Elimination of quantifiers of a theory of real closed rings.

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Abstract

Let T^* be the theory of lattice-ordered subrings, without minimal (non zero) idempontents, convex in von Neumann regular real closed rings that are divisible-proyectable and sc-regular (cf. [12]). In this paper, a local divisibility binary relation is introduced in order to prove the elimination of quantifiers of the theory T^* in the language of lattice-ordered rings adding the divisibility relation, the radical relation associated to the minimal prime spectrum (cf. [20]) and this new local divisibility relation.

1 Introduction.

Real closed rings, in one of its general presentation, were introduced by Niels Schwartz in [23]. Some aspects of the model theory of real closed rings has been studied. Leaving aside the case of real closed fields, the first model theoretic results concerning real closed rings was the model completeness of the von Neumann regular real closed rings without (non zero) minimal idempotents in the language of lattice-ordered rings, proved by Macintyre in [17]. The elimination of quantifiers of this theory has evolved in many different ways. It was firstly proved by Weispfenning in [28] in the language of lattice-ordered rings where an unary function symbol $*$ was added to the language. This $*$ function represents the cuasi-inverse in von Neumann regular rings. This same result was proved later using simpler techniques by Boffa-Cherlin in [5]. In [20], this elimination of quantifiers result was improved by Prestel-Schwartz where this unary function symbol $*$ was replaced by a binary radical relation; precisely the radical relation associated to the minimal prime spectrum. Examples of integral real closed rings that are not fields are real closed valuation rings; this theory was introduced and well studied from the model theoretic point of view by Cherlin and Dickmann in [8]; it was showed there the elimination of quantifiers in the language of ordered rings with an extra binary relation symbol for the divisibility.

It is a well know fact that von Neumann regular rings are Boolean products of fields. Boolean products of real closed valuation rings has been caracterized in [12] as real closed projectable rings satisfying the first convexity property (i.e.: $\forall a \forall b(0 < a < b \to b \mid a))$ that are sc-regular and divisible-projectable. Following the type of elimination of quantifiers for von Neumann regular real closed rings proved by [28] and [5], the author gave in [12] an elimination of quantifiers for this theory (of Boolean products of real closed valuation rings, without non cero minimal idempotents) in the languaje of lattice-ordered rings with an extra binary function symbol $\operatorname{div}(a,b)$ representing (locally) the quotient of b by a if it exists and 0 if not.

In this paper, the elimination of quantifiers for this last mentioned theory is improved as this function symbol $\operatorname{div}(\cdot,\cdot)$ can be replace by the radical relation associated to the minimal prime spectrum (as Prestel and Schwartz did in [20]) and by adjoining to the languaje a binary relation that represents local divisibility (the "global" divisibility is also in the languaje). In the following section, all basic notions and facts needed will be presented. Also, the local divisibility relation will be introduced in this section, and its basics properties. In the third section, the model completeness of the considered theory will be stated in the languaje of lattice-ordered rings considering the radical relation and the local divisibility. In the fourth section, universal theories (considering different languajes) will be giving, and therefore a model companion result will be stated. The local divisibility as introduced at the beginning of this paper it is not sufficient to prove the elimination of quantifiers. The divisible-projectability turns out to be crucial (ones again) to redefine the local divisibility as a *maximal* local divisibility relation, this is the aim of the fifth section. This new maximal local divisibility relation gives an optimal control of the idempotent where the local divisibility is carried out, and this permits us to prove the elimination of quantifiers through the amalgamation property of the universal theory (in the precise languaje); this is carried out in the last section.

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2 First notions and a local divisibility relation.

The principal interest in this section is to introduce the reader the basic facts and notions about the theory T^* of real closed rings considered in this paper.

$\mathcal{L}_{\text{or}} = \{0, 1, +, \cdot, <\}$ will be the languaje of ordered rings and $\mathcal{L}_{\text{lor}} = \{0, 1, +, \cdot, \wedge\}$ will be the languaje of lattice-ordered rings. From now on, all rings will be conmutative with unity.

An f**-ring** is a subdirect product of totally ordered rings. This notion can be expressed by a first-order formula in $\mathcal{L}_{\text{lor}}$ (see [4, 9.1.2]). For an f-ring A, the **absolute value** of $a \in A$ is $|a| = a \vee -a$; two elements $a, b \in A$ are **orthogonal** if $|a| \wedge |b| = 0$ (we denote this by $a \perp b$); the **polar** of $a \in A$ is $a^\perp = \{b \in A : a \perp b\}$ and the **bipolar** of a is $a^{\perp\perp} = \{b \in A : b \perp c \text{ for all } c \in a^\perp\}$. An f-ring A is **projectable** if $A = a^\perp + a^{\perp\perp}$, for all $a \in A$. Note that this notion is expressed by a first order formula in $\mathcal{L}_{\text{lor}}$. A ring is **reduced** if it doesn't have nilpotent elements other than cero. By [4, 9.3.1], if A is a reduced f-ring, then $\forall x \forall y (x \perp y \leftrightarrow xy = 0)$ is valid formula in A, and therefore:

$$b \in a^{\perp\perp} \iff a^{\perp} \subseteq b^{\perp} \iff \mathrm{Ann}(a) \subseteq \mathrm{Ann}(b),$$

for any $a, b \in A$.

Let L be a first-order language, $\{\mathfrak{A}_x : x \in X\}$ a family of L-structures and $\mathfrak{A}$ a L-structure. We say that $\mathfrak{A}$ is a **Boolean product** of $\{\mathfrak{A}_x : x \in X\}$ in L, denoted by $\mathfrak{A} \in \Gamma^{a}_{L}\big(X, (\mathfrak{A}_x)_{x\in X}\big)$, cf. [6], if the following conditions holds:

(i) X is a Boolean space.

(ii) $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}_x : x \in X\}$.

(iii) For every atomic L-formula $\Phi(v_1, \ldots, v_n)$ and every $a_1, \ldots, a_n \in |\mathfrak{A}|$,

$$[\![\, \Phi(a_1, \ldots, a_n) \,]\!] =_{\mathrm{def}} \big\{ x \in X : \mathfrak{A}_x \models \Phi\big(a_1(x), \ldots, a_n(x)\big) \big\}$$

is a clopen subset of X.

(iv) Patchwork property: For every $a, b \in \mathfrak{A}$ and any clopen set N of X, the element $c = a_{\restriction N} \cup b_{\restriction X \setminus N}$ defined by

$$c(y) = \begin{cases} a(y) & if \ y \in N \\ b(y) & if \ y \in X \setminus N, \end{cases}$$

belongs to $|\mathfrak{A}|$. We say that $\mathfrak{A}$ is an **elementary Boolean product** of $\{\mathfrak{A}_x : x \in X\}$ in L, denoted by $\mathfrak{A} \in \Gamma^{e}_{L}\big(X, (\mathfrak{A}_x)_{x\in X}\big)$, if $\mathfrak{A}$ is a Boolean product of $\{\mathfrak{A}_x : x \in X\}$ in L and condition (iii) is verified for **all** L-formulas $\Phi(v_1, \ldots, v_n)$. Those notation comes from [6].

If A is a (unitary) reduced and projectable f-ring, then [16, 6.12] says that:

$$A \in \Gamma^{a}_{\mathcal{L}_{\mathrm{or}}}\big(\pi A, (A/p)_{p\in\pi A}\big),$$

where $\pi A = \{p \in \mathrm{Spec}(A) : p \text{ is a minimal prime ideal}\} = \mathrm{Specmin}(A)$. In that case:

$$\begin{aligned} b \in a^{\perp\perp} &\iff [\![\, b \neq 0 \,]\!] \subseteq [\![\, a \neq 0 \,]\!] \\ &\iff \mathrm{supp}(b) \subseteq \mathrm{supp}(a) \\ &\iff [\![\, a = 0 \,]\!] \subseteq [\![\, b = 0 \,]\!] \\ &\iff \forall p \in \pi A \, (a \in p \Rightarrow b \in p). \end{aligned}$$

Radical relations were introduced in [19] and used in [20] in order to study the model theory of von Neumann regular real closed rings (cf. [23] or [22]) without minimal idempotents different from zero. Radical relations are defined in [20] by:

(1) $a \preceq a$, for all $a \in A$;

(2) if $a \preceq b$ and $b \preceq c$ then $a \preceq c$, for all $a, b, c \in A$;

(3) if $a \preceq c$ and $b \preceq c$ then $a + b \preceq c$, for all $a, b, c \in A$;

(4) if $a \preceq b$ then $ac \preceq bc$, for all $a, b, c \in A$;

(5) $a \preceq 1$, for all $a \in A$ and $1 \not\preceq 0$;

(6) $b \preceq b^2$, for all $b \in A$.

The original definition in [19] was the previous one but reversed. In this context, it is proved in [19] that for any radical relation $\preceq$, there exists a subset $X \subseteq \mathrm{Spec}(A)$ such that:

$$a \preceq b \iff \forall p \in X\,(a \notin p \Rightarrow b \notin p).$$

This radical relation is denoted by $\preceq_X$. The case where $X = \pi A$ is a relevant one studied in [20] and it is proved there that:

$$\begin{aligned} a \preceq_{\pi A} b &\iff \mathrm{Ann}(b) \subseteq \mathrm{Ann}(a) \\ &\iff \forall x(bx = 0 \to ax = 0) \\ &\iff \forall x(ax \neq 0 \to bx \neq 0). \end{aligned}$$

Therefore the radical relation $\preceq_{\pi A}$ has all these possible definitions:

$$\begin{aligned} a \preceq_{\pi A} b &\iff \mathrm{Ann}(b) \subseteq \mathrm{Ann}(a) \\ &\iff \forall x(bx = 0 \to ax = 0) \\ &\iff \forall x(ax \neq 0 \to bx \neq 0) \\ &\iff \forall p \in \pi A\,(a \notin p \Rightarrow b \notin p) \\ &\iff \forall p \in \pi A\,(b \in p \Rightarrow a \in p) \\ &\iff [\![\, b = 0 \,]\!] \subseteq [\![\, a = 0 \,]\!] \\ &\iff [\![\, a \neq 0 \,]\!] \subseteq [\![\, b \neq 0 \,]\!] \\ &\iff \mathrm{supp}(a) \subseteq \mathrm{supp}(b) \\ &\iff b^{\perp} \subseteq a^{\perp} \\ &\iff a \in b^{\perp\perp}. \end{aligned}$$

From now on, this radical relation associated to πA will be denoted by $\preceq$. In [20], the elimination of quantifiers of the theory of von Neumann regular real closed rings without minimal idempotents non-zero is given in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq\}$ of lattice-ordered rings with this radical relation.

Notation 2.1 *Let A be any ring and $a, b \in A$, we will denote $a =_s b$ if $a \preceq b$ and $b \preceq a$.*

According to [8], a **real closed valuation ring** is an ordered domain that satisfies the intermediate value property for polynomials in one variable that it is not a field. In [8], the autors showed that this theory is complete and admits elimination of quantifiers in the languaje $\mathcal{L}_{\mathrm{or}} \cup \{|\}$ of ordered rings with the divisibility relation.

In [12, Definition 2.5], a lattice ordered ring A is called **divisible-projectable** if

$$\forall x \forall y \Big(y \neq 0 \to \exists z \exists w \big(x = z + w \;\&\; z \perp w \;\&\; y \mid z \;\&\; \forall w'(w' \neq 0 \;\&\; w' \perp (w - w') \to y \nmid w') \big) \Big)$$

is a valid in A. In [12, Definition 2.8], a ring A is called **sc-regular** if there exists an element $u \in A$ such that $\mathrm{Ann}(u) = \{0\}$ (or $1 \preceq u$) and $u \nmid e$ for every non-zero idempotent $e \in A$. By [12, Proposition 3.4 (i), Corollary 2.11 and Proposition 2.6], a ring A is a projectable real closed ring with the first convexity property that satisfies the sc-regularity and divisible-projectability if and only if

$$A \in \Gamma^{e}_{\mathcal{L}_{\mathrm{or}} \cup \{|\}}\big(\pi A, (A/p)_{p \in \pi A}\big),$$

where A/p is a real closed valuation ring, for every $p \in \pi A$.

Let T^* be the theory of projectable real closed rings with the first convexity property that satisfies the sc-regularity and divisible-projectability properties, and without minimal idempotents non-zero. By [13, Theorem 10], a ring A is a model of T^* if and only if A is a convex lattice-ordered subring of a von Neumann regular real closed ring, and A satifies the divisible-projectability and sc-regularity properties and it is without minimal idempotents non-zero.

By [12], the theory T^* admits quantifier elimination in $\mathcal{L}_{\text{lor}} \cup \{\text{div}(\cdot,\cdot)\}$, where $\text{div}(\cdot,\cdot)$ is a binary function symbol defined by:

$$T^* \vdash \text{div}(x,y) = c \quad \longleftrightarrow \quad c \in y^{\perp\perp} \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ \forall w'(w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w')\big).$$

Observe that the definition of this binary funcion symbol $\text{div}(\cdot,\cdot)$ can be written using the radical relation $\preceq$ by:

$$T^* \vdash \text{div}(x,y) = c \quad \longleftrightarrow \quad c \preceq y \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ \forall w'(w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w')\big).$$

In order to study the theory T^* from the point of view of existencial formulas or model completeness, it will be usefull to introduce the following binary predicate:

$$R(y,w) \longleftrightarrow \forall w'(w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w'),$$

that express the fact that y does not divide locally w. It will be more pleasant to have it in a positive form and then:

$$y \mid_{\text{loc}} w \longleftrightarrow \neg R(y,w) \longleftrightarrow \exists w'(w' \neq 0 \wedge w' \perp (w - w') \wedge y \mid w').$$

Observe that the last expression on the right is a formula in the languaje $\mathcal{L}_{\text{lor}}$ and it will be preferable to express it just in the languaje of rings by:

$$R(y,w) \longleftrightarrow \forall w'(w' \neq 0 \wedge w'(w - w') = 0 \to y \nmid w'),$$

and

$$y \mid_{\text{loc}} w \longleftrightarrow \neg R(y,w) \longleftrightarrow \exists w'(w' \neq 0 \wedge w'(w - w') = 0 \wedge y \mid w').$$

For the "global" divisibility relation $y \mid w$ one has that $y \mid 0$. But see that if $y \mid_{\text{loc}} 0$ in a reduced ring A then there exists $w' \in A$ with $w' \neq 0$, $w'(-w') = 0$ and $y \mid w'$. Therefore $w' \neq 0$ and ${w'}^2 = 0$; a contradiction because A is reduced. So it is better to redefine:

$$y \mid_{\text{loc}} w \longleftrightarrow w = 0 \vee \exists w'(w' \neq 0 \wedge w'(w - w') = 0 \wedge y \mid w').$$

The following proposition gives some elementary properties of this new local divisibility relation.

Proposition 2.2 *Let A be any ring and let $y, w, c \in A$ and $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The following properties are valid in A.*

(i) *if* $y \mid w$ *then* $y \mid_{\text{loc}} w$*,*

(ii) $y \mid_{\text{loc}} 0$ *and* $1 \mid_{\text{loc}} w$*,*

(iii) *if* $0 \mid_{\text{loc}} w$ *then* $w = 0$*,*

(iv) *if* $cy \mid_{\text{loc}} w$ *then* $y \mid_{\text{loc}} w$*,*

(v) *if* $y^n \mid_{\text{loc}} w$ *then* $y \mid_{\text{loc}} w$*,*

(vi) $y \mid_{\text{loc}} y^n$*,*

(vii) $y \mid_{\text{loc}} w$ *if and only if* $-y \mid_{\text{loc}} w$*, if and only if* $y \mid_{\text{loc}} -w$*, if and only if* $-y \mid_{\text{loc}} -w$*,*

(viii) *if* A *is a domain, then* $y \mid w$ *if and only if* $y \mid_{\text{loc}} w$*.*

Proof: All these properties are showed by routine verifications. ■

One needs to prove a previous lemma in order to prove one more property on "local divisibility".

Lema 2.3 *Let* A *be any lattice-ordered ring and let* $w, w' \in A$ *such that* $w' \perp w - w'$*. Then* $|w'| \leqslant |w|$*.*

Proof: By the definition of $\wedge$ one has that $|w'| \wedge |w| \leqslant |w'|$ and $|w'| \wedge |w| \leqslant |w|$. Observe that one has the following inequality:

$$\begin{array}{rcl} |w'| = |w'| \wedge |w'| = |w'| \wedge |w' - w + w| & \leqslant & |w'| \wedge \big(|w' - w| + |w|\big) \\ & = & \big(|w'| \wedge |w' - w|\big) + \big(|w'| \wedge |w|\big). \end{array}$$

Since $w' \perp w - w'$, then $|w'| \wedge |w - w'| = 0$ and therefore one obtains:

$$|w'| \leqslant 0 + \big(|w'| \wedge |w|\big) = |w'| \wedge |w| \leqslant |w'|.$$

Then $|w'| \wedge |w| = |w'|$, and this shows us that $|w'| \leqslant |w|$. ■

The previous lemma help us to prove the following proposition:

Proposition 2.4 *Let* A *be any lattice-ordered ring and let* $y, w_1, w_2 \in A$*. If* $y \mid_{\text{loc}} w_1$ *and* $y \mid_{\text{loc}} w_2$ *with* $w_1 \perp w_2$ *then* $y \mid_{\text{loc}} w_1 + w_2$*.*

Proof: Let us suppose that $y \mid_{\text{loc}} w_1$ and $y \mid_{\text{loc}} w_2$ with $w_1 \perp w_2$. There are various cases:

- If $w_1 = 0$, since $y \mid_{\text{loc}} w_2$ then $y \mid_{\text{loc}} w_1 + w_2$.
- If $w_2 = 0$, since $y \mid_{\text{loc}} w_1$ then $y \mid_{\text{loc}} w_1 + w_2$.
- if $w_1 \neq 0$ and $w_2 \neq 0$. If $w_1 + w_2 = 0$ then by definition one has that $y \mid_{\text{loc}} w_1 + w_2$.

Let us suppose that $w_1 + w_2 \neq 0$. Since $y \mid_{\text{loc}} w_1$ and $w_1 \neq 0$ then there exists $w_1' \in A$, $w_1' \neq 0$ such that $w_1' \perp w_1 - w_1'$ with $y \mid w_1'$. Since $y \mid_{\text{loc}} w_2$ and $w_2 \neq 0$ then there exists

$w'_2 \in A$, $w'_2 \neq 0$ such that $w'_2 \perp w_2 - w'_2$ and $y \mid w'_2$. Let us see that $w'_1 + w'_2 \neq 0$. If $w'_1 + w'_2 = 0$ then $w'_2 = -w'_1$ and therefore:

$$|w'_1| \wedge |w'_2| = |w'_1| \wedge |-w'_1| = |w'_1| \wedge |w'_1| = |w'_1|.$$

By the lemma 2.3 one has that $|w'_1| \leqslant |w_1|$ and $|w'_2| \leqslant |w_2|$. Then:

$$|w'_1| \wedge |w'_2| \leqslant |w_1| \wedge |w_2|.$$

Since $w_1 \perp w_2$ then $|w_1| \wedge |w_2| = 0$ and by the previous inequality one has $|w'_1| \wedge |w'_2| = 0$. By the asumption one should have that $|w'_1| = 0$, meaning that $w'_1 = 0$; which is impossible since $w'_1 \neq 0$.

Once we stated that $w'_1 + w'_2 \neq 0$, we want to see that:

$$w'_1 + w'_2 \perp (w_1 + w_2) - (w'_1 + w'_2).$$

We have the following inequalities :

$$\begin{aligned}
0 &\leqslant \big|w'_1 + w'_2\big| \wedge \big|(w_1 + w_2) - (w'_1 + w'_2)\big| \\
&= \big|w'_1 + w'_2\big| \wedge \big|(w_1 - w'_1) + (w_2 - w'_2)\big| \\
&\leqslant \big|w'_1 + w'_2\big| \wedge \big(|w_1 - w'_1| + |w_2 - w'_2|\big) \\
&\leqslant \big(|w'_1| + |w'_2|\big) \wedge \big(|w_1 - w'_1| + |w_2 - w'_2|\big) \\
&= \big(|w'_1| \wedge |w_1 - w'_1|\big) + \big(|w'_1| \wedge |w_2 - w'_2|\big) + \big(|w'_2| \wedge |w_1 - w'_1|\big) + \big(|w'_2| \wedge |w_2 - w'_2|\big) \\
&= 0 + \big(|w'_1| \wedge |w_2 - w'_2|\big) + \big(|w'_2| \wedge |w_1 - w'_1|\big) + 0 \\
&= \big(|w'_1| \wedge |w_2 - w'_2|\big) + \big(|w'_2| \wedge |w_1 - w'_1|\big) \\
&\leqslant \big(|w'_1| \wedge \big(|w_2| + |w'_2|\big)\big) + \big(|w'_2| \wedge \big(|w_1| + |w'_1|\big)\big) \\
&= \big(|w'_1| \wedge |w_2|\big) + \big(|w'_1| \wedge |w'_2|\big) + \big(|w'_2| \wedge |w_1|\big) + \big(|w'_2| \wedge |w'_1|\big) \\
&= \big(|w'_1| \wedge |w_2|\big) + 2\big(|w'_1| \wedge |w'_2|\big) + \big(|w'_2| \wedge |w_1|\big).
\end{aligned}$$

Using one more time the lemma 2.3, since $w'_1 \perp (w_1 - w'_1)$ and $w'_2 \perp (w_2 - w'_2)$; one has that $|w'_1| \leqslant |w_1|$ and $|w'_2| \leqslant |w_2|$. Coming back to the inequalities one obtains:

$$\begin{aligned}
0 &\leqslant \big|w'_1 + w'_2\big| \wedge \big|(w_1 + w_2) - (w'_1 + w'_2)\big| \\
&\leqslant \big(|w'_1| \wedge |w_2|\big) + 2\big(|w'_1| \wedge |w'_2|\big) + \big(|w'_2| \wedge |w_1|\big) \\
&\leqslant \big(|w_1| \wedge |w_2|\big) + 2\big(|w_1| \wedge |w_2|\big) + \big(|w_2| \wedge |w_1|\big) \\
&= 4\big(|w_1| \wedge |w_2|\big) \\
&= 4 \cdot 0 \\
&= 0,
\end{aligned}$$

for $w_1 \perp w_2$. This shows that $\big|w'_1 + w'_2\big| \wedge \big|(w_1 + w_2) - (w'_1 + w'_2)\big| = 0$. One then has that $(w'_1 + w'_2) \perp (w_1 + w_2) - (w'_1 + w'_2)$. Since $y \mid w'_1$ and $y \mid w'_2$ then clearly $y \mid w'_1 + w'_2$. Declaring $w' = w'_1 + w'_2$, we had achieved that $w' \neq 0$, $w' \perp (w_1 + w_2) - w'$ and that $y \mid w'$. This means that $\exists w'\big(w' \neq 0 \wedge w'\big(w' - (w_1 + w_2)\big) \wedge y \mid w'\big)$ is a valid formula in A. Precisely one has that $y \mid_{\text{loc}} w_1 + w_2$.

∎

Let A be any reduced f-ring. The sc-regularity of A states the existence of an element $u \in A$ such that $1 \preceq u$ and satisfying that $\forall e(e \neq 0 \wedge e^2 = e \to u \nmid e)$. Observe that:

$$\begin{array}{rcl} u \mid_{\text{loc}} 1 & \longleftrightarrow & \exists w'(w' \neq 0 \wedge w'(w'-1) = 0 \wedge u \mid w') \\ & \longleftrightarrow & \exists w'(w' \neq 0 \wedge w'^2 - w' = 0 \wedge u \mid w') \\ & \longleftrightarrow & \exists e(e \neq 0 \wedge e^2 = e \wedge u \mid e) \end{array} .$$

Therefore:

$$u \nmid_{\text{loc}} 1 \longleftrightarrow \forall e(e \neq 0 \wedge e^2 = e \to u \nmid e).$$

So the condition of sc-regularity can be rewritten as there exists $u \in A$ with $1 \preceq u$ and $u \nmid_{\text{loc}} 1$. Namely, A is sc-regular if and only if $A \models \exists u(1 \preceq u \wedge u \nmid_{\text{loc}} 1)$.

3 Model completeness.

In this sextion the languaje considered is $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq, \mid_{\text{loc}}\}$. Let A and B be two reduced f-rings satisfying the first convexity property and let us suppose that A is a substructure of B in the language $\mathcal{L}$; in particular A is a lattice-ordered subring of B.

Let us denote $i\colon A \hookrightarrow B$ the inclusion and the functorial (continuous) map:

$$\text{Spec}(i)\colon \text{Spec}(B) \to \text{Spec}(A), q \mapsto i^{-1}(q) = q \cap A.$$

Since $A \subseteq_{\mathcal{L}} B$ and the radical relation $\preceq$ belongs to the language then:

$$a \preceq_A a' \Longleftrightarrow i(a) \preceq_B i(a'),$$

for all $a, a' \in A$. Let us denote $\pi B = \text{Specmin}(B) = \{q \in \text{Spec}(B) : q \text{ is a minimal prime ideal }\} \subseteq \text{Spec}(B)$ and similarly $\pi A = \text{Specmin}(A) = \{p \in \text{Spec}(A) : p \text{ is a minimal prime ideal}\} \subseteq \text{Spec}(A)$. Using [20, Theorem, p. 23] and [20, Proposition (a) y (b), p. 22] one has:

$$i^* = \text{Spec}(i)_{\restriction_{\overline{\pi B}^{\text{con}}}} : \overline{\pi B}^{\text{con}} \to \overline{\pi A}^{\text{con}},$$

where $\overline{\pi B}^{\text{con}}$ and $\overline{\pi A}^{\text{con}}$ are the closures of πB and πA in the constructible topology; and i^* is surjective. If one consider that the (unitary) f-rings A and B are projectable, then by [16, 6.11], one should have that the spaces πB and πA are compact (and Hausdorff). By [26, Corollary 2.7], the subspaces πB and πA are proconstructible and therefore $\overline{\pi B}^{\text{con}} = \pi B$ and $\overline{\pi A}^{\text{con}} = \pi A$. In the case that A and B are reduced projectable f-rings, then one has:

$$i^* = \text{Spec}(i)_{\restriction_{\pi B}} : \pi B \to \pi A,$$

and i^* is surjective.

From now on, A and B will be reduced and projectable f-rings. For $q_1, q_2 \in \pi B$, let us declare $q_1 \sim q_2$ if and only if $q_1 \cap A = q_2 \cap A$, if and only if $i^*(q_1) = i^*(q_2)$. Clearly $\sim$ is an equivalence relation on πB. Since the function $i^*\colon \pi B \to \pi A$ is surjective, then πA can be consider with the quotient topology πB induced by i^* or by the equivalence relation $\sim$. By [29, Theorem 9.2, p. 60] one has that the original topology of πA and the induced topology by i^* (or by the equivalence relation $\sim$) coincide if i^* is an open or closed function. Since the f-rings A and B are projectable, then by [16, 6.11], one should have that the spaces πA and πB are compact (and Hausdorff). Since $i^*\colon \pi B \to \pi A$ is a continuous function with πB compact and Hausdorff, then, by [29, p. 120], one has that i^* is a closed function. Therefore the original topology on πA and the quotient topology on πB induced by the equivalence relation $\sim$ are the same. Therefore:

$$j\colon \pi B/{\sim} \to \pi A, q/{\sim} \mapsto i^*(q),$$

is a homeomorphism of topological spaces and Boolean spaces.

Now let $p \in \pi A$ and $q \in (i^*)^{-1}\big(\{p\}\big)$. That is to say that $i^*(q) = q \cap A = p$. Let us consider:

$$h_{pq}\colon A/p \to B/q, a+p \mapsto a+q.$$

Since $p \subseteq q \cap A$, then h_{pq} is well defined for if $a+p = a'+p$ with $a, a' \in A$ then $a - a' \in p$ and $a - a' \in q \cap A$, that carries to $a+q = a'+q$. Since $q \cap A \subseteq p$ then h_{pq} is injective for if $a, a' \in A$ are such that $h_{pq}(a) = h_{pq}(a')$, then $a+q = a'+q$ and $a-a' \in q$, so $a-a' \in q \cap A$; that is to say that $a - a' \in p$. Then $a+p = a'+p$. This proves the injectivity of h_{pq}. It is clear that h_{pq} is a ring homomorphism. Therefore:

$$h_{pq}\colon A/p \to B/q, a+p \mapsto a+q,$$

is a well defined injective ring homomorphism.

Let us see now that h_{pq} respects the order. Let $a, a' \in A$ such that $a+p \leqslant a'+p$ in A/p. Then there exists $c \in p$ such that $c > 0$ and $a + c \leqslant a'$ en A. Since A is an $\mathcal{L}$-sub-structure of B and the order is in the language $\mathcal{L}$ then $a + c \leqslant a'$ in B. Since $p \subseteq q \cap A$ then $c \in q$ with $c > 0$ and $a + c \leqslant a'$ in B. That is to say that $a+q \leqslant a'+q$ in B/q. Then $h_{pq}(a) \leqslant h_{pq}(a')$ in B/q. One should prove the other implication, that is: if $h_{pq}(a) \leqslant h_{pq}(a')$ in B/q then $a+p \leqslant a'+p$ in A/p. But since the orders on A/p and B/q are total then the implication needed to be proved can be immediately deduced from the one we just proved. Therefore $h_{pq}\colon A/p \to B/q$ is an injective homomorphism of ordered rings. In this context, one has the following proposition:

Proposition 3.1 *Let A and B be two reduced projectable f-rings satisfying the first convexity property such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$. If in addition one suppose that A and B are divisible-projectable then for $p \in \pi A$ and $q \in (i^*)^{-1}\big(\{p\}\big)$, the homomorphism of ordered rings $h_{pq}\colon A/p \to B/q, a+p \mapsto a+q$ respects divisibility.*

Proof: We must prove: for any $a, a' \in A$ one has that:

$$a+p \mid a'+p \text{ in } A/p \text{ if and only if } a+q \mid a'+q \text{ in } B/q.$$

($\Rightarrow$) Let us suppose that $a+p \mid a'+p$ en A/p. Then there exists $c+p \in A/p$ such that $(a+p)(c+p) = a'+p$. Therefore $ac - a' \in p$. Since $p \subseteq q \cap A$ then $ac - a' \in q$, what this means is that $(a+q)(c+q) = a'+q$. In fact $a+q \mid a'+q$ en B/q.

($\Leftarrow$) Let us suppose that $a+q \mid a'+q$ in B/q. One has to show that $a+p \mid a'+p$ in A/p.

• If $a' + q = 0$ then $a' \in q$. Since $a' \in A$ then $a' \in q \cap A = p$. So $a' + p = 0$ and therefore $a + p \mid a' + p$ en A/p.

• If $a' + q \neq 0$ then $a' \notin q$. Then $a' \notin p$ and so $a' + p \neq 0$. Let us suppose in this case that $a + p \nmid a' + p$ en A/p. Consider $N = [\![\, a \nmid a' \,]\!]_{\pi A} \cap [\![\, a' \neq 0 \,]\!]_{\pi A}$ which is a clopen set of πA. (Here we using the fact that A is divisible projectable, see [12, Proposition 2.6]). See that $p \in N$ and therefore $N \neq \phi$. Let us define $\alpha' = a'_{\restriction N} \cup 0_{\restriction \pi A \setminus N} \in A$. Since $N \neq \phi$ then $\alpha' \neq 0$.

Now suppose that $A \models a \mid_{\text{loc}} \alpha'$. Since $\alpha' \neq 0$ then:

$$A \models \exists w'(w' \neq 0 \wedge w'(w' - \alpha') = 0 \wedge a \mid w').$$

Since $w' \neq 0$ then there exists $\bar{p} \in \pi A$ such that $w'(\bar{p}) \neq 0$. Since $w'(w' - \alpha') = 0$ then $w'(\bar{p}) = \alpha'(\bar{p})$. By the definition of α' and the fact that $w'(\bar{p}) \neq 0$, one has that $\bar{p} \in N$ and that $\alpha'(\bar{p}) = a'(\bar{p})$. Since $a \mid w'$, there exists $c \in A$ such that $ac = w'$. That is to say that $a(\bar{p})c(\bar{p}) = w'(\bar{p}) = \alpha'(\bar{p}) = a'(\bar{p})$, so $a(\bar{p}) \mid a'(\bar{p})$ in $A/\bar{p}$; but this contradicts the fact that $\bar{p} \in [\![\, a \nmid a' \,]\!]_{\pi A}$. Therefore one has:

$$A \models a \nmid_{\text{loc}} \alpha'.$$

Since A is an $\mathcal{L}$-substructure of B and $\mid_{\text{loc}}$ belongs to the language, then $B \models a \nmid_{\text{loc}} \alpha'$. Since $\alpha' \neq 0$ then:

$$B \models \forall w'(w' \neq 0 \wedge w'(w' - \alpha') = 0 \rightarrow a \nmid w').$$

Our initial assumption was that $a + q \mid a' + q$ in B/q. Therefore $q \in [\![\, a \mid a' \,]\!]_{\pi B}$. We are also in the case that $a' + q \neq 0$, that is to say that $q \in [\![\, a' \neq 0 \,]\!]_{\pi B}$. Since $p \in N$ then $\alpha'(p) = a'(p)$, that is to say that $\alpha' + p = a' + p$. Since $p = q \cap A$ then $\alpha' + q = a' + q$ in B/q and therefore $q \in [\![\, \alpha' = a' \,]\!]_{\pi B}$. Putting $M = [\![\, a \mid a' \,]\!]_{\pi B} \cap [\![\, a' \neq 0 \,]\!]_{\pi B} \cap [\![\, \alpha' = a' \,]\!]_{\pi B}$, one has that M is a clopen set of πB with $q \in M$ and $M \neq \phi$ (here we also used that B is divisible-projectable).

Now let us consider $w'' = \alpha'_{\restriction M} \cup 0_{\restriction \pi B \setminus M} \in B$. Since $M \neq \phi$, for $\bar{q} \in M$ one has that $w''(\bar{q}) = \alpha'(\bar{q}) = a'(\bar{q}) \neq 0$. Then $w'' \neq 0$. Let us see that $w''(w'' - \alpha') = 0$. Let $\bar{q} \in \pi B$. If $\bar{q} \in \pi B \setminus M$ then $w''(\bar{q}) = 0$ and so $\big[w''(w'' - \alpha')\big](\bar{q}) = w''(\bar{q})(w'' - \alpha')(\bar{q}) = 0$. If $\bar{q} \in M$ then $w''(\bar{q}) = \alpha'(\bar{q})$ by the definition of w'', and so $(w'' - \alpha')(\bar{q}) = 0$; that is to say that $\big[w''(w'' - \alpha')\big](\bar{q}) = 0$. In any case we obtain that $\big[w''(w'' - \alpha')\big](\bar{q}) = 0$ (for all $\bar{q} \in \pi B$). Then $w''(w'' - \alpha') = 0$. Since $w'' \in B$ is such that $w'' \neq 0$ and $w''(w'' - \alpha') = 0$, then $a \nmid w''$ en B.

On the other hand, for $\bar{q} \in \pi B$ one has the following:

• if $\bar{q} \in \pi B \setminus M$ then $w''(\bar{q}) = 0$ and therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$.

• if $\bar{q} \in M$ then $\bar{q} \in [\![\, a \mid a' \,]\!]_{\pi B} \cap [\![\, \alpha' = a' \,]\!]_{\pi B}$ and consequently one has $a(\bar{q}) \mid a'(\bar{q}) = \alpha'(\bar{q})$ en $B/\bar{q}$. Therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$.

Therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$ for all $\bar{q} \in \pi B$. For each $\bar{q} \in \pi B$, there exists $c_{\bar{q}} \in B$ such that $a(\bar{q})c_{\bar{q}}(\bar{q}) = w''(\bar{q})$. Then:

$$\pi B = \bigcup_{\bar{q} \in \pi B} [\![\, ac_{\bar{q}} = w'' \,]\!]_{\pi B}.$$

By the compactness of πB, one can distinguish a finite number of $c_{\bar{q}}$'s and by the patchwork property of B, it is easy to construct an element $c \in B$ such that $ac = w''$. Then it has been proved that $a \mid w''$ en B. But we had from below that $a \nmid w''$ en B, a contradiction. Therefore we proved that $a + q \mid a' + q$ in B/q implies that $a + p \mid a' + p$ in A/p. ■

Let A and B be two models of T^* such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq, |_{\text{loc}}\}$. It is known that $i^*\colon \pi B \to \pi A$, $q \mapsto q \cap A$ is a continuous surjective map such that $\pi A \cong \pi B/\sim$ where $\sim$ is the equivalence relation given by $q \sim q'$ if and only if $i^*(q) = q \cap A = q' \cap A = i^*(q')$. Furthermore, for all $p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$, there exists $h_{pq}\colon A/p \to B/q$, $a + p \mapsto a + q$ an injective homomorphism of ordered rings respecting the divisibility.

Let us denote $\mathcal{B}(\pi A)$ and $\mathcal{B}(\pi B)$ the Boolean algebras of clopen sets of πA and πB respectively. Therefore:

$$j = (i^*)^{-1}\colon \mathcal{B}(\pi A) \to \mathcal{B}(\pi B),$$

is an injective homomorphism of Boolean algebras.

We want to show that $A \prec_{\mathcal{L}} B$. Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in A$. By [9, Theorem 1.1], there exists an acceptable sequence $\zeta = \langle \Phi, \theta_1, \ldots, \theta_m \rangle$ of formulas where $\theta_1, \ldots, \theta_m$ are $\mathcal{L}$-formulas with the same free variables of $\phi(x_1, \ldots, x_n)$ and Φ is a formula in the Boolean algebra's language with m free variables such that:

$$A \models \phi(a_1, \ldots, a_n) \Longleftrightarrow \mathcal{B}(\pi A) \models \Phi\Big([\![\theta_1(a_1, \ldots, a_n)]\!]_A, \ldots, [\![\theta_m(a_1, \ldots, a_n)]\!]_A \Big),$$

where $[\![\theta_j(a_1, \ldots, a_n)]\!]_A = \big\{ p \in \pi A : A/p \models \theta_j(a_1 + p, \ldots, a_n + p) \big\}$, for all $j = 1, \ldots, m$.

Since A and B are models of T^* then A/p and B/q are real closed valuation rings, for all $p \in \pi A$ and $q \in \pi B$. Therefore, for $p \in \pi A$ and $q \in (i^*)^{-1}\big(\{p\}\big)$, one has that $h_{pq}\colon A/p \to B/q$, $a + p \mapsto a + q$ is an elementary monomorphism in view of 3.1 and [8]. Therefore:

$$h_{pq}\colon A/p \prec B/q.$$

Then:

$$\begin{aligned} j\Big([\![\theta_l(a_1, \ldots, a_n)]\!]_A \Big) &= \Big\{ q \in \pi B : B/q \models \theta_l\big(h_{pq}(a_1), \ldots, h_{pq}(a_n)\big) \text{ con } p = q \cap A \Big\} \\ &= [\![\theta_l(a_1, \ldots, a_n)]\!]_B. \end{aligned}$$

Since $\mathcal{B}(\pi A)$ are $\mathcal{B}(\pi B)$ are atomless Boolean algebras (A and B are models of T^*) then:

$$j\colon \mathcal{B}(\pi A) \prec \mathcal{B}(\pi B),$$

is an elementary monomorphism. Then one has:

$$\begin{aligned} & \mathcal{B}(\pi A) \models \Phi\Big([\![\theta_1(a_1, \ldots, a_n)]\!]_A, \ldots, [\![\theta_m(a_1, \ldots, a_n)]\!]_A \Big) \\ \Longleftrightarrow\ & \mathcal{B}(\pi B) \models \Phi\Big(j\big([\![\theta_1(a_1, \ldots, a_n)]\!]_A\big), \ldots, j\big([\![\theta_m(a_1, \ldots, a_n)]\!]_A\big) \Big) \\ \Longleftrightarrow\ & \mathcal{B}(\pi B) \models \Phi\Big([\![\theta_1(a_1, \ldots, a_n)]\!]_B, \ldots, [\![\theta_m(a_1, \ldots, a_n)]\!]_B \Big). \end{aligned}$$

By [9, Theorem 1.1] one also has:

$$B \models \phi(a_1,\dots,a_n) \iff \mathcal{B}(\pi B) \models \Phi\Big(\llbracket \theta_1(a_1,\dots,a_n)\rrbracket_B,\dots,\llbracket \theta_m(a_1,\dots,a_n)\rrbracket_B\Big).$$

Therefore we just have seen that:

$$A \models \phi(a_1,\dots,a_n) \text{ if and only if } B \models \phi(a_1,\dots,a_n).$$

This proves that:

$$A \prec_{\mathcal{L}} B.$$

We can therefore state:

Theorem 3.2 *The theory T^* is model complete in $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq, |_{\text{loc}}\}$.* ■

4 Universal theories.

In this section, universal theories of T^* will be given in different languajes. At the beginning of this section, it will be discuss how a reduced projectable f-ring satisfying the first convexity property can be embedded in a model of T^* in the languaje $\mathcal{L}_{\text{lor}} \cup \{\preceq, |, |_{\text{loc}}\}$. The proyectability is not a universal axiom, it will be substitute at the last part of this section by other universal axioms, one for each symbol in $\{|, |_{\text{loc}}\}$.

Let A be a reduced and *projectable* f-ring satisfying the first convexity property. Since A is a reduced f-ring, then $A \subseteq \prod_{p\in\pi A} A/p$, where πA is the space of minimal prime ideals of A and A/p is a totally ordered integral domain, for each $p \in \pi A$. Clearly this inclusion is in the languaje $\mathcal{L}_{\text{lor}}$.

The projectabilitiy of A permits to prove that divisibility is respected. For the first implication the projectability is not neded for if $a, b \in A$ such that $b \mid a$, then there exists $c \in A$ with $bc = a$; therefore $(b + p)(c + p) = bc + p = a + p$, for all $p \in \pi A$. That is $b + p \mid a + p$, for all $p \in \pi A$, or:

$$(b+p)_{p\in\pi A} \mid (a+p)_{p\in\pi A} \text{ in } \prod_{p\in\pi A} A/p.$$

In the other direction, if $a, b \in A$ are such that $(b + p)_{p\in\pi A} \mid (a + p)_{p\in\pi A}$ then there exists $(c_p + p)_{p\in\pi A} \in \prod_{p\in\pi A} A/p$ with $(b+p)_{p\in\pi A} \cdot (c_p + p)_{p\in\pi A} = (a+p)_{p\in\pi A}$. Therefore $bc_p + p = a + p$, for all $p \in \pi A$. Since A is a subdirect product, there exists $\tilde{c_p} \in A$ such that $\tilde{c_p}(p) = c_p$, and this for every $p \in \pi A$. Considering $X_p = \llbracket b \cdot \tilde{c_p} = a \rrbracket$, one has that $p \in X_p$, for all $p \in \pi A$. Then:

$$\pi A = \bigcup_{p\in\pi A} X_p$$

where this is a clopen covering. By compactness of πA and the glueing property of A, there exists $c \in A$ such that $\pi A = \llbracket b \cdot c = a \rrbracket$. Then $(bc) + p = a + p$, for all $p \in \pi A$. That is $bc - a \in \bigcap_{p\in\pi A} p$. Since A is reduced then $\bigcap_{p\in\pi A} p = \{0\}$ and therefore $bc = a$. That is $b \mid a$ in A.

Then $A \subseteq \prod_{p\in\pi A} A/p$ in the languaje $\mathcal{L}_{\rm lor} \cup \{|\}$, where A/p is a totally ordered integral domain. As A satisfies the first convexity property then A/p also satisfies it, for all $p \in \pi A$; see lema 2.3 in [12]. Therefore A/p is a totally ordered domain satisfying the first convexity property, for all $p \in \pi A$. Following the notation in [2], one has A/p is a model of the theory $\mathrm{COVD_D}$ (*Convexely ordered valuation rings*) or a model of $\mathrm{OF_D}$ (*Ordered fields*), for every $p \in \pi A$. By theorem 1(i) in [2], there exists R_p a real closed valuation ring (not beeing a field) such that $A/p \subseteq R_p$ in the languaje $\mathcal{L}_{\rm or} \cup \{|\}$, and for all $p \in \pi A$. Therefore:

$$\prod_{p\in\pi A} A/p \subseteq \prod_{p\in\pi A} R_p,$$

in the languaje $\mathcal{L}_{\rm lor} \cup \{|\}$ since the divisibility is respected coordinated by coordinated. Now for each $p \in \pi A$, consider C_p a copy of the Cantor space and observe that $R_p \subseteq {R_p}^{C_p}$, by $x \mapsto (x)^{C_p}$ the constant inclusion. It is clear that this inclusion can be considered in the languaje $\mathcal{L}_{\rm or} \cup \{|\}$. Therefore (cf. [17]), one has:

$$A \subseteq \prod_{p\in\pi A} A/p \subseteq \prod_{p\in\pi A} R_p \subseteq \prod_{p\in\pi A} {R_p}^{C_p}, \tag{$*$}$$

in the languaje $\mathcal{L}_{\rm lor} \cup \{|\}$. Since the theory of real closed valuation rings, denoted by RCVR, admits elimination of quantifiers in $\mathcal{L}_{\rm or} \cup \{|\}$, then this theory has the Joint Embedding Property (JEP) in the languaje $\mathcal{L}_{\rm or} \cup \{|\}$. Therefore there exists R a real closed valuation ring (not a field) such that:

$$\prod_{p\in\pi A} {R_p}^{C_p} \subseteq R^C, \tag{$**$}$$

where $C = \prod_{p\in\pi A} C_p$ is a product of Cantor spaces. This inclusion can be considered in the languaje $\mathcal{L}_{\rm lor} \cup \{|\}$. By the theorem 2.1.(b) in [6], one has that $R^C \in \Gamma^e_{\mathcal{L}'}(\mathrm{RCVR})$. This means that $B = R^C$ is a model of T^* where $A \subseteq B$.

The inclusions in $(*)$ and in $(**)$ respects the radical relation. Since the radical relation $b \preceq a$ is given by the universal formula $\forall x(ax = 0 \to bx = 0)$, it is clear that this relation goes down in each inclusion. It will be shown that in each of this inclusion, the radical relation goes up. Consider:

$$\iota\colon A \to \prod_{p\in\pi A} A/p,\ a \mapsto (a+p)_{p\in\pi A}.$$

Let $a, b \in A$ and suppose that $A \models a \preceq b$. We want to see that $\prod_{p\in\pi A} A/p \models \iota(a) \preceq \iota(b)$. Let $\tilde{c} = (c_p + p)_{p\in\pi A} \in \prod_{p\in\pi A} A/p$ such that $\iota(b)\tilde{c} = 0$. Then $bc_p + p = 0$, for all $p \in \pi A$. Therefore $bc_p \in p$, for all $p \in \pi A$. Since $a \preceq b$ in A and $c_p \in A$ then $ac_p \preceq bc_p$ for every $p \in \pi A$. Recall that $a \preceq b$ is equivalent to $\forall p \in \pi A (b \in p \Rightarrow a \in p)$. Therefore $ac_p \in p$, for all $p \in \pi A$. Then $ac_p + p = 0$, for all $p \in \pi A$. That is $a\tilde{c} = 0$. It is showed that $\forall \tilde{c}\, (\iota(b)\tilde{c} = 0 \Rightarrow \iota(a)\tilde{c} = 0)$ in $\prod_{p\in\pi A} A/p$. That is $\iota(b) \preceq \iota(a)$ is valid in $\prod_{p\in\pi A} A/p$. [1]

Let's see now that in the second inclusion of $(*)$, the radical relation is respected. Let $\tilde{a} = (a_p + p)_{p\in\pi A}$ and $\tilde{b} = (b_p + p)_{p\in\pi A}$ be in $\prod_{p\in\pi A} A/p$ such that $\tilde{a} \preceq \tilde{b}$, that is: $\forall x(\tilde{b}x = 0 \to \tilde{a}x = 0)$. We want to see that $\tilde{a} \preceq \tilde{b}$ is valid in $\prod_{p\in\pi A} R_p$. In order to see this,

[1] Observe that in this paragraph the only condition used for A is to be a reduced f-ring.

let $\tilde{x} = (x_p)_{p\in\pi A}$ with $x_p \in R_p$ for all $p \in \pi A$, such that $\tilde{b}\tilde{x} = 0$. That is, $(b_p + p)x_p = 0$ for all $p \in \pi A$. Fixing $p \in \pi A$, one has that $b_p + p = 0$ or $x_p = 0$ for R_p is an integral domain. If $x_p = 0$ then clearly one has $(a_p + p)x_p = 0$. If $b_p + p = 0$ then $b_p \in p$. Taking $x \in \prod_{p\in\pi A} A/p$ given by $x(q) = \delta_{pq} = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{cases}$, one has that $\tilde{b}x = 0$ in $\prod_{p\in\pi A} A/p$ and by the hypothesis one has $\tilde{a}x = 0$ in $\prod_{p\in\pi A} A/p$, that is $a_p + p = 0$ and therefore $(a_p + p)x_p = 0$. This is satisfied in all $p \in \pi A$ and therefore one has $\tilde{a}\tilde{x} = 0$. It has been showend that $\forall x(\tilde{b}x = 0 \Rightarrow \tilde{a}x = 0)$ is valid in $\prod_{p\in\pi A} R_p$, i.e.: $\prod_{p\in\pi A} R_p \models \tilde{a} \preceq \tilde{b}$.

For the third inclusion in $(*)$, let us consider $r, s \in \prod_{p\in\pi A} R_p$ given by $r = (r_p)_{p\in\pi A}$ and $s = (s_p)_{p\in\pi A}$, and such that $r \preceq s$ in $\prod_{p\in\pi A} R_p$. We have to take into account that for each $p \in \pi A$, the inclusion $R_p \hookrightarrow R_p^{C_p}$ is given by $r \mapsto (r)^{C_p}$ where $(r)^{C_p}$ is a C_p-uple constantly equal to r. We want to see that $\forall x(sx = 0 \rightarrow rx = 0)$ is true in $\prod_{p\in\pi A} {R_p}^{C_p}$. Let's take $x \in \prod_{p\in\pi A} {R_p}^{C_p}$ such that $sx = 0$ with $x = (x_p)_{p\in\pi A}$ and $x_p = (x_p^i)_{i\in C_p}$. Since $s \in \prod_{p\in\pi A} R_p$ then $s = (s_p)_{p\in\pi A}$, and $sx = 0$ is $s_p x_p^i = 0$,for all $p \in \pi A$ and for all $i \in C_p$. Let's fix $p \in \pi A$, the one has $s_p x_p^i = 0$, for all $i \in C_p$. Since $r \preceq s$ in $\prod_{p\in\pi A} R_p$ then taking $x^p \in \prod_{p\in\pi A} R_p$ given by $x^p(q) = \delta_{pq} = \begin{cases} 1 & \text{if} \quad p = q \\ 0 & \text{if} \quad p \neq q \end{cases}$, one has $(sx^p = 0 \Rightarrow rx^p = 0)$. That is, $(s_p = 0 \Rightarrow r_p = 0)$. Now we have two cases:

- if $s_p = 0$ then $r_p = 0$ and therefore $r_p x_p^i = 0$, for all $i \in C_p$.
- if $s_p \neq 0$ then $x_p^i = 0$ for all $i \in C_p$ and therefore $r_p x_p^i = 0$, for all $i \in C_p$.

We have showed $(s_p x_p^i = 0 \Rightarrow r_p x_p^i = 0)$, for all $i \in C_p$ and for all $p \in \pi A$. This is exactly $r \preceq s$ in $\prod_{p\in\pi A} {R_p}^{C_p}$. [2]

Concerning the last inclusion in $(**)$, one has by model completeness of the real closed valuation rings theory d (cf. [8]) that $R_p \prec R$, for all $p \in \pi A$. By Feferman-Vaught theorem, [11], one has:

$$\prod_{p\in\pi A} {R_p}^{C_p} \prec R^C. \qquad (\dagger)$$

Then clearly the radical relation (and also local divisibility) is preserved by this last inclusion.

We have seen so far that any reduced f-ring satisfying the first convexity property can be embedded in a model of T^* in the languaje $\mathcal{L}_{\text{lor}} \cup \{\preceq\}$. In addition, if the ring A is *projectable*, then the divisibility relation can be added to the languaje.

Now we are going to see, under the assumption that the reduced f-ring is projectable, that the local divisibility relation is preserved by all the inclusions in $(*)$. The local divisibility is preserved under the inclusion $(\dagger)$ as it is an elementary one. The local divisibility is expressed by an existencial formula in the languaje $\mathcal{L}_{\text{lor}}$, and so it goes up in any extension. We only have to prove that the local divisibility goes down in each inclusion from $(*)$.

For the first inclusion in $(*)$, let $a, b \in A$ such that $b \mid_{\text{loc}} a$ is true in $\prod_{p\in\pi A} A/p$. We have to see that it is true in A. This is clear if $a = 0$. Let's suppose that $a \neq 0$. Then there

[2] Observe that the only complication that arise in this third inclusion is about notation, since everything is reduced to prove that for each $p \in \pi A$, the inclusion $R_p \hookrightarrow R_p^{C_p}$ preserves the radical relation. That is obvious since this inclusion is a constant function.

exists $w \in \prod_{p\in\pi A} A/p$ such that $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{p\in\pi A} A/p$. Since $w \neq 0$, there exists $p_0 \in \pi A$ such that $w_{p_0} \neq 0$. Therefore $a(p_0) = w_{p_0} \neq 0$ and $b(p_0) \mid w_{p_0} = a(p_0)$ in A/p_0. Therefore there exists $c_{p_0} \in A/p_0$ such that $b(p_0)c_{p_0} = a(p_0)$. Let $c \in A$ such that $c(p_0) = c_{p_0}$ and then $b(p_0)c(p_0) = a(p_0)$. Therefore $p_0 \in [\![\, bc = a \,]\!] \cap [\![\, a \neq 0 \,]\!] = N \neq \phi$ is a clopen set of πA. Since A is projectable, let $\tilde{w} \in A$ defined by $\tilde{w} = a_{\restriction N} \cup 0_{\restriction \pi A \smallsetminus N}$. Then $\tilde{w} \neq 0$ and clearly $\tilde{w}(\tilde{w} - a) = 0$. It is easy to see that $b \mid \tilde{w}$ in A as you can take $d \in A$ given by $d = c_{\restriction N} \cup 0_{\restriction \pi A \smallsetminus N} \in A$ and satisfying $bd = \tilde{w}$. It has been showed that $A \models \exists w(w \neq 0 \wedge w(w-a) = 0 \wedge b \mid w)$. That is, $A \models b \mid_{\text{loc}} a$.

Now let's consider the second inclusion: $\prod_{p\in\pi A} A/p \subseteq \prod_{p\in\pi A} R_p$. In order to simplify notation let $X = \pi A$ with $A_x = A/p$ adn $R_x = R_p$ for all $x \in X$. Let $a = (a_x)_{x\in X}$ and $b = (b_x)_{x\in X}$ in $\prod_{x\in X} A_x$ such that $b \mid_{\text{loc}} a$ in $\prod_{x\in X} R_x$. If $a = 0$ then clearly $b \mid_{\text{loc}} a$ in $\prod_{x\in X} A_x$. Let's suppose that $a \neq 0$. Then there exists $w = (w_x)_{x\in X} \in \prod_{x\in X} R_x$ such that $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{x\in X} R_x$. For each $x \in X$ one has $w_x = 0$ or $w_x = a_x \in A_x$. Then $w = (w_x)_{x\in X} \in \prod_{x\in X} A_x$. Since $b, w \in \prod_{x\in X} A_x$, then $b \mid w$ in $\prod_{x\in X} R_x$ permits us to see that $b \mid w$ in $\prod_{x\in X} A_x$. We have found $w \in \prod_{x\in X} A_x$ such that $w \neq 0$, $w(w-a) = 0$ with $b \mid w$ in $\prod_{x\in X} A_x$. That is, $b \mid_{\text{loc}} a$ in $\prod_{x\in X} A_x$.

For the last inclusion we should consider $\prod_{x\in X} R_x \subseteq \prod_{x\in X} {R_x}^{C_x}$, let $a = (a_x)_{x\in X}$ and $b = (b_x)_{x\in X}$ in $\prod_{x\in X} R_x$ such that $b \mid_{\text{loc}} a$ in $\prod_{x\in X} {R_x}^{C_x}$. If $a = 0$, then obvioulsy $b \mid_{\text{loc}} a$ in $\prod_{x\in X} R_x$. Let's suppose that $a \neq 0$. Then there exists $w = (w_x^c)_{x\in X, c\in C_x}$ such that $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{x\in X} {R_x}^{C_x}$. For each $x \in X$, if there exists $c \in C_x$ with $w_x^c \neq 0$ then $w_x^c = a_x \neq 0$. In that case, one can redefine $w \in \prod_{x\in X} {R_x}^{C_x}$ in such a way that $w_x^c \neq 0$ by declaring $w_x^c = a_x \neq 0$, for all $c \in C_x$. And if for some $x \in X$ one has $w_x^c = 0$ for all $c \in C_x$ then there is nothing to redefine. Therefore w is in $\prod_{x\in X} R_x$ and one has $w \neq 0$, $w(w-a) = 0$ and $b \mid w$ in $\prod_{x\in X} {R_x}^{C_x}$. Ya se ha visto que también $b \mid w$ en $\prod_{x\in X} R_x$. Now it is clear that $b \mid_{\text{loc}} a$ in $\prod_{x\in X} R_x$.

We have seen that all inclusions in $(*)$ and $(**)$ respects local divisibility. All these previous argumentations can be sumarized up in the following result.

Proposition 4.1 *Let A be a reduced f-ring that satisfies the first convexity property. Then there exists $B \models T^*$ such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\preceq\}$. If in addition A is assumed to be projectable, then this inclusion remains true where $\mathcal{L} = \mathcal{L}_{\text{lor}} \cup \{\mid, \preceq, \mid_{\text{loc}}\}$.*

■

As noted before, the projectability is not adequate for our purposes since it is not a universal axiom. In the sequel of this section, we replace the projectability by a universal axiom for each symbol in $\{\mid, \mid_{\text{loc}}\}$. The following lema goes in that direction.

Lema 4.2 *Let A be a reduced and projectable f-ring, then A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \rightarrow b \mid a\big),$$

for each $n \in \mathbb{N}$.

Proof: By our hypothesis we have that $A \in \Gamma^a_{\mathcal{L}_{\text{or}}}\big(X, (A_x)_{x\in X}\big)$, where X is a Boolean space and $(A_x)_{x\in X}$ is a family of totally ordered integral domains. Let $a, b, c_1, \ldots, c_n \in A$ such that $(bc_1 - a) \cdots (bc_n - a) = 0$. Then we have that:

$$\big(b(x)c_1(x) - a(x)\big) \cdots \big(b(x)c_n(x) - a(x)\big) = 0,$$

for all $x \in X$. For each $i \in \{1, \dots, n\}$, let's declarte:

$$N_i = [\![bc_i - a = 0]\!] = \{x \in X : b(x)c_i(x) - a(x) = 0\},$$

clopen subsets of X. Since the A_x's are integral domain then:

$$X = \bigcup_{i=1}^{n} N_i.$$

Without loosing generality, we can suppose that the N_i's are pairwise disjoint and not empty (in the case some set is empty then eliminate the corresponding c_i). We therefore can write:

$$X = \dot{\bigcup_{i=1}^{n}} N_i.$$

By the patchwork property of A one has:

$$c = c_{1 \restriction_{N_1}} \cup \cdots \cup c_{n \restriction_{N_n}} \in A.$$

Clearly $b(x)c(x) - a(x) = 0$, for all $x \in X$. That is $bc = a$ with $c \in A$. This proves that $b \mid a$ in A.

■

Corollary 4.3 *Let B be a reduced and projectable f-ring, and let A be a substructure of B in the languaje $\mathcal{L}_{\text{lor}} \cup \{|\}$. Then A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \to b \mid a \big),$$

for all $n \in \mathbb{N}$.

Proof: It is clearly deduced by the lema 4.2.

■

In the same sense one has:

Corollary 4.4 *Let $B \models T^*$ and A be a substructure of B in the languaje $\mathcal{L}_{\text{lor}} \cup \{|\}$. Then A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \to b \mid a \big),$$

for each $n \in \mathbb{N}$.

■

In view of our previous results, we established the following definition.

Definition 4.5 *Let A be any ring. We say that A satisfies* **the divisibility glueing axioms** *or* **property** *if A satisfies:*

$$\forall a \forall b \forall c_1 \cdots \forall c_n \big((bc_1 - a) \cdots (bc_n - a) = 0 \to b \mid a \big),$$

for all $n \in \mathbb{N}$.

This *divisibility glueing axioms* is a universal system of formulas and the proposition 4.1 can be proved in the languaje $\mathcal{L}_{\rm lor}\cup\{\preceq,|\}$ if we replace it by the projectability assumption. Since the space πA need not be necessarily compact, we will consider its closure in the constructible topology of some spectral space (and being consistent with the nature of the radical relation $\preceq$, cf. [20]). We will recall some considerations on spectral spaces in general and in the case of reduced f-rings. Let's recall that a topological space X is a spectral space (cf. [10, Definition 1.1.5]) if:

(i) X is quasi-compact and T_0,

(ii) X has a base of quasi-compact subspaces that are closed by finite intersections.

(iii) every non-empty irreducible closed subset Y of X is the closure of a unique point.
A subset is an *irreducible* closed subset if it is not the union of two proper closed subsets.

In [4, (8.4.1)], the irreducibles l-ideals are defined and [4, (10.1.6)] says that if A is an f-ring with unity then $\mathrm{Spec}_l(A)$ the set of irreducibles l-ideals of A is a quasi-compact space. It is clear that $\mathrm{Spec}_l(A)$ is T_0 (for if $p, q \in \mathrm{Spec}_l(A)$ with $p \neq q$ then there exists $a \in A$ with $a \in q$ and $a \notin p$, or $a \notin q$ and $a \in p$; in the first case $p \in S(a)$ and $q \notin S(a)$ and in the second case $q \in S(a)$ and $p \notin S(a)$). By [4, 10.1.4] the base $\{S(a) : a \in A\}$ of open sets of $\mathrm{Spec}_l(A)$ is formed by quasi-compacts subsets that are closed by finite intersections. So the definition of (i) and (ii) of spectral spaces are satisfied by $\mathrm{Spec}_l(A)$. The condition (iii) is essentially [4, 10.1.7], except for the irreducibility. Let's see this in detail. Let F be a non-empty closed irreducible subset of $\mathrm{Spec}_l(A)$. By [4, 10.1.7] one has that $F = H(p)$ with p an l-ideal of A, where $H(p) = \mathrm{Spec}_l(A) \setminus S(p)$ and $S(p) = \{q \in \mathrm{Spec}_l(A) : p \not\subseteq q\}$. We want to see that p is irreducible. Let a and b be two l-ideals of A such that $a \cap b = p$. Then $F = H(p) = H(a) \cup H(b)$, cf. [4, 10.1.11(ii)]. Since F is irreducible then $H(a) = F$ or $H(b) = F$. If $H(a) = F = H(p)$ then $S(a) = S(p)$ and by [4, 10.1.3] one has that $a = p$. Similarly if $H(b) = F = H(p)$, then $b = p$. This shows that $F = H(p) = \overline{\{p\}}$, with $p \in \mathrm{Spec}_l(A)$. In short, section (10.1) of [4, chapter 10] establish that $\mathrm{Spec}_l(A)$ is a spectral space if A is an f-ring with unity.

The result [4, (9.1.5)] says that if A is an f-ring and $p \in \mathrm{Spec}_l(A)$ then A/p is a totally ordered ring. Nevertheless, if A is reduced then not necessarily A/p is an intregral domain. In order that those quotients will be integral domains with restrict to the subspace:

$$Y = \{p \in \mathrm{Spec}_l(A) : p \text{ is prime}\},$$

see the section [4, (9.2)] and in particular [4, (9.2.5)]. Therefore, for each $p \in Y$ one has that A/p is a totally ordered integral domain.

Is Y as a topological subspace of $\mathrm{Spec}_l(A)$ a spectra space ? By [10, 2.1.3], it is sufficient to see that Y is proconstructible in $\mathrm{Spec}_l(A)$. Or equivalently that $\mathrm{Spec}_l(A) \setminus Y$ is an open set in the constructible topology of $\mathrm{Spec}_l(A)$. To see this, let $p_0 \in \mathrm{Spec}_l(A) \setminus Y$. Then p_0 is an irreducible l-ideal that it is not a prime ideal. There exists $a, b \in A$ such that $ab \in p_0$ with $a \notin p_0$ and $b \notin p_0$. Let $\mathcal{O} = V(ab) \cap D(a) \cap D(b)$ an open set in the constructible topology of $\mathrm{Spec}_l(A)$, so that $p_0 \in \mathcal{O}$ and $\mathcal{O} \subseteq \mathrm{Spec}_l(A) \setminus Y$ (none $p \in \mathcal{O}$ will be prime). Therefore $Y = \{p \in \mathrm{Spec}_l(A) : p \text{ is prime}\}$ is also a spectral space.

Summarizing one has that if A is a (reduced) f-ring, there exists $Y \subseteq \mathrm{Spec}_l(A)$ a spectral space such that A/p is a totally ordered integral domain, for all $p \in Y$. Let's observe that by [12, Lemma 2.3], if A is an f-ring satisfying the first convexity property then A/p also satisfies this property, for all $p \in Y$. We are ready to announce and prove the following proposition.

Proposition 4.6 *Let A be a reduced f-ring satisfying the first convexity property and the divisibility glueing axioms. Then there exists $B \models T^*$ such that A is a substructure of B in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$.*

Proof: Let A be a reduced f-ring satisfying the first convexity property and the divisibility glueing property. Let's consider Y any spectral space in such a way that A/p is a totally ordered integral domain, for all $p \in Y$. Let X be any proconstructible subspace of Y containing πA (by [4, 9.3.2] one has that $\pi A \subseteq Y$). In particular X can be $\overline{\pi A}$ the closure of πA in the constructible topology of Y.

Let's consider $\iota\colon A \to \prod_{p\in X} A/p,\ a \mapsto (a+p)_{p\in X}$. Clearly ι is an homomorfism of lattice-ordered rings. Since A is reduced and X contains πA, then ι in an embedding. Therefore $\iota\colon A \hookrightarrow \prod_{p\in X} A/p$ is a monomorfism of lattice-ordered rings. Clearly, if $a, b \in A$ such that $b \mid a$ in A then $\iota(b) \mid \iota(a)$ in $\prod_{p\in X} A/p$. We want to see the inverse implication.

Let $a, b \in A$ such that $\iota(b) \mid \iota(a)$ in $\prod_{p\in X} A/p$. Let's denote $A_x = A/p$ for $x \in X$. Then $b(x)$ divides $a(x)$ in A_x, for all $x \in X$. There exists $c_x \in A_x$ such that $b(x) = c_x = a(x)$, for all $x \in X$. Since A is an f-ring, there exists $\tilde{c}_x \in A$ such that $\tilde{c}_x(x) = c_x$, for all $x \in X$. Then $b(x)\tilde{c}_x(x) = a(x)$, for all $x \in X$. Therefore $x \in [\![\, b\tilde{c}_x = a \,]\!] = N_x$ a clopen set in the constructible topology of X. Therefore:

$$X = \bigcup_{x\in X} N_x,$$

and by compactness of X, there exists $x_1, \ldots, x_n \in X$ such that:

$$X = \bigcup_{i=1}^{N} N_{x_i}.$$

Let's dentote $c_i = \tilde{c}_{x_i}$ and $N_i = N_{x_i}$, for all $i = 1, \ldots, n$. We have that $X = \bigcup_{i=1}^{n} N_i$ and therefore every $x \in X$ satisfies $b(x)c_i(x) = a(x)$ for some $i \in \{1, \ldots, n\}$. Then $\big(b(x)c_1(x) - a(x)\big) \cdots \big(b(x)c_n(x) - a(x)\big) = 0$, for all $x \in X$. That is:

$$(bc_1 - a) \cdots (bc_n - a) = 0.$$

By the divisibility glueing property of A one has that $b \mid a$. We have proved that:

$$A \models b \mid a \text{ if and only if } \prod_{p\in X} A/p \models \iota(b) \mid \iota(a).$$

Therefore ι is a lattice-ordered monomorphism ring that respects the divisibility relation. Note that $\prod_{p\in X} A/p$ is a reduced and projectable f-ring, then by proposition 4.1, there exists $B \models T^*$ such that $\prod_{p\in X} A/p \subseteq B$ in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$. Therefore A can be embedded in B a model of T^* in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$. ∎

In the previous proof, the space X could have been the whole $Y = \{p \in \mathrm{Spec}_l(A) : p \text{ is prime}\}$. Or also it could have been $X = \overline{\pi A}^{\mathrm{con}}$ where πA it is seen as a subspace of Sper(A) the real spectrum of A. Or also, even πA could have been seeing as a subspace of RCVR-Spec(A), cf. [21]. We can now establish the following propostion.

Proposition 4.7 *The universal theory of T^* in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|\}$ is the theory of reduced f-rings satisfying the first convexity property and the divisibility glueing property.*

Proof: This result is deduced by corolary 4.4 and by propostion 4.6. ∎

In the last part of this section, we exhibit the universal theory of T^* adding the local divisibilty and considering as well the radical relation in the languaje. The following proposition goes in that sense.

Proposition 4.8 *Let A be a reduced and projectable f-ring. Then A satisfies:*

$$\forall a \forall b \forall c (a \not\preceq bc - a \rightarrow b \,|_{\mathrm{loc}}\, a). \qquad (\star)$$

Proof: Let A be a reduced and projectable f-ring. Then $A \in \Gamma^a_{\mathcal{L}_{\mathrm{or}}}\big(X, (A_x)_{x\in X}\big)$, where $X = \pi A$ is the space of minimal prime ideals and $(A_x)_{x\in X}$ is a family of totally ordered integral domains. Let $a, b, c \in A$ such that $a \not\preceq bc - a$. By the equivalent way to express the radical relation in termos of X, there exists $x \in X$ such that $a(x) \neq 0$ and $(bc - a)(x) = 0$. Let's declare $N = [\![\, a \neq 0 \,]\!] \cap [\![\, bc - a = 0 \,]\!]$. Therefore $x \in N$ and N is a non-empty clopen set. By the patchwork property in A, there exists $w \in A$ such that $w = a_{\restriction N} \cup 0_{\restriction X \setminus N}$. Let's put $c' = c_{\restriction N} \cup 0_{\restriction X\setminus N} \in A$. By the definition of N one has that $w \neq 0$, $w(w - a) = 0$ and $bc' = w$. We have proved that $\exists w (w \neq 0 \wedge w(w - a) = 0 \wedge b \mid w)$ en A, and this is precisely that $b \,|_{\mathrm{loc}}\, a$. ∎

It is clear that $a \neq 0$ in the context of the previous proof. In general, if the formula $(\star)$ is valid for any radical relation $\preceq$ then $a \neq 0$. For if $a = 0$ then $0 \not\preceq bc$, and this is a contradiction since $0 \preceq d$, for all $d \in A$. We have the following corolaries.

Corollary 4.9 *Let B be reduced and projectable f-ring, with A a substructure of B in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$. Therefore A satisfies:*

$$\forall a \forall b \forall c (a \not\preceq bc - a \rightarrow b \,|_{\mathrm{loc}}\, a).$$

Proof: It is obviously deduced from proposition 4.8. ∎

Corollary 4.10 *Let $B \models T^*$ with A a substructure of B in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{\preceq, |_{\mathrm{loc}}\}$. Therefore A satisfies:*

$$\forall a \forall b \forall c (a \not\preceq bc - a \rightarrow b \,|_{\mathrm{loc}}\, a).$$

∎

The formula $\forall a \forall b \forall c(a \not\preceq bc - a \to b \mid_{\text{loc}} a)$ establish a compatibility between the radical relation $\preceq$ and the local divisibility.

Definition 4.11 *Let A be any ring. We say that A has the* **local divisibility property** *if A satisfies $\forall a \forall b \forall c\big((a \not\preceq bc - a) \to b \mid_{\text{loc}} a\big)$.*

We are in measure to prove the following proposition.

Proposition 4.12 *Let A be any reduced f-ring satisfying the first convexity property and the local divisibility property. Therefore there exists $B \models T^*$ such that $A \subseteq B$ as a substructure in the languaje $\mathcal{L}_{\text{lor}} \cup \{\preceq, \mid_{\text{loc}}\}$.*

Proof: Let A be a reduced f-ring satisfying the first convexity property and the local divisibility property. Let's consider, as in the proposition 4.6, a spectral space $Y \subseteq \text{Spec}_l(A)$ such that A/p is a totally ordered integral domain, for all $p \in Y$. Let's consider also a proconstructible subset X of Y containing πA; in particular X can be $\overline{\pi A}^{\text{con}}$ the closure of πA in the constructible topology of Y.

As in the proposition 4.6, let's consider $\iota\colon A \to \prod_{p\in X} A_p$ the lattice-ordered ring homorphism given by $\iota(a) = (a+p)_{p\in X}$. Since X contains πA and A is reduced, then ι is a monomorphism. Also ι respects the radical relation for X contains πA and therefore defines $\preceq$. We want to see that ι respects the local divisibility. Let $a, b \in A$. If $b \mid_{\text{loc}} a$ in A then clearly $\iota(b) \mid_{\text{loc}} \iota(a)$ in $\prod_{p\in X} A_p$. Let's suppose now that $\iota(b) \mid_{\text{loc}} \iota(a)$ in $\prod_{p\in X} A_p$. If $\iota(a) = 0$ then $a = 0$ and clearly $b \mid_{\text{loc}} a$ in A. Let's suppose that $\iota(a) \neq 0$. Let's denote $\prod_{p\in X} A_p = \prod_{x\in X} A_x$, where A_x is a totally ordered integral domain, for all $x \in X$ and X is a Boolean space (since it is a closed subset in the constructible topology). We have that $\iota(a) = \big(a(x)\big)_{x\in X} \neq 0$. Then there exists $w = (w_x)_{x\in X} \in \prod_{x\in X} A_x$ such that $w \neq 0$ and $w\big(w - \iota(a)\big) = 0$ with $\iota(b) \mid w$ in $\prod_{x\in X} A_x$. Then there exists $c = (c_x)_{x\in X} \in \prod_{x\in X} A_x$ such that $\iota(b)c = w$. Since $w \neq 0$, there exists $x_0 \in X$ such that $w_{x_0} \neq 0$. Therefore $w_{x_0} = a(x_0)$ for $w\big(w - \iota(a)\big) = 0$. There exists $c \in A$ tsuch that $c(x_0) = c_{x_0}$ and then we have $b(x_0)c(x_0) = a(x_0)$ with $a(x_0) \neq 0$. That is $a(x_0) \neq 0$ and $(bc - a)(x_0) = 0$. Since the radical relation is defined by πA or by any proconstructible subset that contains it, one has that $a \not\preceq bc - a$ in A. Since A satisfies the local divisibility property, then $b \mid_{\text{loc}} a$ in A. We have showned that $\iota\colon A \to \prod_{p\in X} A_p$ respects the local divisibility.

Since A satisfies the first convexity property, then A_x satisfies it, for all $x \in X$. Claearly one has that $\prod_{p\in X} A_p$ satisfies the first convexity property as well. Then $\prod_{p\in X} A_p$ is a reduced and projectable f-ring satisfying the first convexity property. By proposition 4.1, there exists $B \models T^*$ such that $\prod_{x\in X} A_x \subseteq B$ in the languaje $\mathcal{L}_{\text{lor}} \cup \{\preceq, \mid_{\text{loc}}\}$. Finaly one has that $A \subseteq B$ in the languaje $\mathcal{L}_{\text{lor}} \cup \{\preceq, \mid_{\text{loc}}\}$. ■

By proposition 4.7, corollary 4.10 and proposition 4.12, it is obvious to deduce the following result.

Proposition 4.13 *The universal theory of T^* in the languaje $\mathcal{L}_{\text{lor}} \cup \{\mid, \preceq, \mid_{\text{loc}}\}$ is the theory of reduced f-rings satisfying the first convextiy property, the divisibility glueing property and local divisibility property.* ■

By theorem 3.2, T^* is also model complete in the language $\mathcal{L}_{\text{lor}} \cup \{\preceq, |, |_{\text{loc}}\}$. In view of proposition 4.13, we have the following result.

Theorem 4.14 *The theory T^* is the model companion of the theory of reduced f-rings satisfying the first convexity property, the divisibility glueing axioms and the local divisibility property in the language $\mathcal{L}_{\text{lor}} \cup \{\preceq, |, |_{\text{loc}}\}$.*

■

5 The maximal local divisibility relation.

In this section, we study equivalent forms of the local divisibility relation valid in the theory of reduced projectable and divisible-projectable f-rings (and so in the theory T^*). One of this forms will be distinguish since it permits us to have the appropiate control over the fibers where the divisibility is carried out, locally speaking. We start with a simple fact.

Fact 5.1 *Let A be any ring. Then for all $a, b \in A$ one has that:*

$$b \,|_{\text{loc}}\, a \leftrightarrow a = 0 \vee \exists w\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big).$$

Proof: ($\Leftarrow$) is clear.

($\Rightarrow$) It is obvious if $a = 0$. If not, there exists $w \neq 0$ such that $w(w-a) = 0$ and $b \mid w$. Let's see that $w \preceq a$, that is: $\forall p \in \pi A(a \in p \Rightarrow w \in p)$. Let $p \in \pi A$ tsuch that $a \in p$. Then $wa \in p$. Since $w(w-a) = 0$ then $w^2 = wa$ and therefore $w^2 \in p$. Then $w \in p$. ■

Observation 5.2 *Let A be any ring and let $a, b \in A$ such that $\exists e(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$ then $\exists w\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big)$.*

Proof: Let's declare $w = ae \neq 0$. Note that $w(a-w) = ae(a-ae) = a^2e(1-e) = 0$. Since for all $x \in A$ one has that $x \preceq 1$, then $e \preceq 1$ and by the (4) in the definition of radical relations $ea \preceq a$, and then $w \preceq a$. Clearly $b \mid w$. ■

Observation 5.3 *Let A be a reduced and projectable f-ring. Let $a, b \in A$ such that $\exists w\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big)$ then $\exists e(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$.*

Proof: Let's suppose there exists $w \in A$ such that $w \neq 0$, $w(a-w) = 0$, $w \preceq a$ and $b \mid w$. The idempotent $e \in A$ is the support of w: it is constructed using the projectability of A, for: $1 = c + d$, where $c \in w^{\perp}$ and $d \in w^{\perp\perp}$, (see [15, Lema 3.3]). Then $c \cdot w = 0$ and $d \preceq w$. Observe that $d \preceq w$ and $w \preceq a$ implies $d \preceq a$. Since $w \cdot c = 0$ then $c \cdot d = 0$. Therefore $c = c \cdot 1 = c(c+d) = c^2 + cd = c^2 + 0 = c^2$. Then c is an idempotent and $d = 1 - c$ is also an idempotent. Let's declare $e = d$. We want to see that $ad \neq 0$. Let's suppose by contradiction that $ad = 0$. Since $w \preceq a$ then $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(w)$, and since $ad = 0$ then $d \in \operatorname{Ann}(a)$, and therefore $d \in \operatorname{Ann}(w)$, that is: $dw = 0$. Since $d \preceq w$, then $\operatorname{Ann}(w) \subseteq \operatorname{Ann}(d)$; therefore $d = d^2 = 0$. Then $c = 1$ and $w = 0$, a contradiction since by hypothesis one has that $w \neq 0$. Therefore $ad \neq 0$. One has initially that $d \preceq w$. Let's see that also $w \preceq d$, that is: $\mathrm{Ann}(d) \subseteq \mathrm{Ann}(w)$. Let $x \in A$ such that $dx = 0$. Since $(1-c)x = 0$ then $x = cx$. Since $wc = 0$ then $wx = wcx = 0$. We have seen that $\forall x(dx = 0 \Rightarrow wx = 0)$, that is: $w \preceq d$. One has that $d \preceq w \wedge w \preceq d$. Then $w =_s d$.

Since $w \preceq a$ then $wd \preceq ad$ and observe that $w = w \cdot 1 = w(c+d) = wc + wd = 0 + wd = wd$. Then $w \preceq ad$. Also $d \preceq w$ and then $ad \preceq aw = w^2$. Therefore $ad \preceq w$, for if $w^2 \preceq w$ then $w \preceq 1$. Then $w \preceq ad$ and $ad \preceq w$, that is: $w =_s ad$.

Now let's see that $w = ad$. For this we are going to use that the ring A is reduced. Let $p \in \pi A$. If $w \notin p$ then $a - w \in p$ for $w(a-w) = 0 \in p$. Then $(a-w)d \in p$. And see that $(w-a)d = wd - ad = w(1-c) - ad = w - ad \in p$. If $w \in p$, since $w =_s ad$ then $ad \in p$, and in particular $w - ad \in p$. Therefore one has $w - ad \in \bigcap\{p : p \in \pi A\} = \{0\}$, because A is reduced. Then $w = ad$ and therefore $b \mid ad$. We have showed that $\exists d(d^2 = d \wedge ad \neq 0 \wedge d \preceq a \wedge b \mid ad)$.

■

Proposition 5.4 *Let A be a reduced and projectable f-ring and $a, b \in A$. The following assertions are equivalent:*

(i) $\exists w\big(w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w\big)$,

(ii) $\exists e(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae)$.

Proof: This is comes from observations 5.2 and 5.3.

■

Proposition 5.5 *Let A be a reduced and projectable f-ring and $a, b \in A$. The following assertions are equivalent on A:*

(i) $\exists w\big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \forall w'(w' \neq 0 \wedge w'(a-w') = 0 \wedge w' \preceq a \wedge b \mid w' \rightarrow w' \preceq w)\big]$,

(ii) $\exists e\big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \rightarrow e' \leqslant e)\big]$.

Proof: Let's first prove that (ii)⇒(i): Let's suppose (ii) and let's declare $w = ae$. Clearly $w \neq 0$ and $w(a-w) = 0$ is the same calculation on previous proposition. It is clear tha $w \preceq a$ for $e \preceq 1$ and multiplying by a one obtains the result. Evidently $b \mid w$. Let $w' \neq 0$ with $w'(a-w') = 0$, $w' \preceq a$ and $b \mid w'$. As in the previous proposition we set $1 = c' + d'$ with $c' \cdot w' = 0$ and $d' \preceq w'$. Then c' are d' idempotents. Since $d' \preceq w'$ and $w' \preceq a$ then $d' \preceq a$. Similarly as in the previous proposition, one sees that $ad' \neq 0$. It can also be showed that $w' \preceq d'$, and therefore $w' =_s d'$. In a similar way it is showed that $w' = ad'$ and evidently $b \mid ad'$. As we are supposing (ii) the one has that $d' \leqslant e$, that is $d' \preceq e$. Then $ad' \preceq ae$, that is $w' \preceq ae = w$.

Let's prove now (i)⇒(ii): Let's suppose (ii) and let $w \in A$ satisfying (i). We set $1 = c+d$ with $cw = 0$ and $d \preceq w$. As before, c and d are idempotents such that $w = ad \neq 0$ and $d \preceq a$ with $b \mid ad$. In fact, one has that $w \preceq d$ and therefore $d =_s w$. The idempotent we are looking for $e = d$. Let e' be an idempotent such that $ae' \neq 0$, $e' \preceq a$ and $b \mid ae'$. Let's put $w' = ae' \neq 0$ and see that $w'(a-w') = 0$ with $w' \preceq a$ and $b \mid w'$. Then $w' \preceq w$. That is $ae' \preceq ad$. We want to show that $e' \leqslant d$. Let $p \in \pi A$ such that $e' \notin p$, that is $e' - 1 \in p$. Since $e' \preceq a$ then $a \notin p$. Therefore $ae' \notin p$. Since $w' = ae'$ then $w' \notin p$ and therefore

$w = ad \notin p$. That means $a \notin p$ and $d \notin p$, in particular $d - 1 \in p$. We have showed for all $p \in \pi A$, if $e' \notin p$ then $d \notin p$. Then $e' \preceq d$, that is $e' \leqslant d$. ■

The following proposition gives differents equivalent assertions of statement (ii) in proposition 5.5:

Proposition 5.6 *Let A be a reduced and projectable f-ring and $a, b \in A$. Then the following assertions are equivalent on A:*

(i) $\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \to b \nmid a(1-e)f\big)\Big)\Big],$$

(ii) $\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge af \neq 0 \wedge f \preceq a(1-e) \to b \nmid af\big)\Big)\Big],$$

(iii) $\exists w\Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a - w = 0 \to b \mid a\big) \wedge$
$$\Big(a - w \neq 0 \to \forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \to b \nmid w'\big)\Big)\Big].$$

Proof: (i)⇒(ii): Let's suppose (i). Let $e \in A$ satisfying (i). Obviously e satisfies the first five conjunctions of (ii). Let's see the last one. Let's suppose that $a(1-e) \neq 0$ and let $f \in A$ be such that $f^2 = f$, $af \neq 0$ and $f \preceq a(1-e)$. Then $ef \preceq a(1-e) \cdot e = 0$ and $ef = 0$. Therefore $a(1-e)f = a(f - ef) = af \neq 0$ and $b \nmid af$. Therefore e satisfies all conjonctions of (ii).

(ii)⇒(iii): Let's suppose there is $e \in A$ satisfying the six conjonctions of (ii). Let's put $w = ae$. Then $w \neq 0$, $w(a-w) = 0$, $w \preceq a$ (cause $e \preceq 1$, and $ae \preceq a$) and $b \mid w$.

If $a - w = 0$ then $a - ae = a(1-e) = 0$ and by (ii) one has $b \mid a$. If $a - w \neq 0$ then $a(1-e) = a - w \neq 0$ and therefore:

$$\forall f\big(f^2 = f \wedge af \neq 0 \wedge f \preceq a(1-e) \to b \nmid af\big).$$

We want to see if $\forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \to b \nmid w'\big)$ is true in A. Let $w' \in A$ be such that $w' \neq 0$, $w'\big((a-w) - w'\big) = 0$ and $w' \preceq a - w$. Since A is projectable then $1 = c' + d'$ with $c' \cdot w' = 0$ and $d' \preceq w'$. Since $w' \preceq a - w$ then $d' \preceq a - w = a - ae = a(1-e)$. Therefore $ed' \preceq ea(1-e) = a(1-e)e = 0$ and consequently $ed' = 0$; that is $d' \leqslant 1 - e$ (or $e \leqslant 1 - d'$). One also has, as before, that $c'd' = 0$ and c', d' are idempotents. Similarly one sees that $w' \preceq d'$ because $\mathrm{Ann}(d') \subseteq \mathrm{Ann}(w')$; for if $x \in A$ is such that $d'x = 0$ then $xc' = x$ and therefore $xw' = xc'w' = x0 = 0$; that is $x \in \mathrm{Ann}(w')$. Then $w' =_s d'$.

Let's see now that $w' = a(1-e)d'$. Let $p \in \pi A$. If $w' + p = 0$ then $w' \in p$, consequently $d' \in p$ and also $a(1-e)d' \in p$; that is $a(1-e)d' + p = 0$. In that case $w' + p = a(1-e)d' + p$. Now, if $w' + p \neq 0$, since $w'\big((a-w) - w'\big) = 0 \in p$ then $w' + p = (a-w) + p = a(1-e) + p$; in that case $d' + p \neq 0$ and therefore $d' + p = 1 + p$. Then $w' + p = a(1-e)d' + p$. We have seen that $w' + p = a(1-e)d' + p$, for all $p \in \pi A$. Since A is reduced, one has $w' = a(1-e)d'$.

We have that $d'^2 = d'$ and see that $0 \neq w' = a(1-e)d' = a(d' - ed') = ad'$, since $ed' = 0$. Then $ad' \neq 0$ and one also has $d' \preceq a(1-e)$. Then $b \nmid ad'$. Since $a(1-e)d' = ad'$ then $b \nmid a(1-e)d'$. That is $b \nmid w'$.

(iii)⇒(i): Let's suppose there exists $w \neq 0$ satisfying (iii). By projectability on the ring A one has that $1 = c + d$ wiht $c \cdot w = 0$ and $d \preceq w$. In fact we also had $w \preceq d$, that is $d =_s w$. It is showed that $d \preceq a$ and $w = ad$ as before (A is reduced). Let's put $e = d$ and one has $w = ad \neq 0$ and $b \mid ad$. Clearly if $a(1-d) = 0$ then $b \mid a$. Now if $a(1-d) \neq 0$, we want to see:

$$\forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f\big)\big)$$

is true in A. Let $f \in A$ satisfying the first tree conjunctions. Let's put $w' = a(1-e)f \neq 0$, and see that $w'\big((a-w)-w'\big) = w'\big((a-ad)-w'\big) = w'\big(a(1-d)-w'\big) = w'\big(a(1-e)-a(1-e)f\big) = w'\big(a(1-e)(1-f)\big) = a(1-e)fa(1-e)(1-f) = \big(a(1-e)\big)^2 f(1-f) = \big(a(1-e)\big)^2 0 = 0$. Now see that $w' = a(1-e)f \preceq a(1-e)$ cause $f \preceq 1$ and multiplying by $a(1-e)$ each side the result is obtained. By (iii) one has that $b \nmid w'$, es decir $b \nmid a(1-e)f$. ■

In the following proposition we will show that the assertions using idempotentes in propositions 5.5 and 5.6 are equivalents.

Proposition 5.7 *Let A be any ring and $a, b \in A$. Then the following assertions on A are equivalent:*

(i) $\exists e\big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \rightarrow e' \leqslant e)\big]$,

(ii) $\exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge \Big(a(1-e) \neq 0 \rightarrow \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f\big)\Big)\Big]$.

Proof: (i)⇒(ii): Let $e \in A$ satisfying (i). We want to show the las two conjunctions of (ii).

• If $a(1-e) = 0$ then $a = ae$ and evidently $b \mid a$.

• If $a(1-e) \neq 0$. We want to see the last conjunction. For this, let $f \in A$ such that $f^2 = f$, $a(1-e)f \neq 0$ and $f \preceq a(1-e)$. We suppose by contradiction that $b \mid a(1-e)f$. Observe that multiplying $f \preceq a(1-e)$ by e on both one has consequently $ef = 0$. Then $a(1-e)f = af$. Therefore $af \neq 0$ and $b \mid af$. Since $f \preceq a(1-e)$, then multiplying by f on both sides ones has that $f^2 \preceq a(1-e)f$, tha is: $f \preceq af$. Since $f \preceq 1$, then $af \preceq a$ and by transitivity one has $f \preceq a$. One gets $f^2 = f$, $af \neq 0$, $b \mid af$ and $f \preceq a$. Therefore by (i) one gets $f \leqslant e$. Then $f^2 \leqslant ef$, that is: $f \leqslant ef = 0$. So $f = 0$. But this contradicts one of our initial assumptions: $af \neq 0$. So, we must have $b \nmid a(1-e)f$.

(ii)⇒(i): Let $e \in A$ satisfying (ii). It is clear that e satisfies the first four conjonctions of (i). We want to see that e satisfies the fifth one. Let $e' \in A$ be such that $e'^2 = e'$, $ae' \neq 0$, $e' \preceq a$ and $b \mid ae'$. We should see $e' \leqslant e$, or equivalently $e'(1-e) = 0$.

• If $a(1-e) = 0$ then $1-e \in \mathrm{Ann}(a)$. Since $e' \preceq a$ then $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(e')$ and therefore $1-e \in \mathrm{Ann}(e')$, that is: $e'(1-e) = 0$.

- If $a(1-e) \neq 0$. We have by (ii):

$$\forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f)\big).$$

Let's suppose by contradiction that $e'(1-e) \neq 0$. Let's see that $a(1-e)e' \neq 0$: for if $a(1-e)e' = 0$ then $(1-e)e' \in \mathrm{Ann}(a)$, and since $e' \preceq a$ then $(1-e)e' \in \mathrm{Ann}(e')$, and therefore $(1-e)e' \cdot e' = (1-e)e'^2 = (1-e)e' = 0$. Then we have that $a(1-e)e' \neq 0$. Let's put $f = (1-e)e'$, obviously $f^2 = f$. See also that $a(1-e)f = a(1-e)e' \neq 0$. Since $e' \preceq a$, then $e'(1-e) \preceq a(1-e)$, that is $f \preceq a(1-e)$. Then we have $b \nmid a(1-e)f$. But $a(1-e)f = a(1-e)(1-e)e' = a(1-e)e' = ae' - aee'$. Since $b \mid ae'$ then $b \mid aee'$ and since $b \mid ae'$, then $b \mid ae' - aee' = a(1-e)e'$. We have come to that $b \mid a(1-e)e'$ and $b \nmid a(1-e)e'$, a contradiction. Therefore $e'(1-e) = 0$ and $e' \leqslant e$. ■

Observation 5.8 Proposition 5.7 can be formulated in the following terms: in any ring A and for any $a, b, e \in A$, the following assertions are equivalents:

(i) $e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \forall e'(e'^2 = e' \wedge ae' \neq 0 \wedge e' \preceq a \wedge b \mid ae' \rightarrow e' \leqslant e)$,

(ii) $e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \rightarrow b \mid a\big) \wedge$
$\Big(a(1-e) \neq 0 \rightarrow \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \rightarrow b \nmid a(1-e)f)\big)\Big)$.

Observe by the proof of proposition 5.7 that the element $e \in A$ satisfying (i) and (ii) is the same one. And observe also that the element e given by proposition 5.7 (i) is unique; for if $\bar{e}_1$ and $\bar{e}_2$ are two idempotents satisfying (i), then $\bar{e}_1 \leqslant \bar{e}_2$ and $\bar{e}_2 \leqslant \bar{e}_1$. Therefore $\bar{e}_1 = \bar{e}_2$. [In fact, by condition (i) it is evidently that this element $e \in A$ is unique.] ■

Corollary 5.9 *Let A be a reduced and projectable f-ring, and let and $a, b \in A$. Then the following assertions are equivalents:*

(i) $\exists w\big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \forall w'(w' \neq 0 \wedge w'(a-w') = 0 \wedge w' \preceq a \wedge b \mid w' \rightarrow w' \preceq w)\big]$,

(ii) $\exists w\Big[w \neq 0 \wedge w(a-w) = 0 \wedge w \preceq a \wedge b \mid w \wedge \big(a - w = 0 \rightarrow b \mid a\big) \wedge \Big(a - w \neq 0 \rightarrow \forall w'\big(w' \neq 0 \wedge w'\big((a-w) - w'\big) = 0 \wedge w' \preceq a - w \rightarrow b \nmid w'\big)\Big)\Big]$.

Proof: It is deduced by propositions 5.6, 5.7 y 5.5. ■

The previous corollary 5.9 was deduced easily using some propositions under the hypothesis of projectability and reducibility of the ring. The author thinks this equivalence may be proved directly under more general assumptions. The following proposition uses the divisible-projectability of the ring and proves that the local divisibility is equivalent to one of it previous "stroger" forms.

Proposition 5.10 *Let B be a reduced and divisible-projectable f-ring, and let $a, b \in B$. The following assertions are equivalents:*

(i) $b \mid_{\mathrm{loc}} a$ *and* $a \neq 0$,

(ii) $\exists w\Big[w\neq 0 \wedge w(a-w)=0 \wedge w\preceq a \wedge b\mid w \wedge \big(a-w=0\to b\mid a\big)\wedge$

$$\Big(a-w\neq 0\to \forall w'\big(w'\neq 0\wedge w'\big((a-w)-w'\big)=0\wedge w'\preceq a-w\to b\nmid w'\big)\Big)\Big].$$

Proof: (ii)⇒(i) is obvious. Let's do (i)⇒(ii). Let's suppose that $b\mid_{\mathrm{loc}} a$ and $a\neq 0$. Then $b\neq 0$. Since B is divisible-projectable then there exists $a_1,a_2\in B$ such that $a=a_1+a_2$, $a_1\cdot a_2=0$, $b\mid a_2$ and, $b\nmid_{\mathrm{loc}} a_2$ or $a_2=0$. Observe that if $a_1=0$ then $a_2=a$ and then $b\nmid_{\mathrm{loc}} a$, since $a\neq 0$. Therefore $a_1\neq 0$ and $a_1\cdot a_2=0$ says that $a_1(a-a_1)=0$ with $b\mid a_1$. From $a_1(a-a_1)=0$ one has that $a_1\preceq a$. Then there exists $w=a_1\neq 0$ such that $w(a-w)=0$, $w\preceq a$ and $b\mid w$. If $a_2=0$ then $a-a_1=a-w=0$ and $a_1=a$ with $b\mid a$. If $a_2\neq 0$ then $a_2=a-a_1\neq 0$ and one gets $b\nmid_{\mathrm{loc}} a-a_1$. This non local divisibility means by fact 5.1 that $\forall w'\big(w'\neq 0\wedge w'\big((a-w)-w'\big)=0\wedge w'\preceq a-w\to b\nmid w'\big)$. We have showned (ii).
■

The following proposition resumes all equivalent forms of local divisibility in the reduced projectables and divisible-projectables f-rings.

Proposition 5.11 *Let A be a reduced projectable and divisible-projectable f-ring ; and let $a,b\in A$. The following assertions are equivalent:*

(i) $b\mid_{\mathrm{loc}} a\wedge a\neq 0$,

(ii) $\exists w\big(w\neq 0\wedge w(a-w)=0\wedge w\preceq a\wedge b\mid w\big)$,

(iii) $\exists e(e^2=e\wedge ae\neq 0\wedge e\preceq a\wedge b\mid ae)$,

(iv) $\exists e\big[e^2=e\wedge ae\neq 0\wedge e\preceq a\wedge b\mid ae\wedge$
$$\forall e'(e'^2=e'\wedge ae'\neq 0\wedge e'\preceq a\wedge b\mid ae'\to e'\leqslant e)\big],$$

(v) $\exists w\big[w\neq 0\wedge w(a-w)=0\wedge w\preceq a\wedge b\mid w\wedge$
$$\forall w'(w'\neq 0\wedge w'(a-w')=0\wedge w'\preceq a\wedge b\mid w'\to w'\preceq w)\big],$$

(vi) $\exists e\big[e^2=e\wedge ae\neq 0\wedge e\preceq a\wedge b\mid ae\wedge$
$$\forall e'(e'^2=e'\wedge ae'\neq 0\wedge e'\preceq a\wedge b\mid ae'\to e'\leqslant e)\big],$$

(vii) $\exists e\Big[e^2=e\wedge ae\neq 0\wedge e\preceq a\wedge b\mid ae\wedge\big(a(1-e)=0\to b\mid a\big)\wedge$
$$\Big(a(1-e)\neq 0\to\forall f\big(f^2=f\wedge a(1-e)f\neq 0\wedge f\preceq a(1-e)\to b\nmid a(1-e)f\big)\Big)\Big],$$

(viii) $\exists w\Big[w\neq 0\wedge w(a-w)=0\wedge w\preceq a\wedge b\mid w\wedge\big(a-w=0\to b\mid a\big)\wedge$
$$\Big(a-w\neq 0\to\forall w'\big(w'\neq 0\wedge w'\big((a-w)-w'\big)=0\wedge w'\preceq a-w\to b\nmid w'\big)\Big)\Big].$$

Proof: It is deduced by all previous propositions and observations.
■

6 Quantifier elimination.

In the previous section we saw different equivalents ways of expressing the local divisibility in reduced projectable and divisible-projectables f-rings, and in particular in models of the theory T^*. Hopefully, one of this maximal local divisibility expression will permit us more control over the fibers where the divisibility is carried out in order to prove a quantifier elimination result. In that sense, we define:

Definition 6.1 *Let A be any ring and let $a, b \in A$. We define a binary relation called* **maximal local divisibility** *by:*

$$b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a \leftrightarrow a = 0 \vee \exists e\Big[e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae \wedge \big(a(1-e) = 0 \to b \mid a\big) \wedge$$
$$\Big(a(1-e) \neq 0 \to \forall f\big(f^2 = f \wedge a(1-e)f \neq 0 \wedge f \preceq a(1-e) \to b \nmid a(1-e)f\big)\Big)\Big].$$

If A is any ring, then clearly:

$$A \models \forall a \forall b\Big(b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a \to a = 0 \vee \exists e\big(e^2 = e \wedge ae \neq 0 \wedge e \preceq a \wedge b \mid ae\big)\Big),$$

and by observation 5.2, one has that:

$$A \models \forall a \forall b(b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a \to b \,|_{\mathrm{loc}}\, a).$$

If in addition A is a reduced, projectable and divisible-projectable f-ring, then the equivalence between (i) and (vii) of proposition 5.11 permits us to write:

$$A \models \forall a \forall b(b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a \leftrightarrow b \,|_{\mathrm{loc}}\, a).$$

It's worth noting that the previous formula is valid in any model of T^*, that is:

$$T^* \vdash \forall a \forall b(b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a \leftrightarrow b \,|_{\mathrm{loc}}\, a).$$

In this section we are considering the languaje $\mathcal{L} = \mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$. First of all, we are going to adapt the characterizations of the universal theories in the differents languajes given in section 4 to this new languaje $\mathcal{L}$. In that sense we have:

Proposition 6.2 *Let B be reduced, projectable and divisible-projectable f-ring. Then:*

$$B \models \forall a \forall b \forall c(a \not\preceq bc - a \to b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a).$$

Proof: Observe that this follows by previous comments and by proposition 4.8. ■

Corollary 6.3 *Let B be a reduced, projectable and divisible-projectable f-ring and let A be any ring such that $A \subseteq B$ as a substructure in the languaje $\mathcal{L}$. Then:*

$$A \models \forall a \forall b \forall c(a \not\preceq bc - a \to b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a).$$

Proof: It is evidently since the formula in question is universal. ■

Corollary 6.4 *Let $B \models T^*$ and A a ring such that $A \subseteq B$ as a substructure in the languaje $\mathcal{L}$. Then:*

$$A \models \forall a \forall b \forall c(a \not\preceq bc - a \to b \,|^{\mathrm{m}}_{\mathrm{loc}}\, a).$$

Proof: In particular B satisfies the hypothesis of corollary 6.3. ■

This formula is going to be suficient, with all the previous ones, in order to prove the inclusion in models of T^* and in this new languaje.

Definition 6.5 *Let A be any ring. We say that A has the* **maximal local divisibility property** *if:*

$$A \models \forall a \forall b \forall c (a \not\preceq bc - a \to b \mid^{\mathrm{m}}_{\mathrm{loc}} a).$$

Proposition 6.6 *Let A be a reduced f-ring satisfying the first convexity property, the divisibility glueing axioms and the maximal local divisibility property. Then there exists $B \models T^*$ such that $A \subseteq B$ in the languaje $\mathcal{L}$.*

Proof: Since A satisfies the maximal local divisibility property and $\forall a \forall b (b \mid^{\mathrm{m}}_{\mathrm{loc}} a \to b \mid_{\mathrm{loc}} a)$, then A satisfies the local divisibility property. By proposiction 4.13, there exists $B \models T^*$ such that $A \subseteq B$ in the languaje $\mathcal{L}_{\mathrm{an-re}} \cup \{ \preceq, \mid, \mid_{\mathrm{loc}} \}$. We want to show that for all $a, b \in A$, one has that $A \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$ if and only if $B \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$.

($\Rightarrow$) Let's suppose $A \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$. Then $A \models b \mid_{\mathrm{loc}} a$. By the construction of B, one has $B \models b \mid_{\mathrm{loc}} a$. Since B is a model of T^*, then $B \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$.

($\Leftarrow$) Let's suppose $B \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$. If $a = 0$, evidently $A \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$. Now let's suppose $a \neq 0$. Since $B \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$ then $B \models b \mid_{\mathrm{loc}} a$. Since A is a substructure of B in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{ \preceq, \mid, \mid_{\mathrm{loc}} \}$ then $A \models b \mid_{\mathrm{loc}} a$. Since $a \neq 0$, there exists $w \in A$, $w \neq 0$, $w(a - w) = 0$ and $b \mid w$. Then there exists $c \in A$ such that $bc = w$. Since $w \neq 0$ and A is reduced, there exists $p \in \pi A$ such that $w + p \neq 0$. Then $w + p = a + p \neq 0$. Therefore $bc + p = w + p = a + p$. That is $bc - a \in p$ and $a \notin p$. Then $a \not\preceq bc - a$ in A for $a, b, c \in A$. Since A satisfies the maximal local divisibility property then $A \models b \mid^{\mathrm{m}}_{\mathrm{loc}} a$. ■

Proposition 6.7 *The universal theory of T^* in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{ \mid, \preceq, \mid^{\mathrm{m}}_{\mathrm{loc}} \}$ is the theory of reduced f-rings satisfying the first convexity property, the divisibility glueing axioms and maximal local divisibility property.*

Proof: This comes from the propositions 6.6, 6.4 and 4.13. ■

The following definition has the purpose of expressing in a consice way the relation an idempotent e has in terms of a and b in the proposition 5.7 and in the observation 5.8:

$$\begin{aligned} \mathrm{Divloc}(b, a, e) \quad \leftrightarrow_{\mathrm{def}} \quad & (e^2 = e \wedge ae \neq 0 \wedge b \mid ae \wedge e \preceq a) \\ & \wedge \forall e' (e'^2 = e' \wedge ae' \neq 0 \wedge b \mid ae' \wedge e' \preceq a \to e' \leqslant e). \end{aligned}$$

The following proposition reinforces the fact that an idempotent e satisfying $\mathrm{Divloc}(b, a, e)$ is unique. Nevertheless, what we are interested in is to show that this idempotent controls where the divisibility (locally speaking) is carried out.

Proposition 6.8 *Let B be a reduced, projectable and divisible-projectable f-ring. Let $a, b, e \in B$ be such that $B \models \mathrm{Divloc}(b, a, e)$. Let $B \in \Gamma^a_{\mathcal{L}_{\mathrm{or}} \cup \{\mid\}}(X, (B_x)_{x \in X})$, where X is a Boolean space and $(B_x)_{x \in X}$ is a family of totally ordered integral domains. Then:*

$$[\![e = 1]\!] = [\![b \mid a]\!] \cap [\![a \neq 0]\!].$$

Proof: Let $x \in X$ such that $x \in [\![e = 1]\!]$. Then $e(x) = 1$, and since $e \preceq a$ then $a(x) \neq 0$. Since $b \mid ae$ then $b(x) \mid a(x)e(x) = a(x)$. Therefore $x \in [\![b \mid a]\!] \cap [\![a \neq 0]\!]$. We proved that $[\![e = 1]\!] \subseteq [\![b \mid a]\!] \cap [\![a \neq 0]\!]$. Let's suppose $[\![e = 1]\!] \subsetneq [\![b \mid a]\!] \cap [\![a \neq 0]\!]$. Then:

$$M = [\![b \mid a]\!] \cap [\![a \neq 0]\!] \setminus [\![e = 1]\!] \neq \emptyset.$$

Let $e' \in B$ defined by $e' = 1_{\restriction M} \cup 0_{\restriction X \setminus M}$. Clearly $e'^2 = e'$, $ae' \neq 0$ since $M \neq \emptyset$. One has that $b \mid ae'$ by the compactness of X and by the patchwork property of B. Clearly $e' \preceq a$. Since $B \models \mathrm{Divloc}(b, a, e)$ then $e' \leqslant e$. But for every $x_0 \in M$ one has that $e'(x_0) = 1$ and $e(x_0) \neq 1$, i.e.: $e(x_0) = 0$. Then we can not have $e' \leqslant e$, contradicting the strict inclusion. So we have showed:

$$[\![e = 1]\!] = [\![b \mid a]\!] \cap [\![a \neq 0]\!].$$

■

The following proposition shows the interest of introducing the $|^{\mathrm{m}}_{\mathrm{loc}}$ relation symbol in the languaje since this permits to show that the divisibility is respected by the local morphisms.

Proposition 6.9 *Let $A \models T^*_\forall$ in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$ and let B be a reduced, projectable and divisible-projectable f-ring. Let $f\colon A \to B$ be a monomorphism in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$. For $q \in \overline{\pi B}^{\mathrm{con}}$ and $p \in \overline{\pi A}^{\mathrm{con}}$ such that $\tilde{f}(q) = p = f^{-1}(q)$, one has that $f_{pq}\colon A/p \to B/q$ is a monomorphism in the languaje $\mathcal{L}' = \{0, 1, +, \cdot, \leqslant, |\}$.*

Dm: Clearly $f_{pq}\colon A/p \to B/q$, $f_{pq}(a+p) = f(a)+q$, is a monomorphism of totally ordered integral domains. We only have to show that f_{pq} respects the divisibility relation. Let $a, b \in A$, we have to see that $A/p \models b + p \mid a + p$ if and only if $B/q \models f(b) + q \mid f(a) + q$. The direction ($\Rightarrow$) is clear. Therefore we are going to see ($\Leftarrow$). If $f(a) + q = 0$ then $f(a) \in q$ and $a \in f^{-1}(q) = p$, therefore $a + p = 0$. In that case, evidently $b + p \mid a + p$ in A/p. Let's suppose that $f(a) + q \neq 0$. Since $B/q \models f(b) + q \mid f(a) + q$ then there exists $c \in B$ such that $(f(b) + q)(c + q) = f(a) + q$. Therefore $f(b)c - f(a) \in q$. So we have $f(a) \notin q$ and $f(b)c - f(a) \in q$. Then $f(a) \not\preceq f(b)c - f(a)$. By proposition 6.2 one has that $f(b) \;|^{\mathrm{m}}_{\mathrm{loc}}\; f(a)$ and $f(a) \neq 0$. By definition 6.1, there exists $\bar{e} \in B$ such that $\bar{e}^2 = \bar{e}$, $f(a)\bar{e} \neq 0$, $f(b) \mid f(a)\bar{e}$ and $\bar{e} \preceq f(a)$; also $\big(a(1 - \bar{e}) = 0 \to b \mid a\big)$ and $\Big(a(1 - \bar{e}) \neq 0 \to \forall f \big(f^2 = f \wedge a(1 - \bar{e})f \neq 0 \wedge f \preceq a(1 - \bar{e}) \to b \nmid a(1 - \bar{e})f\big)\Big)$. Observe that the conclusion of this last implication is that $f(b) \nmid_{\mathrm{loc}} f(a)(1 - \bar{e})$ is satisfied in B. Since $B/q \models f(b) + q \mid f(a) + q \wedge f(a) + q \neq 0$, by the proposition 6.8 one has $\bar{e} - 1 \in q$, that is $\bar{e} + q = 1 + q$. Since $|^{\mathrm{m}}_{\mathrm{loc}}$ is in the languaje then $A \models b \;|^{\mathrm{m}}_{\mathrm{loc}}\; a$ and $a \neq 0$. Then there exists $e_A \in A$ an idempotent such that ${e_A}^2 = e_A$, $ae_A \neq 0$, $b \mid ae_A$ y $e_A \preceq a$; besides that $\big(a(1 - e_A) = 0 \to b \mid a\big)$ and

$$\Big(a(1 - e_A) \neq 0 \to \forall e' \big(e'^2 = e' \wedge a(1 - e_A)e' \neq 0 \wedge e' \preceq a(1 - e_A) \to b \nmid a(1 - e_A)e'\big)\Big).$$

Observe that last formula can be written as $a(1 - e_A) \neq 0 \to b \nmid_{\mathrm{loc}} a(1 - e_A)$, in A. Aplying the monomorphism f to this five first formulas, we obtain $f(e_A)^2 = f(e_A)$, $f(a)f(e_A) \neq 0$, $f(e_A) \preceq f(a)$, $f(b) \mid f(a)f(e_A)$ and $f(a)\big(1 - f(e_A)\big) = 0 \to f(b) \mid f(a)$. Now let's see that we also have $f(a)\big(1 - f(e_A)\big) \neq 0 \to f(b) \nmid_{\mathrm{loc}} f(a)\big(1 - f(e_A)\big)$ in B. If not, we

should have $f(a)\big(1-f(e_A)\big) \neq 0$ and $f(b) \mid_{\rm loc} f(a)\big(1-f(e_A)\big)$ in B. Since B is a reduced, projectable and divisible-projectable f-ring, by proposition 5.11 one should have $f(b) \mid^{\rm m}_{\rm loc} f(a)\big(1-f(e_A)\big)$ in B. Since the $\mid^{\rm m}_{\rm loc}$ relation symbol is in the languaje then $b \mid^{\rm m}_{\rm loc} a(1-e_A)$ in A. And therefore $b \mid_{\rm loc} a(1-e_A)$ should be valid in A with $a(1-e_A) \neq 0$. That contradicts that the formula $a(1-e_A) \neq 0 \to b \nmid_{\rm loc} a(1-e_A)$ is true in A. Therefore we have that the formula $f(a)\big(1-f(e_A)\big) \neq 0 \to f(b) \nmid_{\rm loc} f(a)\big(1-f(e_A)\big)$ is valid in B. We have showed that $\bar{e}$ and $f(e_A)$ are elements of B satisfying the same formula (ii) in the observation 5.8, that is equivalent to the predicate Divloc, with respect to $f(a)$ and $f(b)$. That is, $\mathrm{Divloc}\big(f(b), f(a), \bar{e}\big)$ and $\mathrm{Divloc}\big(f(b), f(a), f(e_A)\big)$ are true in B. Therefore $f(e_A) = \bar{e}$. Since $\bar{e}-1 \in q$, then $f(e_A)-1 = f(e_A-1) \in q$. And then $e_A - 1 \in f^{-1}(q) = p$. That is: $e_A + p = 1 + p$. Remind that we have $b \mid ae_A$ in A.Therefore $(b+p) \mid (a+p)(e_A+p) = (a+p)(1+p) = a+p$ en A/p. We have achieved to prove that $A/p \models b+p \mid a+p$.
■

The proposition 6.9 will permit us to achieve the proof of the elimination of quantifiers of the theory T^* in the languaje $\mathcal{L}_{\rm lor} \cup \{\mid, \preceq, \mid^{\rm m}_{\rm loc}\}$, that will be made using the amalgamation property in the models of the universal theory $T^*_\forall$. But fisrt it will be needed a well known fact that we state as a lemma, and give its proof although it is quite obvious.

Lema 6.10 *If $(A_i)_{i\in I}$ is a family of totally ordered integral domains, then*

$$A = \prod_{i\in I} A_i$$

is a reduced, projectable and divisible-projectable f-ring.

Proof: Clearly A is an f-ring. Since the A_i's are integral domains, for all $i \in I$; then A is reduced. Let's see now that A s projectable. Let $a = (a_i)_{i\in I} \in A$ and $b = (b_i)_{i\in I} \in A$. Let's define $I_0 = \{i \in I : a_i = 0\}$ and $I_1 = I \setminus I_0$. Let's also define c and d in A by

$$c(i) = \begin{cases} b(i) & \text{si} \quad i \in I_0 \\ 0 & \text{si} \quad i \in I_1, \end{cases}$$

and

$$d(i) = \begin{cases} 0 & \text{si} \quad i \in I_0 \\ b(i) & \text{si} \quad i \in I_1. \end{cases}$$

It is obvious that $b = c + d$ and that $c \cdot a = 0$. It is easy to see that $d \preceq a$.

Now let's see that A is divisible-projectable. Let $x, y \in A$ with $y \neq 0$. Let's declare $I_0 = \{i \in I : y(i) \mid x(i)\}$ and $I_1 = I \setminus I_0 = \{i \in I : y(i) \nmid x(i)\}$. Let's define now $z, w \in A$ by:

$$z(i) = \begin{cases} x(i) & \text{si} \quad i \in I_0 \\ 0 & \text{si} \quad i \in I_1, \end{cases}$$

and

$$w(i) = \begin{cases} 0 & \text{si} \quad i \in I_0 \\ x(i) & \text{si} \quad i \in I_1, \end{cases}$$

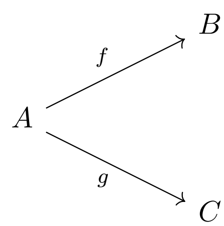
Clearly one has that $z \cdot w = 0$, $x = z + w$ and $y \mid z$. Now let's see that $\forall (w' \neq 0 \wedge w'(w' - w) = 0 \to y \nmid w')$. Let $w' \in A$ such that $w' \neq 0$ and $w'(w' - w) = 0$. There exists $i \in I$

such that $w'(i) \neq 0$. Then $w'(i) = w(i) \neq 0$, since A_i is an integral domain. Then $i \in I_1$ since $w(i) \neq 0$. Therefore $y(i) \nmid x(i) = w(i)$. This implies that $y \nmid w'$; if not there exists $c \in A$ such that $yc = w'$ and therefore $y(i)c(i) = w'(i) = w(i)$, giving a contradiction. Therefore $y \nmid w'$ and A is divisible-projectable.

■

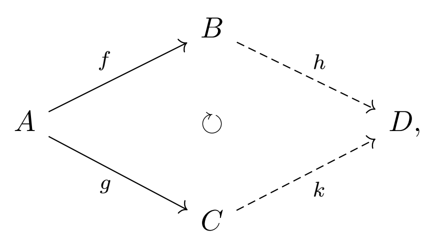
Proposition 6.11 *The theory $T^*_\forall$ has the amalgamation property in $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$.*

Proof: Let $A, B, C \models T^*_\forall$ such that there exists $f\colon A \to B$ and $g\colon A \to C$ monomorphisms in the languaje $\mathcal{L}_{\text{lor}} \cup \{|, \preceq, |^{\text{m}}_{\text{loc}}\}$. Iniatilly we have the following diagram:



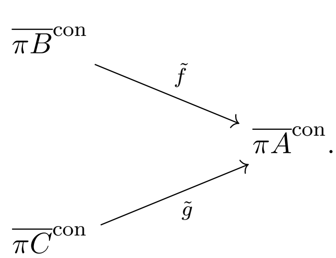
We may replace B and C by extensions that are models of T^*. That is important since we are going to use proposition 6.9 and we need that B and C will be reduced, projectables and divisible-projectables f-rings.

And we want to show that there exists $D \models T^*_\forall$ and monomorphisms $h\colon B \to D$ and $k\colon C \to D$ such that the following diagram is conmutative:



that is: $h \circ f = k \circ g$.

In section 4 we have been able to observe that models A of $T^*_\forall$ are described by $\overline{\pi A}^{\text{con}}$ and the family of totally ordered integral domains $\left(A/p\right)_{p \in \overline{\pi A}^{\text{con}}}$. As the radical relation is in the languaje, by [20, Theorem, pág. 23] and [20, Proposition (a) y (b), pág. 22], there exists $\tilde{f}\colon \overline{\pi B}^{\text{con}} \to \overline{\pi A}^{\text{con}}$ and $\tilde{g}\colon \overline{\pi C}^{\text{con}} \to \overline{\pi A}^{\text{con}}$ continuous surjectives functions; that is:



Since we are asuming that B and C are models of T^*, then B and C are in particular (unitary) projectables f-rings; by [16, 6.12] one has that πB are πC are Boolean spaces.

In particular πB and πC are quasi-compacts. Then πB and πC are proconstructibles, cf. [26, Corollary 2.7]. Then $\overline{\pi B}^{\text{con}} = \pi B$ and $\overline{\pi C}^{\text{con}} = \pi C$. By this we have:

$$\begin{array}{ccc} \pi B & & \\ & \searrow^{\tilde{f}} & \\ & & \overline{\pi A}^{\text{con}}. \\ & \nearrow_{\tilde{g}} & \\ \pi C & & \end{array}$$

In order to complete (dually) this diagram, we use the pullback of πB and πC over $\overline{\pi A}^{\text{con}}$, given by:

$$X = \pi B \times_{\overline{\pi A}^{\text{con}}} \pi C = \big\{(q_1, q_2) \in \pi B \times \pi C : \tilde{f}(q_1) = \tilde{g}(q_2)\big\}.$$

We then have the following diagram:

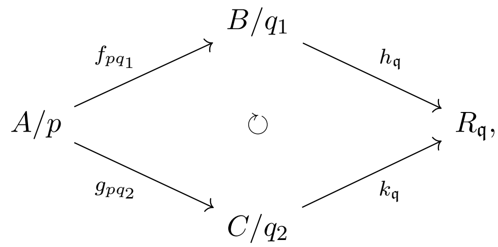
$$\begin{array}{ccccc} & & \pi B & & \\ & \nearrow^{\pi_1} & & \searrow^{\tilde{f}} & \\ X & & \circlearrowleft & & \overline{\pi A}^{\text{con}}. \\ & \searrow_{\pi_2} & & \nearrow_{\tilde{g}} & \\ & & \pi C & & \end{array}$$

The spaces $\overline{\pi A}^{\text{con}}, \pi B$ and πC are all of them Boolean spaces. Therefore $\pi B \times \pi C$ is a Boolean space, cf [18, Corollary 3.14, page 1249]. It is straighforward to prove that $\pi B \times_{\overline{\pi A}^{\text{con}}} \pi C$ is a Boolean space. [3] Summarizing we have that the space $X = \pi B \times_{\overline{\pi A}^{\text{con}}} \pi C$ is a Boolean space. Then $\mathfrak{q} \in X$ is an element of the form $\mathfrak{q} = (q_1, q_2) \in \pi B \times \pi C$ such that $\tilde{f}(q_1) = \tilde{g}(q_2)$; where $\tilde{f} \colon \pi B \to \overline{\pi A}^{\text{con}}$ and $\tilde{g} \colon \pi C \to \overline{\pi A}^{\text{con}}$ are continuous and surjective functions. Let's put $p = \tilde{f}(q_1) = \tilde{g}(q_2) \in \overline{\pi A}^{\text{con}}$.

Any $p \in \overline{\pi A}^{\text{con}} \subseteq \text{Spec}(A)$ is a prime ideal. In order to see that p is an l-ideal, it is suficient by [4, (8.2.1) and (2.2.1)(5)] to see that p is convex and closed by absolute value. Since A satisfies the first convexity property, then p is convex. It is easy to see that p is closed by absolute value: let $x \in p$ and $\big(x - |x|\big)\big(x + |x|\big) = 0$ is true in A since A is an f-ring. Since p is prime then $x - |x| \in p$ or $x + |x| \in p$. In any case, $|x| \in p$. Therefore p is a prime l-ideal of A. By [4, (9.2.5)] one has that A/p is a totally ordered integral domain.

By the proposition 6.9, one has that $f_{pq_1} \colon A/p \to B/q_1$ and $g_{pq_2} \colon A/p \to C/q_2$ are monomorphisms in the languaje $\mathcal{L}_{\text{or}} \cup \{|\}$. Since A is a reduced f-ring satisfying the first convexity property, then $A/p \models \text{COVR} \cup \text{OF}$, where COVR is the theory tof convexely ordered valuation rings and OF is the theory of ordered fields, both in the languajes $\mathcal{L}_{\text{or}} \cup \{|\}$ (cf. [2, Theorem 1]). Observe that B/q_1 and C/q_2 are real closed valuation rings (RCVR), as B and C are models of T^* (cf. [12, Corollary 2.11 and Proposition 2.4]). By the elimination of quantifiers of RCVR (cf. [8]) in $\mathcal{L}_{\text{or}} \cup \{|\}$, there exist $R_{\mathfrak{q}}$ a real closed valuation ring and there exists $h_{\mathfrak{q}} \colon B/q_1 \to R_{\mathfrak{q}}$ and $k_{\mathfrak{q}} \colon C/q_2 \to R_{\mathfrak{q}}$ two monomorphism in the languaje $\mathcal{L}_{\text{or}} \cup \{|\}$ such that:

[3]Even in the general case, it is possible to prove that $\overline{\pi B}^{\text{con}} \times_{\overline{\pi A}^{\text{con}}} \overline{\pi C}^{\text{con}} \subseteq \overline{\pi B}^{\text{con}} \times \overline{\pi C}^{\text{con}}$ is a Boolean space.



the diagram is conmutative; that is: $h_{\mathfrak{q}} \circ f_{pq_1} = k_{\mathfrak{q}} \circ g_{pq_2}$.

Now let's consider:

$$D = \prod_{\mathfrak{q} \in X} R_{\mathfrak{q}},$$

and let $h\colon B \to D$ and $k\colon C \to D$ given by:

$$h(b) = \big(h_{\mathfrak{q}}(b + q_1)\big)_{\mathfrak{q} \in X} \quad \text{and} \quad k(c) = \big(k_{\mathfrak{q}}(c + q_2)\big)_{\mathfrak{q} \in X},$$

for all $b \in B$ and $c \in C$, where $\mathfrak{q} = (q_1, q_2) \in X$.

Let's see that h is a monomorphism in the languaje $\mathcal{L}_{\mathrm{lor}} \cup \{|, \preceq, |^{\mathrm{m}}_{\mathrm{loc}}\}$. By lema 6.10, D is a reduced, projectable and divisible-projectable f-ring and it is easy seen h is a morphism of lattice-ordered rings. Let's observe also that when $\mathfrak{q}$ runs over X then q_1 runs over πB, as $\tilde{f}$ is surjective. Therefore h is injective, and therefore a monomorphism of lattice-ordered rings (or f-rings).

Now let's see that h preserves the radical relation, namely: for all $b, b' \in B$, we should see that $b \preceq b'$ if and only if $h(b) \preceq h(b')$. The direction $(\Leftarrow)$ is clear as $\preceq$ it is expressed by a universal formula. Let's suppose that $b \preceq b'$. Let $x = (x_{\mathfrak{q}})_{\mathfrak{q} \in X}$ such that $h(b')x = 0$. Then $h_{\mathfrak{q}}(b' + q_1)x_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in X$. Therefore $h_{\mathfrak{q}}(b' + q_1) = 0$ or $x_q = 0$, for all $\mathfrak{q} \in X$. Since $h_{\mathfrak{q}}$ is injective, then $b' \in q_1$ or $x_{\mathfrak{q}} = 0$, for all $\mathfrak{q} \in X$. Since $b \preceq b'$ in B, then $b \in q_1$ or $x_{\mathfrak{q}} = 0$, for all $\mathfrak{q} \in X$. That is $h_{\mathfrak{q}}(b + q_1)x_{\mathfrak{q}} = 0$, for all $\mathfrak{q} \in X$. Then $h(b)x = 0$. We have showed that $h(b) \preceq h(b')$ in D.

Let's recall that B is a reduced, projectable and divisible-projectable f-ring. We want to see that the divisibility and the maximal local divisibility are respected by h. Let's first see that the divisibility is rspected. For this, let $b, b' \in B$, we want to see that $B \models b \mid b'$ if and only if $D \models h(b) \mid h(b')$. The implication $(\Rightarrow)$ is clear since the formula definying the divisibility is existential. Now let's suppose that $h(b) \mid h(b')$ in D. This means that $h_{\mathfrak{q}}(b + q_1) \mid h_{\mathfrak{q}}(b' + q_1)$ in $R_{\mathfrak{q}}$, for all $\mathfrak{q} \in X$; or for all $q_1 \in \pi B$. Since each $h_{\mathfrak{q}}$ respects the divisibility, then we have $b + q_1 \mid b' + q_1$ in B/q_1, for all $q_1 \in \pi B$. Since B is projectable, by the compacity of πB and the patchwork property of B one has that $b \mid b'$ in B.

Now let's see that the maximal local divisibility is respected by h, that is: for $b, b' \in B$ one should see that $B \models b \ |^{\mathrm{m}}_{\mathrm{loc}}\ b'$ if and only if $D \models h(b) \ |^{\mathrm{m}}_{\mathrm{loc}}\ h(b')$. Let's first see that if $D \models h(b) \ |^{\mathrm{m}}_{\mathrm{loc}}\ h(b')$ then $B \models b \ |^{\mathrm{m}}_{\mathrm{loc}}\ b'$. Let's first observe that, since $h(b) \ |^{\mathrm{m}}_{\mathrm{loc}}\ h(b')$ then $h(b) \ |_{\mathrm{loc}}\ h(b')$. If $h(b') = 0$ then $b' = 0$ and therefore $b \ |^{\mathrm{m}}_{\mathrm{loc}}\ b'$ in B. If $h(b') \neq 0$, then there exists $w \in D$ with $w = (w_{\mathfrak{q}})_{\mathfrak{q} \in X} \neq 0$ such that $w\big(h(b') - w\big) = 0$ and $h(b) \mid w$ in D. Therefore there exists $\mathfrak{q} \in X$ such that $w_{\mathfrak{q}} \neq 0$. Then $w_{\mathfrak{q}} = h(b')_{\mathfrak{q}} = h_{\mathfrak{q}}(b + q_1)$, where $\mathfrak{q} = (q_1, q_2) \in X$. Since $h(b) \mid w$ in D, then there exists $c = (c_{\mathfrak{q}})_{\mathfrak{q} \in X}$ such that

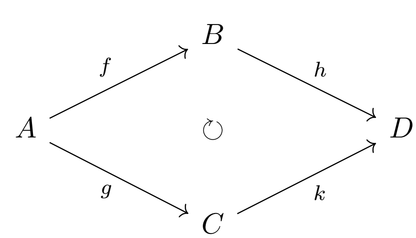
$h(b)c = w$. Then $h(b)_{\mathfrak{q}}c_{\mathfrak{q}} = w_{\mathfrak{q}} = h(b')_{\mathfrak{q}}$, that is: $h_{\mathfrak{q}}(b+q_1)c_{\mathfrak{q}} = h_{\mathfrak{q}}(b'+q_1)$. Therefore $h_{\mathfrak{q}}(b+q_1) \mid h_{\mathfrak{q}}(b'+q_1)$ in $R_{\mathfrak{q}}$. Since $h_{\mathfrak{q}}$ respects the divisiblity then $b+q_1 \mid b'+q_1$ in B/q_1. Then there exists $c' \in B$ such that $bc'-b' \in q_1$. See that $b' \notin q_1$. Therefore $b' \not\preceq bc'-b'$ in B. Since $B \models T_{\forall}^*$, then B has the maximal local divisibilidad property: i.e. B satisfies $\forall a \forall b \forall c (a \not\preceq bc - a \to b \mid_{\text{loc}}^{\text{m}} a)$. Then $b \mid_{\text{loc}}^{\text{m}} b'$ in B. We have showed that the maximal local divisibility goes down from D to B. In particular we showed that $b \mid_{\text{loc}} b'$ in B.

Let's observe that in the previous paragraph we showed that if $D \models h(b) \mid_{\text{loc}} h(b')$ then $B \models b \mid_{\text{loc}} b'$. Now let's see that the maximal local divisibility goes up from B to D. Let's suppose that $b \mid_{\text{loc}}^{\text{m}} b'$ in B. We want to see that $h(b) \mid_{\text{loc}}^{\text{m}} h(b')$ in D. By hypothesis, there exists $e \in B$ such that $e^2 = e$, $b'e \neq 0$, $e \preceq b'$, $b \mid b'e$ and if $b'(1-e) = 0$ then $b \mid b'$; besides that if $b'(1-e) \neq 0$ then $b \nmid_{\text{loc}} b'(1-e)$. Since h is an f-ring monomorphism that preserves the radical relation $\preceq$ and the divisibility $\mid$, one has that $h(e)^2 = h(e)$, $h(b')h(e) \neq 0$, $h(e) \preceq h(b')$, $h(b) \mid h(b')h(e)$ and if $h(b')\big(1-h(e)\big) = 0$ then $h(b) \mid h(b')$. For the last implication, recall that we saw that the local divisibility goes down from D to B, consequently the negation of the lcoal divisibility goes up from B to D. Then we have that if $h(b')\big(1-h(e)\big) \neq 0$ then $h(b) \nmid_{\text{loc}} h(b')\big(1-h(e)\big)$. We have showed that $h(b) \mid_{\text{loc}}^{\text{m}} h(b')$. Therefore we have showed that h respects the maximal local divisibility.

Similarly, $k\colon C \to D$ is a monomorphism in the languaje $\mathcal{L}_{\text{lor}} \cup \{\mid, \preceq, \mid_{\text{loc}}^{\text{m}}\}$. We then have that $h\colon B \to D$ and $k\colon C \to D$ are monomorphisms in the languaje $\mathcal{L}_{\text{lor}} \cup \{\mid, \preceq, \mid_{\text{loc}}^{\text{m}}\}$. Let's see that $h \circ f = k \circ g$. Let $a \in A$, then:

$$\begin{aligned}
(h \circ f)(a) = h\big(f(a)\big) = \Big(h_{\mathfrak{q}}\big(f(a)+q_1\big)\Big)_{\mathfrak{q}\in X} &= \Big(h_{\mathfrak{q}}\big(f_{pq_1}(a+p)\big)\Big)_{\mathfrak{q}\in X} \\
&= \Big(\big(h_{\mathfrak{q}} \circ f_{pq_1}\big)(a+p)\Big)_{\mathfrak{q}\in X} \\
&= \Big(\big(k_{\mathfrak{q}} \circ g_{pq_2}\big)(a+p)\Big)_{\mathfrak{q}\in X} \\
&= \Big(k_{\mathfrak{q}}\big(g_{pq_2}(a+p)\big)\Big)_{\mathfrak{q}\in X} \\
&= \Big(k_{\mathfrak{q}}\big(g(a)+q_2\big)\Big)_{\mathfrak{q}\in X} \\
&= k\big(g(a)\big) \\
&= (k \circ g)(a).
\end{aligned}$$

By the lema 6.10, one has that D is a reduced, projectable and divisible-projectable f-ring. It is easy to see that D satisfies the first convexity property. By the lema 4.2, one has that D satisfies the divisibility glueing axioms. By the proposition 6.2 one has that D satisfies the maximal local divisibility property. Therefore we have that $D \models T_{\forall}^*$ with the following conmutative diagram:



We have showed the amalgamation property of $T^*_\forall$ in the languaje $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$. ■

We can then state our main result:

Theorem 6.12 *T^* admits elimination of quantifiers in the languaje $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$.*

Proof: By the theorem 3.2, T^* is model complete in the languaje $\mathcal{L}_{\rm lor} \cup \{\preceq, |_{\rm loc}\}$. Then T^* is model complete in the languaje $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |_{\rm loc}\}$. Since $T^* \vdash \forall a \forall b (b \,|_{\rm loc}\, a \leftrightarrow b \,|^{\rm m}_{\rm loc}\, a)$, then T^* is model complete in $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$. By proposition 6.11, $T^*_\forall$ has the amalgamation property in $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$. By [7, Proposition 3.5.19.], the theory T^* admits elimination of quantifiers in $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$. ■

Corollary 6.13 *T^* admits elimination of quantifiers in the languaje $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |_{\rm loc}\}$.*

Proof: Since $T^* \vdash \forall a \forall b (b \,|_{\rm loc}\, a \leftrightarrow b \,|^{\rm m}_{\rm loc}\, a)$, replace each appearance of $|_{\rm loc}$ by $|^{\rm m}_{\rm loc}$. Then uses the theorem 6.12 to find an equivalent quanfitier-free formula in the languaje $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |^{\rm m}_{\rm loc}\}$. Now replace each appearance of the $|^{\rm m}_{\rm loc}$ by $|_{\rm loc}$, in order to obtain a quantifier-free formula in the languaje $\mathcal{L}_{\rm lor} \cup \{|, \preceq, |_{\rm loc}\}$. ■

Corollary 6.14 *The theory T^* is the model completion of the theory of reduced f-rings satisfying the first convexity property, the divisibility glueing axioms and the local divisibility property.*

Proof: This is deduced by [7, Proposition 3.5.19.]. ■

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