Local divisibility and model completeness of a theory of real closed rings.

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Abstract

Let T^* be the theory of lattice-ordered rings convex in von Neumann regular real closed f-rings, without minimal idempotents (non zero) that are divisible-projectable and sc-regular. I introduce a binary relation describing local divisibility. If this relation is added to the language of lattice ordered rings with the radical relation associated to the minimal prime spectrum (cf. [12]), it can be shown the model completeness of T^*.

1 Introduction.

The theory T^* can also be described as the theory of real closed, reduced, projectable f-rings that are divisible-projectable, sc-regular, satisfying the first convexity property, and without minimal idempotents (non zero), cf. [8, Theorem 10].

By [7], T^* admits elimination of quantifiers in $\mathcal{L}^* = \{0, 1, +, -, \cdot, \wedge, \operatorname{div}\}$, the language of lattice-ordered rings where $\operatorname{div}(\cdot,\cdot)$ is a binary function symbol defined by:

$$\begin{aligned} T^* \vdash \operatorname{div}(x, y) = c \quad \longleftrightarrow \quad & c \in y^{\perp\perp} \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ & \forall w'(w' \neq 0 \wedge w' \perp (w - w') \to y \nmid w')\big). \end{aligned}$$

If A is a reduced f-ring, it is known by [3] that $\forall x \forall y (x \perp y \leftrightarrow xy = 0)$ is a valid formula in A. For $a \in A$, the polar of a is defined by $a^\perp = \{b \in A : b \perp a\}$ and the bipolar by $a^{\perp\perp} = \{b \in A : b \perp c \ \forall c \perp a\}$. It is also known by [3] that:

$$b \in a^{\perp\perp} \Longleftrightarrow a^\perp \subseteq b^\perp \Longleftrightarrow \operatorname{Ann}(a) \subseteq \operatorname{Ann}(b).$$

If A is a projectable reduced f-ring, then [9] says that:

$$A \in \Gamma^a_{\mathcal{L}}\big(\pi A, (A/p)_{p \in \pi A}\big),$$

where $\mathcal{L}$ is the language of ordered rings (see notations in [4]) and,

$$\pi A = \{p \in \operatorname{Spec}(A) : p \text{ is a minimal prime ideal}\} = \operatorname{Specmin}(A).$$

In this case:

$$\begin{aligned} b \in a^{\perp\perp} &\iff [\![b \neq 0]\!] \subseteq [\![a \neq 0]\!] \\ &\iff \operatorname{supp}(b) \subseteq \operatorname{supp}(a) \\ &\iff [\![a = 0]\!] \subseteq [\![b = 0]\!] \\ &\iff \forall p \in \pi A \, (a \in p \Rightarrow b \in p). \end{aligned}$$

In [12], the authors used radical relations, introduced in [11], in order to study the model theory of von Neumann regular real closed f-rings without minimal idempotents (non zero). Radical relations are given, cf [11], by a subset $X \subseteq \operatorname{Spec}(A)$ through:

$$b \preceq_X a \iff \forall p \in X \, (b \notin p \Rightarrow a \notin p).$$

The case $X = \pi A$ is relevant and studied in [12] and we then have:

$$\begin{aligned} b \preceq_{\pi A} a &\iff \forall p \in \pi A \, (b \notin p \Rightarrow a \notin p) \\ &\iff \forall p \in \pi A \, (a \in p \Rightarrow b \in p) \\ &\iff b \in a^{\perp\perp} \\ &\iff a^{\perp} \subseteq b^{\perp} \\ &\iff \operatorname{Ann}(a) \subseteq \operatorname{Ann}(b). \end{aligned}$$

Following [12], let us extend the language of lattice-ordered rings $\mathcal{L} = \{0, 1, +, \cdot, \wedge\}$ introducing a binary relation symbol $\preceq$ defined by:

$$b \preceq a \iff b \in a^{\perp\perp} \iff \operatorname{Ann}(a) \subseteq \operatorname{Ann}(b).$$

In fact, a radical relation $\preceq$ is a binary relation defined in [12] by:

(1) $a \preceq a$, for all $a \in A$;

(2) if $a \preceq b$ and $b \preceq c$ then $a \preceq c$, for all $a, b, c \in A$;

(3) if $a \preceq c$ and $b \preceq c$ then $a + b \preceq c$, for all $a, b, c \in A$;

(4) if $a \preceq b$ then $ac \preceq bc$, for all $a, b, c \in A$;

(5) $a \preceq 1$, for all $a \in A$ and $1 \not\preceq 0$;

(6) $b \preceq b^2$, for all $b \in A$.

In the theory of real closed valuation rings the divisibility plays a key role (see [5]), it is therefore interesting to ask if the divisibility relation can be given by a radical relation. Looking the defining properties (1) to (6) of a radical relation, let us set:

$$a \preceq b \iff b \mid a.$$

Let us see if in this case $\preceq$ is in fact a radical relation. The first five conditions are easily seen to be satisfied. But for the sixth condition, it is seen that:

$$b \preceq b^2 \iff b^2 \mid b \iff \exists x (b^2 x = b) \iff \exists x (bxb = b),$$

that is precisely the definition of a von Neumann regular ring. Then $a \preceq b \iff b \mid a$ is a radical relation if and only if the ring is von Neumann regular.

In fact, if A is a von Neumann regular f-ring, then the relation given by:

$$a \preceq_{\pi A} b \iff b^{\perp} \subseteq a^{\perp} \iff [\![b = 0]\!] \subseteq [\![a = 0]\!],$$

is the divisibility. For $a, b \in A$:

- If $b \mid a$ then there exists $x \in A$ with $bx = a$. Then $[\![b = 0]\!] \subseteq [\![a = 0]\!]$.
- If $[\![b = 0]\!] \subseteq [\![a = 0]\!]$, consider $x \in A$ defined by:

$$x = 0_{\restriction [\, b=0 \,]} \cup \left(\frac{a}{b}\right)_{\restriction [\, b \neq 0 \,]} \in A,$$

and it is such that $bx = a$. Then $b \mid a$.

This is an indication that the divisibility relation can not be consider in the context of models of T^* as a radical relation. For this reason and by the definition of the binary function symbol $\operatorname{div}(\cdot,\cdot)$ is that I will introduce a binary relation symbol of local divisibility. First of all, observe that the definition of the $\operatorname{div}(\cdot,\cdot)$ symbol can be written using the radical relation $\preceq$ associated to the minimal prime spectrum:

$$\begin{aligned} T^* \vdash \operatorname{div}(x,y) = c \quad \longleftrightarrow \quad & c \preceq y \wedge \exists z \exists w \big(x = z + w \wedge z \perp w \wedge cy = z \wedge \\ & \forall w' (w' \neq 0 \wedge w' \perp (w - w') \rightarrow y \nmid w') \big). \end{aligned}$$

In order to study the theory T^* from an existential formula or model completeness point of view, I introduce a binary relation given by:

$$R(y,w) \longleftrightarrow \forall w' (w' \neq 0 \wedge w' \perp (w - w') \rightarrow y \nmid w'),$$

that express the fact that y does not divide locally w. It will more pleasant to have in a positive form:

$$y \mid_{\mathrm{loc}} w \longleftrightarrow \neg R(y,w) \longleftrightarrow \exists w' (w' \neq 0 \wedge w' \perp (w - w') \wedge y \mid w').$$

Observe that this two last expressions in the language of lattice-ordered rings can be reformulated in the language of rings by:

$$R(y,w) \longleftrightarrow \forall w' (w' \neq 0 \wedge w'(w - w') = 0 \rightarrow y \nmid w'),$$

and

$$y \mid_{\mathrm{loc}} w \longleftrightarrow \neg R(y,w) \longleftrightarrow \exists w' (w' \neq 0 \wedge w'(w - w') = 0 \wedge y \mid w').$$

For the "global" divisibility relation $y \mid w$ it is obvious that $y \mid 0$. But observe that if $y \mid_{\mathrm{loc}} 0$ in a reduced ring A then there exists $w' \in A$ with $w' \neq 0$ and $w'(-w') = 0$ such that $y \mid w'$. Therefore $w' \neq 0$ and $w'^2 = 0$, that gives a contradiction in the reduced ring A. That is why I redefined:

$$y \mid_{\mathrm{loc}} w \longleftrightarrow \exists w' (w' \neq 0 \wedge w'(w - w') = 0 \wedge y \mid w') \vee w = 0.$$

2 Local divisibility relation.

This section is related to the study of this "local divisibility" relation in the general theory of rings. A list of properties is given, they will be stated as general as possible. Some of them will be stated in the theory of reduced (f)-rings.

Proposition 2.1 *Let A be any ring and $y, w \in A$. If $y \mid w$ then $y \mid_{\text{loc}} w$.*

Proof: If $w = 0$ then clearly $y \mid_{\text{loc}} w$. If $w \neq 0$, then take $w' = w$. Clearly $w' \neq 0$ and $w'(w - w') = w \cdot 0 = 0$ with $y \mid w$. Then the formula $\exists w'(w' \neq 0 \wedge w'(w' - w) = 0 \wedge y \mid w')$ is valid in A. In both cases: $y \mid_{\text{loc}} w$. ∎

Proposition 2.2 *Let A be any ring and $y, w \in A$. For $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, if $y^n \mid_{\text{loc}} w$ then $y \mid_{\text{loc}} w$.*

Proof: Let us suppose that $y^n \mid_{\text{loc}} w$ for $n \in \mathbb{N}^*$. If $w = 0$ then clearly $y \mid_{\text{loc}} w$. If $w \neq 0$, then exists $w' \in A$ with $w' \neq 0$, $w'(w - w') = 0$ and $y^n \mid w'$. Since $y \mid y^n$ then $\exists w'(w' \neq 0 \wedge w'(w' - w) = 0 \wedge y \mid w')$ is a valid formula in A. In this case we also have $y \mid_{\text{loc}} w$. ∎

Proposition 2.3 *Let A be any ring and $w \in A$. Then $1 \mid_{\text{loc}} w$.*

Proof: If $w = 0$ then clearly $1 \mid_{\text{loc}} w$. If $w \neq 0$, declaring $w' = w$ we obtain $1 \mid w'$ and then $\exists w'(w' \neq 0 \wedge w'(w' - w) = 0 \wedge 1 \mid w')$ is valid in A. In this case one also have $1 \mid_{\text{loc}} w$. ∎

Proposition 2.4 *Let A be any ring and $y \in A$. Then $y \mid_{\text{loc}} 0$.*

Proof: By definition. ∎

Proposition 2.5 *Let A any ring and $c, y, w \in A$. If $cy \mid_{\text{loc}} w$ then $y \mid_{\text{loc}} w$.*

Proof: If $w = 0$ then by definition $y \mid_{\text{loc}} w$. If $w \neq 0$, as we have $cy \mid_{\text{loc}} w$ then exists $w' \in A \setminus \{0\}$such that $w'(w' - w) = 0$ and $cy \mid w'$. Since $y \mid cy$ then by transitivity $y \mid w'$. Then the formula $\exists w'(w' \neq 0 \wedge w'(w' - w) = 0 \wedge y \mid w')$ is valid in A. In this case we also have $y \mid_{\text{loc}} w$. ∎

Proposition 2.6 *Let A be any ring and $w \in A$. If $0 \mid_{\text{loc}} w$ then $w = 0$.*

Proof: Let us suppose that $w \neq 0$. Since $0 \mid_{\text{loc}} w$ there exists $w' \in A$ with $w' \neq 0$, $w'(w' - w) = 0$ and $0 \mid w'$. But $0 \mid w'$ gives us $w' = 0$, a contradiction. Then $w = 0$. ∎

Proposition 2.7 *Let A be any ring and $y, w \in A$. Then $y \mid_{\text{loc}} w$ if and only if $-y \mid_{\text{loc}} w$.*

Proof: ($\Leftarrow$) Let us suppose that $-y \mid_{\text{loc}} w$. If $w = 0$ then by definition we have that $y \mid_{\text{loc}} w$. If $w \neq 0$, then there exists $w' \in A$ con $w' \neq 0$, $w'(w' - w) = 0$ and $-y \mid w'$. Evidently $y \mid w'$. The formula $\exists w'(w' \neq 0 \wedge w'(w' - w) = 0 \wedge y \mid w')$ is valid in A and therefore in this case $y \mid_{\text{loc}} w$. We have proved that if $-y \mid_{\text{loc}} w$ then $y \mid_{\text{loc}} w$, for all $y, w \in A$.

($\Rightarrow$) This implication can be deduced by the previous one interchanging y by $-y$. ■

Fact 2.8 Observe that if the ring A is unitary, then the previous property can be proved using the proposition 2.5 with $c = -1$.

Proposition 2.9 *Let A be any ring and $y, w \in A$. Then $y \mid_{\text{loc}} w$ if and only if $y \mid_{\text{loc}} -w$.*

Proof: ($\Rightarrow$) Let us suppose that $y \mid_{\text{loc}} w$. We want to show that $y \mid_{\text{loc}} -w$. If $w = 0$ then $-w = 0$ and $y \mid_{\text{loc}} -w$ by definition. If $w \neq 0$, since $y \mid_{\text{loc}} w$ there exists $w' \in A$, $w' \neq 0$ with $w'(w' - w) = 0$ and $y \mid w'$. Declaring $w'' = -w' \in A$ one has clearly that $w'' \neq 0$. Since $w'(w' - w) = 0$ then $(-w')(-w' + w) = 0$. Observe that since $y \mid w'$ then $y \mid_{\text{loc}} -w' = w''$. Then the formula $\exists w''(w'' \neq 0 \wedge w''(w'' - (-w)) = 0 \wedge y \mid w'')$ is valid in A. This says that $y \mid_{\text{loc}} -w$. It is been shown that if $y \mid_{\text{loc}} w$ then $y \mid_{\text{loc}} -w$, for $y, w \in A$.

($\Leftarrow$) This implication is deduced from the previous one replacing w by $-w$. ■

Proposition 2.10 *Let A be any ring and $y, w \in A$. Then $y \mid_{\text{loc}} -w$ if and only if $-y \mid_{\text{loc}} w$.*

Proof: This property is deduced immediately from previous propositions 2.7 and 2.9. ■

Proposition 2.11 *Let A be any ring, $y \in A$ and $n \in \mathbb{N}^*$. Then $y \mid_{\text{loc}} y^n$.*

Proof: • If $y^n = 0$ then by proposition 2.4 one has $y \mid_{\text{loc}} 0$.

• If $y^n \neq 0$. Declaring $w' = y^n$ one obtains $w' \neq 0$, $y^n(w' - y^n) = 0$ and clearly $y \mid y^n = w'$ for $n \geqslant 1$. Then the formula $\exists w'(w' \neq 0 \wedge y^n(w' - y^n) = 0 \wedge y \mid w')$ is valid in A; one then has in this case $y \mid_{\text{loc}} y^n$. ■

One needs to prove a previous lemma in order to prove one more property on "local divisibility".

Lema 2.12 *Let A be any lattice-ordered ring and let $w, w' \in A$ such that $w' \perp w - w'$. Then $|w'| \leqslant |w|$.*

Proof: By the definition of $\wedge$ one has that $|w'| \wedge |w| \leqslant |w'|$ and $|w'| \wedge |w| \leqslant |w|$. Observe that one has the following inequality:

$$\begin{aligned} |w'| = |w'| \wedge |w'| = |w'| \wedge |w' - w + w| &\leqslant |w'| \wedge \big(|w' - w| + |w|\big) \\ &= \big(|w'| \wedge |w' - w|\big) + \big(|w'| \wedge |w|\big). \end{aligned}$$

Since $w' \perp w - w'$, then $|w'| \wedge |w - w'| = 0$ and therefore one obtains:

$$|w'| \leqslant 0 + \big(|w'| \wedge |w|\big) = |w'| \wedge |w| \leqslant |w'|.$$

Then $|w'| \wedge |w| = |w'|$, and this shows us that $|w'| \leqslant |w|$. ■

The previous lemma help us to prove the following proposition:

Proposition 2.13 *Let A be any lattice-ordered ring and let $y, w_1, w_2 \in A$. If $y \,|_{\text{loc}}\, w_1$ and $y \,|_{\text{loc}}\, w_2$ with $w_1 \perp w_2$ then $y \,|_{\text{loc}}\, w_1 + w_2$.*

Proof: Let us suppose that $y \,|_{\text{loc}}\, w_1$ and $y \,|_{\text{loc}}\, w_2$ with $w_1 \perp w_2$. There are various cases:

- If $w_1 = 0$, since $y \,|_{\text{loc}}\, w_2$ then $y \,|_{\text{loc}}\, w_1 + w_2$.
- If $w_2 = 0$, since $y \,|_{\text{loc}}\, w_1$ then $y \,|_{\text{loc}}\, w_1 + w_2$.
- if $w_1 \neq 0$ and $w_2 \neq 0$. If $w_1 + w_2 = 0$ then by definition one has that $y \,|_{\text{loc}}\, w_1 + w_2$.

Let us suppose that $w_1 + w_2 \neq 0$. Since $y \,|_{\text{loc}}\, w_1$ and $w_1 \neq 0$ then there exists $w_1' \in A$, $w_1' \neq 0$ such that $w_1' \perp w_1 - w_1'$ with $y \mid w_1'$. Since $y \,|_{\text{loc}}\, w_2$ and $w_2 \neq 0$ then there exists $w_2' \in A$, $w_2' \neq 0$ such that $w_2' \perp w_2 - w_2'$ and $y \mid w_2'$. Let us see that $w_1' + w_2' \neq 0$. If $w_1' + w_2' = 0$ then $w_2' = -w_1'$ and therefore:

$$|w_1'| \wedge |w_2'| = |w_1'| \wedge |-w_1'| = |w_1'| \wedge |w_1'| = |w_1'|.$$

By the lemma 2.12 one has that $|w_1'| \leqslant |w_1|$ and $|w_2'| \leqslant |w_2|$. Then:

$$|w_1'| \wedge |w_2'| \leqslant |w_1| \wedge |w_2|.$$

Since $w_1 \perp w_2$ then $|w_1| \wedge |w_2| = 0$ and by the previous inequality one has $|w_1'| \wedge |w_2'| = 0$. By the asumption one should have that $|w_1'| = 0$, meaning that $w_1' = 0$; which is impossible since $w_1' \neq 0$.

Once we stated that $w_1' + w_2' \neq 0$, we want to see that:

$$w_1' + w_2' \perp (w_1 + w_2) - (w_1' + w_2').$$

We have the following inequalities :

$$\begin{aligned}
0 &\leqslant \big|w_1' + w_2'\big| \wedge \big|(w_1 + w_2) - (w_1' + w_2')\big| \\
&= \big|w_1' + w_2'\big| \wedge \big|(w_1 - w_1') + (w_2 - w_2')\big| \\
&\leqslant \big|w_1' + w_2'\big| \wedge \big(|w_1 - w_1'| + |w_2 - w_2'|\big) \\
&\leqslant \big(|w_1'| + |w_2'|\big) \wedge \big(|w_1 - w_1'| + |w_2 - w_2'|\big) \\
&= \big(|w_1'| \wedge |w_1 - w_1'|\big) + \big(|w_1'| \wedge |w_2 - w_2'|\big) + \big(|w_2'| \wedge |w_1 - w_1'|\big) + \big(|w_2'| \wedge |w_2 - w_2'|\big) \\
&= 0 + \big(|w_1'| \wedge |w_2 - w_2'|\big) + \big(|w_2'| \wedge |w_1 - w_1'|\big) + 0 \\
&= \big(|w_1'| \wedge |w_2 - w_2'|\big) + \big(|w_2'| \wedge |w_1 - w_1'|\big) \\
&\leqslant \big(|w_1'| \wedge \big(|w_2| + |w_2'|\big)\big) + \big(|w_2'| \wedge \big(|w_1| + |w_1'|\big)\big) \\
&= \big(|w_1'| \wedge |w_2|\big) + \big(|w_1'| \wedge |w_2'|\big) + \big(|w_2'| \wedge |w_1|\big) + \big(|w_2'| \wedge |w_1'|\big) \\
&= \big(|w_1'| \wedge |w_2|\big) + 2\big(|w_1'| \wedge |w_2'|\big) + \big(|w_2'| \wedge |w_1|\big).
\end{aligned}$$

Using one more time the lemma 2.12, since $w_1' \perp (w_1 - w_1')$ and $w_2' \perp (w_2 - w_2')$; one has that $|w_1'| \leqslant |w_1|$ and $|w_2'| \leqslant |w_2|$. Coming back to the inequalities one obtains:

$$\begin{aligned}
0 &\leqslant \big|w_1' + w_2'\big| \wedge \big|(w_1 + w_2) - (w_1' + w_2')\big| \\
&\leqslant \big(|w_1'| \wedge |w_2|\big) + 2\big(|w_1'| \wedge |w_2'|\big) + \big(|w_2'| \wedge |w_1|\big) \\
&\leqslant \big(|w_1| \wedge |w_2|\big) + 2\big(|w_1| \wedge |w_2|\big) + \big(|w_2| \wedge |w_1|\big) \\
&= 4\big(|w_1| \wedge |w_2|\big) \\
&= 4 \cdot 0 \\
&= 0,
\end{aligned}$$

for $w_1 \perp w_2$. This shows that $\left|w_1' + w_2'\right| \wedge \left|(w_1 + w_2) - (w_1' + w_2')\right| = 0$. One then has that $(w_1' + w_2') \perp (w_1 + w_2) - (w_1' + w_2')$. Since $y \mid w_1'$ and $y \mid w_2'$ then clearly $y \mid w_1' + w_2'$. Declaring $w' = w_1' + w_2'$, we had achieved that $w' \neq 0$, $w' \perp (w_1 + w_2) - w'$ and that $y \mid w'$. This means that $\exists w'\big(w' \neq 0 \wedge w'\big(w' - (w_1 + w_2)\big) \wedge y \mid w'\big)$ is a valid formula in A. Precisely one has that $y \mid_{\text{loc}} w_1 + w_2$.

■

Proposition 2.14 *Let A be any domain and $y, w \in A$. Then $y \mid w$ if and only if $y \mid_{\text{loc}} w$.*

Proof: ($\Rightarrow$) This implication is proposition 2.1.

($\Leftarrow$) Let us suppose that $y \mid_{\text{loc}} w$. If $w = 0$ then clearly $y \mid w$. If $w \neq 0$, there exists $w' \in A$, $w' \neq 0$ with $w'(w' - w) = 0$ and $y \mid w'$. Since A is a domain then $w' - w = 0$. Therefore $w' = w$ and then $y \mid w$.

■

Let A be any reduced f-ring. In [7], the ring A is sc-regular if there exist an element $u \in A$ such that $u^{\perp} = \{0\}$ and satisfying that $\forall e(e \neq 0 \wedge e^2 = e \rightarrow u \nmid e)$. The condition $u^{\perp} = \{0\}$ can be rewritten as $\mathrm{Ann}(u) = \{0\}$. Since $1 \neq 0$ then observe that:

$$\begin{aligned} u \mid_{\text{loc}} 1 &\longleftrightarrow \exists w'(w' \neq 0 \wedge w'(w' - 1) = 0 \wedge u \mid w') \\ &\longleftrightarrow \exists w'(w' \neq 0 \wedge w'^2 - w' = 0 \wedge u \mid w') \\ &\longleftrightarrow \exists w'(w' \neq 0 \wedge w'^2 = w' \wedge u \mid w') \\ &\longleftrightarrow \exists e(e \neq 0 \wedge e^2 = e \wedge u \mid e) \end{aligned} .$$

Therefore:

$$u \nmid_{\text{loc}} 1 \longleftrightarrow \forall e(e \neq 0 \wedge e^2 = e \rightarrow u \nmid e).$$

So the condition of sc-regularity can be rewritten as there exists $u \in A$ with $\mathrm{Ann}(u) = \{0\}$ and $u \nmid_{\text{loc}} 1$. That is to say: A is sc-regular if and only if,

$$\exists u(\mathrm{Ann}(u) = \{0\} \wedge u \nmid_{\text{loc}} 1)$$

is valid in A.

3 Model completeness.

Let A and B two reduced f-rings satisfying the first convexity property and

$$\mathcal{L} = \{0, 1, +, \cdot, <, \wedge, \preceq, \mid_{\text{loc}}\},$$

be the language of lattice-ordered rings with the radical relation given by the minimal prime spectrum and the relation of local divisibility.

Let us recall that $a \preceq b$ if and only if $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)$. Let us suppose that $A \subseteq_{\mathcal{L}} B$, that is to say: A is a substructure of B in the language $\mathcal{L}$; in particular A is a lattice-ordered subring of B.

Let us denote $i \colon A \hookrightarrow B$ the inclusion and the functorial (continuous) map:

$$\mathrm{Spec}(i) \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A), q \mapsto i^{-1}(q) = q \cap A.$$

Since $A \subseteq_{\mathcal{L}} B$ and the radical relation $\preceq$ belongs to the language then:

$$a \preceq_A a' \iff i(a) \preceq_B i(a'),$$

for all $a, a' \in A$. Let us denote $\pi B = \mathrm{Specmin}(B) = \{q \in \mathrm{Spec}(B) : q$ is a minimal prime ideal $\} \subseteq \mathrm{Spec}(B)$ and similarly $\pi A = \mathrm{Specmin}(A) = \{p \in \mathrm{Spec}(A) : p$ is a minimal prime ideal$\} \subseteq \mathrm{Spec}(A)$. Using [12, Theorem, p. 23] and [12, Proposition (a) y (b), p. 22] one has:

$$i^* = \mathrm{Spec}(i)_{\restriction_{\pi B}} : \pi B \to \pi A$$

and i^* is surjective. This means that if q is a minimal prime ideal of B then $q \cap A$ is a minimal prime ideal of A and that if p is a minimal prime ideal of A, then there exists at least one minimal prime ideal q of B such that $q \cap A = p$.

For $q_1, q_2 \in \pi B$, let us declare $q_1 \sim q_2$ if and only if $q_1 \cap A = q_2 \cap A$, if and only if $i^*(q_1) = i^*(q_2)$. Clearly $\sim$ is an equivalence relation on πB. Since the function $i^* : \pi B \to \pi A$ is surjective, then πA can be consider with the quotient topology πB induced by i^* or by the equivalence relation $\sim$. By [15, Theorem 9.2, p. 60] one has that the original topology of πA and the induced topology by i^* (or by the equivalence relation $\sim$) coincide if i^* is an open or closed function. If one consider that the f-rings A and B are projectable, then by [3] and [9], one should have that the spaces πA and πB are compact (and Hausdorff). Since $i^* : \pi B \to \pi A$ is a continuous function with πB compact and Hausdorff, then, by [15, p. 120], one has that i^* is a closed function. Therefore the original topology on πA and the quotient topology on πB induced by the equivalence relation $\sim$ are the same. Therefore:

$$j : \pi B/\sim \to \pi A, q/\sim \mapsto i^*(q),$$

is a homeomorphism of topological spaces and Boolean spaces.

Now let $p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$. That is to say that $i^*(q) = q \cap A = p$. Let us consider:

$$h_{pq} : A/p \to B/q, a + p \mapsto a + q.$$

Since $p \subseteq q \cap A$, then h_{pq} is well defined for if $a + p = a' + p$ with $a, a' \in A$ then $a - a' \in p$ and $a - a' \in q \cap A$, that carries to $a + q = a' + q$. Since $q \cap A \subseteq p$ then h_{pq} is injective for if $a, a' \in A$ are such that $h_{pq}(a) = h_{pq}(a')$, then $a + q = a' + q$ and $a - a' \in q$, so $a - a' \in q \cap A$; that is to say that $a - a' \in p$. Then $a + p = a' + p$. This proves the injectivity of h_{pq}. It is clear that h_{pq} is a ring homomorphism. Therefore:

$$h_{pq} : A/p \to B/q, a + p \mapsto a + q,$$

is a well defined injective ring homomorphism.

Let us see now that h_{pq} respects the order. Let $a, a' \in A$ such that $a + p \leqslant a' + p$ in A/p. Then there exists $c \in p$ such that $c > 0$ and $a + c \leqslant a'$ en A. Since A is an $\mathcal{L}$-sub-structure of B and the order is in the language $\mathcal{L}$ then $a + c \leqslant a'$ in B. Since $p \subseteq q \cap A$ then $c \in q$ with $c > 0$ and $a + c \leqslant a'$ in B. That is to say that $a + q \leqslant a' + q$ in B/q. Then $h_{pq}(a) \leqslant h_{pq}(a')$ in B/q. One should prove the other implication, that is: if $h_{pq}(a) \leqslant h_{pq}(a')$ in B/q then $a + p \leqslant a' + p$ in A/p. But since the orders on A/p and B/q are total then the implication needed to be proved can be immediately deduced from the one we just proved. Therefore $h_{pq} : A/p \to B/q$ is an injective homomorphism of ordered rings. In this context, one has the following proposition:

Proposition 3.1 *Let A and B be two reduced projectable f-rings satisfying the first convexity property such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \{0, 1, +, \cdot, <, \wedge, \preceq, |_{\text{loc}}\}$ is the language of lattice-ordered rings with the radical relation $\preceq$ given by the minimal prime spectrum and the local divisibility relation $|_{\text{loc}}$. If in addition one suppose that A and B are divisible-projectable then for $p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$, the homomorphism of ordered rings $h_{pq}\colon A/p \to B/q, a+p \mapsto a+q$ respects divisibility.*

Proof: Let us see that for $p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$, the injective homomorphism of ordered rings $h_{pq}\colon A/p \to B/q, a+p \mapsto a+q$ respects divisibility. That is to say that for $a, a' \in A$ one has that:

$$a + p \mid a' + p \text{ in } A/p \text{ if and only if } a + q \mid a' + q \text{ in } B/q.$$

($\Rightarrow$) Let us suppose that $a+p \mid a'+p$ en A/p. Then there exists $c+p \in A/p$ such that $(a+p)(c+p) = a'+p$. So $ac+p = a'+p$ and therefore $ac - a' \in p$. Since $p \subseteq q \cap A$ then $ac - a' \in q$, what this means is that $(a+q)(c+q) = a'+q$. In fact $a+q \mid a'+q$ en B/q.

($\Leftarrow$) Let us suppose that $a+q \mid a'+q$ in B/q. One has to show that $a+p \mid a'+p$ in A/p.

• If $a'+q = 0$ then $a' \in q$. Since $a' \in A$ then $a' \in q \cap A = p$. So $a'+p = 0$ and therefore $a+p \mid a'+p$ en A/p.

• If $a'+q \neq 0$ then $a' \neq 0$ and $a' \notin q$. Then $a' \notin p$ and so $a'+p \neq 0$. Let us suppose in this case that $a+p \nmid a'+p$ en A/p. Consider $N = [\![\, a \nmid a' \,]\!]_{\pi A} \cap [\![\, a' \neq 0 \,]\!]_{\pi A}$ which is a clopen set of πA. (Here we using the fact that A is divisible projectable, see [7]). See that $p \in N$ and therefore $N \neq \phi$. Let us define $\alpha' = a'_{\upharpoonright N} \cup 0_{\upharpoonright \pi A \smallsetminus N} \in A$. Since $N \neq \phi$ then $\alpha' \neq 0$.

Now suppose that $A \models a \,|_{\text{loc}}\, \alpha'$. Since $\alpha' \neq 0$ then:

$$A \models \exists w'(w' \neq 0 \wedge w'(w' - \alpha') = 0 \wedge a \mid w').$$

Let then be $w' \in A$, $w' \neq 0$ with $w'(w' - \alpha') = 0$ and $a \mid w'$. Since $w' \neq 0$ then there exists $\bar{p} \in \pi A$ such that $w'(\bar{p}) \neq 0$. Since $w'(w' - \alpha') = 0$ then $w'(\bar{p}) = \alpha'(\bar{p})$. By the definition of α' and the fact that $w'(\bar{p}) \neq 0$, one has that $\bar{p} \in N$ and that $\alpha'(\bar{p}) = a'(\bar{p})$. Since $a \mid w'$, there exists $c \in A$ such that $ac = w'$. That is to say that $a(\bar{p})c(\bar{p}) = w'(\bar{p}) = \alpha'(\bar{p}) = a'(\bar{p})$, so $a(\bar{p}) \mid a'(\bar{p})$ in $A/\bar{p}$; but this contradicts the fact that $\bar{p} \in [\![\, a \nmid a' \,]\!]_{\pi A}$. Therefore one has:

$$A \models a \nmid_{\text{loc}} \alpha'.$$

Since A is an $\mathcal{L}$-substructure of B and $|_{\text{loc}}$ belongs to the language,, then $B \models a \nmid_{\text{loc}} \alpha'$. Since $\alpha' \neq 0$ then:

$$B \models \forall w'(w' \neq 0 \wedge w'(w' - \alpha') = 0 \rightarrow a \nmid w').$$

Our initial assumption was that $a+q \mid a'+q$ in B/q. Therefore $q \in [\![\, a \mid a' \,]\!]_{\pi B}$. We are also in the case that $a'+q \neq 0$, that is to say that $q \in [\![\, a' \neq 0 \,]\!]_{\pi B}$. Since $p \in N$ then $\alpha'(p) = a'(p)$, that is to say that $\alpha'+p = a'+p$. Since $p = q \cap A$ then $\alpha'+q = a'+q$ in B/q and therefore $q \in [\![\, \alpha' = a' \,]\!]_{\pi B}$. Putting $M = [\![\, a \mid a' \,]\!]_{\pi B} \cap [\![\, a' \neq 0 \,]\!]_{\pi B} \cap [\![\, \alpha' = a' \,]\!]_{\pi B}$, one has that M is a clopen set of πB with $q \in M$ and $M \neq \phi$ (here we also used that B is divisible-projectable).

Now let us consider $w'' = \alpha'_{\restriction M} \cup 0_{\restriction \pi B \smallsetminus M} \in B$. Since $M \neq \phi$, for $\bar{q} \in M$ one has that $w''(\bar{q}) = \alpha'(\bar{q}) = a'(\bar{q}) \neq 0$. Then $w'' \neq 0$. Let us see that $w''(w'' - \alpha') = 0$. Let $\bar{q} \in \pi B$. If $\bar{q} \in \pi B \smallsetminus M$ then $w''(\bar{q} = 0$ and so $\big[w''(w'' - \alpha')\big](\bar{q}) = w''(\bar{q})(w'' - \alpha')(\bar{q}) = 0$. If $\bar{q} \in M$ then $w''(\bar{q}) = \alpha'(\bar{q})$ by the definition of w'', and so $(w'' - \alpha')(\bar{q}) = 0$; that is to say that $\big[w''(w'' - \alpha')\big](\bar{q}) = 0$. In any case we obtain that $\big[w''(w'' - \alpha')\big](\bar{q}) = 0$ (for all $\bar{q} \in \pi B$). Then $w''(w'' - \alpha') = 0$. Since $w'' \in B$ is such that $w'' \neq 0$ and $w''(w'' - \alpha') = 0$, then $a \nmid w''$ en B.

On the other hand, for $\bar{q} \in \pi B$ one has the following:

- if $\bar{q} \in \pi B \smallsetminus M$ then $w''(\bar{q}) = 0$ and therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$.
- if $\bar{q} \in M$ then $\bar{q} \in [\![\, a \mid a' \,]\!]_{\pi B} \cap [\![\, \alpha' = a' \,]\!]_{\pi B}$ and consequently one has $a(\bar{q}) \mid a'(\bar{q}) = \alpha'(\bar{q})$ en $B/\bar{q}$. Therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$.

Therefore $a(\bar{q}) \mid w''(\bar{q})$ in $B/\bar{q}$ for all $\bar{q} \in \pi B$. For each $\bar{q} \in \pi B$, there exists $c_{\bar{q}} \in B$ such that $a(\bar{q})c_{\bar{q}}(\bar{q}) = w''(\bar{q})$. Then:

$$\pi B = \bigcup_{\bar{q} \in \pi B} [\![\, ac_{\bar{q}} = w'' \,]\!]_{\pi B}.$$

By the compactness of πB, one can distinguish a finite number of $c_{\bar{q}}$'s and by the patchwork property of B, it is easy to construct an element $c \in B$ such that $ac = w''$. Then it has been proved that $a \mid w''$ en B. But we had from below that $a \nmid w''$ en B, giving a contradiction. Therefore we can not suppose that $a + p \nmid a' + p$ in A/p and then we have in this case that $a + q \mid a' + q$ in B/q implies that $a + p \mid a' + p$ in A/p. ∎

Let A and B be two models of T^* such that $A \subseteq_{\mathcal{L}} B$ where $\mathcal{L} = \{0, 1, +\cdot, \wedge, \preceq, |_{\text{loc}}\}$ is the language of lattice-ordered rings with the radical relation $\preceq$ given by the minimal prime spectrum and $|_{\text{loc}}$ is our local divisibility relation.

It is known that $i^*\colon \pi B \to \pi A$, $q \mapsto q \cap A$ is a continuous surjective map such that $\pi A \cong \pi B/\sim$ where $\sim$ is the equivalence relation given by $q \sim q'$ if and only if $i^*(q) = q \cap A = q' \cap A = i^*(q')$. Furthermore, for all $p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$, there exists $h_{pq}\colon A/p \to B/q$, $a + p \mapsto a + q$ an injective homomorphism of ordered rings respecting the divisibility .

Let us denote $\mathcal{B}(\pi A)$ and $\mathcal{B}(\pi B)$ the Boolean algebras of clopen sets of πA and πB respectively. Therefore:

$$j = (i^*)^{-1}\colon \mathcal{B}(\pi A) \to \mathcal{B}(\pi B),$$

is an injective homomorphism of Boolean algebras.

We want to show that $A \prec_{\mathcal{L}} B$. Let $\phi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in A$. By [6, Theorem 1.1], there exists an acceptable sequence $\zeta = \langle \Phi, \theta_1, \ldots, \theta_m \rangle$ of formulas where $\theta_1, \ldots, \theta_m$ are $\mathcal{L}$-formulas with the same free variables of $\phi(x_1, \ldots, x_n)$ and Φ is a formula in the Boolean algebra's language with m free variables such that:

$$A \models \phi(a_1, \ldots, a_n) \Longleftrightarrow \mathcal{B}(\pi A) \models \Phi\Big([\![\, \theta_1(a_1, \ldots, a_n) \,]\!]_A, \ldots, [\![\, \theta_m(a_1, \ldots, a_n) \,]\!]_A \Big),$$

where $[\![\theta_j(a_1,\ldots,a_n)]\!]_A = \{p \in \pi A : A/p \models \theta_j(a_1+p,\ldots,a_n+p)\}$, for all $j = 1,\ldots,m$.

Since A and B are models of T^* then A/p and B/q are real closed valuation rings, for all $p \in \pi A$ and $q \in \pi B$. Therefore, for $p \in \pi A$ and $q \in (i^*)^{-1}(\{p\})$, one has that $h_{pq}\colon A/p \to B/q$, $a+p \mapsto a+q$ is an elementary monomorphism in view of 3.1 and [5]. Therefore:

$$h_{pq}\colon A/p \prec B/q.$$

Then:

$$\begin{aligned} j\Big([\![\theta_l(a_1,\ldots,a_n)]\!]_A\Big) &= \Big\{q \in \pi B : B/q \models \theta_l\big(h_{pq}(a_1),\ldots,h_{pq}(a_n)\big) \text{ con } p = q \cap A\Big\} \\ &= [\![\theta_l(a_1,\ldots,a_n)]\!]_B. \end{aligned}$$

Since $\mathcal{B}(\pi A)$ are $\mathcal{B}(\pi B)$ are atomless Boolean algebras (A and B are models of T^*) then:

$$j\colon \mathcal{B}(\pi A) \prec \mathcal{B}(\pi B),$$

is an elementary monomorphism. Then one has:

$$\begin{aligned} & \mathcal{B}(\pi A) \models \Phi\Big([\![\theta_1(a_1,\ldots,a_n)]\!]_A,\ldots,[\![\theta_m(a_1,\ldots,a_n)]\!]_A\Big) \\ \iff\ & \mathcal{B}(\pi B) \models \Phi\Big(j\big([\![\theta_1(a_1,\ldots,a_n)]\!]_A\big),\ldots,j\big([\![\theta_m(a_1,\ldots,a_n)]\!]_A\big)\Big) \\ \iff\ & \mathcal{B}(\pi B) \models \Phi\Big([\![\theta_1(a_1,\ldots,a_n)]\!]_B,\ldots,[\![\theta_m(a_1,\ldots,a_n)]\!]_B\Big). \end{aligned}$$

By [6, Theorem 1.1] one also has:

$$B \models \phi(a_1,\ldots,a_n) \iff \mathcal{B}(\pi B) \models \Phi\Big([\![\theta_1(a_1,\ldots,a_n)]\!]_B,\ldots,[\![\theta_m(a_1,\ldots,a_n)]\!]_B\Big).$$

Therefore we just have seen that:

$$A \models \phi(a_1,\ldots,a_n) \text{ if and only if } B \models \phi(a_1,\ldots,a_n).$$

This proves that:

$$A \prec_{\mathcal{L}} B.$$

We can therefore state:

Theorem 3.2 *The theory T^* is model complete in $\mathcal{L} = \{0, 1, +\cdot, \wedge, \preceq, |_{\rm loc}\}$.* ■

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