Groups definable in partial differential fields with an automorphism

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Abstract

In this paper we study groups definable in existentially closed partial differential fields of characteristic 0 with an automorphism which commutes with the derivations. In particular, we study Zariski dense definable subgroups of simple algebraic groups, and show an analogue of Phyllis Cassidy's result for partial differential fields. We also show that these groups have a smallest definable subgroup of finite index.

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1 Introduction

Fields with operators appear everywhere in mathematics, and are particularly present in areas close to algebra. The development of differential and difference algebra dates back to J. Ritt ([27]) in the 1950's, and was then further expanded by E. Kolchin ([15], [16]) and R. Cohn ([11]) in the 1960's. The study of differential and difference fields has been important in mathematics since the 1940's and has applications in many areas of mathematics.

One can also mix the operators, this gives the notion of differential-difference fields, i.e., a field equipped with commuting derivations and automorphisms. These fields were first studied from the point of view of algebra by Cohn in [12].

Model theorists have long been interested in fields with operators, until recently mainly on fields of characteristic 0 with one or several commuting derivations (ordinary or partial differential fields), and on fields with one automorphisms (difference fields). The first author

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also started in [5] the model-theoretic study of the existentially closed difference-differential fields of characteristic 0, around 2005 (one derivation, one automorphism). Then work of D. Pierce ([23]) and of O. León-Sánchez ([18]) brought back the model-theory of differential fields with several commuting derivations in the forefront of research in the area, as well as when a generic automorphism is added to these fields. Unlike in the pure ordinary differential case, and in the pure difference case, little is known on the possible interactions between definable subsets of existentially closed differential fields with several derivations, nor with an added automorphism.

In this paper we study groups definable in existentially closed differential-difference fields. We were motivated by the following result of Phyllis Cassidy (Theorem 19 in [8]):

Theorem. *Let $\mathcal{U}$ be a differentially closed field of characteristic* 0 *(with m commuting derivations), let H be a simple algebraic group, and $G \leq H(\mathcal{U})$ a Δ-algebraic subgroup of $H(\mathcal{U})$ which is Zariski dense in H. Then G is definably isomorphic to $H(L)$, where L is the constant field of a set Δ' of commuting derivations. Furthermore, the isomorphism is given by conjugation by an element of $H(\mathcal{U})$.*

She has similar results for Zariski dense Δ-closed subgroups of semi-simple algebraic groups. A version of her result for (existentially closed) difference fields was also proved by Chatzidakis, Hrushovski and Peterzil (Proposition 7.10 of [10]):

Theorem. *Let $(\mathcal{U}, \sigma)$ be a model of* ACFA. *Let H be an almost simple algebraic group defined over $\mathcal{U}$, and let G be a Zariski dense definable subgroup of $H(\mathcal{U})$. If $SU(G)$ is infinite then $G = H(\mathcal{U})$. If $SU(G)$ is finite, there are an isomorphism $f : H \to H'$ of algebraic groups, and integers $m > 0$ and n such that some subgroup of $f(G)$ of finite index is conjugate to a subgroup of $H'(\mathrm{Fix}(\sigma^m Frob^n))$. In particular, the generic types of G are non-orthogonal to the formula $\sigma^m(x) = x^{p^{-n}}$. If H is defined over $\mathrm{Fix}(\sigma)^{alg}$, then we may take $H = H'$ and f to be conjugation by an element of $H(\mathcal{U})$.*

In this paper, we generalise Cassidy's results to the theory $\mathrm{DCF}_m\mathrm{A}$, the model companion of the theory of fields of characteristic 0 with m derivations and an automorphism which commute, and one of our main results is:

Theorem 4.1. *Let $\mathcal{U}$ be a model of $\mathrm{DCF}_m\mathrm{A}$, let H be a simple algebraic group defined over $\mathbb{Q}$, and G a definable subgroup of $H(\mathcal{U})$ which is Zariski dense in H.*

Then G has a definable subgroup G_0 of finite index, the Kolchin closure of which is conjugate to $H(L)$, where L is an $\mathcal{L}_\Delta$-definable subfield of $\mathcal{U}$, say by an element g. Furthermore, either $G_0^g = H(L)$, or $G_0^g \subseteq H(\mathrm{Fix}(\sigma^\ell)(L))$ for some integer $\ell \geq 1$. In the latter case, if H is centerless, we are able to describe precisely the subgroup G_0^g as $\{g \in H(L) \mid \sigma^r(g) = \varphi(g)\}$ for some r and algebraic automorphism φ of $H(L)$.

We have analogous results for Zariski dense definable subgroups of semi-simple centerless algebraic groups (Theorem 4.7). Using an isogeny result (Proposition 3.4), and introducing the correct notion of *definably quasi-(semi-)simple* definable group, gives then slightly more general results, see Theorem 4.2.

Inspired by results of Hrushovski and Pillay on groups definable in pseudo-finite fields, we then endeavour to show that definable groups which are definably quasi-semi-simple have a smallest definable subgroup of finite index (this smallest definable subgroup is called the *connected component*). This is done in Corollary 5.8, and follows from several intermediate results. We first show the result for Zariski dense definable subgroups of simply connected algebraic simple groups, give a precise description of the connected component (Theorem 5.6), and show that every definable Zariski dense subgroup of $H(\mathcal{U})$ is quantifier-free definable. We then show the existence of a smallest definable subgroup of finite index for an arbitrary simple algebraic group H (Theorem 5.7), to finally reach the conclusion. Part of the study involves giving a description of definable subgroups of algebraic groups and we obtain the following result, of independent interest:

Theorem 5.1. *Let H be an algebraic group, $G \leq H(\mathcal{U})$ a Zariski dense definable subgroup. Then there are an algebraic group H', a quantifier-free definable subgroup R of $H'(\mathcal{U})$, together with a quantifier-free definable $f : R \to G$, with $f(R)$ contained and of finite index in G, and* $\mathrm{Ker}\,(f)$ *finite central in R.*

We conclude the paper with some results on the model theory of the fixed subfield $\mathrm{Fix}(\sigma) = \{a \in \mathcal{U} \mid \sigma(a) = a\}$ and of its finite algebraic extensions.

The paper is organised as follows. Section 2 contains the algebraic and model-theoretic preliminaries. Section 3 introduces the notions of definably quasi-(semi)simple groups and shows the isogeny result (3.4). Section 4 contains the main results of the paper: description of Zariski dense definable subgroups of simple and semi-simple algebraic groups (4.1, 4.2 and 4.7). Section 5 gives the results on definable subgroups of algebraic groups which are not quantifier-free definable (5.1) and shows that definably quasi-semi-simple definable groups have a definable connected component. Section 6 gives the results on the fixed field.

2 Preliminaries

This section is divided in four subsections: 2.1 - Differential and difference algebra; 2.2 - Model theory of differential and difference fields; 2.3 - The results of Cassidy; 2.4 - Quantifier-free canonical bases.

Notation and conventions: All rings are commutative, all fields are commutative of characteristic 0.

If K is a field, then K^{alg} denotes an algebraic closure of K (in the sense of the theory of fields).

2.1 Differential and difference algebra

Definition 2.1. For more details, please see [16], [11] and [12].

(1) Recall that a *derivation* on a ring R is a map $\delta : R \to R$ which satisfies $\delta(a+b) = \delta(a)+\delta(b)$ and $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$.

(2) A *differential ring*, or Δ*-ring*, is a ring equipped with a set $\Delta = \{\delta_1, \dots, \delta_m\}$ of commuting derivations. A *differential field* is a differential ring which is a field.

(3) A *difference ring* is a ring equipped with a distinguished automorphism, which we denote by σ. (This differs from the usual definition which only requires σ to be an endomorphism.) A *difference field* is a difference ring which is a field.

(4) A *difference-differential ring* is a differential ring equipped with an automorphism σ (which commutes with the derivations). A *difference-differential field* is a difference-differential ring which is a field.

Notation 2.2. (1) If Δ' is a set of derivations on the field K, then $K^{\Delta'}$ denotes the field of Δ'-constants, i.e., $\{a \in K \mid \delta(a) = 0\,\forall \delta \in \Delta'\}$.

(2) Similarly, if K is a difference field, then $\text{Fix}(\sigma)(K)$, or $\text{Fix}(\sigma)$ if there is no ambiguity, denotes the *fixed field* of K, $\{a \in K \mid \sigma(a) = a\}$.

(3) Let $K \subset \mathcal{U}$ be difference-differential fields, and $A \subset \mathcal{U}$. Then $K(A)_\Delta$ denotes the differential field generated by A over K, $K(A)_\sigma$ the difference field generated by A over K, and $K(A)_{\sigma,\Delta}$ the difference-differential field generated by A over K. (Note that we require $K(A)_\sigma$ and $K(A)_{\sigma,\Delta}$ to be closed under σ^{-1}.)

Polynomial rings and the corresponding ideals and topologies

Definition 2.3. Let K be a difference-differential ring, $y = (y_1, \ldots, y_n)$ a tuple of indeterminates.

- Then $K\{y\}$ (or $K\{y\}_\Delta$) denotes the ring of polynomials in the variables $\delta_1^{i_1} \cdots \delta_m^{i_m} y_j$, where $1 \leq j \leq n$, and the superscripts i_k are non-negative integers. It becomes naturally a differential ring, by setting $\delta_k(\delta_1^{i_1} \cdots \delta_m^{i_m} y_j) = \delta_1^{j_1} \cdots \delta_m^{j_m} y_j$, where $i_\ell = j_\ell$ if $\ell \neq k$, and $j_k = i_k + 1$. The elements of $K\{y\}$ are called *differential polynomials*, or Δ*-polynomials*.
- $K[y]_\sigma$ denotes the ring of polynomials in the variables $\sigma^i(y_j)$, $1 \leq j \leq n$, $i \in \mathbb{Z}$, with the obvious action of σ; thus it is also a difference ring. They are called *difference polynomials*, or σ*-polynomials*.
- $K\{y\}_\sigma$ denotes the ring of polynomials in the variables $\sigma^i \delta_1^{i_1} \cdots \delta_m^{i_m} y_j$, with the obvious action of σ and derivations. They are called *difference-differential polynomials*, or σ-Δ-*polynomials*.
- A Δ-*ideal* of a differential ring R is an ideal which is closed under the derivations in Δ and it is called *linear* if it is generated by homogeneous linear Δ-polynomials.
  Similarly, a σ-*ideal* I of a difference ring R is an ideal closed under σ; if it is also closed under σ^{-1}, we will call it *reflexive*; if whenever $a\sigma^n(a) \in I$ then $a \in I$, it is *perfect*. Finally, a σ-Δ-*ideal* is an ideal which is closed under σ and Δ.

Remarks 2.4. As with the Zariski topology, if K is a difference-differential field, the set of zeroes of differential polynomials, σ-polynomials and (σ, Δ)-polynomials in some K^n are the basic closed sets of a Noetherian topology on K^n, see Corollary 1 of Theorem III in [12]. We will call these sets Δ-*closed* (or *Kolchin closed*, or Δ-*algebraic*), σ-*closed/algebraic* and σ-Δ-*closed/algebraic* respectively. These topologies are called the *Kolchin topology* (or Δ-*topology*), σ-*topology* and (σ, Δ)-*topology* respectively. There are natural notions of closures and of irreducible components.

Remarks 2.5. The following results are certainly classical and well-known, but we did not know any reference. Recall that we are in characteristic 0, this result is false in positive characteristic. We let K be a differential subfield of the differentially closed field $\mathcal{U}$. Consider the commutative monoid Θ (with 1) generated by $\delta_1, \ldots, \delta_m$, and let $K\Theta$ be the K-vector space with basis Θ. It can be made into a ring, using the commutation rule $\delta_i \cdot a = a\delta_i + \delta_i(a)$, $i = 1, \ldots, m$. Each element f of $\mathcal{U}\Theta$ defines a *linear differential operator* $L_f : \mathbb{G}_a \to \mathbb{G}_a$, defined by $a \mapsto f(a)$. One has $L_{f \cdot g} = L_f \circ L_g$. Every Δ-closed subgroup of $\mathbb{G}_a(\mathcal{U})$ is defined as the set of zeroes of linear differential operators in $\mathcal{U}\Theta$, and for $n \geq 1$, every Δ-closed subgroup of $\mathbb{G}_a^n(\mathcal{U})$ is defined by conjunctions of equations of the form $L_1(x_1) + \cdots + L_n(x_n) = 0$, with the L_i in $\mathcal{U}\Theta$, and with the L_i in $K\Theta$ if the subgroup is defined over K, see e.g. Proposition 11 in [7].

Let S be a K-subspace of $K\Theta$, and assume that it is closed under δ_i, $i = 1, \ldots, m$, and that it does not contain 1. Then the differential ideal I generated by the set $S(x) := \{f(x) \mid f \in S\} \subset K\{x\}_\Delta$ does not contain x, and is prime.

Note that I is simply the $K\{x\}_\Delta$-module generated by $S(x)$, i.e., an element of I is a finite $K\{x\}_\Delta$-linear combination of elements of $S(x)$. Moreover, all elements of I have constant term 0. Every element f in $K\{x\}_\Delta$ can be written uniquely as $f_0 + f_1 + f_{>1}$, with f_0 the constant term, f_1 the sum of the linear terms, and $f_{>1}$ the sum of terms of f of total degree ≥ 2. Note that $(f + g)_i = (f + g)_i$ for $i \in \{0, 1, > 1\}$. Moreover

$$(fg)_0 = f_0 g_0, \quad (fg)_1 = f_0 g_1 + f_1 g_0, \quad (fg)_{>1} = f_{>1} g + f g_{>1} + f_1 g_1.$$

This easily implies that if $f \in I$, then $f_1 \in S(x)$: if $g \in I$, then $g_0 = 0$ and so $(fg)_1 = f_0 g_1$. As $1 \notin S$, this gives that $x \notin I$. Furthermore, the primeness of I follows from the fact that it is generated by linear differential polynomials, so that, as a ring, $K\{x\}_\Delta / I$ is isomorphic to a polynomial ring (in maybe infinitely many indeterminates) over $\mathcal{U}$.

2.2 Model theory of differential and difference fields

Notation 2.6. We consider the language $\mathcal{L}$ of rings, let $\Delta = \{\delta_1, \ldots, \delta_m\}$. We define $\mathcal{L}_\Delta = \mathcal{L} \cup \Delta$, $\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}$ and $\mathcal{L}_{\sigma,\Delta} = \mathcal{L}_\Delta \cup \{\sigma\}$ where the δ_i and σ are unary function symbols.

2.7. The theory DCF_m
The model theoretic study of differential fields (with one derivation, in characteristic 0) started with the work of Abraham Robinson ([28]) and of Lenore Blum ([2]). For several commuting derivations, Tracey McGrail showed in [21] that the $\mathcal{L}_\Delta$-theory of differential fields of characteristic zero with m commuting derivations has a model companion, which we denote by DCF_m. The $\mathcal{L}_\Delta$-theory DCF_m is complete, ω-stable and eliminates quantifiers and imaginaries. Its models are called *differentially closed*. Differentially closed fields had appeared earlier in the work of Ellis Kolchin ([15]).

2.8. Definable and algebraic closure, independence. Let $(\mathcal{U}, \Delta)$ be a differentially closed field. If $A \subset \mathcal{U}$, then $\mathrm{dcl}_\Delta(A)$ and $\mathrm{acl}_\Delta(A)$ denote the definable and algebraic closure in the sense of the theory DCF_m. Then $\mathrm{dcl}_\Delta(A)$ is the smallest differential field containing A, and $\mathrm{acl}_\Delta(A)$

is the field-theoretic algebraic closure of $\mathrm{dcl}_\Delta(A)$. Independence is given by independence in the sense of the theory ACF (of algebraically closed fields) of the algebraic closures, i.e., $A \perp_C B$ iff $\mathrm{acl}_\Delta(CA)$ and $\mathrm{acl}_\Delta(CB)$ are linearly disjoint over $\mathrm{acl}_\Delta(C)$.

2.9. The theories ACFA and $\mathrm{DCF}_m\mathrm{A}$

The $\mathcal{L}_\sigma$-theory of difference fields has a model companion denoted ACFA ([20], see also [9] and [10]). Omar León-Sánchez showed that the $\mathcal{L}_{\sigma,\Delta}$-theory of difference-differential fields admits a model companion, $\mathrm{DCF}_m\mathrm{A}$, and he gave an explicit axiomatisation of this theory in [18]. (When $m = 1$, the theory was extensively studied by the third author, in [6], see also [5]).

The theories ACFA and $\mathrm{DCF}_m\mathrm{A}$ have similar properties, they are model-complete, supersimple and eliminate imaginaries, but they are not complete and do not eliminate quantifiers. The completions of both theories are obtained by describing the isomorphism type of the difference subfield $\mathbb{Q}^{alg}$. In what follows we will view ACFA as $\mathrm{DCF}_m\mathrm{A}$ with $m = 0$, and we fix a (sufficiently saturated) model $\mathcal{U}$ of $\mathrm{DCF}_m\mathrm{A}$.

2.10. The fixed field

The fixed field of $\mathcal{U}$, $\mathrm{Fix}(\sigma) := \{x \in \mathcal{U} : \sigma(x) = x\}$, is a pseudo-finite field. Then $\mathrm{Fix}(\sigma^k)$ is the unique extension of $\mathrm{Fix}(\sigma)$ of degree k.

Theorem 2.11. *([18], Propositions 3.1, 3.3 and 3.4). Let a, b be tuples in $\mathcal{U}$ and let $A \subseteq \mathcal{U}$. We will denote by* $\mathrm{acl}(A)$ *the model theoretic closure of A in the $\mathcal{L}_{\sigma,\Delta}$-structure $\mathcal{U}$. Then:*

1. $\mathrm{acl}(A)$ *is the (field-theoretic) algebraic closure of the difference-differential field generated by A.*
2. *If $A = \mathrm{acl}(A)$, then the union of the quantifier-free diagramme of A and of the theory* $\mathrm{DCF}_m\mathrm{A}$ *is a complete theory in the language $\mathcal{L}_{\sigma,\Delta}(A)$.*
3. *$tp(a/A) = tp(b/A)$ if and only if there is an $\mathcal{L}_{\sigma,\Delta}(A)$-isomorphism* $\mathrm{acl}(Aa) \to \mathrm{acl}(Ab)$ *sending a to b.*
4. *Every $\mathcal{L}_{\sigma,\Delta}$-formula $\varphi(x)$ is equivalent modulo* $\mathrm{DCF}_m\mathrm{A}$ *to a disjunction of formulas of the form $\exists y\, \psi(x, y)$, where ψ is quantifier-free (positive), and such that for every tuples a and b (in a difference-differential field of characteristic 0), if $\psi(a, b)$ holds, then $b \in \mathrm{acl}(a)$.*
5. *Every completion of* $\mathrm{DCF}_m\mathrm{A}$ *is supersimple (of SU-rank ω^{m+1}). Independence is given by independence (in the sense of* ACF*) of algebraically closed sets:*
   a and b are independent over C if and only if the fields $\mathrm{acl}(Ca)$ *and* $\mathrm{acl}(Cb)$ *are linearly disjoint over* $\mathrm{acl}(C)$*.*
6. *Every completion of* $\mathrm{DCF}_m\mathrm{A}$ *eliminates imaginaries.*
7. *If $k \geq 1$, and $\mathcal{U} \models \mathrm{DCF}_m\mathrm{A}$, then the difference-differential field $\mathcal{U}[k] = (\mathcal{U}, +, \cdot, \Delta, \sigma^k)$ is also a model of* $\mathrm{DCF}_m\mathrm{A}$*, and the algebraic closure of* $\mathrm{Fix}(\sigma)$ *is a model of* DCF_m.

Remarks 2.12. (a) Item (4) is stated in a slightly different way in [18]. Here we prefer to have our set defined positively, at the cost of y consisting of maybe several elements. This gives us that every definable subset of $\mathcal{U}^n$ is the projection of a σ-Δ-algebraic set W by a projection with finite fibers.

(b) Recalling that independence in DCF_m is given by independence (in the sense of ACF) of algebraically closed sets, it follows that another way of phrasing (5) is to say that independence is given by independence (in the sense of DCF_m) of algebraically closed sets. This shows in particular that $\mathrm{DCF}_m\mathrm{A}$ is *one-based over* DCF_m, a notion which was introduced by Thomas Blossier, Amador Martin-Pizarro and Frank O. Wagner in [1].

(c) As with ACFA, it then follows that if G is a definable subgroup of some algebraic group H, and if one defines the *prolongations* $p_n : H(\mathcal{U}) \to H(\mathcal{U}) \times \sigma(H(\mathcal{U})) \times \cdots \times \sigma^n(H(\mathcal{U}))$, $g \mapsto (g, \sigma(g), \ldots, \sigma^n(g))$, and let $G_{(n)}$ be the Kolchin closure of $p_n(G)$, then an element $g \in G$ is a generic if and only if for each n, $p_n(g)$ is a generic of the Δ-closed subgroup $G_{(n)}$ of $H(\mathcal{U}) \times \sigma(H(\mathcal{U})) \times \cdots \times \sigma^n(H(\mathcal{U}))$. In particular, G will have finite index in its σ-Δ-closure.

(d) Let $A \subset \mathcal{U}$ be a difference-differential subfield, and let L be a difference-differential field extending A. Assume that $L \cap A^{alg} = A$. Then there is an A-embedding of L into $\mathcal{U}$. Indeed, our assumption implies that $L \otimes_A A^{alg}$ is an integral domain, and because $A^{alg} = \mathrm{acl}(A)$, the conclusion follows.

(e) This has the following consequence, which we will use:
Let q be a quantifier-free type over a difference-differential subfield A of $\mathcal{U}$, and suppose that q is stationary, i.e., if a realises q, then $A(a)_{\sigma,\Delta} \cap A^{alg} = A$. Let $f : A \to A' \subset \mathcal{U}$ be an isomorphism; then $f(q)$ is realised in $\mathcal{U}$.

(f) When $m = 0$, all these results appear in [9]. When $m = 1$, they appear in [5], [6].

2.3 The results of Cassidy

Let $\mathcal{U}$ be a (sufficiently saturated) differentially closed field of characteristic 0. An (affine) Δ-*algebraic group* is a subset of affine space which is both a differential variety in the sense of Kolchin and Ritt, and whose group laws are morphisms of differential varieties. By quantifier elimination of the theory DCF_m, they correspond to definable groups, see [7]. This context was extended to the non affine setting, see e.g. [16] chapter 1 §2. An affine Δ-algebraic group is then just a group definable in DCF_m (by quantifier-elimination). Cassidy shows that a connected semi-simple differential group surjects (with finite kernel) onto a linear differential algebraic group, i.e., a subgroup of some $\mathrm{GL}_n(\mathcal{U})$ (see Corollary 3 of Theorem 13 in [7]). A result of Pillay (Theorem 4.1 and Corollary 4.2 in [24]; it is proved for one derivation, but the proof adapts immediately to several commuting derivations) also tells us that every differential algebraic group embed into an algebraic group, so putting the two together tells us that a connected semi-simple differential group embeds into a semi-simple algebraic group.

Definition 2.13. A Δ-algebraic group G is Δ-*simple* if G is non-commutative and has no proper connected normal Δ-closed subgroup. Thus a finite center is allowed.
Similarly, a Δ-algebraic group G is Δ-*semi-simple* if it has no non-trivial connected normal commutative Δ-closed subgroup.

The following results were shown by Phyllis Cassidy in [8]:

Theorem 2.14. *(Cassidy, [8], Theorem 15) Let G be a Zariski dense Δ-closed subgroup of a semi-simple algebraic group $A \leq \mathrm{GL}(n, \mathcal{U})$, with simple components $A_1, \ldots, A_t$. Then there exist connected nontrivial Δ-simple normal Δ-closed subgroups $G_1, \ldots, G_t$ of G such that*

(1) *If* $i \neq j$, *then* $[G_i, G_j] = 1$.
(2) *The product morphism* $G_1 \times \cdots \times G_t \to G$ *is a* Δ*-isogeny (i.e., is onto, with finite kernel).*
(3) G_i *is the identity component of* $G \cap A_i$, *and is Zariski dense in* A_i.
(4) G *is* Δ*-semi-simple.*

Theorem 2.15. *(Cassidy, [8], Theorem 19). Let* H *be a simple algebraic group defined and split over* $\mathbb{Q}$, *and* $G \leq H(\mathcal{U})$ *be a* Δ*-algebraic subgroup which is Zariski dense in* H. *Then* G *is definably isomorphic to* $H(L)$, *where* L *is the constant field of a set* Δ' *of commuting derivations. Furthermore, the isomorphism is given by conjugation by an element of* $H(\mathcal{U})$.

Remarks 2.16. Cassidy's results are stated in different terms. Instead of speaking of *simple algebraic groups, defined and split over* $\mathbb{Q}$ in [8], she speaks about *simple Chevalley groups*. In fact, all her results are stated in terms of Chevalley groups, but we chose not to do that. Recall that any simple algebraic group is isomorphic to one which is defined and split over the prime field, $\mathbb{Q}$ in our case.
When the field F is algebraically closed, a *Chevalley group* is $G(F)$, where G is a semi-simple connected algebraic group G which is defined over $\mathbb{Q}$ and is split over $\mathbb{Q}$. When the field F is not algebraically closed, with G as above, it is defined as the subgroup of $G(F)$ generated by the unipotent subgroups, and thus may be strictly smaller than $G(F)$. Since we will consider fields which are not algebraically closed, we preferred using the "simple" terminology.
Note also that the field L of Theorem 2.15 is algebraically closed. We will therefore be able to use Fact 2.17 below.

Fact 2.17. Let G be a simple algebraic group defined and split over $\mathbb{Q}$, let K be an algebraically closed field of characteristic 0. Then
(a) $G(K)$ has no infinite normal subgroup;
(b) The field K is definable in the pure group $G(K)$.

Both assertions are well-known, but we were not able to find easy references. The first assertion follows from the fact that if $g \in G(K) \setminus Z(G(K))$, then the infinite irreducible Zariski closed set $(g^{G(K)}g^{-1})$ is connected, contains 1, and therefore generates a Zariski closed normal subgroup of $G(K)$, which must equal $G(K)$. The second is also well-known, see for instance Theorem 3.2 in [17].

2.4 Quantifier-free canonical bases

As $\mathrm{DCF}_m\mathrm{A}$ is supersimple there is a notion of canonical basis for complete types which is defined as a sort of amalgamation basis, and is not easy to describe. In our case, we will focus on an easier concept: canonical bases of quantifier-free types. They are defined as follows:

We work in a model $(\mathcal{U}, \sigma, \Delta)$ of $\mathrm{DCF}_m\mathrm{A}$. Let a be a finite tuple in $\mathcal{U}$, and $K \subset \mathcal{U}$ a difference-differential field. We define the *quantifier-free canonical basis* of $tp(a/K)$, denoted by qf-Cb(a/K), as the smallest difference-differential subfield k of K such that $k(a)_{\sigma,\Delta}$ and K are linearly disjoint over k. Another way of viewing this field is as the smallest difference-differential subfield of K over which the smallest K-definable σ-Δ-closed set containing a is defined (this set is called the σ-Δ-*locus of* a *over* K). Analogous notions exist for DCF_m and ACFA. We were not able to find explicit statements of the following easy consequences of the Noetherianity of the σ-Δ-topology, so we will indicate a proof.

Lemma 2.18. *Let $a, K \subset \mathcal{U}$ be as above.*
 (1) *qf-Cb(a/K) exists and is unique; it is finitely generated as a difference-differential field.*
 (2) *Let $K \subset M \subset K(a)_{\sigma,\Delta}$. Then $M = K(b)_{\sigma,\Delta}$ for some finite tuple b in M.*

Proof. (1) Let $n = |a|$, and write $K\{y\}_\sigma = \bigcup_{r\in\mathbb{N}} K[r]$, where

$$K[r] = K[\sigma^i \delta_1^{i_1} \delta_2^{i_2} \cdots \delta_m^{i_m} y_j \mid 1 \le j \le n, |i| + \sum_j i_j \le r].$$

Then each $K[r]$ is finitely generated over K as a ring, and is Noetherian. For each r, consider the ideal $I[r] = \{f \in K[r] \mid f(a) = 0\}$, and the corresponding σ-Δ-closed subset $X[r]$ of $\mathcal{U}^n$ defined by $I[r]$. Then the sets $X[r]$ form a decreasing sequence of σ-Δ-closed subsets of $\mathcal{U}^n$, which stabilises for some r, which we now fix. Note that the ideal $I[r]$ is a prime ideal (of the polynomial ring $K[r]$), and as such has a smallest field of definition, say k_0, and that k_0 is finitely generated as a field, and is unique. We now let k be the difference-differential field generated by k_0.

Claim 1. $k(a)_{\sigma,\Delta}$ and K are linearly disjoint over k.

Proof. This follows from the fact that $X[s] = X[r]$ for every $s \ge r$. ☐

(2) Consider $B :=$ qf-Cb(a/M). By (1), B is finitely generated as a difference-differential field.

Claim 2. $KB = M$.

Proof. Indeed, by definition, $B(a)_{\sigma,\Delta}$ and M are linearly disjoint over B. Hence, $KB(a)_{\sigma,\Delta}$ and M are linearly disjoint over KB. But this is only possible if $KB = M$. ☐

Remarks 2.19. Given fields $K \subset L$ (of characteristic 0), the field L is a regular extension of $L_0 := K^{alg} \cap L$. So, if $L = K(a)_{\sigma,\Delta}$ for some (maybe infinite) tuple a, then qf-Cb(a/K^{alg}) is contained in L_0, and we have qf-Cb$(a/K^{alg})K = L_0$.

3 The isogeny result

We work in a sufficiently saturated model $(\mathcal{U}, \Delta, \sigma)$ of $\mathrm{DCF}_m\mathrm{A}$. We will often work in its reduct to $\mathcal{L}_\Delta$. Unless otherwise mentioned, definable will mean $\mathcal{L}_{\sigma,\Delta}$-definable.

Definition 3.1. Let G be a definable group. We say that G is *definably quasi-simple* if G has no abelian subgroup of finite index and if whenever H is a definable infinite subgroup of G of infinite index, then its normaliser $N_G(H)$ has infinite index in G. We say that G is *definably quasi-semi-simple* if G has no abelian subgroup of finite index and if whenever H is a definable infinite commutative subgroup of G of infinite index, then its normaliser $N_G(H)$ has infinite index in G.

Remark 3.2. In our context (of a supersimple theory), a definable group will in general have infinitely many definable subgroups of finite index, so it will not have a smallest one. Note that our definition takes care of that problem, as both notions are preserved when going to definable subgroups of finite index and quotients by finite normal subgroups.

Lemma 3.3. *Let G be a group, G_0 a definable subgroup of G of finite index, and Z a finite normal subgroup of G.*

(1) *G is definably quasi-simple if and only if G_0 is definably quasi-simple.*
(2) *G is definably quasi-simple if and only if G/Z is definably quasi-simple.*
(3) *The same assertions hold with "quasi-semi-simple" in place of quasi-simple.*

Proof. (1) Suppose G_0 is definably quasi-simple, let H be an infinite subgroup of G of infinite index, and assume that $N_G(H)$ has finite index in G. Then $N_G(H) \cap G_0$ has finite index in G_0; but $H \cap G_0$ has finite index in H, hence is infinite, and of infinite index in G_0, and we get the desired contradiction.
For the other direction, assume H is an infinite subgroup of G_0 of infinite index in G_0, and with $N_{G_0}(H)$ of finite index in G_0; then $N_G(H)$ has finite index in G, which gives us the desired contradiction.

(2) By (1), going to a definable sugroup of G of finite index, we may assume that Z is central in G. Assume G/Z is definably quasi-simple, and let H be an infinite definable subgroup of G of infinite index. Then HZ/Z is infinite and has infinite index in G/Z, so its normalizer N has infinite index in G/Z, and if $N' \supset Z$ is such that $N'/Z = N$, then N' has infinite index in G, and normalizes HZ. But HZ is a finite union of cosets of H, N' permutes these cosets, which implies that $N_G(H)$ has infinite index in G. The other direction is immediate because Z is central.

(3) Reason as in (1) and (2).

Proposition 3.4. *Let G be a group definable in $\mathcal{U}$, and assume that G is definably quasi-simple (resp. definably quasi-semi-simple). Then there are a definable subgroup G_0 of finite index in G, a Δ-simple (resp. Δ-semi-simple) Δ-algebraic group H defined and split over $\mathbb{Q}$, and a definable homomorphism $\phi : G_0 \to H(\mathcal{U})$, with finite kernel and Kolchin dense image.*

Proof. By Remark 2.12(b), and by Theorem 4.9 and Corollary 4.10 of [1], there is a homomorphism ϕ of some definable subgroup G_0 of finite index in G into a group $\bar{G}$ which is definable in the differential field $\mathcal{U}$, and with $\mathrm{Ker}\,(\phi)$ finite. We may assume that the image of G_0 is Kolchin dense in $\bar{G}$ and, going to a subgroup of G_0 of finite index, that $\bar{G}$ is connected (as a Δ-algebraic group).
Moreover, if G is definably quasi-simple, we may assume that $\bar{G}$ is a Δ-simple group: if N is an $\mathcal{L}_\Delta$-definable connected normal subgroup of $\bar{G}$, then $\phi^{-1}(N) \cap G_0$ is a normal subgroup of $\phi(G_0)$. Our hypothesis on G implies that $\phi^{-1}(N) \cap G_0$ is finite, and so is $N \cap \phi(G_0)$. We may therefore compose ϕ with the projection $\bar{G} \to \bar{G}/N$.
If G is definably quasi-semi-simple, the same reasoning allows us to assume that $\bar{G}$ is Δ-semi-simple, i.e., that it has no proper connected abelian normal Δ-definable subgroup. Then

Theorem 17 of [8] and its corollary give us the result in the simple case, and Theorem 18 of [8] in the semi-simple case.

4 Definable subgroups of semi-simple algebraic groups

In this section we give a description of Zariski dense definable subgroups of simple and semi-simple algebraic groups. We work in a sufficiently saturated model $(\mathcal{U}, \Delta, \sigma)$ of $\mathrm{DCF}_m\mathrm{A}$. Unless otherwise mentioned, definable will mean $\mathcal{L}_{\sigma,\Delta}$-definable.

Theorem 4.1. *Let H be a simple algebraic group defined over $\mathbb{Q}$, and G a definable subgroup of $H(\mathcal{U})$ which is Zariski dense in H.*

Then G has a definable subgroup G_0 of finite index, the Kolchin closure of which is conjugate to $H(L)$, where L is an $\mathcal{L}_\Delta$-definable subfield of $\mathcal{U}$, say by an element g. Furthermore, either $G_0^g = H(L)$, or $G_0^g \subseteq H(\mathrm{Fix}(\sigma^\ell)(L))$ for some integer $\ell \geq 1$. In the latter case, if H is centerless, we are able to describe precisely the subgroup G_0^g as $\{g \in H(L) \mid \sigma^n(g) = \varphi(g)\}$ for some n and algebraic automorphism φ of $H(L)$.

Proof. Replacing G by a subgroup of finite index, we may assume that the Kolchin closure $\bar{G}$ of G is connected. Then $\bar{G}$ is also Zariski dense in H, and by Theorem 2.15, $\bar{G}$ is conjugate to $H(L)$, for some $\mathcal{L}_\Delta$-definable subfield L of $\mathcal{U}$.

The strategy is the same as in the proof of Proposition 7.10 in [10]. Going to the σ-closure of G within $H(L)$, and then to a subgroup of finite index, we may assume that G is quantifier-free definable, and that it is connected for the σ-Δ-topology. If $G = H(L)$, then we are done, because $H(L)$ has no proper definable subgroup of finite index, since it is simple (see Fact 2.17). Assume therefore that $G \neq H(L)$. We will first do the case where H is centerless.

In the notation of Remark 2.12(c), let n be the smallest integer such that $G_{(n)}$ is not equal to $H(L) \times \sigma(H(L)) \times \cdots \times \sigma^n(H(L))$. If π is the projection on the last factor $\sigma^n(H(L))$, then $\pi(G_{(n)}) = \sigma^n(H(L))$.

Write $G_{(n)} \cap ((1)^n \times \sigma^n(H(L))) = (1)^n \times S_0$. Because $G_{(n)}$ projects onto $\sigma^n(H(L))$, it follows that S_0 is a normal subgroup of $\sigma^n(H(L))$: Let $s \in S_0$ and $g \in \sigma^n(H(L))$. Since $\pi(G_{(n)}) = \sigma^n(H(L))$, there is $h \in H(L) \times \cdots \times \sigma^{n-1}(H(L))$ such that $(h, g) \in G_{(n)}$. Then $(h, g)^{-1}(1, s)(h, g) = (1, g^{-1}sg) \in G_{(n)}$, so $g^{-1}sg \in S_0$.

Since $G_{(n)}$ projects onto $G_{(n-1)} = H(L) \times \cdots \times \sigma^{n-1}(H(L))$ and is not equal to $H(L) \times \cdots \times \sigma^n(H(L))$, the normal subgroup S_0 must equal (1) (because $Z(H) = (1)$). So $G_{(n)}$ is the graph of a group epimorphism $\theta : H(L) \times \cdots \times \sigma^{n-1}(H(L)) \to \sigma^n(H(L))$. As all $\sigma^i(H(L))$ are simple, it follows that $\mathrm{Ker}\,(\theta)$ is a product of some of the factors, and by minimality of n, the first factor $H(L)$ is not contained in $\mathrm{Ker}\,(\theta)$. Hence, $\mathrm{Ker}\,(\theta) = \sigma(H(L)) \times \cdots \times \sigma^{n-1}(H(L))$, and $G_{(n)}$ is in fact defined by the equation $\sigma^n(g) = \theta'(g)$, where θ' is the morphism $H(L) \to \sigma^n(H(L))$ induced by θ. Note that θ' is $\mathcal{L}_\Delta$-definable, and defines an isomorphism between the groups $H(L)$ and $H(\sigma^n(L))$.

The Theorem of Borel-Tits (see Theorem A in [4], or 2.7, 2.8 in [30], or Theorem 4.17 in

[26]) which describes abstract isomorphisms between simple algebraic groups, tells us that there are an algebraic automorphism φ of the algebraic group $H(L)$ and a field isomorphism $\psi : L \to \sigma^n(L)$, such that $\theta' = \bar{\psi}\varphi$, where $\bar{\psi}$ is the obvious isomorphism $H(L) \to H(\sigma^n(L))$ induced by ψ. Since θ' and φ are $\mathcal{L}_\Delta$-definable, so is ψ, by Fact 2.17(b).

Claim. $L = \sigma^n(L)$ and $\psi = id$.

Proof. The graph of ψ defines an additive subgroup S of $L \times \sigma^n(L) \leq \mathcal{U} \times \mathcal{U}$.
By Remark 2.5 there are linear differential polynomials $F_i(x)$ and $G_i(y)$, $i = 1, \ldots, s$, such that S is defined by the equations $F_i(x) = G_i(y)$, $i = 1, \ldots, s$. Because S is the graph of an isomorphism, we have $\bigcap_{i=1}^s \mathrm{Ker}\,(F_i) = \{0\} = \bigcap_{i=1}^s \mathrm{Ker}\,(G_i)$. Hence, x belongs to the differential ideal generated by the $F_i(x)$, and this implies (see Remark 2.5) that there are linear differential polynomials $L_1, \ldots, L_s$ such that $\sum_{i=1}^s L_i(F_i(x)) = x$; letting $G(y) = \sum_{i=1}^s L_i(G_i(y))$, we get $x = G(y)$. Since S is the graph of a field automorphism, we must then have $G(y) = y$, i.e.: $\psi = id$. □

An alternate proof is to quote Sonat Suer (Theorem 3.38 in [31]) to deduce that $L = \sigma^n(L)$, and then show that $\psi = id$.

In other words, we have shown that θ' is an algebraic group automorphism of $H(L)$, and in particular shown the last assertion: when $G < H(L)$, then G is defined by

$$\{h \in H(L) \mid \sigma^n(h) = \theta'(h)\}.$$

By Proposition 14.9 of [3], the group $\mathrm{Inn}(H)$ of inner automorphisms of $H(L)$ has finite index in the group $\mathrm{Aut}(H)$ of algebraic automorphisms of $H(L)$. Moreover σ^n induces a permutation of $\mathrm{Aut}(H)/\mathrm{Inn}(H)$, and hence there are some $r \in \mathbb{N}^*$ and $h \in H(L)$ such that

$$\sigma^{n(r-1)}(\theta') \circ \sigma^{n(r-2)}(\theta') \circ \cdots \circ \theta' = \lambda_h,$$

where λ_h is conjugation by h. I.e., our group G is contained in the group G' defined by $\sigma^{nr}(g) = \lambda_h(g)$.
By DCF$_m$A, there is some $u \in H(L)$ such that $\sigma^{nr}(u) = h^{-1}u$. So, if $g \in G'$, then

$$\begin{aligned} \sigma^{nr}(u^{-1}gu) &= \sigma^{nr}(u^{-1})\lambda_h(g)\sigma^{nr}(u) \\ &= h(h^{-1}gh)(h^{-1}u) \\ &= u^{-1}gu. \end{aligned}$$

I.e., $u^{-1}G'u \subset H(\mathrm{Fix}(\sigma^{nr}) \cap L))$.

This does the case when H is centerless. Assume that the center Z of H is non-trivial. By the first part we know that there are $u \in H(\mathcal{U})$ and $\ell \geq 1$ such that $(u^{-1}GZu)/Z \subseteq (H/Z)(\mathrm{Fix}(\sigma^\ell(L))$. Since Z is finite and characteristic, there is some $s \in \mathbb{N}$ such that for all $a \in Z$, we have $\prod_{i=0}^{s-1} \sigma^i(a) = 1$. If $g \in u^{-1}Gu$, then $\sigma^\ell(g)g^{-1} \in Z$; hence $\sigma^{\ell s}(g)g^{-1} = 1$, and $u^{-1}Gu \subset H(\mathrm{Fix}(\sigma^{\ell s}))$. □

Corollary 4.2. *Let G be an infinite group definable in a model $\mathcal{U}$ of* $\mathrm{DCF}_m\mathrm{A}$*, and suppose that G is definably quasi-simple. Then there are a simple algebraic group H defined and split over $\mathbb{Q}$, a definable subgroup G_0 of G of finite index, and a definable group homomorphism $\phi : G_0 \to H(\mathcal{U})$, with the following properties:*

(1) $\mathrm{Ker}\,(\phi)$ *is finite.*

(2) *The Kolchin closure of $\phi(G_0)$ is $H(L)$ for some $\mathcal{L}_\Delta$-definable subfield L of the differential field $\mathcal{U}$.*

(3) *Either $\phi(G_0) = H(L)$, or for some integer ℓ, $\phi(G_0)$ is a subgroup of $H(\mathrm{Fix}(\sigma^\ell) \cap L)$.*

Proof. By Proposition 3.4 we can reduce to the case where G is a definable subgroup of a simple algebraic group H. Then apply Proposition 4.1 to conclude. □

Lemma 4.3. *Let H be a simple algebraic group, defined and split over $\mathbb{Q}$, let $L \leq \mathcal{U}$ be a field of constants, and let φ be an algebraic automorphism of H. Let $\ell \geq 1$, and consider the subgroup $G \leq H(L)$ defined by $\sigma^\ell(g) = \varphi(g)$. Then G is definably quasi-simple.*

Proof. By Lemma 3.3, we may assume that $Z(H) = (1)$. Let U be an infinite definable subgroup of G of infinite index, and assume by way of contradiction that its normalizer N has finite index in G.
Consider p_ℓ as defined in Remark 2.12(c), and $U_{(\ell)} \leq G_{(\ell)}$. Then $U_{(\ell)} \trianglelefteq N_{(\ell)} = G_{(\ell)}$ (the latter equality because $[G : N]$ is finite). In particular, $U_{(0)} \trianglelefteq G_{(0)} = H(L)$, and as the group $H(L)$ is simple (by Fact 2.17(a)), the Kolchin closure of U must be $H(L)$.
Moreover, as every generic of U is a generic of its σ-Δ-closure $\tilde{U}$, it follows that G normalizes $\tilde{U}$. So, we may replace U by $\tilde{U}$; then G also normalises the connected component of $\tilde{U}$ (for the σ-Δ-topology), and so we may assume that U is σ-Δ-closed and connected. By Theorem 4.1, for some $r \leq \ell$ and algebraic automorphism ψ of $H(L)$, the group $\tilde{U}$ is defined within $H(L)$ by the equation $\sigma^r(g) = \psi(g)$. We will show that this is impossible unless $r = \ell$ (and $\psi = \varphi$). Indeed, suppose that $r < \ell$, take a generic (u, g) of $U \times G$. Consider now $(u, \sigma^r(u))$, and $(g, \sigma^r(g))$. The elements u, g and $\sigma^r(g)$ are independent generics of the algebraic group H. Since $u \in \tilde{U}$, we have

$$\sigma^r(g^{-1}ug) = \sigma^r(g)^{-1}\psi(u)\sigma^r(g) = \psi(g^{-1}ug) = \psi(g)^{-1}\psi(u)\psi(g).$$

I.e., $\sigma^r(g)\psi(g)^{-1} \in C_H(\psi(u))$. As ψ is an automorphism of H, the elements $\sigma^r(g)$, $\psi(g)$ and $\psi(u)$ are independent generics of H; this gives us the desired contradiction, as $\sigma^r(g)\psi(g)^{-1}$ and $\psi(u)$ are independent generics of the non-commutative algebraic group H. □

4.4. The semi-simple case needs some additional lemmas. Indeed, Zariski denseness and the previous results do not suffice to give a complete description. Here is a simple example: Let H be a simple algebraic group defined and split over $\mathbb{Q}$, and consider the subgroup G of $H(\mathcal{U})^2$ defined by

$$G = \{(g_1, g_2) \in H(\mathcal{U})^2 \mid \sigma(g_1) = g_2\}.$$

Then G is Kolchin dense in $H(\mathcal{U})^2$, however G is isomorphic to $H(\mathcal{U})$, via the projection on the first factor. We will now prove several lemmas which will allow us to take care of this problem.

Lemma 4.5. *Let $G_1, \dots, G_t$ be centerless simple Δ-algebraic groups, with G_i Zariski dense in some algebraic group H_i, and $G \leq G_1 \times \cdots \times G_t$ a Δ-definable subgroup, which projects via the natural projections onto each G_i. Then there are a set $\Psi \subset \{1, \dots, t\}^2$, algebraic isomorphisms $\psi_{i,j} : H_i \to H_j$ whenever $(i, j) \in \Psi$, such that*

$$G = \{(g_1, \dots, g_t) \in \prod_{i=1}^{t} G_i \mid g_j = \psi_{i,j}(g_i), (i, j) \in \Psi\}.$$

Moreover, if $(i, j), (j, k) \in \Psi$ with $k \neq i$, then $(j, i), (i, k) \in \Psi$, $\psi_{j,i} = \psi_{i,j}^{-1}$, and $\psi_{i,k} = \psi_{j,k}\psi_{i,j}$.

Proof. Let us first remark the following result, which is implicit in the proof of Theorem 4.1 (paragraphs 4 to 6, and the Claim): assume in addition that G projects onto $\prod_{i=2}^{t} G_i$, but $G \neq \prod_{i=1}^{t} G_i$. Then there is some index $i \geq 2$, and an algebraic isomorphism $\psi_{1,i} : H_1 \to H_i$ such that

$$G = \{(g_1, \dots, g_t) \in \prod G_i \mid g_i = \psi_{1,i}(g_1)\}.$$

We let Ψ be the set of pairs $(i, j) \in \{1, \dots, t\}^2$ such that the image $G_{i,j}$ of G under the natural projection $\prod_{\ell=1}^{t} H_\ell \to H_i \times H_j$ is a proper subgroup of $G_i \times G_j$. By the above, if $(i, j) \in \Psi$, then $G_{i,j}$ is the graph of an isomorphism $G_i \to G_j$, restriction of some algebraic isomorphism $\psi_{i,j} : H_i \to H_j$. Then the set $(\Psi, \psi_{i,j})$ satisfies the moreover part of the conclusion, and we have

$$G \leq \{(g_1, \dots, g_t) \in \prod_{i=1}^{t} G_i \mid g_j = \psi_{i,j}(g_i), (i, j) \in \Psi\}.$$

To prove equality, we let $T \subset \{1, \dots, t\}$ be maximal such that whenever $i, j \in T$, then $(i, j) \notin \Psi$; then the natural projection $\prod_{\ell=1}^{t} H_\ell \to \prod_{\ell \in T} H_\ell$ defines an injection on G, and sends G to a subgroup G' of $\prod_{\ell \in T} G_\ell$, with the property that whenever $k \neq \ell \in T$, then G' projects onto $G_k \times G_\ell$. By the first case and an easy induction, this implies that $G' = \prod_{\ell \in T} G_\ell$, and finishes the proof of the lemma. □

Lemma 4.6. *Let $H_1, \dots, H_r$ be simple centerless algebraic groups defined and split over $\mathbb{Q}$, $L_1, \dots, L_r$ $\mathcal{L}_\Delta$-definable subfields of $\mathcal{U}$, and $G \leq \prod_{i=1}^{r} H_i(L_i)$ a Kolchin dense quantifier-free definable subgroup, which is connected for the σ-Δ-topology. Let $\tilde{G}_i \leq H_i(L_i)$ be the σ-Δ-closure of the projection of G on the i-th factor $H_i(L_i)$.*
Then there is a partition of $\{1, \dots, r\}$ into subsets $I_1, \dots, I_s$, such that for each $1 \leq k \leq s$, the following holds:
If $i \neq j \in I_k$, then there are an integer $n_{ij} \in \mathbb{Z}$ and an algebraic isomorphism $\theta_{ij} : H_i(L_i) \to H_j(\sigma^{n_{ij}}(L_j))$ such that if π_{I_k} is the projection $\prod_{j=1}^{r} H_j(L_j) \to \prod_{j \in I_k} H_j(L_j)$, and $i \in I_k$ is fixed, then

$$\pi_{I_k}(G) = \{(g_j)_{j \in I_k} \in \prod_{j \in I_k} H_j(L_j) \mid \theta_{ij}(g_i) = \sigma^{n_{ij}}(g_j) \text{ if } j \neq i\}.$$

Moreover, $G \simeq \prod_{k=1}^{s} \pi_{I_k}(G)$, and G projects onto each $\tilde{G}_i$.

Proof. We use the prolongations p_n defined in 2.12, and choose N large enough so that $G = \{\bar{g} \in \prod_{i=1}^r H_i(L_i) \mid p_N(\bar{g}) \in G_{(N)}\}$. Then $G_{(N)}$ is a Δ-algebraic subgroup of

$$\prod_{i=1}^r (\tilde{G}_i)_{(N)} \leq \prod_{1\leq i\leq r, 0\leq k\leq N} H_i(\sigma^k(L_i)).$$

Let $\Psi \subset (\{1,\dots,r\} \times \{0,\dots,N\})^2$ be the set of pairs given by Lemma 4.5, and $\psi_{(i,k),(j,\ell)}$, $((i,k),(j,\ell)) \in \Psi$, the corresponding set of algebraic isomorphisms

$$\psi_{(i,k),(j,\ell)} : H_i(\sigma^k(L_i)) \to H_j(\sigma^\ell(L_j)).$$

So, if $(g_1,\dots,g_r) \in G$, then

$$\psi_{(i,k),(j,\ell)}(\sigma^k(g_i)) = \sigma^\ell(g_j). \tag{1}$$

Note the following, whenever $((i,k),(j,\ell)) \in \Psi$:

- If $k+1, \ell+1 \leq N$, then $((i,k+1),(j,\ell+1)) \in \Psi$, with $\psi_{(i,k+1),(j,\ell+1)} = {\psi_{(i,k),(j,\ell)}}^\sigma$ (here, ${\psi_{(i,k),(j,\ell)}}^\sigma$ denotes the isomorphism obtained by applying σ to the coefficients of the isomorphism $\psi_{(i,k),(j,\ell)}$);
- If $k, \ell \geq 1$, then $((i,k-1),(j,\ell-1)) \in \Psi$, with $\psi_{(i,k-1),(j,\ell-1)} = {\psi_{(i,k),(j,\ell)}}^{\sigma^{-1}}$;
- If $k \leq \ell$, then applying σ^{-k} to equation (1) gives

$$((i,0),(j,\ell-k)) \in \Psi, \text{ and } \psi_{(i,k),(j,\ell)} = {\psi_{(i,0),(j,\ell-k)}}^{\sigma^k}.$$

- Finally, if $i = j$ and $k < \ell$, then $\tilde{G}_i$ is defined by an equation $\sigma^{n_i}(g) = \varphi_i(g)$ within $H_i(L_i)$ for some integer n_i and algebraic automorphism φ_i of $H(L)$, $((i,0),(i,n_i)) \in \Psi$ with associated isomorphism $\psi_{(i,0),(i,n_i)} = \varphi_i$, and $\ell - k$ is a multiple of the integer n_i. This is because G projects onto a subgroup of finite index of $\tilde{G}_i$, and therefore $G_{(N)}$ projects onto $(\tilde{G}_i)_{(N)}$.

By Lemma 4.5, we know that the set Ψ and the $\psi_{i,j}$ completely determine G, and by the above observations, each condition $\sigma^k(g_i) = \psi_{(i,k),(j,\ell)}(\sigma^\ell(g_j))$ is implied by

$$\sigma^{k-\ell}(g_i) = {\psi_{(i,k),(j,\ell)}}^{\sigma^{-\ell}}(g_j). \tag{2}$$

The set Ψ defines a structure of graph on $\{1,\dots,r\}\times\{0,\dots,N\}$, which in turn induces a graph structure on $\{1,\dots,r\}$, by $E(i,j)$ iff there are some k,ℓ such that $((i,k),(j,\ell)) \in \Psi$. If $E(i,j)$, then the isomorphism $\tilde{G}_i \to \tilde{G}_j$ is given by equation (2). Then $(\{1,\dots,r\}, E)$ has finitely many connected components, say $I_1,\dots,I_s$, and for every k, if $i \in I_k$, then $I_k = \{i\} \cup \{j \mid E(i,j)\}$. Lemma 4.6 shows that $G = \prod_{k=1}^s \pi_{I_k}(G)$, and gives the desired description of $\pi_{I_k}(G)$, with $\theta_{i,j} = {\psi_{(i,k),(j,\ell)}}^{\sigma^{-\ell}}$ and $n_{i,j} = k - \ell$, if $((i,k),(j,\ell)) \in \Psi$. ☐

Theorem 4.7. *Let G be a definable subgroup of $H(\mathcal{U})$, where H is a semi-simple algebraic group defined and split over $\mathbb{Q}$, and with trivial center. Assume that G is Zariski dense in H.*

(1) *Assume that the σ-Δ-closure of G is connected (for the σ-Δ-topology). Then there are s and simple normal algebraic subgroups $H_1, \ldots, H_s$ of H, a projection $\pi : H \to H_1 \times \cdots \times H_s$ which restricts to an injective map on G, $\mathcal{L}_\Delta$-definable subfields L_i of $\mathcal{U}$, definable subgroups G_i and G'_i of $H_i(L_i)$ for $1 \le i \le s$, and $h \in \pi(H)(\mathcal{U})$, such that*

$$G_1 \times \ldots \times G_s \le h^{-1}\pi(G)h \le G'_1 \times \cdots \times G'_s,$$

and each G_i is a normal subgroup of finite index of G'_i.

(2) *Assumptions as in (1). If in addition G is σ-Δ-closed, then $h^{-1}\pi(G)h = G_1 \times \cdots \times G_s$, and for each i, either $G_i = H_i(L_i)$, or for some integer ℓ_i and automorphism φ_i of $H_i(L_i)$, G_i is defined within $H_i(L_i)$ by $\sigma^{\ell_i}(g) = \varphi_i(g)$.*

Proof. By Theorem 2.14, if $H_1, \ldots, H_r$ are the simple algebraic components of H, and $\bar{G}$ is the Kolchin closure of G, then $\bar{G}$ is Δ-semi-simple; if $\bar{G}_i$ is the connected (for the Δ-topology) component of $\bar{G} \cap H_i(\mathcal{U})$, then the morphism $\rho : \bar{G}_1 \times \cdots \times \bar{G}_r \to \bar{G}$ is an isogeny, and because H is centerless, is an isomorphism.
By Theorem 2.15, we know that there are Δ-definable subfields L_i of $\mathcal{U}$, such that each $\bar{G}_i$ is conjugate to $H_i(L_i)$ within $H_i(\mathcal{U})$. But as $[H_i, H_j] = 1$ for $i \neq j$, there is $h \in H(\mathcal{U})$ such that $h^{-1}\bar{G}_i h \le H_i(L_i)$ for all i. We will replace G by $h^{-1}Gh$, so that $\bar{G}_i = H_i(L_i)$ for every i.

(1) For each i, consider the projection π_i on the i-th factor $H_i(L_i)$, and let $G'_i = \pi_i(G)$. Further, let $G_i = H_i(L_i) \cap G$. So, $G_1 \times \cdots \times G_r$ is a subgroup of G.

Claim 1. G'_i is Kolchin dense in $H_i(L_i)$, for $i = 1, \ldots, r$.

Proof. Since G is Kolchin dense in $\bar{G}$, any generic $g := (g_1, \ldots, g_r)$ of G is a generic of the Δ-algebraic group $\bar{G}$. Then g_i is a generic of $H_i(L_i)$ for all i, and the claim is proved. □

Claim 2. For all $i \in \{1, \ldots, r\}$, $G_i \trianglelefteq G'_i$.

Proof. Let $q : H \to H_2 \times \cdots \times H_r$ be the projection on the last $r-1$ factors. Then $G \cap \mathrm{Ker}\,(q)$ is normal in G, contained in $H_1(L_1) \times (1)^{r-1}$, and equals $G_1 \times (1)^{r-1}$. As G projects onto G'_1, we get $G_1 \trianglelefteq G'_1$. The proof for the other indices is similar. □

Claim 3. If $G_i \neq (1)$, then $[G'_i : G_i] < \infty$. If moreover G is quantifier-free definable, then $G_i = G'_i$.

Proof. Both G_i and G'_i are definable subgroups of the simple Δ-algebraic group $H_i(L_i)$ and G'_i is Kolchin dense in $H_i(L_i)$.
If $G'_i = H_i(L_i)$, then $G_i = G'_i$ since $H_i(L_i)$ is a simple (abstract) group (by 2.17, and because $Z(H) = (1)$). If $G'_i \neq H_i(L_i)$, then by Theorem 4.1, Claim 1 and Lemma 4.3, G'_i is definably quasi-simple. Hence, Claims 1 and 2 give the result when G is definable.
If G is quantifier-free definable, so is every G_i, and therefore G_i is closed in the σ-Δ-topology. This implies that $G_i = G'_i$, because G, and therefore also G'_i, is connected for the σ-Δ-topology. □

If all G_i are non-trivial, we have shown that our group G is squeezed between $G_1 \times \cdots \times G_r$ and $G'_1 \times \cdots \times G'_r$. And that if G is quantifier-free definable, then $G = \prod_{i=1}^r G_i$.

Assume now that some G_i are trivial. If $\tilde{G}$ denotes the σ-Δ-closure of G, and $\tilde{G}_i$ the σ-Δ-closure of G_i', then these groups are connected for the σ-Δ-topology, quantifier-free definable, and $\tilde{G}$ is a proper subgroup of $\prod_{i=1}^r \tilde{G}_i$. Hence Lemma 4.6 applies, and gives a subset T of $\{1, \ldots, r\}$ such that the natural projection π_T defines an isomorphism $\tilde{G} \to \prod_{i \in T} \tilde{G}_i$, which restricts to an embedding $G \to \prod_{i \in T} G_i'$ with $[\prod_{i \in T} G_i' : \pi_T(G)] < \infty$. Moreover, applying Claim 3 to $G_i'' := \pi_T(G) \cap H_i(L_i)$, $i \in T$, we get

$$\prod_{i \in T} G_i'' \leq \pi_T(G) \leq \prod_{i \in T} G_i',$$

with G_i'' a normal subgroup of G_i' of finite index. This finishes the proof of (1) (modulo a change of notation).
We showed that $\pi_T(\tilde{G}) = \prod_{i \in T} \tilde{G}_i$, which, together with Theorem 4.1, proves (2). □

Remarks 4.8. In the general case of $Z(H) \neq (1)$, we can obtain a similar result in a particular case: let $H_i(L_i)$ are the subgroups of $\bar{G}$ given by Theorem 2.14, and define $G_i = G \cap H_i(L_i)$ as above. Then if all G_i are infinite or trivial, the same proof gives some subset T of $\{1, \ldots, r\}$, and an isogeny $\prod_{i \in T} G_i$ onto a subgroup of finite index of G.
In the general case, however, we can only obtain such a representation of a proper quotient of G: the problem arises from the fact that the groups G_i may be finite non-trivial, so that the projection π_T defined in the proof will restrict to an isogeny on G. So, we might as well work with the image of G in $H/Z(H)$.

5 Definable subgroups of finite index

We work in a sufficiently saturated model $(\mathcal{U}, \sigma, \Delta)$ of DCF$_m$A. Unless otherwise mentioned, definable will mean $\mathcal{L}_{\sigma,\Delta}$-definable.
The aim of this section is to show that a definably quasi-simple group definable in $\mathcal{U}$ has a definable connected component. To do that, we investigate definable subgroups of algebraic groups which are not quantifier-free definable, and obtain a description similar to the one obtained by Hrushovski and Pillay in Proposition 3.3 of [13].

Theorem 5.1. *Let H be an algebraic group, $G \leq H(\mathcal{U})$ a Zariski dense definable subgroup. Then there are an algebraic group H', a quantifier-free definable subgroup R of $H'(\mathcal{U})$, together with a quantifier-free definable $f : R \to G$, with $f(R)$ contained and of finite index in G, and* Ker (f) *finite central in R.*

Proof. We follow the proof of Hrushovski-Pillay given in [13, Prop. 3.3], but with a slight simplification due to characteristic 0. Passing to a subgroup of G of finite index, we may assume that $\tilde{G}$ is connected for the σ-Δ-topology. We work over some small $F_0 = \mathrm{acl}(F_0) \subset \mathcal{U}$ over which G is defined. By Theorem 2.11(4), we know that there is some quantifier-free definable set W, and a projection $\pi : W \to \tilde{G}$, with finite fibers and such that $G = \pi(W)$.

Let b, c be independent generics of G, let $a \in G$ be such that $ab = c$, and let $\hat{b}, \hat{c} \in \mathcal{U}$ be such that $(b, \hat{b}), (c, \hat{c}) \in W$. So $\hat{b} \in \mathrm{acl}(F_0 b)$, and $\hat{c} \in \mathrm{acl}(F_0 c)$.

We let $a_1 \in \mathcal{U}$ be such that $\mathrm{acl}(F_0a) \cap F_0(b, \hat{b}, c, \hat{c})_{\sigma,\Delta} = F_0(a, a_1)_{\sigma,\Delta,}$. Note that because $a = cb^{-1}$ and $\mathrm{acl}(F_0a)$ is Galois over $F_0(a)_{\sigma,\Delta}$, $F_0(b, \hat{b}, c, \hat{c})_{\sigma,\Delta}$ is a regular extension of $\mathrm{acl}(F_0a) \cap F_0(b, \hat{b}, c, \hat{c})_{\sigma,\Delta}$, which is finitely generated algebraic over $F_0(a)_{\sigma,\Delta}$. Hence a_1 can be chosen finite by Lemma 2.18. Moreover, $qftp(b, \hat{b}, c, \hat{c}/F_0(a, a_1)_{\sigma,\Delta})$ is stationary (see Remark 2.12(e)), and $F_0(a, a_1)_{\sigma,\Delta}$ contains qf-Cb$(b, \hat{b}, c, \hat{c}/\mathrm{acl}(F_0a))$ (the quantifier-free canonical basis, see subsection 2.4).

Observe that $qftp(c, \hat{c}, a, a_1/F_0(b, \hat{b})_{\sigma,\Delta})$ is stationary: this is because $qftp(c, \hat{c}/F(b, \hat{b})_{\sigma,\Delta})$ is stationary, and $(a, a_1) \in F_0(b, \hat{b}, c, \hat{c})_{\sigma,\Delta}$. Hence, if b_1 is such that $\mathrm{acl}(F_0b) \cap F_0(a, a_1, c, \hat{c})_{\sigma,\Delta} = F_0(b, b_1)_{\sigma,\Delta}$, then $b_1 \in F_0(b, \hat{b})_{\sigma,\Delta}$. Similarly, if c_1 is such that $\mathrm{acl}(F_0c) \cap F_0(a, a_1, b, b_1)_{\sigma,\Delta} = F_0(c, c_1)_{\sigma,\Delta}$, then $c_1 \in F_0(c, \hat{c})_{\sigma,\Delta}$. So we obtain qf-Cb$(a, a_1, c, \hat{c}/\mathrm{acl}(F_0b)) \subseteq F_0(b, b_1)_{\sigma,\Delta}$ and qf-Cb$(qftp(a, a_1, b, b_1/\mathrm{acl}(F_0c))) \subseteq F_0(c, c_1)_{\sigma,\Delta}$. This implies that $b_1 \in F_0(a, a_1, c, c_1)_{\sigma,\Delta}$ and $a_1 \in F_0(b, b_1, c, c_1)_{\sigma,\Delta}$. I.e., we have

$$F_0(a, a_1, c, c_1)_{\sigma,\Delta} = F_0(a, a_1, b, b_1)_{\sigma,\Delta} = F_0(b, b_1, c, c_1)_{\sigma,\Delta}.$$

As in [13], (a, a_1) defines the germ of a generically defined, invertible, σ-Δ-rational map g_{a,a_1} from (the set of realisations of) $q_1 = qftp(b, b_1/F_0)$ to $q_2 = qftp(c, c_1/F_0)$. (In our setting, this means: there are $\mathcal{L}_\Delta$-definable sets U_1 and U_2, with U_i intersecting the set of realisations of q_i in a Kolchin dense subset, and such that g_{a,a_1} defines a Δ-rational invertible map $U_1 \to U_2$. We may shrink the U_i if necessary to relatively Kolchin dense subsets.)

Choose $(\tilde{a}, \tilde{a}_1) \in \mathcal{U}$ realising $qftp(a, a_1/F_0)$ and independent from (b, c) over F_0. Let $F_0' \prec \mathcal{U}$ contain $F_0(\tilde{a})$ and such that (a, b, c) is independent from F_0' over F_0. Let (b', b_1') be such that $qftp(a, a_1, b, b_1, c, c_1/F_0) = qftp(\tilde{a}, \tilde{a}_1, b', b_1', c, c_1/F_0)$; note that $(b', b_1') \in F_0(\tilde{a}, \tilde{a}_1, c, c_1)_{\sigma,\Delta}$, and let $d = (\tilde{a})^{-1}a$. Let $r = qftp(a, a_1/F_0')$ (the unique non-forking extension of $qftp(a, a_1/F_0)$ to F_0').

Claim 1.
(i) $F_0'(b, \hat{b}, c, \hat{c})_{\sigma,\Delta} \cap \mathrm{acl}(F_0'd) = F_0'(a, a_1)_{\sigma,\Delta}$.
(ii) $qftp(b, b_1/F_0') = qftp(b', b_1'/F_0') =: q_1'$ is the unique non-forking extension of q_1 to F_0'.
(iii) (a, a_1) defines over F_0' the germ of an invertible generically defined function from q_1' to q_1'.
(iv) $d \in F_0'(a, a_1)_{\sigma,\Delta}$.
(v) $db = b'$.
(vi) $(a, a_1) \in F_0'(b, b_1, b', b_1')_{\sigma,\Delta}$.

Proof. This follows immediately from the fact that (a, b, c) is independent from F_0' over F_0, that $F_0'(a)_{\sigma,\Delta} = F_0'(d)_{\sigma,\Delta}$, and the definition of a_1. □

Claim 2. r is closed under generic composition.

Proof. Let (a', a_1') realise r in $\mathcal{U}$, and independent from (a, b, b') over F_0'. If $(b'', b_1'') \in \mathcal{U}$ is such that

$$qftp(a', a_1', b', b_1', b'', b_1''/F_0') = qftp(a, a_1, b, b_1, b', b_1'/F_0'),$$

then from the fact that

$$F_0'(a, a_1, b, b_1)_{\sigma,\Delta} = F_0'(a, a_1, b', b_1')_{\sigma,\Delta} = F_0'(b, b_1, b', b_1')_{\sigma,\Delta},$$

we obtain that (b, b_1) and (b'', b''_1) are independent over F'_0, and that $qftp(b, b_1, b'', b''_1/F'_0) = qftp(b, b_1, b', b'_1/F'_0)$; hence if $(a'', a''_1) \in F'_0(b, b_1, b'', b''_1)_{\sigma,\Delta}$ is such that $qftp(a'', a''_1, b, b_1, b'', b''_1/F'_0) = qftp(a, a_1, b, b_1, b', b'_1/F'_0)$, then $qftp(a'', a''_1/F'_0) = r$ as desired. $\square$

Furthermore, note that $a'' \in F'_0(a, a')$, and, unravelling the definitions, that

$$(a'', a''_1) \in F'_0(b, b_1, a, a_1, a', a'_1)_{\sigma,\Delta}.$$

Hence $(a'', a''_1) \in F'_0(a, a')^{alg}_{\sigma,\Delta} \cap F'_0(b, b_1, a, a_1, a', a'_1)_{\sigma,\Delta} = F'_0(a, a_1, a', a'_1)_{\sigma,\Delta}$ because (b, b_1) is independent from (a, a_1, a', a'_1) over F'_0. Similarly, using the fact that the first part of the tuple lives in the algebraic group H, one gets that the group law which to $((a, a_1), (a', a'_1))$ associates (a'', a''_1) as above, is associative. Hence we are in presence of a normal group law as in [33] (page 359), involving however infinite tuples.

We now will reason as in [24] (Lemma 2.3 and Propositions 3.1 and 4.1 in [24]), use the fact that the σ-Δ-topology is Noetherian, and obtain that r is the generic type of a quantifier-free definable subgroup R of some algebraic group H'.
More precisely: as in Lemma 2.3 of [24], we replace (a, a_1) by the infinite tuple obtained by closing (a, a_1) under σ, σ^{-1} and the δ_i. This allows to represent the normal group law as a normal group law on some inverse limit of algebraic sets, together with a (σ-Δ-rational) map from the set of realisations of r to this inverse limit. Then Proposition 3.1 of [24] shows how to replace this inverse limit by an inverse limit of algebraic groups. And finally, as in Theorem 4.1 of [24], the Noetherianity of the σ-Δ-topology guarantees that the map from the set of realizations of r to this inverse limit of groups must yield an injection at some finite stage.

Observe also that $qftp(b, b_1, b', b'_1/F'_0) = qftp(b', b'_1, b, b_1/F'_0)$, and so we get a realisation of r which is the germ of the inverse of (a, a_1); as the first coordinate of this germ belongs to $F'_0(a)$, it follows that it belongs to $F'_0(a, a_1)_{\sigma,\Delta}$.

Let us now look at $p = qftp(a, a_1, d/F'_0)$, and recall that $F'_0(a)_{\sigma,\Delta} = F'_0(d)_{\sigma,\Delta}$, and let K be the subgroup of $(H' \times H)(\mathcal{U})$ generated by the realisations of p. It is definable by a quantifier-free $\mathcal{L}_{\sigma,\Delta}$-formula.

As in [13], it follows that K is the graph of a group epimorphism $f : R \to \tilde{G}$, with finite kernel. Because R is connected for the σ-Δ-topology, the kernel is central.

Claim 3. $f(R) \leq G$.

Proof. Let (g, g_1) be a generic of R, i.e., a realisation of r. Then $g \in \tilde{G}$. We know that $qftp(b, \hat{b}, c, \hat{c}/F'_0(a, a_1)_{\sigma,\Delta})$ is stationary, and therefore so is its image under any F'_0-automorphism of the differential field $\mathcal{U}$ sending (a, a_1) to (g, g_1), so that there are $(h, \hat{h}, u, \hat{u})$ in $\mathcal{U}$ such that

$$qftp(a, a_1, b, \hat{b}, c, \hat{c}/F'_0) = qftp(g, g_1, h, \hat{h}, u, \hat{u}/F'_0).$$

Thus $h, u \in G$, and so does $g = uh^{-1}$.

Observe that $f(R)$ has finite index in G, because it has the same generics. $\square$

Remark 5.2. In the notation of Theorem 5.1, consider $R_{(n)}$ and $G_{(n)}$, as well as the natural $\mathcal{L}_\Delta$-map $f_{(n)} : R_{(n)} \to G_{(n)}$. While the map f is clearly not surjective in the difference-differential field $\mathcal{U}$, the map $f_{(n)}$ is surjective for all $n \geq 0$ (in the differential field $\mathcal{U}$). This follows from quantifier-elimination in DCF_m. Moreover, the image of R in G is dense for the σ-Δ-topology, i.e., this is the appropriate notion of a *dominant map* between difference varieties.

Definition 5.3. Let H be an algebraic group. It is *simply connected* if it is connected and whenever $f : H' \to H$ is an isogeny from the connected algebraic group H' onto H, then f is an isomorphism.
The *universal covering of the connected algebraic group* H is a simply connected algebraic group $\hat{H}$, together with an isogeny $\pi : \hat{H} \to H$. It satisfies the following universal property (see 18.8 in [22]): if $\varphi : H' \to H$ is an isogeny of connected algebraic groups, then there is a unique algebraic homomorphism $\psi : \hat{H} \to \hat{H}'$ such that $\varphi\psi = \pi$.

Remark 5.4. (1) The definition of simply connected in arbitrary characteristic is a little more complicated. The algebraic groups we will consider will be semi-simple algebraic groups, defined and split over $\mathbb{Q}$, and we will be considering their rational points in some algebraically closed field K.

(2) Every simple algebraic group has a universal covering, see section 5 in [29] for properties, or Chapter 19 in [22].
(3) Note that if H is a simple algebraic group and K is algebraically closed, then $H(K)/Z(H(K))$ is simple as an abstract group.
(4) Moreover, since a semi-simple algebraic group is isogenous to the product of its simple factors, it follows that the universal covering of a semi-simple algebraic group is simply the product of the universal coverings of its simple factors.

Lemma 5.5. *Let H be a simple algebraic group defined over the algebraically closed field L of characteristic 0, and $\pi : \hat{H} \to H$ its universal covering. Then any algebraic automorphism of $H(L)$ lifts to one of $\hat{H}(L)$.*

Proof. Let φ be an algebraic automorphism of $H(L)$, and consider the map $p : \hat{H}(L) \to H(L)$ defined by $\varphi \circ \pi$. Then there is a map $\psi : \hat{H}(L) \to \hat{H}(L)$ such that $\pi = \varphi \circ \pi \circ \psi$. It then follows easily that ψ is an isomorphism: $\psi(\hat{H}(L))$ is a subgroup of $\hat{H}(L)$ which projects onto $H(L)$ via π, hence must equal $\hat{H}(L)$. So ψ is onto, and because $\mathrm{Ker}\,(\pi)$ is finite, it must be injective. □

Theorem 5.6. *Let H be a simply connected simple algebraic group defined and split over $\mathbb{Q}$, and $G \leq H(\mathcal{U})$ a proper Zariski dense definable subgroup. Then G is quantifier-free definable. Equivalently, G has a smallest definable subgroup G^0 of finite index, and G^0 is quantifier-free definable.*
Furthermore, there is an $\mathcal{L}_\Delta$-definable subfield L of $\mathcal{U}$, such that $h^{-1}G^0h \leq H(L)$ for some $h \in H(\mathcal{U})$, and either $h^{-1}G^0h = H(L)$, or

$$h^{-1}G^0h = \{g \in H(L) \mid \sigma^n(g) = \theta(g)\}$$

for some integer n and algebraic automorphism θ of $H(L)$.

Proof. Let us first discuss the equivalence of the two assertions. If any Zariski dense definable subgroup of $H(\mathcal{U})$ is quantifier-free definable, then every definable subgroup of G of finite index is quantifier-free definable, and the Noetherianity of the σ-Δ topology implies that there is a smallest one, G^0. Conversely, let G be a Zariski dense definable subgroup of $H(\mathcal{U})$, and assume it has a smallest definable subgroup of finite index, G^0, and that G^0 is quantifier-free definable. Then so is G, since it is a finite union of cosets of G^0.

By Theorem 4.1, G has a definable subgroup of finite index G_0 which is conjugate to a Kolchin dense subgroup of $H(L)$, for some definable subfield L of $\mathcal{U}$. So, without loss of generality, we will assume that $G \leq H(L)$ is connected for the σ-Δ-topology, is quantifier-free definable, and we will show that G has no proper definable subgbroup of finite index.
First note that if $G = H(L)$, then G has no definable subgroups of finite index (by Fact 2.17(a)), and the result is proved. Assume therefore that G is a proper subgroup of $H(L)$. Let $H' = H/Z(H)$. By Theorem 4.1, there are an integer $n \geq 1$ and an algebraic automorphism θ' of $H'(L)$ such that the σ-Δ-closure G' of GZ/Z (in $H'(L)$) is defined by

$$G' = \{g \in H'(L) \mid \sigma^n(g) = \theta'(g)\}.$$

As H is simply connected, $H \to H/Z$ is the universal covering of H/Z. By Lemma 5.5, there is an algebraic automorphism θ of $H(L)$ which lifts θ'.

Claim. $G = \{g \in H(L) \mid \sigma^n(g) = \theta(g)\}$.

Proof. The group on the right hand side is clearly quantifier-free definable, connected for the σ-Δ-topology, and projects onto a subgroup of finite index of GZ/Z, with finite kernel. As G is the connected component of the group GZ (for the σ-Δ-topology), the conclusion follows. □

Assume by way of contradiction that G has a definable subgroup of finite index > 1. By Proposition 5.1, there are a quantifier-free definable group R (living in some algebraic group S) and a (quantifier-free) definable map $f : R \to G$ with finite non-trivial kernel, and image of finite index > 1 in G. We may assume that R is connected for the σ-Δ-topology, so that $\mathrm{Ker}\,(f)$ is central.
For every $r \geq 1$, the map f induces a dominant Δ-map $f_{(r)} : R_{(r)} \to G_{(r)}$, and for $r \geq n-1$, this map has finite central kernel, since for $r \geq n-1$, the natural map $G_{(r)} \to G_{(n-1)}$ has trivial kernel. Fix $r \geq n-1$, and consider the map $f_{(r)} : R_{(r)} \to G_{(r)} \simeq H(L)^n$. Because H is simply connected, so is H^n, and therefore $R_{(r)} \simeq H(L)^n \times \mathrm{Ker}\,(f_r)$. Since $H(L)$ equals its commutator subgroup, it follows that $[R_{(r)}, R_{(r)}]$ ($\simeq H(L)^n$) is a Δ-definable normal subgroup of $R_{(r)}$ which projects via $f_{(r)}$ onto $G_{(r)} \simeq H(L)^n$. As R is connected for the σ-Δ-topology, $R_{(r)}$ is connected for the Δ-topology, and we must therefore have $\mathrm{Ker}\,(f_{(r)}) = (1)$. □

Theorem 5.7. *Let H be a simple algebraic group, $G \leq H(\mathcal{U})$ be a definable subgroup which is Zariski dense in H. Then G has a smallest definable subgroup G^0 of finite index. Let $\pi : \hat{H} \to H$ be the universal finite central extension of H, and let $\tilde{G}$ be the connected component of the σ-Δ-closure of $\pi^{-1}(G)$. Then $G^0 = \pi(\tilde{G})$.*

Proof. By Lemma 5.6, $\tilde{G}$ has no definable subgroup of finite index. Hence, neither does $\pi(\tilde{G})$, which is therefore the smallest definable subgroup of finite index of G. □

Corollary 5.8. *Let G be a definably quasi-semi-simple definable group. Then G has a definable connected component.*

Proof. Let P be the property "having a definable connected component". The result follows easily from Proposition 3.4, Proposition 4.7, Theorem 5.7, and the following remarks:

(a) If G_0 is a definable subgroup of finite index of G, then G_0 has P if and only if G has P;

(b) If the group G is the direct product of its definable subgroups G_1, G_2, and G_1, G_2 have P, then so does G;

(c) Let $f : G \to G_1$ be a definable onto map, with $\mathrm{Ker}\,(f)$ finite. Then G_1 has P if and only if G has P. One direction is clear, for the other, we may assume that G_1 is connected, so that $\mathrm{Ker}\,(f)$ is central, finite. If G_0 is a subgroup of finite index of G, then $f(G_0) = G_1$, so that $G_0\mathrm{Ker}\,(f) = G$; hence $[G : G_0] \leq |\mathrm{Ker}\,(f)|$. Let $G_0 < G$ be definable, of finite index, and with $|G_0 \cap \mathrm{Ker}\,(f)|$ minimal. Then G_0 has no proper definable subgroup of finite index.

□

6 The fixed field

Definition 6.1. Let M be a $\mathcal{L}$-structure. A definable subset D of M is *stably embedded* if every M-definable subset of D^n is definable with parameters from D, for any $n \geq 1$.

Notations and Conventions 6.2. Let $(\mathcal{U}, \sigma, \Delta)$ be a sufficiently saturated model of $\mathrm{DCF}_m\mathrm{A}$. For $\ell \geq 1$, we consider the difference-differential field $F_\ell = \mathrm{Fix}(\sigma^\ell)$.

Lemma 6.3. *Fix $\ell \geq 1$. Then F_ℓ is stably embedded, and its induced structure is that of the pure difference-differential field. If $\ell = 1$, it is the pure differential field.*

Proof. The first part follows from elimination of imaginaries (Prop. 3.3 in [18]): if c is a code for a definable subset S of F_ℓ^n, then $\sigma^\ell(c) = c$. So every definable subset of F_ℓ^n is definable using parameters from F_ℓ.

By the description of types in $\mathrm{DCF}_m\mathrm{A}$, every formula $\varphi(x)$ is equivalent (modulo $\mathrm{DCF}_m\mathrm{A}$) to a formula of the form $\exists y\, \psi(x, y)$, where $\psi(x, y)$ is quantifier-free, and whenever (a, b) realises ψ, then $b \in \mathrm{acl}(a)$. But if $a \in F_\ell$, then $b \in F_\ell^{alg}$. Let d be a bound on the degree of b over a, and $N(d)$ the least common multiple of all integers $\leq d$.

Let $F_0 \prec F_\ell^\Delta$ be small, and let $\alpha \in F_0^{alg}$ generate the unique extension of F_0 of degree $N(d)$. Note that it also generates the unique extension of F_ℓ of degree $N(d)$. So, if $(a, b) \in F_\ell^{alg}$ satisfies ψ as above, then $b \in F_\ell[\alpha]$. If u is the $N(d)$-tuple of coefficients of the minimal polynomial of α over F_0, one sees that the differential field $(F_\ell(\alpha), \sigma)$ is interpretable in F_ℓ (with parameters in F_0, or even in $\mathbb{Q}(u)$). Thus there is an $\mathcal{L}_{\sigma,\Delta}(F_0)$-formula $\theta(x, z)$ such that for any tuples $a \in F_\ell$ and $b \in F_\ell(\alpha)$, if $b = \sum_{i=0}^{N(d)-1} c_i\alpha^i$ with the c_i in F_ℓ, then

$$(F_\ell(\alpha), \sigma) \models \psi(a, b) \iff (F_\ell, \sigma) \models \theta(a, c).$$

To prove the last statement, it suffices to notice that if $b \in F_0(a, \alpha)$, then the tuple c belongs to $F_0(a, \alpha, b)$, and it also belongs to F_ℓ. As both α and b are algebraic over $F_0(a)$, it follows that so is c. This finishes the proof. □

Corollary 6.4. *If $A \subset F_\ell$, then $\mathrm{acl}_{F_\ell}(A) = \mathrm{acl}(A) \cap F_\ell$, and independence is given by independence (in the sense of ACF) of algebraic closures.*

Proof. This follows directly from Lemma 6.3. □

Corollary 6.5. *Same hypotheses as in 5.1, and assume that $G \leq H(F_\ell)$. Then the group R can be taken to be quantifier-free definable in the $\mathcal{L}_{\sigma,\Delta}$-structure F_ℓ.*

Proof. Inspection of the proof of Theorem 5.1 shows that if the tuples a, b, c are in F_ℓ, then by Lemma 6.7, so are the tuples $\hat{a}$, $\hat{b}$ et $\hat{c}$, and therefore also the tuples a_1, b_1 et c_1. I.e., the whole reasoning can be done inside F_ℓ. □

Definition 6.6. Let F be a differential field. We say that F is Δ-*PAC* if whenever L is a differential field extending F and which is regular over F (i.e., $L \cap F^{alg} = F$), then F is existentially closed in L.

Remarks 6.7. This definition coincides with the notion of PAC-substructure of a model of DCF_m, which was given by Pillay and Polkowska in [25] .
Consider the theory of the differential field $F = \mathrm{Fix}(\sigma)$, in the language $\mathcal{L}_\Delta$ augmented by the constant symbols needed to define all algebraic extensions of F_0. We know that F^{alg} is a model of DCF_m, by [18, Prop. 3.4(vi)].

Proposition 6.8. *The differential field F is a model of the theory UC_m introduced by Tressl in [32]. In particular,*

(1) $\mathrm{Th}(F)$ *is model-complete in the language $\mathcal{L}_\Delta(F_0)$.*
(2) *F is Δ-PAC.*

Proof. The theory UC_m has the following property (Thm 7.1 in [32]): if a theory T of fields of characteristic 0 is model complete, then $T \cup \mathrm{UC}_m$ is the model companion of the theory $T \cup \mathrm{DF}_m$, where DF_m is the theory of differential fields with m commuting derivations. We know that F is large as a pure field (all PAC fields are large), and that its theory in the language of rings augmented by constant symbols for F_0 is model-complete. Hence it has a regular extension F^* which is a model of UC_m (Thm 6.2 in [32]). Consider the differential field $\mathrm{Frac}(\mathcal{U} \otimes_F F^*)$, and extend σ to F^* by setting it to be the identity. As $\mathcal{U}$ is existentially closed in $\mathrm{Frac}(\mathcal{U} \otimes_F F^*)$, it follows that F is existentially closed in F^*, and therefore must be a model of UC_m. This proves the first part, and the same proof gives (2).
(1) follows from [32, thm 7.1]. □

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