from (7.11). Moreover, since $[J, XY] = X[J, Y] + [J, X]Y$, the sum in (7.13) telescopes to

$$
\begin{aligned}
b\omega(X_0, \dots, X_{n+1}) &= \operatorname{Tr}(J X_0 [J, X_1] \dots [J, X_{n+1}]) - \operatorname{Tr}(J[J, X_0] \dots [J, X_n] X_{n+1}) \\
&\qquad + \operatorname{Tr}(J[J, X_{n+1} X_0] \dots [J, X_n]) \\
&= -\operatorname{Tr}(J[J, X_{n+1}] X_0 [J, X_1] \dots [J, X_n]) - \operatorname{Tr}(J X_{n+1} [J, X_0][J, X_1] \dots [J, X_n]) \\
&\qquad + \operatorname{Tr}(J[J, X_{n+1} X_0] \dots [J, X_n]) = 0,
\end{aligned}
$$

because J anticommutes with every $[J, X]$.

The bosonic Schwinger term α is thus (the restriction to $\mathfrak{sp}'(V)$ of) a cyclic 1-cocycle. The introduction of cyclic cohomology [27] is a stepping stone to noncommutative geometry, which allows for a far-reaching development of new methods in the foundations of quantum field theory. We shall not discuss these matters here, except to say that a convenient first step is to produce a supersymmetric formulation; it is seen in [1] that the fermionic Schwinger term similarly yields a cyclic cocycle.

The relation of cyclic cohomology to Lie-algebraic cohomology, that we have exemplified, is a general result, established in [29, 30]. Recall that a Lie-algebra $(n+1)$-cocycle is an *alternating* $(n+1)$-linear form, i.e., it satisfies the analogue of (7.12) for an arbitrary (rather than a cyclic) permutation of the arguments. If $\mathbb{A}$ denotes skewsymmetrization of the arguments, the relation between $\delta\alpha = 0$ and $b\alpha = 0$ may be extended and succinctly expressed as: $\mathbb{A}(b\alpha) = \delta(\mathbb{A}\alpha)$.

The remark that $[J, \cdot]$ is a derivation allows one to lift cyclic cocycles to linear forms on a universal differential graded algebra $\Omega^\bullet\mathcal{A}$ [27, 31, 32]; for example, (7.14) can be written in the form $\omega(X_0, X_1, \dots, X_n) = \tau(X_0\, dX_1\, dX_2 \cdots dX_n)$ where τ is a graded trace and d is the differential which lifts $[J, \cdot]$. The starting point of "noncommutative geometry" is that the exterior derivative of differential forms can be similarly lifted to a universal differential; one can then use d to define noncommutative generalizations of connections and curvatures, from which ordinary connections and curvatures may be recovered by suitable projections [27, 32].

It has been pointed out in [33] that the cyclic cocycle α of (7.10) can be viewed as a curvature form representing the first Chern class of a complex line bundle; when $\mathrm{Sp}'(V)$ is replaced by the restricted general linear group of V, this is the determinant bundle over the unitary Grassmannian [6].

Formula (7.14) obviously works for any element of the Schatten class $\mathcal{L}^{n+1}$; on the other hand (7.10) must be modified when X, Y belong to $\mathcal{L}^{n+1}$, for $n > 1$. A recipe for that is given by Mickelsson in [34].

8 The Virasoro subgroup of the extended symplectic group

Thus far, we have considered general symplectic vector spaces and compatible complex structures. To go further, we must understand how particular complex structures arise in specific examples. One wishes in general that the chosen complex structure be invariant under a given "free dynamics". The detailed construction of unique *preferred* complex structures is given in the Appendix; a short summary of the procedure will suffice for the moment.

One usually starts with a linear Hamiltonian system (V_0, s_0, A_0) where V_0 is a real Banach space (with a suitable norm), s_0 is a symplectic form on V_0, and A_0 is an (unbounded) densely defined operator on V_0, skewadjoint with respect to s_0; and such that the classical energy function $v \mapsto s_0(v, A_0 v)$ satisfies a positivity condition of the type $s_0(v, A_0 v) \geqslant \varepsilon \|v\|^2$, where $\|\cdot\|$ denotes the original Banach norm on V_0. Then $d_0(u, v) := s_0(u, A_0 v)$ is a positive form making $\operatorname{Dom} A_0$ a real prehilbert space, whose completion V_1 is a Hilbert space densely embedded in A_0. The sought-after complex structure J is the polar part of the restriction of A_0 to V_1; we write $d_J(u, v) := s_0(u, Jv)$ and complete V_1 again with respect to the new scalar product d_J to obtain the final Hilbert space V: see Theorem A.3.

Before tackling the standard Klein–Gordon field, it is instructive to consider the basic example of function spaces on the circle, which leads to the action of the Virasoro group on a boson Fock space. Besides its intrinsic interest, the Virasoro example gives us a clearer picture of how the various strands of the field construction are delicately intertwined.

8.1 A rotation-invariant complex structure

This example, motivated by string theory, arises in the study of the loop group $\mathrm{Map}(\mathbb{S}^1, \mathbb{T})$ of the circle [5, 6]. It blends itself agreeably with pieces of classical analysis. The Lie algebra of $\mathrm{Map}(\mathbb{S}^1, \mathbb{T})$ is the vector space $\mathrm{Map}(\mathbb{S}^1, \mathbb{R})$ of smooth real-valued maps of the circle $\mathbb{S}^1$. The Banach space V_0 is the space $L^2(\mathbb{S}^1, \mathbb{R})/\mathbb{R}$ obtained by enlarging this space to include all square-integrable functions and quotienting by the constant maps; which can be identified with the space of periodic square-integrable functions on the interval $0 \leqslant \theta \leqslant 2\pi$ whose Fourier expansions have vanishing constant term. The symplectic form s_0 is then given by

$$s_0(f, h) := \frac{1}{2\pi} \int_0^{2\pi} f'(\theta)\, h(\theta)\, d\theta. \tag{8.1}$$

which is nondegenerate on V_0 (in the weak sense). For A_0 we take the generator of the rotations of the circle: $A_0 = d/d\theta$, with $\operatorname{Dom} A_0 := \{\, f \in V_0 : f' \in L^2(\mathbb{S}^1, \mathbb{R}) \,\}$.

The energy norm d_0 – see Eq. (A.3) – satisfies the estimate (A.1) with $\varepsilon = 1$, since

$$\begin{aligned} d_0(f,f) := s_0(f,f') = \frac{1}{2\pi}\int_0^{2\pi} |f'(\theta)|^2\,d\theta = \sum_{n\neq 0} n^2 |\hat{f}(n)|^2 \\ \geqslant \sum_{n\neq 0} |\hat{f}(n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2\,d\theta, \end{aligned} \tag{8.2}$$

where $\hat{f}(n)$ denotes the nth Fourier coefficient of f. Moreover, since

$$s_0(f,h') = \frac{1}{2\pi}\int_0^{2\pi} f'(\theta)\,h'(\theta)\,d\theta = s_0(h,f') = -s_0(f',h) \tag{8.3}$$

for $f,h \in \operatorname{Dom} A_0$, the operator A_0 is skewsymmetric with respect to s_0; indeed, it is clear from (8.3) that f lies in the domain of the s_0-adjoint $A_0^\ddagger$ if and only if $f' \in L^2(\mathbb{S}^1,\mathbb{R})$, so that A_0 is in fact skewadjoint with respect to s_0.

Thus the hypotheses of Lemma A.2 are verified. From (8.2), it is clear that $V_1 = \operatorname{Dom} A_0$ (which in this case is already complete for the energy norm) and that, if A denotes the restriction of A_0 to $\operatorname{Dom} A := \{\, f \in V_0 : f', f'' \in L^2(\mathbb{S}^1,\mathbb{R}) \,\}$, then A maps this domain onto V_1. Since $(-d^2/d\theta^2)(\sin k\theta) = k^2 \sin k\theta$ and $(-d^2/d\theta^2)(\cos k\theta) = k^2 \cos k\theta$, we see that

$$J := \frac{d}{d\theta}\biggl(-\frac{d^2}{d\theta^2}\biggr)^{-1/2}$$

is given by

$$J(\sin k\theta) := \cos k\theta, \qquad J(\cos k\theta) := -\sin k\theta,$$

for k positive. In other words, J is the classical operator that associates to a periodic function its conjugate periodic function. This is known to be representable by a *Hilbert transform* [35]:

$$Jf(\theta) = \frac{1}{2\pi}\ \mathrm{PV}\!\int_0^{2\pi} \cot\biggl(\frac{\theta'-\theta}{2}\biggr) f(\theta')\,d\theta' \tag{8.4a}$$

That is therefore the unique rotation-invariant positive compatible complex structure on V_1 .

Now the real Hilbert space (V, d_J) is determined by

$$d_J(f,h) := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\biggl(-\frac{d^2}{d\theta^2}\biggr)^{1/2} h(\theta)\,d\theta,$$

i.e., V is the space of real "half-densities" on $\mathbb{S}^1$.

Let us note that on the complexification $V_{\mathbb{C}}$, there holds

$$-iJ(e^{ik\theta}) = \varepsilon_k e^{ik\theta}, \tag{8.4b}$$

with $\varepsilon_k = +1$ or -1 according as k is positive or negative. Thus the polarization $W_0 = (1 - iJ)V$ consists of complex-valued functions on the circle whose Fourier series $f(\theta) = \sum_{k>0} a_k e^{ik\theta}$ satisfy $\sum_{k>0} k|a_k|^2 < \infty$: note that isotropy is directly checked by Cauchy's theorem! These lie in the Hardy space $H^2(\mathbb{D})$ of holomorphic functions on the unit disk $\mathbb{D}$ which extend to square-integrable functions on the boundary $\mathbb{S}^1$, and moreover vanish at the origin; similarly, W_0^* may be considered as a subspace of the Hardy space of functions holomorphic outside $\mathbb{S}^1$, square-integrable on the circle and vanishing at infinity. Elements $f, g \in W_0$ have the scalar product

$$\langle\!\langle f \mid g \rangle\!\rangle = \frac{1}{\pi i} \int_{\mathbb{D}} df^* \wedge dg.$$

In summary, there exists on V a unique positive symplectic complex structure that commutes with rotations, given by (8.4); the operator $-i\, d/d\theta$ is *positive* on the complex Hilbert space determined by J; that will ensure, by means of the theory developed in subsection 4.1, that the corresponding representation of the Virasoro group is a "positive energy" representation in the sense of [5, 6].

8.2 The Schwinger term for the Virasoro group

Let us now see how this rotation-invariant complex structure, and the full quantization which follows therefrom, together with the Schwinger term which we have derived from the metaplectic representation, allows us to compute "from first principles" the well-known anomalous term of the Virasoro Lie algebra.

The group $\mathrm{Diff}^+(\mathbb{S}^1)$ of orientation-preserving diffeomorphisms of the circle acts on the space V of the previous subsection by $(g_\phi f)(\theta) := f(\phi^{-1}(\theta))$ for $\phi \in \mathrm{Diff}^+(\mathbb{S}^1)$. In view of (8.1) and the fundamental theorem of integration theory we conclude that $g_\phi \in \mathrm{Sp}(V)$ for each ϕ.

In fact, the g_ϕ belong to the *restricted* symplectic group $\mathrm{Sp}'(V)$, i.e., $[J, g_\phi]$ is a Hilbert-Schmidt operator on V. To see that [6], we compute the integral kernel of $[J, g_\phi]$:

$$\begin{aligned} K(\theta_1, \theta_2) &= \frac{1}{2\pi} \int_0^{2\pi} \delta(\phi^{-1}(\theta_1) - \theta) \cot\Big(\frac{\theta_2 - \theta}{2}\Big) - \cot\Big(\frac{\theta - \theta_1}{2}\Big)\, \delta(\phi^{-1}(\theta) - \theta_2)\, d\theta \\ &= \cot\Big(\frac{\theta_2 - \phi^{-1}(\theta_1)}{2}\Big) - \cot\Big(\frac{\phi(\theta_2) - \theta_1}{2}\Big)\phi'(\theta_2), \end{aligned}$$

which is continuous except perhaps when $\theta_1 = \phi(\theta_2)$. Since $\cot x - 1/x$ vanishes at $x = 0$, we need only observe that

$$\frac{2}{\theta_2 - \phi^{-1}(\theta_1)} - \frac{2\phi'(\theta_2)}{\phi(\theta_2) - \theta_1} \to \frac{\phi''(\theta_2)}{\phi'(\theta_2)} \quad \text{as } \theta_1 \to \phi(\theta_2)$$

to conclude that $[J, g_\phi]$ has a continuous kernel. By the same token, it is seen that K is continuously differentiable – indeed smooth – and hence is Hilbert–Schmidt. Our proof is superficially different from the arguments given in [6].

The metaplectic representation of $\mathrm{Sp}'(V)$ thus gives rise to a projective unitary representation of $\mathrm{Diff}^+(\mathbb{S}^1)$. This lifts to a linear unitary representation of a one-dimensional central extension of $\mathrm{Diff}^+(\mathbb{S}^1)$ by $\mathrm{U}(1)$. This extension (i.e., the Virasoro group) could be developed from scratch, as is done very instructively in [36], for example; but in our case a more powerful approach is available: we use the $\mathrm{U}(1)$ extension of $\mathrm{Sp}'(V)$ already constructed from the metaplectic representation and identify the Virasoro group as the subgroup generated by $\mathrm{Diff}^+(\mathbb{S}^1)$ and $\mathrm{U}(1)$. Since $\mathrm{Diff}^+(\mathbb{S}^1)$ is simple [37], the metaplectic representation is the only unitary representation of the Virasoro group intertwining with the given action of $\mathrm{Diff}^+(\mathbb{S}^1)$.

At the infinitesimal level, the derived metaplectic representation $\dot{\nu}$ carries the Virasoro Lie algebra into an algebra of operators on the boson Fock space $\mathcal{B}(V)$. The Lie algebra of $\mathrm{Diff}^+(\mathbb{S}^1)$ consists of vector fields $\xi(\theta)\frac{d}{d\theta} \in \mathfrak{X}(\mathbb{S}^1)$ for which $\xi(\theta)$ is smooth. The Lie bracket is of course $[\xi\frac{d}{d\theta}, \eta\frac{d}{d\theta}] = (\xi\eta' - \xi'\eta)\frac{d}{d\theta}$. A basis for the (complexified) Lie algebra is given by the vector fields

$$X_k := ie^{-ik\theta}\frac{d}{d\theta}. \tag{8.5}$$

It is clear that they verify the Lie algebra relations:

$$[X_k, X_m] = (m-k)X_{k+m}.$$

Write $A_k := \frac{1}{2}(X_k + JX_kJ)$ to denote the antilinear part of X_k. Then from (8.4) and (8.5) we get at once:

$$A_k(e^{in\theta}) = \tfrac{1}{2}n(\varepsilon_n\varepsilon_{n-k} - 1)\, e^{i(n-k)\theta}.$$

Notice that the coefficient vanishes unless n lies between 0 and k, so that A_k is of finite rank. We see that $[A_k, A_m](e^{in\theta})$ is a multiple of $(e^{i(n-k-m)\theta})$, and so $\mathrm{Tr}_{\mathbb{C}}([A_k, A_m]) = 0$ unless $m = -k$. Moreover,

$$\begin{aligned}[A_k, A_{-k}](e^{in\theta}) &= \tfrac{1}{4}n\big\{(n+k)(\varepsilon_n\varepsilon_{n+k} - 1)^2 - (n-k)(\varepsilon_n\varepsilon_{n-k} - 1)^2\big\}\, e^{in\theta} \\ &= \tfrac{1}{2}n\big\{2k - \varepsilon_n(\varepsilon_{n+k}(n+k) - \varepsilon_{n-k}(n-k))\big\}\, e^{in\theta}. \end{aligned} \tag{8.6}$$

The Schwinger term acts as the generator of the nontrivial second cohomology space of the Lie algebra $\mathfrak{X}(\mathbb{S}^1)$. It is now easy to compute: $\mathrm{Tr}_{\mathbb{C}}([A_k, A_{-k}])$ is just the sum of the (diagonal) coefficients in (8.6) for $n > 0$; and these coefficients vanish for $n \geqslant |k|$. Thus, if $k > 0$,

$$\begin{aligned}\alpha(X_k, X_{-k}) = \frac{1}{2}\mathrm{Tr}_{\mathbb{C}}([A_k, A_{-k}]) &= \frac{1}{4}\sum_{n=1}^{k-1}(2nk - n(n+k) - n(n-k)) \\ &= \frac{1}{2}\sum_{n=1}^{k-1} n(k-n) = \frac{k^3 - k}{12}\,. \end{aligned}$$

If $X = \xi\frac{d}{d\theta} = -i\sum_k \hat{\xi}(-k)X_k$ and $Y = \eta\frac{d}{d\theta}$, we therefore find that

$$\begin{aligned}\alpha(X, Y) = \frac{1}{12}\sum_k (k - k^3)\hat{\xi}(-k)\hat{\eta}(k) &= -\frac{i}{24\pi}\int_0^{2\pi}(\xi'(\theta) + \xi'''(\theta))\eta(\theta)\, d\theta \\ &= \frac{i}{24\pi}\int_0^{2\pi}(\xi(\theta) + \xi''(\theta))\eta'(\theta)\, d\theta, \end{aligned} \tag{8.7}$$

which is the Gelfand–Fuchs cocycle [38] determining the Virasoro Lie algebra as a central extension of $\mathfrak{X}(\mathbb{S}^1)$. Notice that the term $\int \xi\,\eta'$ is a Lie algebra coboundary which could be dropped without altering the extension.

The unitary representation of the Virasoro algebra we have been dealing with has central charge $c = 1$. For a discussion of the properties of the irreducible subrepresentations, we refer to [5, 6].

8.3 The Virasoro anomaly

The anomaly arising from the adjoint representation of the Virasoro group can in principle be computed directly from the general expression (7.8) for the bosonic anomaly. However, a shorter path is afforded by (7.5). We shall use the equality $[\mathfrak{X}(\mathbb{S}^1), \mathfrak{X}(\mathbb{S}^1)] = \mathfrak{X}(\mathbb{S}^1)$. The adjoint action of $\mathrm{Diff}^+(\mathbb{S}^1)$ on $\mathfrak{X}(\mathbb{S}^1)$ is easy to determine [39]; indeed, $\mathrm{Ad}(g_\phi)Xf = X(f \circ \phi) \circ \phi^{-1} = (\phi_* X)f$, so

$$[\mathrm{Ad}(g_\phi^{-1})Xf](\theta) = X(f \circ \phi^{-1}(\theta))(\phi(\theta)) = \xi(\phi(\theta))\,(f \circ \phi^{-1})'(\phi(\theta)) = \frac{\xi(\phi(\theta))}{\phi'(\theta)} f'(\theta),$$

or, more simply, $\mathrm{Ad}(g_\phi^{-1})\big(\xi \frac{d}{d\theta}\big) = (\xi \circ \phi)/\phi' \frac{d}{d\theta}$. Therefore,

$$\alpha(\mathrm{Ad}(g_\phi^{-1})X, \mathrm{Ad}(g_\phi^{-1})Y) = \frac{i}{24\pi} \int_0^{2\pi} \bigg(\frac{\xi \circ \phi}{\phi'} + \Big(\frac{\xi \circ \phi}{\phi'}\Big)''\bigg)\Big(\frac{\eta \circ \phi}{\phi'}\Big)' \, d\theta. \tag{8.8}$$

With the notation $\theta = \theta(\phi)$ for $\phi^{-1} \in \mathrm{Diff}^+(\mathbb{S}^1)$, the first term of (8.8) simplifies thus:

$$\begin{aligned} \frac{i}{24\pi} \int_0^{2\pi} \Big(\frac{\xi \circ \phi}{\phi'}\Big) \frac{d}{d\theta}\Big(\frac{\eta \circ \phi}{\phi'}\Big) \, d\theta &= \frac{i}{24\pi} \int_0^{2\pi} (\xi\theta')(\phi) \frac{d}{d\phi}(\eta\theta')(\phi) \, d\phi \\ &= \frac{i}{24\pi} \int_0^{2\pi} (\xi\eta')(\phi)\theta'(\phi)^2 + (\xi\eta)(\phi)\theta'(\phi)\theta''(\phi) \, d\phi \\ &= \frac{i}{48\pi} \int_0^{2\pi} (\xi\eta' - \xi'\eta)(\phi)\theta'(\phi)^2 \, d\phi. \end{aligned} \tag{8.9}$$

If we write $h(\phi(\theta)) := \phi''(\theta)/\phi'(\theta)^2$, we get $\frac{d}{d\theta}((\xi \circ \phi)/\phi') = (\xi' - h\xi) \circ \phi$, so the second term of (8.8) gives

$$\begin{aligned} \frac{i}{24\pi} \int_0^{2\pi} \Big(\frac{\xi \circ \phi}{\phi'}\Big)'' \Big(\frac{\eta \circ \phi}{\phi'}\Big)' \, d\theta &= \frac{i}{24\pi} \int_0^{2\pi} ((\xi' - h\xi) \circ \phi)'((\eta' - h\eta) \circ \phi) \, d\theta \\ &= \frac{i}{24\pi} \int_0^{2\pi} (\xi' - h\xi)'(\eta' - h\eta) \, d\phi \\ &= \frac{i}{48\pi} \int_0^{2\pi} (\xi' - h\xi)'(\eta' - h\eta) - (\xi' - h\xi)(\eta' - h\eta)' \, d\phi \\ &= \frac{i}{48\pi} \int_0^{2\pi} (\xi''\eta' - \xi'\eta'') - (2h' + h^2)(\xi\eta' - \xi'\eta) \, d\phi. \end{aligned} \tag{8.10}$$

Now we note that

$$\phi'(\theta)^2(h' + \tfrac{1}{2}h^2)(\phi(\theta)) = \frac{\phi'''(\theta)}{\phi'(\theta)} - \frac{3}{2}\Big(\frac{\phi''(\theta)}{\phi'(\theta)}\Big)^2 =: S(\phi)(\theta),$$

where $S(\phi)$ is the Schwarzian derivative of ϕ. We think that the following identity is well known:

$$\frac{S(\phi)(\theta)}{\phi'(\theta)^2} = -S(\theta)(\phi).$$

Combining then (8.9) and (8.10), we arrive at

$$\alpha(\mathrm{Ad}(g_\phi^{-1})X,\ \mathrm{Ad}(g_\phi^{-1})Y) - \alpha(X,Y) = \frac{i}{48\pi}\int_0^{2\pi} (\xi\eta' - \xi'\eta)(\phi)\big(\theta'(\phi)^2 - 1 + 2S(\theta)(\phi)\big)\,d\phi.$$

Interchanging ϕ and ϕ^{-1}, replacing $[X,Y]$ by X and using (7.5), the Virasoro anomaly is thereby obtained:

$$\gamma(g_\phi, X) = \frac{i}{48\pi}\int_0^{2\pi} \xi(\theta)\big(2S(\phi)(\theta) + \phi'^2(\theta) - 1\big)\,d\theta, \tag{8.11}$$

where X was $\xi\,\frac{d}{d\theta}$.

We can understand this formula in the following way. The dual of the Lie algebra $\mathfrak{X}(\mathbb{S}^1)$ – actually, the regular part of the dual in Kirillov's terminology [39] – is the space of quadratic differentials $q(\theta)\,d\theta^2$ on the circle. The duality is given by

$$\langle q, X\rangle = \frac{1}{2\pi}\int_0^{2\pi} q(\theta)\,\xi(\theta)\,d\theta,$$

which is invariant under reparametrizations $\phi \in \mathrm{Diff}^+(\mathbb{S}^1)$: this can be seen, at the infinitesimal level, from the Lie derivative $\eta\,\frac{d}{d\theta}(q(\theta)\,d\theta^2) = (2\eta' q + \eta q')d\theta^2$. Thus the Virasoro coalgebra consists of pairs $(q, -it)$ with $t \in \mathbb{R}$, and the coadjoint action of the Virasoro group is given by

$$\langle \widetilde{\mathrm{Coad}}(g_\phi^{-1})(q,-it), (X, ir)\rangle := \langle (q,-it), (\mathrm{Ad}(g_\phi)X, ir + \gamma(g_\phi, X))\rangle,$$

which reduces to

$$\widetilde{\mathrm{Coad}}(g_\phi^{-1})(q,-it) = (q\circ\phi + \tfrac{t}{12}(S(\phi) + \phi'^2 - 1), -it).$$

This is the starting point for the classification of the coadjoint orbits of the Virasoro group, which has been studied by several authors [39–41]. (Our formulas have some differences with those of Witten [41], who uses the alternative version $(i/24\pi)\int \xi''\eta'\,d\theta$ of the Gelfand–Fuchs cocycle, yielding a cohomologous extension.)

Remark. The Virasoro group can also be extracted as a subgroup of a one dimensional extension of the restricted *orthogonal* group, if one starts from the one-particle space of a fermion theory [1] and replaces the metaplectic representation by the spin representation on the fermion Fock space. An approach in this spirit has been given in an important paper by Maderner [36], who develops the anomalous terms in the context of a 2-dimensional conformal field theory: his representation of the Virasoro group is given *a priori*, as a twisted version of that proposed by G. Segal [5], which is essentially the one developed here. The Schwinger term for the spin representation is given by (7.4) or (7.10) but with the opposite – fermionic – sign (see [7], for example) and so one arrives by a parallel route at the Gelfand–Fuchs cocycle (8.7). The expression (8.11) for the Virasoro anomaly is also obtained by Maderner (with central charge equal to $\frac{1}{2}$); his procedure of exponentiating the action of the Virasoro Lie algebra seems a bit circuitous, but works well in practice. He identifies it as the energy-momentum tensor anomaly of a conformal field theory.

9 The neutral scalar field

9.1 The complex structure for the Klein–Gordon equation

Let us take for V_0 a space of real solutions of the Klein–Gordon equation, which we rewrite as a first-order system:

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} =: A_0 \begin{pmatrix} f \\ g \end{pmatrix}. \tag{9.1}$$

The corresponding symplectic form is:

$$s_0\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) := \int_{\mathbb{R}^3} (f_2(\boldsymbol{x})g_1(\boldsymbol{x}) - f_1(\boldsymbol{x})g_2(\boldsymbol{x}))\, d^3\boldsymbol{x}, \tag{9.2}$$

where we write $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{k}, \dots$ for 3-vectors and $x, y, k, \dots$ for 4-vectors.

We recall that the Sobolev space $H^s(\mathbb{R}^3)$, for s real, is defined as the completion of $C_c^\infty(\mathbb{R}^3)$, say, in the norm

$$\|f\|_s^2 := \int_{\mathbb{R}^3} (1 + \boldsymbol{k}^2)^{s/2} |\hat{f}(\boldsymbol{k})|^2\, d^3\boldsymbol{k}.$$

The operator $\omega := (m^2 - \Delta)^{1/2}$ is positive from $H^s(\mathbb{R}^3)$ to $H^{s-2}(\mathbb{R}^3)$ with bounded inverse of norm m^{-1}.

The energy norm on V_0 is given by:

$$d_0\begin{pmatrix} f \\ g \end{pmatrix} = \frac{1}{2} s_0\left(\begin{pmatrix} f \\ g \end{pmatrix}, A_0\begin{pmatrix} f \\ g \end{pmatrix}\right) = \frac{1}{2}\int_{\mathbb{R}^3} \big((g(\boldsymbol{x}))^2 + (\nabla f(\boldsymbol{x}))^2 + m^2 (f(\boldsymbol{x}))^2\big)\, d^3x. \tag{9.3}$$

The completion of $C_c^\infty(\mathbb{R}^3) \oplus C_c^\infty(\mathbb{R}^3)$ in this norm is the real Hilbert space $H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. Note that the integrand is the usual Hamiltonian density of classical Lagrangian field theory.

The formal solution of (9.1) with Cauchy data $f(\cdot, 0)$, $g(\cdot, 0)$ is:

$$\begin{pmatrix} f(\cdot, t) \\ g(\cdot, t) \end{pmatrix} = \begin{pmatrix} \cos \omega t\, f(\cdot, 0) + \omega^{-1} \sin \omega t\, g(\cdot, 0) \\ -\omega \sin \omega t\, f(\cdot, 0) + \cos \omega t\, g(\cdot, 0) \end{pmatrix}. \tag{9.4}$$

We obtain the appropriate solution space applying the machinery developed in the Appendix. We could as well start with the Banach space $V_0 := L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ with s_0 given by (9.2), and take A_0 as in (9.1) with domain $\operatorname{Dom} A_0 := H^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. The energy norm (9.3) gives $V_1 = H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. It is readily seen that $\operatorname{Dom} A_0^\ddagger = \operatorname{Dom} A_0$ and that $A_0^\ddagger = -A_0$. Remark that condition (A.2) holds. Thus we may proceed to apply Lemma A.2.

In order that the restriction A of A_0 have range in V_1, we must take $\operatorname{Dom} A := H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3)$. Then A is skewadjoint with respect to d_0 and to s, and the complex structure $J := A(-A^2)^{-1/2}$ is given by

$$J = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{9.5}$$

Note that $J = e^{\pi A/2}$. This is a bounded operator on $H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ – or on $H^s(\mathbb{R}^3) \oplus H^{s-1}(\mathbb{R}^3)$, for that matter. Use of $s(\cdot, J\cdot)$ takes us finally to $V := H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3)$. In this final space, s is strongly symplectic.

On $H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3)$ the functional calculus allows us to make sense of (9.4) as defining a one-parameter group of *unitary* transformations that solves the initial-value problem for the Klein–Gordon equation.

The moral of the story might be that complex structures are associated to the dynamics itself, they do not come from quantum considerations. Once they have been properly chosen, quantization can proceed.

Nor was the choice of s given by (9.2) arbitrary. It is well known that it is the only continuous – on $H^{1/2}(\mathbb{R}^3) \oplus H^{-1/2}(\mathbb{R}^3)$, say – Poincaré-invariant skewsymmetric form, apart from multiplication by a constant.

9.2 Quantization of the Klein–Gordon equation

We can now complete the casting of the theory of the neutral scalar field into the metaplectic mold. From (9.2) and (9.5) we derive

$$d_J\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) = \int_{\mathbb{R}^3} \big(f_1(\boldsymbol{x})\,\omega f_2(\boldsymbol{x}) + g_1(\boldsymbol{x})\,\omega^{-1} g_2(\boldsymbol{x})\big)\,d^3x.$$

The polarization projector which we must use is

$$v := \begin{pmatrix} f \\ g \end{pmatrix} \longmapsto \frac{1}{2}\begin{pmatrix} f - i\omega^{-1}g \\ i\omega(f - i\omega^{-1}g) \end{pmatrix} =: P_+v,$$

and one may check that $JP_+v = iP_+v$. Now define

$$c(\boldsymbol{k}) := \mathcal{F}^{-1}(\omega f - ig)(\boldsymbol{k}), \tag{9.6}$$

where $\mathcal{F}$ denotes the standard (unitary) Fourier transform on $H^s(\mathbb{R}^3)$. Denote $\omega(\boldsymbol{k}) := \sqrt{m^2 + \boldsymbol{k}^2}$. Consider the Hilbert space $\mathcal{H}_m^{0,+}$ of square summable functions over the forward mass hyperboloid H_m^+ with the Lorentz-invariant measure $d\mu(k) := d^3\boldsymbol{k}/2\omega(\boldsymbol{k})$. This space carries the unitary irreducible representation of the Poincaré group corresponding to massive particles of zero spin, as described by Wigner [42]. It is clear now that there is a unitary map $(V, s, J) \longrightarrow \mathcal{H}_{m,0}^+$ given by $\binom{f}{g} \mapsto c$, with inverse given by:

$$f(\boldsymbol{x}) = (2\pi)^{-3/2} \int \big(c(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}} + c^*(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}\big)\,d\mu(k),$$
$$g(\boldsymbol{x}) = i(2\pi)^{-3/2} \int \omega(\boldsymbol{k})\big(c(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}} - c^*(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}\big)\,d\mu(k).$$

For some purposes it is convenient to work with the column vector $\binom{c}{c^*}$. We shall commit in the following a slight *abus de notation*, not distinguishing between ω and the multiplication operator $\mathcal{F}^{-1}\omega\mathcal{F}$. Since

$$\begin{pmatrix} c \\ c^* \end{pmatrix} = \begin{pmatrix} \omega\mathcal{F}^{-1} & -i\mathcal{F}^{-1} \\ \omega\mathcal{F} & i\mathcal{F} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix},$$

the Hamiltonian is thus given by

$$\frac{1}{2}\begin{pmatrix} \omega\mathcal{F}^{-1} & -i\mathcal{F}^{-1} \\ \omega\mathcal{F} & i\mathcal{F} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} \omega^{-1}\mathcal{F} & \omega^{-1}\mathcal{F}^{-1} \\ i\mathcal{F} & -i\mathcal{F}^{-1} \end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix},$$

and the evolution is given by

$$c(\boldsymbol{k}) \mapsto c(\boldsymbol{k})\, e^{i\omega(\boldsymbol{k})t}.$$

Therefore, we can write (9.4) in covariant form:

$$\begin{pmatrix} f(\boldsymbol{x},t) \\ g(\boldsymbol{x},t) \end{pmatrix} = (2\pi)^{-3/2} \begin{pmatrix} \int (c(\boldsymbol{k})e^{ikx} + c^*(\boldsymbol{k})e^{-ikx})\, d\mu(k) \\ i\int \omega(\boldsymbol{k})(c(\boldsymbol{k})e^{ikx} - c^*(\boldsymbol{k})e^{-ikx})\, d\mu(k) \end{pmatrix},$$

where $kx := k^\mu x_\mu$ with metric tensor $\mathrm{diag}(1,-1,-1,-1)$.

At last the stage is set. Now, the *standard* Bargmann–Fock construction, as performed in the previous sections, effected over (V,s,J) or equivalently over $H^+_{m,0}$, gives the correct quantization of the real Klein–Gordon equation. There has been no need to mention the infamous "positive-energy" or "negative-energy" solutions. In the process we have uncovered an affinity with the method of quantization based on Wigner's classification of Poincaré group representations [43].

We finally remark that

$$dG(A) = -i\, a^\dagger A a = a^\dagger \omega a.$$

This is the rigorous counterpart in our treatment of the quantized Hamiltonian operator usually written as [44]:

$$\frac{1}{2}\int :\Big(\frac{\partial\phi}{\partial t}\Big)^2 + (\nabla\phi)^2 + m^2\phi^2 :\, d^3\boldsymbol{x}.$$

9.3 The Feynman propagator and the generating functional

Henceforth we shall use the notation $|0\rangle$ to denote the vacuum. In this section we shall witness the natural appearance of the Feynman propagator in the quantized theory. Consider the Klein–Gordon equation with an external source S:

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = A\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ S \end{pmatrix}.$$

The solution of this equation is

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = e^{At}\left[\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} + \int_0^t e^{-A\tau}\begin{pmatrix} 0 \\ S(\tau) \end{pmatrix} d\tau\right] =: e^{At}\left(\begin{pmatrix} f_0 \\ g_0 \end{pmatrix} + \alpha(t,0)\right).$$

The quantities of physical interest are related to the scattering by the source. Write α for $\alpha(+\infty,-\infty)$. A classical solution v of the Klein–Gordon equation will be classically scattered into the solution $v+\alpha$. The incoming vacuum is scattered into $|0_{\mathrm{out}}\rangle = \beta(\alpha)\,|0_{\mathrm{in}}\rangle$.

In this case the vacuum persistence amplitude is nothing but the vacuum state functional we encountered in Section 4:

$$\langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}}\rangle_S = \exp(-\tfrac{1}{4}\langle\alpha \mid \alpha\rangle).$$

We can compute easily the probability p_1 that one particle is created out of the vacuum. Consider an arbitrary orthonormal basis $\{e_k\}$ of V; then $P_1 = \sum_k |e_k\rangle\langle e_k|$ is the projector on the one-particle subspace of $\mathcal{B}(V)$, so:

$$\begin{aligned} p_1 = \langle 0_{\mathrm{out}} \mid P_1 \mid 0_{\mathrm{out}}\rangle_S &= \sum_k |\langle e_k \mid \beta(\alpha) \mid 0\rangle|^2 = \frac{1}{2}e^{-\frac{1}{2}\langle\alpha|\alpha\rangle}\sum_k |\langle e_k \mid \alpha\rangle|^2 \\ &= \frac{1}{2}\langle\alpha \mid \alpha\rangle e^{-\frac{1}{2}\langle\alpha|\alpha\rangle}, \end{aligned}$$

where we have used $\beta(\alpha)\,|0\rangle = e^{-\langle\alpha|\alpha\rangle/4}\exp\big(\tfrac{i}{\sqrt{2}}a^\dagger(\alpha)\big)\,|0\rangle$ from Section 4.

More generally, the probability of creation of n particles out of the vacuum is given by

$$\begin{aligned} p_n = \langle 0_{\rm out} \mid P_n \mid 0_{\rm out}\rangle_S &:= \sum_{k_1,\dots,k_n} \frac{1}{n!}\big|\langle e_{k_1}\vee\cdots\vee e_{k_n} \mid 0_{\rm out}\rangle\big|^2 \\ &= \frac{1}{2^n n!} e^{-\frac12\langle\alpha|\alpha\rangle} \sum_{k_1,\dots,k_n} \big|\langle e_{k_1}\mid\alpha\rangle\cdots\langle e_{k_n}\mid\alpha\rangle\big|^2 = \frac{(\frac12\langle\alpha\mid\alpha\rangle)^n}{n!}\, e^{-\frac12\langle\alpha|\alpha\rangle}, \end{aligned}$$

yielding the Poisson distribution with mean $\frac12\langle\alpha\mid\alpha\rangle$.

From (9.4) and (9.6) it is clear that one can rewrite the functional in momentum space in the following form:

$$\langle 0_{\rm in} \mid 0_{\rm out}\rangle_S = \exp\biggl\{-\frac{\pi}{2}\int |\widehat{S}(\boldsymbol{k},\omega(\boldsymbol{k}))|^2\,d\mu(k)\biggr\},$$

where $\widehat{S}$ denotes the 4-dimensional Fourier transform of S. Equivalently,

$$\langle 0_{\rm in} \mid 0_{\rm out}\rangle_S = \exp\biggl\{\frac{i}{2}\iint S(x)\,D_F(x-y)\,S(y)\,d^4x\,d^4y\biggr\} =: e^{\frac{i}{2}\langle SD_FS\rangle},$$

where

$$D_F(x-y) := \frac{1}{(2\pi)^4}\int (-p^2+m^2-i0)^{-1}e^{-ip(x-y)}\,d^4p$$

is the Feynman propagator.

9.4 Covariant description and the Feynman propagator

Denote by $D(\boldsymbol{x},t;\boldsymbol{y},0)$ the (distributional) kernel of the operator $-\omega^{-1}\sin\omega t$:

$$(-\omega^{-1}\sin\omega t)g(\boldsymbol{x},t) = \int_{\mathbb{R}^3} D(\boldsymbol{x},t;\boldsymbol{y},0)g(\boldsymbol{y},0)\,d^3\boldsymbol{y}.$$

Note that D is skewsymmetric in its arguments $(\boldsymbol{x},t)$ and $(\boldsymbol{y},0)$. We can write the solution of the Klein–Gordon equation as

$$f(\boldsymbol{x},t) = \int_{\mathbb{R}^3}\biggl(f(\boldsymbol{y},0)\frac{\partial}{\partial s}\bigg|_{s=0} D(\boldsymbol{x},t;\boldsymbol{y},s) - D(\boldsymbol{x},t;\boldsymbol{y},0)g(\boldsymbol{y},0)\biggr)\,d^3\boldsymbol{y}.$$

By a standard argument, using that D solves the Klein–Gordon equation, the hyperplane $s=0$ in Minkowski space M^4 can be replaced by any spacelike hypersurface Σ. One obtains:

$$f(x) = \int_\Sigma (f(y)\,\partial_y^\rho D(x,y) - D(x,y)\,\partial^\rho f(y))\,d\sigma_\rho(y), \tag{9.7}$$

where $d\sigma_\rho$ denotes the volume element on Σ. Also the symplectic form s of (9.2) can be covariantly written:

$$s(f_1,f_2) = \int_\Sigma (f_2(x)\partial^\rho f_1(x) - f_1(x)\partial^\rho f_2(x))\,d\sigma_\rho(x), \tag{9.8}$$

(making apparent its Poincaré invariance). More elegantly:

$$s(f_1, f_2) = \int_\Sigma f_2 \star df_1 - f_1 \star df_2,$$

where $\star$ is the Hodge operator.

Now we want to express elements of V as 4-dimensional integrals. Let h be a smooth function on M^4 of compact support. Then

$$f(x) = \int D(x, y)h(y)\, d^4y \tag{9.9}$$

corresponds to an element of V, because $D(\cdot, y)$ is a solution of the Klein–Gordon equation. Reciprocally, any element $f \in V$ with compact support can be represented in this way. For we may take any four spacelike surfaces $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ subject to $\Sigma_1 < \Sigma_2 < \Sigma_3 < \Sigma_4$ and write

$$h_f(y) := (\Box + m^2)\phi(y)f(y), \tag{9.10}$$

where ϕ is a smooth function with $\phi(y) = 0$ before Σ_2 and $\phi(y) = 1$ after Σ_3. Then (9.9) with $h = h_f$ of (9.10) gives a solution of the Klein–Gordon equation which coincides on Σ_4 with f and hence equals f.

Of course, such an h_f is far from unique. We can add to the right hand side of (9.9) any function of the form $(\Box + m^2)k$, where k is a smooth function of compact support but otherwise arbitrary. In fact, in so doing we are identifying elements of V with residue classes of functions on Minkowski space, modulo the range of the Klein–Gordon operator $\Box + m^2$.

Next we rewrite s and J in terms of the representation (9.9). Substituting in (9.8) we get:

$$s(f_1, f_2) = \iint h_{f_2}(x)D(x, y)h_{f_1}(y)\, d^4x\, d^4y.$$

The previous arguments show that such an apparently hugely degenerate form is actually well defined. This can be verified also by 4-dimensional Fourier transformation: it is then seen that only the on-mass-shell harmonics of h_{f_1}, h_{f_2} contribute.

Now let $D^1(x, y)$ be the kernel of $\omega^{-1}\cos\omega t$, a different solution of the Klein–Gordon equation, which obeys $D^1(x, y) = D^1(y, x)$. Then we find that:

$$Jf(x) = -\int_\Sigma \big(f(y)\, \partial_y^\rho D^1(x, y) - D^1(x, y)\, \partial^\rho f(y)\big)\, d\sigma_\rho(y),$$

because, by (9.4) and (9.5), this is true when Σ is the hypersurface $y^0 = 0$. Substituting (9.9) for f and applying the propagation formula (9.7) to D^1 – as we may, again because D^1 is a solution – one gets simply:

$$Jf(x) = -\int D^1(x, y)h_f(y)\, d^4y. \tag{9.11}$$

The complex structure property $J^2 = -1$ is equivalent [45,46] to the following distributional identity for the kernels D and D^1:

$$D(x, y) = \int_\Sigma \big(D^1(x, z)\, \partial_z^\rho D^1(z, y) - D^1(z, y)\, \partial_z^\rho D^1(x, z)\big)\, d\sigma_\rho(z).$$

From (9.7), (9.8) and (9.11) it follows easily that

$$s(f_1, Jf_2) = \iint h_{f_1}(x)\, D^1(x,y)\, h_{f_2}(y)\, d^4x\, d^4y.$$

Assume that $\operatorname{supp} h_1 \cap$ (past of $\operatorname{supp} h_2$) $= \emptyset$. Recall that $D_F = \frac{1}{2}(D_{\mathrm{ret}} + D_{\mathrm{adv}} + iD^1)$ and also $D = D_{\mathrm{adv}} - D_{\mathrm{ret}}$. If $v_1 = \binom{f_1}{g_1}$ with $h_1 = h_{f_1}$ and similarly for v_2, we get

$$\begin{aligned}\langle v_1 \mid v_2\rangle &= s(v_1, Jv_2) + is(v_1, v_2)\\ &= \int h_1(x)\, [D^1 + iD](x,y)\, h_2(y)\, d^4x\, d^4y = -2i\, \langle h_{f_1} D_F h_{f_2}\rangle.\end{aligned}$$

Analogously, if h_1 is to the past of h_2, we obtain:

$$\langle v_2 \mid v_1\rangle = \int h_1(x)\, [D^1 - iD](x,y)\, h_2(y)\, d^4x\, d^4y = -2i\, \langle h_{f_1} D_F h_{f_2}\rangle.$$

This is of course consistent with the results of the previous subsection: it suffices to remark that α is a solution of the Klein–Gordon equation given by $-\int D(\cdot, y)\, S(y)\, d^4y$.

In order to obtain transition amplitudes from test functions on M^4, we must smear them out with the Feynman propagator. In other words, D_F, which plays no classical role, is related to the choice of quantization; hence its inevitability.

Part of the previous treatment can be immediately generalized to equations of Klein–Gordon type in globally hyperbolic Lorentzian spacetimes, with no other change than substituting the invariant volume element $\sqrt{-g}\, d^4x$ for d^4x. Whereas D for time-dependent field theories can still be defined merely from the dynamics, the definition of D_F and of J are linked; and the (difficult) problems of figuring out what are the correct complex structures and the Feynman propagator are essentially the same. There is a large – and continually growing – literature on the subject. Consult, as well as [45, 46], the now-classic [47] for static spacetimes (where no major difficulties arise) and [48] for conformally, asymptotically Minkowskian spacetimes. Also the book by Fulling [49] and the review [11] are pertinent.

10 The scattering matrix for boson fields

10.1 The out vacuum

In analyzing a scattering experiment of a Klein–Gordon particle by an external field, it makes sense to keep the complex structure associated to the *free* motion as the preferred one; with respect to the corresponding quantization, the evolution of the system under the full Hamiltonian is interpreted as creating or annihilating particles. To fix ideas, consider the scalar coupling of the Klein–Gordon equation to an external potential:

$$\frac{\partial^2}{\partial t^2} f = (\Delta - m^2 + V(\boldsymbol{x}, t))f,$$

that we rewrite as:

$$\frac{d}{dt}\begin{pmatrix} f \\ g\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix} =: (A + \widetilde{V})\begin{pmatrix} f \\ g\end{pmatrix}. \tag{10.1}$$

The space of solutions of this equation is again a symplectic space and the symplectic form s has the same form as for the free equation. The vector field $A + \widetilde{V}$ is still Hamiltonian and the (total) energy function is:

$$d_0\begin{pmatrix} f \\ g \end{pmatrix} = s\left(\begin{pmatrix} f \\ g \end{pmatrix}, (A+\widetilde{V})\begin{pmatrix} f \\ g \end{pmatrix}\right) = \frac{1}{2}\int g(\boldsymbol{x})^2 + \nabla f(\boldsymbol{x})^2 + (m^2 + V(\boldsymbol{x}))f(\boldsymbol{x})^2\, d^3\boldsymbol{x}$$

when V is time-independent.

This system, or its momentum space equivalent, can be dealt with perturbatively following the Dirac–Dyson strategy: if $U(t,s)$ denotes the solution of (10.1), introduce the "interaction picture" propagator

$$g(t,s) := \exp(-At)\, U(t,s)\, \exp(As).$$

Thus

$$\frac{d}{dt}g(t,s) = \exp(-At)\, \widetilde{V}(t) \exp(At)\, g(t,s),$$

and we solve the attendant integral equation by iteration; under appropriate restrictions for V, the procedure is wholly rigorous. It is well known that then the classical scattering matrix is $S_{\mathrm{cl}} = g(\infty, -\infty)$. We shall simply write g for S_{cl}.

The quantum scattering transformation will have to intertwine between the boson field $\phi(\cdot)$ and the boson field corresponding to the scattered solution $\phi(g\,\cdot)$. Thus it coincides – except perhaps for the phase – with the metaplectic representation. Since we have computed ν explicitly in the Segal–Bargmann presentation, we already possess the exact form of the quantum scattering matrix. All we need to do is to translate our results in the usual Fock space language of quantum scattering theory.

Simple considerations allow us to find immediately the form of the out-vacuum. The out vacuum is characterized, up to a phase factor, by the equation $a_g(gv)\,|0_{\mathrm{out}}\rangle = 0$, for all $v \in V$. In view of (5.15), replacing v by $p_g^{-1}v$, this condition is:

$$(a(v) + a^\dagger(T_g v))\,|0_{\mathrm{out}}\rangle = 0, \quad \text{for all} \quad v \in V. \tag{10.2}$$

We already know that this equation has an essentially unique solution $c_g f_{T_g}$ in $\mathcal{B}(V)$. It is, however, instructive to rederive it in a more concrete fashion.

If B is an antilinear symmetric operator on V, the operator $a^\dagger B a^\dagger$ given by (4.19), when applied to the vacuum vector $|0\rangle$, produces the antiholomorphic function $-\frac{1}{2}\langle u \mid Bu\rangle$. Thus

$$\exp(-\tfrac{1}{2}a^\dagger T_g a^\dagger)\,|0\rangle(u) = \exp(\tfrac{1}{4}\langle u \mid T_g u\rangle) = f_{T_g}(u), \tag{10.3}$$

so that $|0_{\mathrm{out}}\rangle \propto \exp(-\frac{1}{2}a^\dagger T_g a^\dagger)\,|0\rangle$ is indeed a solution to the equation (10.2).

Formal computations with the CCR (4.14) indicate that $[a(v), a^\dagger B a^\dagger] = 2a^\dagger(Bv)$ and thus that

$$[a(v), \exp(-\tfrac{1}{2}a^\dagger B a^\dagger)] = -\exp(-\tfrac{1}{2}a^\dagger B a^\dagger)\, a^\dagger(Bv),$$

which serves as a heuristic derivation of the solution from the CCR alone.

The absolute value of the vacuum persistence amplitude is now given by

$$|\langle 0_{\mathrm{in}} \mid 0_{\mathrm{out}}\rangle| = c_g\, f_{T_g}(0) = c_g.$$

There is no reason to suppose that the imaginary part of the vacuum persistence amplitude is zero. Nevertheless, the phase of the quantum scattering matrix may in principle be determined by a reasoning similar to the one used in subsection 6.1. It is thus intimately related to the metaplectic cocycle and anomaly.

To compute the phase factor, we assume that the quantum evolution operator given by

$$U(t,s) := e^{i\theta(t,s)}\,\nu(g(t,s))$$

exists; this is the case for tame enough external potentials. From $U(t,r) = U(t,s)\,U(s,r)$ for $t \geqslant s \geqslant r$, we obtain

$$e^{i\theta(t,r)} = e^{i\theta(t,s)}e^{i\theta(s,r)}\,c(g(t,s),g(s,r)). \tag{10.4}$$

We may as well suppose also that $\frac{\partial}{\partial t}\big|_{t=s}\theta(t,s) = 0$ – which is a kind of "normal ordering" rule, analogous to (6.6). Differentiating (10.4) with respect to t at $t = s$ and solving the resulting equation for $\theta(t,r)$ then yields

$$\theta(t,r) = -i\int_r^t \frac{\partial}{\partial\tau}\bigg|_{\tau=s} c(g(\tau,s),g(s,r))\,ds.$$

As in the proof of Proposition 7.2, we get

$$\begin{aligned}\frac{\partial}{\partial\tau}\bigg|_{\tau=s} c(g(\tau,s),g(s,r)) &= \frac{\partial}{\partial\tau}\bigg|_{\tau=s} \exp(i\arg\det{}^{-1/2}(1-T_{g(s,r)}\widehat{T}_{g(\tau,s)}))\\ &= -\frac{1}{4}\operatorname{Tr}_{\mathbb{C}}\big([\tfrac{\partial}{\partial\tau}\big|_{\tau=s}\widehat{T}_{g(\tau,s)},T_{g(s,r)}]\big).\end{aligned}$$

We thus find, for the phase of the scattering matrix:

$$e^{i\theta} = e^{i\theta(+\infty,-\infty)} = \exp\bigg\{\frac{1}{8}\int_{-\infty}^{\infty}\operatorname{Tr}_{\mathbb{C}}\big([e^{-At}(\widetilde{V}(t)+J\widetilde{V}(t)J)e^{At},T_{g(t,-\infty)}]\big)\,dt\bigg\}.$$

10.2 The scattering matrix in the boson Fock space

We effect now the promised translation, following [50]. Let us recall the form of the kernel of the metaplectic representation (5.5). We may factorize it as follows:

$$\nu(g)\,E_v = c_g\exp(\tfrac14\langle\widehat{T}_g v \mid v\rangle)\,f_{T_g}E_{p_g^{-t}v}.$$

Thus we seek to factorize $\nu(g)$ as

$$\nu(g) = c_g S_1 S_2 S_3, \tag{10.5a}$$

where the S_i, for $i = 1,2,3$, are operators on $\mathcal{B}(V)$ such that

$$\begin{aligned} S_3E_v &= \exp\tfrac14(\langle\widehat{T}_g v \mid v\rangle)E_v,\\ S_2E_v &= E_{p_g^{-t}v},\\ S_1E_w &= f_{T_g}E_w. \end{aligned} \tag{10.5b}$$

First of all let us note that, because the $a^\dagger(v)$ act as multiplication operators, the result of (10.3) extends immediately to give $\exp(-\frac12 a^\dagger T_g a^\dagger)F = f_{T_g}F$ for any F in the domain of this exponential. Since the principal vectors are smooth vectors for $(a^\dagger T_g a^\dagger)$, we may take $F = E_w$ and thereby obtain

$$S_1 = \exp(-\tfrac12 a^\dagger T_g a^\dagger). \tag{10.6}$$

From (4.21) we also see that

$$S_3 = \exp(-\tfrac{1}{2} a\widehat{T}_g a). \tag{10.7}$$

More precisely, (4.21) shows that the right hand side defines an operator whose domain includes all E_v, and hence is dense, and that both sides of (10.7) coincide on all E_v.

To obtain an expression for S_2, we must mix creation and annihilation operators. Now the vanishing of vacuum expectations (6.6), which has been adopted as the quantization rule for the derived metaplectic representation, forces us to take the *normal ordering* in our explicit expressions for $\nu(g)$. We may thus expect S_2 to be a *Wick-ordered exponential*:

$$\begin{aligned} S_2 &= {:}\exp(a^\dagger C a){:} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1\dots k_n \\ l_1\dots l_n}} a^\dagger(f_{k_1}) \cdots a^\dagger(f_{k_n}) \langle f_{k_1} \mid C e_{l_1}\rangle \cdots \langle f_{k_n} \mid C e_{l_n}\rangle a(e_{l_n}) \cdots a(e_{l_1}), \end{aligned} \tag{10.8}$$

for some bounded linear operator C on V. This expression makes sense as a quadratic form whose domain includes every E_v. Indeed:

$$\langle E_w \mid S_2 E_v\rangle = \sum_{p=0}^{\infty} \frac{1}{2^p (p!)^2} \langle 0 \mid a(w)^p S_2 a^\dagger(v)^p \mid 0\rangle,$$

with

$$\begin{aligned} &\langle 0 \mid a(w)^p S_2 a^\dagger(v)^p \mid 0\rangle \\ &\quad = \sum_{n=0}^{p} \frac{1}{n!} \sum_{\substack{k_1\dots k_n \\ l_1\dots l_n}} \Big\langle 0 \,\Big|\, a(w)^p a^\dagger(f_{k_1}) \cdots a^\dagger(f_{k_n}) \prod_{j=1}^{n} \langle f_{k_j} \mid C e_{l_j}\rangle a(e_{l_1}) \cdots a(e_{l_n}) a^\dagger(v)^p \,\Big|\, 0 \Big\rangle \\ &\quad = \sum_{n=0}^{p} \binom{p}{n} \langle w \mid v\rangle^{p-n} \sum_{\substack{k_1\dots k_n \\ l_1\dots l_n}} \prod_{j=1}^{n} \langle w \mid f_{k_j}\rangle \, \langle f_{k_j} \mid C e_{l_j}\rangle \, \langle e_{l_j} \mid v\rangle \\ &\quad = \sum_{n=0}^{p} \binom{p}{n} \langle w \mid v\rangle^{p-n} \langle w \mid C v\rangle^n = \langle w \mid v + C v\rangle^p, \end{aligned}$$

so that $\langle E_w \mid S_2 E_v\rangle$ converges and equals $\langle E_w \mid E_{(1+C)v}\rangle$. Therefore, ${:}\exp(a^\dagger C a){:}\, E_v = E_{(1+C)v}$; comparing with (10.5), we arrive at

$$S_2 := {:}\exp(a^\dagger (p_g^{-t} - 1) a){:}$$

and in particular, we notice also that ${:}\exp(-a^\dagger a){:} = |0\rangle\langle 0|$.

Let us take stock of the explicit form of the scattering matrix:

$$\mathbb{S} = e^{i\theta} \nu(g) = \langle 0_{\rm in} \mid 0_{\rm out}\rangle \, \exp(-\tfrac{1}{2} a^\dagger T_g a^\dagger) \; {:}\exp(a^\dagger (p_g^{-t} - 1) a){:} \; \exp(-\tfrac{1}{2} a \widehat{T}_g a). \tag{10.9}$$

The (p, T) parametrization of the restricted symplectic group is revealed here as the nursery for a useful calculus, lending itself for a very explicit expression of the S-matrix. We saw already that c_g could be interpreted as the absolute value of the vacuum persistence amplitude. Many other parameters of the symplectic group and the metaplectic representation acquire a physical meaning when the latter is reinterpreted as a scattering matrix. For example, the total number of particles created in the scattering process is easily computed:

$$\begin{aligned}\langle 0_{\text{out}} \mid N \mid 0_{\text{out}}\rangle &= \langle 0 \mid \nu(g^{-1})\, a^\dagger a\, \nu(g) \mid 0\rangle = \Big\langle 0 \,\Big|\, \sum_k a^\dagger(g^{-1} f_k) a(g^{-1} f_k) \,\Big|\, 0 \Big\rangle \\ &= \Big\langle 0 \,\Big|\, \sum_k a(q_{g^{-1}} f_k) a^\dagger(q_{g^{-1}} f_k) \,\Big|\, 0 \Big\rangle = \|q_{g^{-1}}\|^2_{HS} = \|q_g\|^2_{HS}.\end{aligned}$$

Thus Shale's theorem may be paraphrased as saying that the classical symplectic transformation is unitarily implementable if and only if the average number of particles produced is finite. This will certainly happen if the total energy of the external field – after integration over the whole of spacetime – is finite. Note however, that this condition is not necessary. There could be an infinite expectation value of the quantum Hamiltonian $dG(A + \widetilde{V})$ in the final state. Already this points to the somewhat conventional character, from the physical point of view, of Shale's restriction. This is one reason why, as Fulling [49] indicates, there may be situations in which the particle *density* and other local observables remain finite, in the presence of a non-Fock final state. Our formulas keep a heuristic value in such "infrared-divergent" contexts, that should not be regarded *a priori* as physically pathological. Moreover, even prior to the introduction of the generalized metaplectic representation, it was clear, for purely algebraic reasons, that the dynamics of operators that can be expressed as finite sums of creation and annihilation operators is unconditionally computable in the present formalism.

The reducibility of the metaplectic representation shows that coupling with quadratic Hamiltonians will always result in creation of particles only in pairs, even for a neutral field; whereas we saw in the previous section that coupling to a source gives rise to states containing contributions from both odd and even particle-number states.

► We close this section by considering the effect of the factorized S-matrix on the Gaussians f_T. This is done in order to better compare the bosonic S-matrix with the fermionic equivalent, wherein a good analogue of the E_v is not available.

Lemma 10.1. *If $R \in \mathcal{D}'(V)$, then*

$$\begin{aligned} S_1 f_R &= f_{T_g + R}\,, \\ S_2 f_R &= f_{p_g^{-t} R p_g^{-1}}\,, \\ S_3 f_R &= \det{}^{-1/2}(1 - R\widehat{T}_g)\, f_{R(1-\widehat{T}_g R)^{-1}}\,, \end{aligned} \tag{10.10}$$

whenever f_R lies in the domain of S_1, S_2 or S_3, respectively.

Proof. Since $f_{T_g+R}(u) = \exp \frac14 \langle u \mid (T_g + R)u\rangle = f_{T_g}(u) f_R(u)$, the relation $S_1 f_R = f_{T_g+R}$ is just a special case of $S_1 F = f_{T_g} F$ for $F \in \operatorname{Dom} S_1$; and it is evident that $f_R \in \operatorname{Dom} S_1$ whenever $T_g + R \in \mathcal{D}'(V)$.

If $v \in V$, then $S_2^\dagger E_v = {:}\exp(a^\dagger(p_g^{-1} - 1)a){:}\, E_v = E_{p_g^{-1} v}$. From this we obtain

$$\begin{aligned} S_2 f_R(v) &= \langle E_v \mid S_2 f_R\rangle = \langle S_2^\dagger E_v \mid f_R\rangle = \langle E_{p_g^{-1} v} \mid f_R\rangle \\ &= f_R(p_g^{-1} v) = \exp \tfrac14 \langle p_g^{-1} v \mid R p_g^{-1} v\rangle = f_{p_g^{-t} R p_g^{-1}}(v). \end{aligned}$$

We thereby see that $f_R \in \operatorname{Dom} S_2$ whenever $p_g^{-t} R p_g^{-1} \in \mathcal{D}'(V)$.

Since the Gaussians f_S generate a dense subspace of $\mathcal{B}_0(V)$, which is preserved by S_3, we conclude from

$$\begin{aligned}
\langle f_S \mid S_3 f_R\rangle &= \langle S_3^\dagger f_S \mid f_R\rangle = \langle \exp(-\tfrac{1}{2} a^\dagger \widehat{T}_g a^\dagger) f_S \mid f_R\rangle \\
&= \langle f_{S+\widehat{T}_g} \mid f_R\rangle = \det{}^{-1/2}(1 - R(S + \widehat{T}_g)) \\
&= \det{}^{-1/2}(1 - R\widehat{T}_g) \det{}^{-1/2}(1 - R(1 - \widehat{T}_g R)^{-1} S) \\
&= \det{}^{-1/2}(1 - R\widehat{T}_g) \langle f_S \mid f_{R(1-\widehat{T}_g R)^{-1}}\rangle
\end{aligned}$$

that $f_R \in \operatorname{Dom} S_3$ whenever $R(1 - \widehat{T}_g R)^{-1} \in \mathcal{D}'(V)$, and then $S_3 f_R = \det^{-1/2}(1 - R\widehat{T}_g) f_{R(1-\widehat{T}_g R)^{-1}}$ follows. □

We remark that, on applying S_1, S_2, S_3 in turn to f_R, the index of the Gaussian undergoes the transformation

$$R \mapsto T_g + p_g^{-t} R(1 - \widehat{T}_g R)^{-1} p_g^{-1} = g \cdot R$$

by (2.19), and so $(c_g S_1 S_2 S_3) f_R = c_g \det^{-1/2}(1 - R\widehat{T}_g) f_{g\cdot R} = \nu(g) f_R$ by (5.9). This provides a second proof of the factorization (10.5) of the S-matrix.

11 The scattering matrix for a charged boson field

11.1 The charge operator

So far, our arguments have dealt principally with neutral fields; no charge operator has been manifested. We now take up the case of a charged field, to obtain a system where particles and antiparticles differ. Classically, the starting point is simply a pair of real Klein–Gordon equations, but it is technically convenient to work with a complex equation, although multiplication by i there is by no means *the* adequate complex structure.

We write the equation in the form

$$i \frac{d}{dt} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

where f, g are now complex-valued functions, $g := i \frac{\partial f}{\partial t}$. The symplectic form on the space of Cauchy data of this equation is now

$$s\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) = \Im \int \big(f_1^*(\boldsymbol{x})\, g_2(\boldsymbol{x}) + g_1^*(\boldsymbol{x})\, f_2(\boldsymbol{x})\big)\, d^3\boldsymbol{x}, \qquad (11.1)$$

which is invariant under the equations of motion. One adopts as the Hilbert space of solutions the complexification $V_\mathbb{C}$ of the space V of solutions of the real Klein–Gordon equation, and the complex structure J on $V_\mathbb{C}$ is just the complex amplification of the operator (9.5). It follows that $J = i(P_+ - P_-)$, where

$$P_\pm = \frac{1}{2} \begin{pmatrix} 1 & \pm\omega^{-1} \\ \pm\omega & 1 \end{pmatrix}.$$

On $V_{\mathbb{C}}$, one can then write

$$d_J\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) = \Re \int \big(f_1^*(\boldsymbol{x})\, \omega\, f_2(\boldsymbol{x}) + g_1^*(\boldsymbol{x})\, \omega^{-1}\, g_2(\boldsymbol{x})\big)\, d^3\boldsymbol{x}.$$

Now the (indefinite) sesquilinear form

$$q\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) := \Re \int (f_1^*(\boldsymbol{x})\, g_2(\boldsymbol{x}) + g_1^*(\boldsymbol{x})\, f_2(\boldsymbol{x}))\, d^3\boldsymbol{x} \tag{11.2}$$

is also conserved by the equations of motion; the associated operator Q, determined by:

$$s(u, Qv) := q(u, v),$$

is the *charge* operator. Its presence is directly related to the invariance of the complex Klein–Gordon equation under transformations $f \mapsto e^{i\alpha} f$; this is the symmetry that becomes gauged. On comparing (11.1) and (11.2), it is immediate that Q acts on $V_{\mathbb{C}}$ as multiplication by i, so Q is J-linear and moreover $s(Qu, v) = -s(u, Qv)$, that is, $Q \in \mathfrak{sp}(V_{\mathbb{C}})$, with $V_{\mathbb{C}}$ regarded as a real symplectic space under (11.1).

► As in the real case, it is possible to pass to a momentum-space representation, given essentially by (9.6) but with $\begin{pmatrix} c \\ c^* \end{pmatrix}$ replaced by $\begin{pmatrix} b \\ d^* \end{pmatrix}$, with $b, d \in \mathcal{H}_m^{0,+}$ not necessarily equal. This transformation is inverted by:

$$f(\boldsymbol{x}) = (2\pi)^{-3/2} \int \big(b(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}} + d^*(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}\big)\, d\mu(k),$$
$$g(\boldsymbol{x}) = i(2\pi)^{-3/2} \int \omega(\boldsymbol{k})\big(b(\boldsymbol{k})e^{-i\boldsymbol{k}\boldsymbol{x}} - d^*(\boldsymbol{k})e^{i\boldsymbol{k}\boldsymbol{x}}\big)\, d\mu(k).$$

On this space J goes over to multiplication by i and Q goes over to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Evolution is trivial. For the charge form, we obtain:

$$q\left(\begin{pmatrix} b_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} b_2 \\ d_2 \end{pmatrix}\right) := \Re \int \big(b_1^*(\boldsymbol{k})\, b_2(\boldsymbol{k}) - d_1(\boldsymbol{k})\, d_2^*(\boldsymbol{k})\big)\, d\mu(k).$$

11.2 The scattering matrix for a charged field

In view of the above, we shall regard the complex space $V_{\mathbb{C}} = V \oplus iV$, for a general (V, s, J), as the classical phase space for a field with particles and antiparticles. In this subsection, the operators g, T_g, etc. will thus be complex-linear operators on $V_{\mathbb{C}}$. Our strategy is very simple: it is to adapt the general formulas of the previous section. As before, we denote by P_+ and P_- the projectors on the respective subspaces W_0 and W_0^*, which are the spaces of one-particle solutions of positive or negative energy. Let us write, with respect to the decomposition $V_{\mathbb{C}} = W_0 \oplus W_0^* = W_0 \oplus W_0^\perp$,

$$g = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}. \tag{11.3}$$

It is immediate that

$$p_g = \begin{pmatrix} S_{++} & 0 \\ 0 & S_{--} \end{pmatrix} \quad \text{and} \quad q_g = \begin{pmatrix} 0 & S_{+-} \\ S_{-+} & 0 \end{pmatrix}, \tag{11.4}$$

from which

$$T_g = \begin{pmatrix} 0 & S_{+-}S_{--}^{-1} \\ S_{-+}S_{++}^{-1} & 0 \end{pmatrix} \quad \text{and} \quad \widehat{T}_g = \begin{pmatrix} 0 & -S_{++}^{-1}S_{+-} \\ -S_{--}^{-1}S_{-+} & 0 \end{pmatrix}. \tag{11.5}$$

The conditions for $g \in \mathrm{Sp}'(V_\mathbb{C})$ are that $g(P_+ - P_-)g^\dagger = (P_+ - P_-)$ and that $S_{+-} \in \mathrm{HS}$.

Since $T_g^\dagger = T_g$ on $V_\mathbb{C}$, we obtain

$$\begin{aligned} |\langle 0_{\text{in}} \mid 0_{\text{out}}\rangle| &= \det{}^{1/4}(1 - T_g^2) = \det{}^{1/2}(1 - (S_{+-}S_{--}^{-1})^\dagger S_{+-}S_{--}^{-1}) \\ &= \det{}^{1/2}\big((S_{--}^\dagger)^{-1}(S_{--}^\dagger S_{--} - S_{+-}^\dagger S_{+-})S_{--}^{-1}\big) \\ &= \det{}^{1/2}\big((S_{--}^\dagger)^{-1}S_{--}^{-1}\big) = \det{}^{-1/2}(S_{--}S_{--}^\dagger), \end{aligned}$$

since $S_{--}^\dagger S_{--} - S_{+-}^\dagger S_{+-} = 1$ by combining (2.7) and (11.4). Recall [7] that $\det(1-AB) = \det(1-BA)$ whenever both determinants exist. On the other hand,

$$|\langle 0_{\text{in}} \mid 0_{\text{out}}\rangle| = \det{}^{-1/4}(p_g p_g^\dagger) = \det{}^{-1/4}(S_{++}S_{++}^\dagger)\ \det{}^{-1/4}(S_{--}S_{--}^\dagger),$$

and hence both factors on the right-hand side are equal.

We thus arrive at the simplified form

$$\begin{aligned} |\langle 0_{\text{in}} \mid 0_{\text{out}}\rangle| &= \det{}^{-1/2}(S_{--}S_{--}^\dagger) = \det{}^{-1/2}(S_{++}S_{++}^\dagger) \\ &= \det{}^{-1/2}(1 + S_{+-}S_{+-}^\dagger) = \det{}^{-1/2}(1 + S_{-+}S_{-+}^\dagger), \end{aligned} \tag{11.6}$$

on again using (2.7) to obtain the last two expressions.

Let $\{\phi_1, \phi_2, \dots\}$ and $\{\psi_1, \psi_2, \dots\}$ be orthonormal bases for W_0 and W_0^*, with respect to the scalar product (2.11) on $V_\mathbb{C}$. In view of (2.12), two orthonormal bases $\{f_k\}$, $\{e_k\}$ of V are determined by $P_+(f_k) = \varphi_k$, $P_-(e_k) = \psi_k$. We can now distinguish the positive and negative energy sectors by setting $b^\dagger(\varphi_k) := a^\dagger(f_k)$, $d^\dagger(\psi_k) := a^\dagger(e_k)$ and similarly for the annihilation operators. Since $T_g^\dagger = T_g$, bearing in mind the relations (2.12), we find:

$$\begin{aligned} -\tfrac{1}{2}a^\dagger T_g a^\dagger &= -\frac{1}{2}\sum_{j,k} a^\dagger(f_k)\,\langle f_k \mid T_g e_j\rangle\, a^\dagger(e_j) + a^\dagger(e_j)\,\langle e_j \mid T_g f_k\rangle\, a^\dagger(f_k) \\ &= -\frac{1}{2}\sum_{j,k} b^\dagger(\varphi_k)\,\langle\!\langle \varphi_k \mid T_g\psi_j\rangle\!\rangle\, d^\dagger(\psi_j) + d^\dagger(\psi_j)\,\langle\!\langle T_g\varphi_k \mid \psi_j\rangle\!\rangle\, b^\dagger(\varphi_k) \\ &= -\frac{1}{2}\sum_{j,k} b^\dagger(\varphi_k)\,\langle\!\langle \varphi_k \mid S_{+-}S_{--}^{-1}\psi_j\rangle\!\rangle\, d^\dagger(\psi_j) + d^\dagger(\psi_j)\,\langle\!\langle S_{-+}S_{++}^{-1}\varphi_k \mid \psi_j\rangle\!\rangle\, b^\dagger(\varphi_k) \\ &= -\sum_{j,k} b^\dagger(\varphi_k)\,\langle\!\langle \varphi_k \mid S_{+-}S_{--}^{-1}\psi_j\rangle\!\rangle\, d^\dagger(\psi_j) =: -b^\dagger S_{+-}S_{--}^{-1}d^\dagger, \\ -\tfrac{1}{2}a\widehat{T}_g a &= -\frac{1}{2}\sum_{j,k} a(f_k)\,\langle \widehat{T}_g f_k \mid e_j\rangle\, a(e_j) + a(e_j)\,\langle \widehat{T}_g e_j \mid f_k\rangle\, a(f_k) \\ &= -\frac{1}{2}\sum_{j,k} b(\varphi_k)\,\langle\!\langle \psi_j \mid \widehat{T}_g\varphi_k\rangle\!\rangle\, d(\psi_j) + d(\psi_j)\,\langle\!\langle \widehat{T}_g\psi_j \mid \varphi_k\rangle\!\rangle\, b(\varphi_k) \\ &= \frac{1}{2}\sum_{j,k} b(\varphi_k)\,\langle\!\langle \psi_j \mid S_{--}^{-1}S_{-+}\varphi_k\rangle\!\rangle\, d(\psi_j) + d(\psi_j)\,\langle\!\langle S_{++}^{-1}S_{+-}\psi_j \mid \varphi_k\rangle\!\rangle\, b(\varphi_k) \\ &= \sum_{j,k} d(\psi_j)\,\langle\!\langle \psi_j \mid S_{--}^{-1}S_{-+}\varphi_k\rangle\!\rangle\, b(\varphi_k) =: dS_{--}^{-1}S_{-+}b. \end{aligned} \tag{11.7}$$

The Wick-ordered product $:\exp(a^\dagger(p_g^{-t}-1)a):$ contains terms of type $b^\dagger(\varphi_k)\,b(\varphi_l)$ and $d(\psi_r)\,d^\dagger(\psi_s)$, but no $b^\dagger d$ or $d^\dagger b$ terms, from the block diagonal form of $(p_g^{-t}-1)$. Moreover, the $b^\dagger(\varphi_k)\,b(\varphi_l)$ and $d(\psi_r)\,d^\dagger(\psi_s)$ commute, so $:\exp(a^\dagger(p_g^{-t}-1)a): = S_{2b}\,S_{2d}$. On account of $\langle f_k \mid (p_g^{-t}-1)f_l\rangle = \langle\!\langle \varphi_k \mid ((S_{++}^\dagger)^{-1}-1)\varphi_l\rangle\!\rangle$, the series expansion for S_{2b} is

$$\begin{aligned} S_{2b} &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{k_1\dots k_n\\ l_1\dots l_n}} b^\dagger(\varphi_{k_1})\cdots b^\dagger(\varphi_{k_n})\prod_{j=1}^{n}\langle\!\langle \varphi_{k_j} \mid ((S_{++}^\dagger)^{-1}-1)\varphi_{l_j}\rangle\!\rangle\, b(\varphi_{l_n})\cdots b(\varphi_{l_1}) \\ &= :\exp(b^\dagger((S_{++}^\dagger)^{-1}-1)b): \end{aligned}$$

and similarly we obtain

$$\begin{aligned} S_{2d} &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{k_1\dots k_n\\ l_1\dots l_n}} d^\dagger(\psi_{k_1})\cdots d^\dagger(\psi_{k_n})\prod_{j=1}^{n}\langle\!\langle ((S_{--}^\dagger)^{-1}-1)\psi_{l_j} \mid \psi_{k_j}\rangle\!\rangle\, d(\psi_{l_n})\cdots d(\psi_{l_1}) \\ &= :\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{k_1\dots k_n\\ l_1\dots l_n}} d(\psi_{l_1})\cdots d(\psi_{l_n})\prod_{j=1}^{n}\langle\!\langle \psi_{l_j} \mid (S_{--}^{-1}-1)\psi_{k_j}\rangle\!\rangle\, d^\dagger(\psi_{k_n})\cdots d^\dagger(\psi_{k_1}): \\ &= :\exp(d(S_{--}^{-1}-1)d^\dagger): . \end{aligned}$$

In summary, the S-matrix for the charged boson field has the explicit form:

$$\begin{aligned} \mathbb{S} = e^{i\theta}\nu(g) = \langle 0_{\text{in}} \mid 0_{\text{out}}\rangle\ &\exp(-b^\dagger S_{+-}S_{--}^{-1}d^\dagger) \\ &\times :\exp(b^\dagger((S_{++}^\dagger)^{-1}-1)b + d(S_{--}^{-1}-1)d^\dagger):\ \exp(dS_{--}^{-1}S_{-+}b). \end{aligned} \tag{11.8}$$

In fine, the full scattering matrix for charged boson fields may be explicitly derived from the general theory of the infinite-dimensional metaplectic representation.

► The quantized charge operator is $\mathbb{Q} := dG(Q) = -i\,\dot\nu(Q) = -i\,a^\dagger Qa$ from (6.9) and J-linearity. Now $\langle f_k \mid Qf_k\rangle = \langle\!\langle \varphi_k \mid i\varphi_k\rangle\!\rangle = i$ whereas $\langle e_j \mid Qe_j\rangle = \langle\!\langle i\psi_j \mid \psi_j\rangle\!\rangle = -i$, which leads to

$$\mathbb{Q} = :b^\dagger b - dd^\dagger: = b^\dagger b - d^\dagger d.$$

Conservation of charge at the quantum level now follows from the anomaly formula (7.8). Since Q is linear, we get $A_Q = 0$, $C_Q = Q$. The classical charge conservation $\mathrm{Ad}(g)Q = Q$ also yields $[Q, \widehat{T}_g] = 0$, and so $\gamma(g,Q) = 0$, leading at once to $\nu(g)\,\mathbb{Q}\,\nu(g)^{-1} = \mathbb{Q}$; taking $g = S_{\text{cl}}$ shows that the scattering transformation leaves $\mathbb{Q}$ invariant, without further calculation.

A The choice of complex structures

In subsection 4.1 we remarked that there exists essentially one full quantization for each complex structure defined on a symplectic vector space. In this Appendix we address the matter of how a preferred J may be chosen in the first place. It is here that physics intervenes. In general, we start from a given classical linear dynamical system:

$$\frac{dv}{dt} = Av.$$

Assume that we have solved it:

$$v(t) = e^{At}\, v_0,$$

using, say, semigroup theory. We would like the evolution operator e^{At} to be *unitary* in the one-particle space (V, s, J); in other words, we inquire whether it is possible to choose some J commuting with A. This cannot always be done; but when it is possible, the procedure given below singles out the needed complex structure in a completely satisfactory way.

Most material in this appendix can be found in [51]. We have streamlined it to suit our needs. Of course, the matter is bound up with the general question of hilbertizability touched on in subsection 2.1. We first prove the assertion made there.

Proposition A.1. *Let (V, d_0) be a real Hilbert space and let s be a symplectic form on V which is continuous with respect to d_0. Then (V, s) is hilbertizable.*

Proof. Continuity of s means that $s(u, v) = d_0(Bu, v)$ for some bounded operator B on (V, d_0). Since s is nondegenerate, B is injective, and since s is an antisymmetric form, $B^t = -B$ is also injective (the transpose here being taken with respect to d_0). Thus the range of B is dense in (V, d_0). Hence the polar part J of the polar decomposition $B =: J(-B^2)^{1/2}$ is a d_0-isometry. The point to note is that B is normal, so the three operators B, J and $(-B^2)^{1/2}$ commute. It follows that $J^2 = -1$, $s(Ju, Jv) = d_0(JBu, Jv) = s(u, v)$ and $s(v, Jv) = d_0(v, (-B^2)^{1/2}v) > 0$ for $v \neq 0$, so that J is a compatible complex structure.

Define $d(u, v) := s(u, Jv) = d_0(Bu, Jv) = d_0((-B^2)^{1/2}u, v)$; this is a positive definite symmetric bilinear form on V. With the scalar product $d + is$ as in (2.4), V becomes a prehilbert space. It is not complete in general, because the inverse of B can be unbounded; as is indeed the case unless s is strongly symplectic. □

We return to the main issue. Classically, one is given a linear Hamiltonian system (V_0, s_0, A_0). Some extra topological structure is needed in practice; to fix ideas, we shall assume that V_0 is a Banach space under some suitable norm $\|\cdot\|$. Since A_0 is unbounded in all interesting cases, a little care is necessary. We shall assume that A_0 is a densely defined operator on V_0, skewadjoint with respect to s_0, i.e., $A_0^\ddagger = -A_0$, where $A_0^\ddagger$ denotes the s_0-adjoint of A_0, with domain

$$\operatorname{Dom} A_0^\ddagger := \{\, v \in V_0 : s_0(v, A_0 u) = s_0(w, u) \text{ whenever } u \in \operatorname{Dom} A_0, \text{ for some } w \in V_0 \,\},$$

setting $A_0^\ddagger v := w$.

Remark. We can show that $A_0^\ddagger = -A_0$ if A_0 is the generator of a strongly continuous group $U(t)$ of linear canonical transformations in the Banach space V_0; skewsymmetry follows from

$$\frac{d}{dt}\, s_0(U(t)v_1, U(t)v_2) = s_0(A_0 U(t)v_1, U(t)v_2) + s_0(U(t)v_1, A_0 U(t)v_2), \tag{A.1}$$

so that $A_0 \subseteq -A_0^\ddagger$. We remark that $U^\ddagger(t) = U(-t)$. Now if $v \in \operatorname{Dom} A_0^\ddagger$ with $A_0^\ddagger v = w$, then for any $u \in \operatorname{Dom} A_0$, $U(t)u = u + \int_0^t A_0 U(t)u\, dt$, so that

$$s_0(u, U(-t)v) = s_0(U(t)u, v) = s_0(u, v) + \int_0^t s_0(u, U(-\tau)w)\, d\tau,$$

the interchange of s_0 and the integral being permissible by continuity of s_0 and well-known properties of the Bochner integral. Now, $\operatorname{Dom} A_0$ is dense and s_0 is nondegenerate, yielding the relation $U(-t)v = v + \int_0^t U(-\tau)w\, d\tau$. By differentiation, $v \in \operatorname{Dom} A_0$ and $-A_0 v = w = A_0^\ddagger v$.

In order to proceed we require a symmetric form d_0, and there is no other raw material to fabricate it than s_0 and A_0 themselves! Suppose that the classical energy function $v \mapsto s_0(v, A_0 v)$ obeys the following positivity condition:

$$s_0(v, A_0 v) \geqslant \varepsilon \|v\|^2 \quad \text{when} \quad v \in V_0, \quad \text{for some} \quad \varepsilon > 0. \tag{A.2}$$

Then we can *define*

$$d_0(u, v) := s_0(u, A_0 v), \quad \text{for} \quad u, v \in \operatorname{Dom} A_0; \tag{A.3}$$

and (A.2) shows that d_0 is a positive definite (real) scalar product on V_0.

Lemma A.2. *Let V_0 be a Banach space with norm $\|\cdot\|$, let s_0 be a weakly symplectic form on V_0, and let A_0 be a densely defined linear operator on V_0 satisfying* (A.2) *for some $\varepsilon > 0$. Suppose moreover that A_0 is skewadjoint with respect to s_0. Then there is a hilbertizable symplectic space (V_1, s) and a densely defined linear operator A on V_1 such that*

$$\operatorname{Dom} A \subseteq \operatorname{Dom} A_0 \subseteq V_1 \subseteq V_0$$

with dense inclusions, s being the restriction of s_0 to V_1 and A a restriction of A_0 that is skewadjoint with respect to s.

Proof. Denote by V_1 the completion of $\operatorname{Dom} A_0$ with respect to energy norm, i.e., that arising from the scalar product d_0 of (A.3). The inclusion $\operatorname{Dom} A_0 \hookrightarrow V_0$ extends to a continuous map $m \colon V_1 \to V_0$. This map is one-to-one, since if $h \in H$ with $m(h) = 0$, then $h = \lim_{n\to\infty} v_n$ for some Cauchy sequence (in the d_0-norm) $\{v_n\} \subseteq \operatorname{Dom} A_0$; the continuity of m gives $v_n \to 0$ in V_0 and $d_0(h, u) = \lim_{n\to\infty} d_0(v_n, u) = \lim_{n\to\infty} s_0(v_n, A_0 u) = 0$ for $u \in \operatorname{Dom} A_0$, so that $h = 0$. Hence we can identify V_1 with the subspace $m(V_1)$ of V_0.

Now let A denote the restriction of A_0 to $\operatorname{Dom} A := \{\, v \in \operatorname{Dom} A_0 : A_0 v \in V_1 \,\}$, and let s be the restriction of s_0 to V_1. Then

$$s(u, Av) = d_0(u, v) = d_0(v, u) = s(v, Au) = -s(Au, v), \quad \text{for} \quad u, v \in \operatorname{Dom} A, \tag{A.4}$$

so A is skewsymmetric with respect to s. In fact, A is also skewsymmetric with respect to d_0, since

$$d_0(Au, v) = s(Au, Av) = -s(Av, Au) = -d_0(Av, u) = -d_0(u, Av), \quad \text{for} \quad u, v \in \operatorname{Dom} A.$$

If we now consider A as an operator on the real Hilbert space V_1, it generates a strongly continuous group of *isometries*. This follows from Stone's theorem once we verify that A is skewadjoint for d_0. To see this, notice that for any $v \in V_0$, the linear functional $u \mapsto s_0(v, u)$ is continuous on V_0 and *a fortiori* on V_1, so by the Riesz theorem there is a vector $z \in V_1$ such that $s_0(v, h) = d_0(z, h)$ for all $h \in V_1$. But then $s_0(v, h) = s_0(z, A_0 h)$ for $h \in \operatorname{Dom} A_0$, which shows that $z \in \operatorname{Dom} A_0^\ddagger = \operatorname{Dom} A_0$ and $A_0 z = -A_0^\ddagger z = -v$. In other words, A_0 is surjective, hence A is surjective with a bounded inverse, and therefore is skewadjoint with respect to d_0.

Let $U(t) := e^{At}$ denote this strongly continuous group of isometries. Now (A.4) shows that $\frac{d}{dt}\, s(U(t)u, U(t)v) = 0$, so that $U(t)$ is also a group of canonical transformations of (V_1, s). We can thus conclude that A is skewadjoint with respect to s. ∎

Theorem A.3. *Under the hypotheses of Lemma A.2, there is a unique complex structure J on V_1 which is compatible with s and positive, and which commutes with A. If V denotes the completion of V_1 with respect to $d(u, v) := s(u, Jv)$, then V is a complex Hilbert space under the scalar product $d + is$, on which $-JA$ is a positive selfadjoint operator with bounded inverse.*

Proof. This J is none other than the orthogonal part of the polar decomposition of the skewadjoint operator A constructed in Lemma A.2, i.e., $A = J(-A^2)^{1/2}$; in other words, J is the closure of the operator $A(-A^2)^{-1/2}$ on V_1. As in the proof of Proposition A.1, J is a complex structure, compatible with the symplectic form s and positive.

The final step in the definition of a canonical setting for the dynamical system (V_0, s_0, A_0) is to drop the energy norm d_0 and extend V_1 to its completion V with respect to $d(u, v) := s(u, Jv)$. Since J commutes with A, the group $U(t)$ extends by continuity to a group of *unitary* operators on V whose generator is A (regarded as an operator on V) which is (complex) skewadjoint, with bounded inverse. Also, $-JA$ is a positive selfadjoint operator on V without zero eigenvalue, which may be used to verify the existence of a full quantization of (V, s, J) in the sense of Definition 4.1.

This is the *only* complex structure commuting with A, since any other would commute also with J and hence would coincide with J on account of (2.17). □

There is an equivalent procedure to obtain J, which amounts to showing that $-iA$ is selfadjoint on $V_{\mathbb{C}}$; then the spectral projections $P_\pm$ on the positive and negative parts of its spectrum give rise to polarizations and J may be defined as $i(P_+ - P_-)$ restricted to V; then $-iA$ is positive on (V, J). Both procedures are clearly illustrated in the Virasoro example in Section 8.

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