STARK UNITS AND SPECIAL GAMMA VALUES

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ABSTRACT. In this paper we develop an effective procedure for expressing Stark units in real quadratic extensions of totally real fields as values of the Barnes multiple Gamma function at algebraic points. This procedure is used to explicitly generate non-abelian extensions of $\mathbb{Q}$ by special Gamma values. As a main component of our work, we develop an algorithm to compute Shintani sets in all dimensions.

1. INTRODUCTION

1.1. **Overview.** It is a fundamental problem in number theory to explicitly generate number fields by values of transcendental functions at algebraic points. To give a basic example of this, we recall how a real quadratic field can be generated over $\mathbb{Q}$ by values of Euler's Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \mathrm{Re}(z) > 0$$

at rational numbers.

Let K be a real quadratic field of discriminant $D > 0$. Let $\chi_D(n) = (D/n)$ be the Kronecker symbol and $L(\chi_D, s)$ be the Dirichlet L–function of χ_D. Further, let $h(D)$ be the class number and $\varepsilon_D > 1$ be the fundamental unit. Then the Dirichlet class number formula states that

$$L'(\chi_D, 0) = \frac{h(D)}{2}\log(\varepsilon_D). \tag{1.1}$$

Now, the Hurwitz zeta function is defined by

$$\zeta(s, z) := \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}, \quad \mathrm{Re}(s) > 1, \quad \mathrm{Re}(z) > 0.$$

It has a meromorphic continuation in s to the complex plane $\mathbb{C}$ with only a simple pole at $s = 1$. We have the decomposition

$$L(\chi_D, s) = D^{-s}\sum_{k=1}^{D}\chi_D(k)\zeta(s, k/D). \tag{1.2}$$

Moreover, Lerch [10] evaluated the second term in the Taylor expansion of $\zeta(s, z)$ at $s = 0$ as

$$\zeta'(0, z) = \frac{\partial}{\partial s}\zeta(s, z)\bigg|_{s=0} = \log\left(\frac{\Gamma(z)}{\sqrt{2\pi}}\right). \tag{1.3}$$

Differentiating (1.2) and substituting (1.3) yields

$$L'(\chi_D, 0) = -\log(D)L(\chi_D, 0) + \sum_{k=1}^{D}\chi_D(k)\log\left(\frac{\Gamma(k/D)}{\sqrt{2\pi}}\right).$$

Since χ_D is even, the functional equation implies that $L(\chi_D, 0) = 0$. Then using the orthogonality relations we get

$$L'(\chi_D, 0) = \sum_{k=1}^{D} \chi_D(k) \log(\Gamma(k/D)). \tag{1.4}$$

Finally, by combining (1.1) and (1.4) we obtain the following identity for the fundamental unit:

$$\varepsilon_D = \prod_{\substack{k=1 \\ (k,D)=1}}^{D} \Gamma(k/D)^{2\chi_D(k)/h(D)}. \tag{1.5}$$

Since $K/\mathbb{Q}$ is quadratic, we have $K = \mathbb{Q}(\varepsilon_D)$ and thus K is generated over $\mathbb{Q}$ by products of special Gamma values.

Example 1.1. The identity (1.5) can be illustrated as follows. Let $K = \mathbb{Q}(\sqrt{5})$. Then the fundamental unit ε_5 equals the golden ratio

$$\varepsilon_5 = \frac{1+\sqrt{5}}{2},$$

and a short calculation yields the elegant identity

$$\frac{1+\sqrt{5}}{2} = \frac{\Gamma(1/5)\,\Gamma(4/5)}{\Gamma(2/5)\,\Gamma(3/5)}.$$

In particular,

$$K = \mathbb{Q}\left(\frac{\Gamma(1/5)\,\Gamma(4/5)}{\Gamma(2/5)\,\Gamma(3/5)}\right).$$

The primary objective of this paper is to extend the identity (1.5) to Stark units in certain real quadratic extensions of totally real fields and to develop an effective procedure for computing both sides of this identity. The quadratic extensions K/F we will consider are determined by the following condition.

Condition 1.2. Assume that (K, F) is a pair of number fields such that:

- F is a totally real number field of degree n over $\mathbb{Q}$ with real embeddings $\sigma_1 = \mathrm{id}_F, \sigma_2, \ldots, \sigma_n$.
- $K = F(\sqrt{\Delta})$ is a quadratic extension of F such that $\sigma_1(\Delta) > 0$ and $\sigma_i(\Delta) < 0$ for $i = 2, \ldots, n$. In other words, K has signature $\mathrm{sig}(K) = (2, n-1)$.

Assume that (K, F) is a pair of number fields satisfying Condition 1.2.

Let $\chi_{K/F}$ be the quadratic Hecke character of conductor $\mathfrak{D}_{K/F}$ associated to K/F by class field theory where $\mathfrak{D}_{K/F}$ is the relative discriminant of K/F. Let $L(\chi_{K/F}, s)$ be the Hecke L–function of $\chi_{K/F}$. Let $h(K)$ be the class number of K and $h(F)$ be the class number of F.

Write $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ where σ is the embedding of K defined by

$$\sigma : \sqrt{\Delta} \longmapsto -\sqrt{\Delta}.$$

Let $\mathcal{O}_F$ be the ring of integers of F and $\mathcal{O}_F^\times$ be the group of units of F. Define

$$v := \begin{cases} 1, & \text{if } \Delta \in \mathcal{O}_F^\times (F^\times)^2 \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{O}_F^\times (F^\times)^2$ is the set units in $F^\times$ which can be expressed as the product of a unit in $\mathcal{O}_F^\times$ and a square in $(F^\times)^2$. Choose any unit $\epsilon \in \mathcal{O}_K^\times$ satisfying $[\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \epsilon \rangle] = 2^v$ (see e.g. Proposition 3.1). Then the *Stark unit* $\varepsilon_{K/F,S}$ in K is defined by

$$\varepsilon_{K/F,S} := \max \left\{ \left| \frac{\epsilon}{\sigma(\epsilon)} \right|, \left| \frac{\epsilon}{\sigma(\epsilon)} \right|^{-1} \right\} > 1.$$

In [14, Theorem 2], Stark proved that

$$L'(\chi_{K/F}, 0) = 2^{n-2-v} \frac{h(K)}{h(F)} \log(\varepsilon_{K/F,S}). \tag{1.6}$$

In particular,

$$\varepsilon_{K/F,S} = \exp \left(2^{2+v-n} \frac{h(F)}{h(K)} L'(\chi_{K/F}, 0) \right). \tag{1.7}$$

The formula (1.6) is a generalization of the Dirichlet class number formula (1.1).

Remark 1.3. In analogy with cyclotomic fields and their maximal totally real subfields, Stark [14] proved that K and F are generated over $\mathbb{Q}$ by $\varepsilon_{K/F,S}$ and $\varepsilon_{K/F,S} + \varepsilon_{K/F,S}^{-1}$, respectively. Stark also remarked that the same holds for any nonzero integral power of $\varepsilon_{K/F,S}$, that is, for all nonzero integers $\ell \in \mathbb{Z}$ we have

$$K = \mathbb{Q}(\varepsilon_{K/F,S}^\ell) \quad \text{and} \quad F = \mathbb{Q}(\varepsilon_{K/F,S}^\ell + \varepsilon_{K/F,S}^{-\ell}). \tag{1.8}$$

By (1.6) we have

$$\varepsilon_{K/F,S}^\alpha = \exp \left(L'(\chi_{K/F}, 0) \right),$$

where $\alpha := 2^{n-2-v} h(K)/h(F)$. Since the relative class number $h(K)/h(F)$ is an integer (see e.g. [17, Proposition 4.11]), if $n \geq 3$ then α is a nonzero integer. Hence it follows from (1.8) that if $n \geq 3$, then

$$K = \mathbb{Q}(\exp \left(L'(\chi_{K/F}, 0) \right)).$$

Starting with the identity (1.7), we need an algorithm to compute the Stark unit $\varepsilon_{K/F,S}$ and an n-dimensional generalization of Lerch's identity (1.4) for $L'(\chi_{K/F}, 0)$ in which all quantities involved can be effectively computed. We will develop an algorithm to compute the Stark unit $\varepsilon_{K/F,S}$. Now, Shintani [12] gave an effective generalization of Lerch's identity in dimension 2. He also [13] gave similar (but much more complicated) identities in dimension $n \geq 3$, however, the quantities appearing in these identities are *not* effectively computable. Roughly speaking, Shintani's argument relies on the existence of a fundamental domain for the action of the group of totally positive units of F on $\mathbb{R}^n_{>0}$ consisting of polyhedral cones of varying dimensions, but there is no effective way to construct these cones. Diaz y Diaz and Friedman [5], and Charollois, Dasgupta, and Greenberg [4] independently (and by different methods) constructed a "signed" fundamental domain for this group action which is effective. By building on the works [13, 5, 4], we will give an effective generalization of (1.5). We will then develop an algorithm to compute the algebraic points at which the Barnes multiple

Gamma function is evaluated in this identity. These sets of algebraic points, which we call *Shintani sets*, have a rich structure which we investigate extensively. In particular, we will show that there is a strong connection between the combinatorial geometry of these sets and the algebraicity of special Gamma values.

1.2. **The Barnes multiple Gamma function.** In this section we define the Barnes multiple Gamma function [1, 2] and summarize some of its basic properties.

To motivate the definition of the Barnes multiple Gamma function, consider the identity (1.3). We can reverse things and view (1.3) as *defining* Euler's Gamma function,

$$\Gamma(z) := \sqrt{2\pi}\exp\left(\frac{\partial}{\partial s}\zeta(s,z)\bigg|_{s=0}\right). \tag{1.9}$$

Since $\Gamma(z)$ has a simple pole at $z=0$ with residue 1, (1.9) shows that

$$\exp\left(\frac{\partial}{\partial s}\zeta(s,z)\bigg|_{s=0}\right)$$

has a simple pole at $z=0$ with residue $1/\sqrt{2\pi}$.

To define the Barnes multiple Gamma function, we require an n-dimensional generalization of the Hurwitz zeta function defined by

$$\zeta_n(s,z,\mathbf{w}) := \sum_{(r_1,r_2,\dots,r_n)\in\mathbb{Z}^n_{\geq 0}} \frac{1}{(z+r_1w_1+r_2w_2+\dots+r_nw_n)^s}, \quad \mathrm{Re(s)}>n, \ \mathrm{Re}(z)>0$$

for $\mathbf{w}=(w_1,w_2,\dots,w_n)\in\mathbb{R}^n_{>0}$. This function has a meromorphic continuation in s to the complex plane $\mathbb{C}$ with simple poles at $s=1,\dots,n$. In particular, since $\zeta_n(s,z,\mathbf{w})$ is holomorphic at $s=0$, the function

$$\exp\left(\frac{\partial}{\partial s}\zeta_n(s,z,\mathbf{w})\bigg|_{s=0}\right) \tag{1.10}$$

is defined for $\mathrm{Re}(z)>0$.

The function (1.10) has a meromorphic continuation in z to the whole complex plane with a simple pole at $z=0$. Let $\rho_n(\mathbf{w})^{-1}$ denote the residue of (1.10) at $z=0$. Then the n-dimensional *Barnes multiple Gamma function* is defined by

$$\Gamma_n(z,\mathbf{w}) := \rho_n(\mathbf{w})\exp\left(\frac{\partial}{\partial s}\zeta_n(s,z,\mathbf{w})\bigg|_{s=0}\right).$$

Observe that when $n=1$ and $\mathbf{w}=(1)\in\mathbb{R}_{>0}$, the identity (1.9) shows that $\Gamma_1(z,(1))=\Gamma(z)$. Hence, Euler's Gamma function is a special case of the Barnes multiple Gamma function.

1.3. **Shintani sets.** In this section we define the special sets of algebraic points at which the Barnes multiple Gamma function will be evaluated in the identity for the Stark unit $\varepsilon_{K/F,S}$. The following discussion is adapted from the setup in [5].

Let $\mathcal{O}_F^{\times,+}$ denote the group of totally positive units of F. Since F has signature $(n,0)$, Dirichlet's unit theorem implies that both $\mathcal{O}_F^{\times}$ and $\mathcal{O}_F^{\times,+}$ have rank $n-1$. Fix a choice of generators $\epsilon_1,\dots,\epsilon_{n-1}$ of $\mathcal{O}_F^{\times,+}$.

Let $\iota : F \hookrightarrow \mathbb{R}^n$ be the embedding of F in $\mathbb{R}^n$ given by $\iota(x) := (\sigma_1(x), \ldots, \sigma_n(x)) \in \mathbb{R}^n$ for any $x \in F$. For each $1 \leq i \leq n$ and any permutation $\tau \in S_{n-1}$, define the totally positive unit

$$f_{\tau,i} := \epsilon_{\tau(1)}\epsilon_{\tau(2)} \cdots \epsilon_{\tau(i-1)} = \prod_{j=1}^{i-1} \epsilon_{\tau(j)} \in \mathcal{O}_F^{\times,+}. \tag{1.11}$$

Note that for $i = 1$ this gives the empty product, so $f_{\tau,1} = 1$.

For any $\tau \in S_{n-1}$, define the weight

$$w_\tau := \frac{(-1)^{n-1}\mathrm{sgn}(\tau) \cdot \mathrm{sign}(\det(\sigma_i(f_{\tau,j})))}{\mathrm{sign}(\det(\log|\sigma_i(\epsilon_j)|))} \in \{0, \pm 1\}.$$

Note that $w_\tau \neq 0$ if and only if $\det(\sigma_i(f_{\tau,j})) \neq 0$. Thus $w_\tau \neq 0$ if and only if the vectors $\iota(f_{\tau,j}) \in \mathbb{R}^n_{>0}$ for $j = 1, \ldots, n$ form a basis for $\mathbb{R}^n$. In particular, if $w_\tau \neq 0$, then we can write the n-th standard basis vector in $\mathbb{R}^n$ as

$$e_n := (0, 0, \ldots, 1) = \sum_{i=1}^n c_i \iota(f_{\tau,i})$$

for some unique real numbers $c_i \in \mathbb{R}$. Define the following intervals in terms of the sign of c_i,

$$I_{\tau,i} := \begin{cases} [0,1) & \text{if } c_i > 0 \\ (0,1] & \text{otherwise.} \end{cases}$$

Similarly, observe that if $w_\tau \neq 0$, then the algebraic numbers $\{f_{\tau,i}\}_{i=1}^n$ form a $\mathbb{Q}$-basis for F. In particular, given a nonzero integral ideal $\mathfrak{f}$ of F, every element $z \in \mathfrak{f}^{-1}$ can be expressed as a linear combination of the form

$$z = \sum_{i=1}^n t_{z,\tau,i} f_{\tau,i}$$

for some unique rational numbers $t_{z,\tau,i} \in \mathbb{Q}$. Let

$$\mathbf{t}_{z,\tau} := (t_{z,\tau,1}, \ldots, t_{z,\tau,n}) \in \mathbb{Q}^n$$

be the coordinate vector of z with respect to the $\mathbb{Q}$-basis $\{f_{\tau,i}\}_{i=1}^n$.

For $\tau \in S_{n-1}$ such that $w_\tau \neq 0$ and for $\mathfrak{f}$ a nonzero integral ideal of F, we define the *Shintani set* associated to $\mathfrak{f}$ by

$$\mathcal{R}^\tau(\mathfrak{f}) = \mathcal{R}^\tau(\mathfrak{f}; \epsilon_1, \ldots, \epsilon_{n-1}) := \left\{ z \in \mathfrak{f}^{-1} \;\middle|\; \mathbf{t}_{z,\tau} \in I_{\tau,1} \times \cdots \times I_{\tau,n} \right\}.$$

Similarly, we define the *restricted Shintani set* associated to $\mathfrak{f}$ by

$$\widetilde{\mathcal{R}^\tau}(\mathfrak{f}) = \widetilde{\mathcal{R}^\tau}(\mathfrak{f}; \epsilon_1, \ldots, \epsilon_{n-1}) := \left\{ z \in \mathfrak{f}^{-1} \;\middle|\; \mathbf{t}_{z,\tau} \in I_{\tau,1} \times \cdots \times I_{\tau,n},\ \mathfrak{f}\langle z \rangle \text{ coprime to } \mathfrak{f} \right\}.$$

The Shintani set $\mathcal{R}^\tau(\mathfrak{f})$ is finite, and embeds via ι as a subset of the F-rational cone

$$C_F^\tau = C_F^\tau(\epsilon_1, \ldots, \epsilon_{n-1}) := \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^n t_i \iota(f_{\tau,i}),\ t_i \geq 0\} \subset \mathbb{R}^n_{\geq 0}.$$

Finally, we define the set of *boundary points* in the Shintani set by

$$\partial\mathcal{R}^\tau(\mathfrak{f}) := \{z \in \mathcal{R}^\tau(\mathfrak{f}) \big| \text{ at least one entry of } \mathbf{t}_{z,\tau} \text{ is 0 or 1}\}$$

and the set of *interior points* in the Shintani set by

$$\mathrm{int}\,(\mathcal{R}^\tau(\mathfrak{f})) := \mathcal{R}^\tau(\mathfrak{f}) \setminus \partial\mathcal{R}^\tau(\mathfrak{f}).$$

1.4. **Identities for Stark units.** Let

$$A = (a_{ij}) \in M_n(\mathbb{R}_{>0})$$

be an $n \times n$ matrix with positive real entries. Then for any vector $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_{\geq 0}^n$ and any pair of integers $j, \ell \in \mathbb{Z}$ with $1 \leq j, \ell \leq n$ and $j \neq \ell$, define the constants

$$C_{\mathbf{h},j,\ell}(A) := \int_0^1 \left(\prod_{t=1}^n (a_{jt} + a_{\ell t}u)^{h_t - 1} - \prod_{t=1}^n a_{jt}^{h_t - 1} \right) \frac{du}{u}$$

and

$$C_{\mathbf{h}}(A) := \sum_{\substack{(j,\ell)\in\mathbb{Z}^2 \\ 1\leq j,\ell\leq n \\ j\neq \ell}} C_{\mathbf{h},j,\ell}(A).$$

Define the matrix

$$A^\tau := (\sigma_i(f_{\tau,j})) \in M_n(\mathbb{R}_{>0}), \quad \tau \in S_{n-1}$$

and let

$$A_i^\tau := (\sigma_i(f_{\tau,1}), \ldots, \sigma_i(f_{\tau,n}))$$

denote the i-th row of A^τ. Let $B_k(x) \in \mathbb{Q}[x]$ denote the k-th Bernoulli polynomial. Then define the constant $C(K/F) = C(K/F; \epsilon_1, \ldots, \epsilon_{n-1})$ by

$$C(K/F) := \frac{(-1)^n}{n} \sum_{\tau \in S_{n-1}} \sum_{z \in \partial\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} c_{K/F,\tau}(z) \sum_{\substack{\mathbf{h}=(h_1,\ldots,h_n)\in\mathbb{Z}_{\geq 0}^n \\ \sum_{i=1}^n h_i = n}} C_{\mathbf{h}}(A^\tau) \prod_{i=1}^n \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} \tag{1.12}$$

where

$$c_{K/F,\tau}(z) := \frac{w_\tau \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) h(F)}{2^{n-v-2} h(K)}. \tag{1.13}$$

Remark 1.4. In Proposition 7.2 we show that the constants $C_{\mathbf{h}}(A^\tau)$ appearing in (1.12) can be explicitly evaluated when the degree $[F : \mathbb{Q}] = n$ is prime.

Finally, we define the product $\mathbf{\Gamma}_{K/F,n}$ of special Gamma values

$$\mathbf{\Gamma}_{K/F,n} = \mathbf{\Gamma}_{K/F,n}(\epsilon_1, \ldots, \epsilon_{n-1}) := \prod_{\substack{\tau \in S_{n-1} \\ w_\tau \neq 0}} \prod_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} \prod_{i=1}^n \Gamma_n(\langle \mathbf{t}_{z,\tau}, A_i^\tau \rangle, A_i^\tau)^{c_{K/F,\tau}(z)} \tag{1.14}$$

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Theorem 1.5. *Let (K, F) be a pair of number fields satisfying Condition 1.2 and assume that $\mathfrak{D}_{K/F}$ is principal with a totally positive generator. Then the Stark unit $\varepsilon_{K/F,S}$ is given by*

$$\varepsilon_{K/F,S} = \exp\left(C(K/F)\right) \cdot \mathbf{\Gamma}_{K/F,n}, \tag{1.15}$$

where $C(K/F)$ *is defined by* (1.12) *and* $\mathbf{\Gamma}_{K/F,n}$ *is defined by* (1.14). *Furthermore, if* $\partial\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}) = \varnothing$*, then*

$$\varepsilon_{K/F,S} = \mathbf{\Gamma}_{K/F,n}$$

and we have $\mathbf{\Gamma}_{K/F,n} \in \overline{\mathbb{Q}}$ *and* $K = \mathbb{Q}(\mathbf{\Gamma}_{K/F,n})$.

1.5. **An example.** In this section we give an example which illustrates the type of explicit identities for Stark units in terms of special Gamma values that the procedure developed in this paper can yield (see Section 2 for the proof). In particular, this gives the first identity of this type for a pair of number fields (K, F) satisfying Condition 1.2 with $[K : \mathbb{Q}] \geq 6$. Identities of this type for $[K : \mathbb{Q}] = 4$ were given by Shintani in [12].

Let Γ_3 denote the Barnes triple Gamma function (see Section 1.2) and let $\{x\} = x - [x]$ denote the fractional part of $x \in \mathbb{R}$.

Theorem 1.6. (1) *Let* r *be the root of the irreducible polynomial* $p(X) = X^3 - X^2 - 2X + 1$ *which is approximately equal to* $-1.2469\ldots$. *Let* $F = \mathbb{Q}(r)$ *and* $K = F(\sqrt{\Delta})$ *where* $\Delta := r^2 - 2r - 3 > 0$. *Then* (K, F) *is a pair of number fields satisfying Condition 1.2 where* F *is a totally real abelian cubic field with discriminant* $d_F = 49$ *and narrow class number 1.*

(2) *The Stark unit* $\varepsilon_{K/F,S}$ *is the root of the monic irreducible polynomial*

$$X^6 - 2X^5 + 2X^4 - 3X^3 + 2X^2 - 2X + 1 \tag{1.16}$$

which is approximately equal to $1.6355\ldots$.

(3) *We have*

$$\varepsilon_{K/F,S} = \mathbf{\Gamma}_{K/F,3} := \mathbf{\Gamma}^1_{K/F,3} \cdot \mathbf{\Gamma}^2_{K/F,3} \cdot \mathbf{\Gamma}^3_{K/F,3} \cdot \mathbf{\Gamma}^4_{K/F,3},$$

where

$$\mathbf{\Gamma}^1_{K/F,3} := \prod_{i=1}^{3} \prod_{\substack{m=1 \\ m\neq 5}}^{13} \Gamma_3\left(\left\{\frac{29}{39}(3m-2)\right\} + \frac{1}{3}\alpha_i + \left\{\frac{3m-2}{39}\right\}\beta_i, (1, \alpha_i, \beta_i)\right)^{\frac{c_1(m)}{2}}$$

$$\mathbf{\Gamma}^2_{K/F,3} := \prod_{i=1}^{3} \prod_{\substack{m=1 \\ m\neq 5}}^{13} \Gamma_3\left(\left\{\frac{3m-2}{39}\right\} + \frac{2}{3}\alpha_i + \left\{\frac{35}{39}(3m-2)\right\}\beta_i, (1, \alpha_i, \beta_i)\right)^{\frac{c_1(m)}{2}}$$

$$\mathbf{\Gamma}^3_{K/F,3} := \prod_{i=1}^{3} \prod_{m=1}^{12} \Gamma_3\left(\left\{\frac{m}{13}\right\} + \alpha_i + \left\{\frac{9}{13}m\right\}\beta_i, (1, \alpha_i, \beta_i)\right)^{\frac{c_2(m)}{2}}$$

$$\mathbf{\Gamma}^4_{K/F,3} := \prod_{i=1}^{3} \prod_{m=1}^{12} \Gamma_3\left(\left\{\frac{m}{13}\right\} + \left\{\frac{9}{13}m\right\}\beta_i, (1, \gamma_i, \beta_i)\right)^{\frac{c_2(m)}{2}}$$

with

$$\alpha_i := \begin{cases} -r^2 + r + 3, & i = 1 \\ -r + 2, & i = 2 \\ r^2, & i = 3 \end{cases} \quad \beta_i := \begin{cases} (r+1)^2, & i = 1 \\ -3r^2 + r + 8, & i = 2 \\ 2r^2 - 3r + 1, & i = 3 \end{cases} \quad \gamma_i := \begin{cases} r^2 + r, & i = 1 \\ -2r^2 + r + 5, & i = 2 \\ (r-1)^2, & i = 3 \end{cases}$$

and

$$c_1(m) := \begin{cases} 1, & m \in \{1,2,4,6,8,9\} \\ -1, & m \in \{3,7,10,11,12,13\} \end{cases} \qquad c_2(m) := \begin{cases} 1, & m \in \{1,3,4,9,10,12\} \\ -1, & m \in \{2,5,6,7,8,11\}. \end{cases}$$

(4) *We have* $\mathbf{\Gamma}_{K/F,3} \in \overline{\mathbb{Q}}$ *and* $K = \mathbb{Q}(\mathbf{\Gamma}_{K/F,3})$.

Remark 1.7. Part (2) of Theorem 1.6 highlights the fact that any Stark unit lying over a totally real base field F with $[F:\mathbb{Q}] \geq 2$ satisfies a monic palindromic polynomial in $\mathbb{Z}[X]$. Also, the only other real root of (1.16) is $\varepsilon_{K/F,S}^{-1}$, and the four non-real roots of (1.16) all have absolute value equal to 1. See [15, p. 74].

Remark 1.8. In [18], Yamamoto expressed every Stark unit lying over a totally real field as a finite product of special values of the multiple sine function.

1.6. **Generating non-abelian extensions of $\mathbb{Q}$ by special Gamma values.** Given a pair of number fields (K,F) satisfying Condition 1.2 where $\mathfrak{D}_{K/F}$ is principal with a totally positive generator, it would be very desirable to know whether the extension $K/\mathbb{Q}$ can always be generated by special Gamma values. As we have seen in Theorem 1.5, this is true if the set of boundary points is empty (which forces the constant $C(K/F)$ to vanish). An important feature of our work is an algorithm to compute the restricted Shintani set $\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})$, which in turn allows us to compute the constant $C(K/F)$. Based on our computation of $C(K/F)$ for many pairs (K,F), we believe there is always a choice of generators $\epsilon_1,\dots,\epsilon_{n-1}$ of $\mathcal{O}_F^{\times,+}$ such that $C(K/F)=0$. If true, then it is always possible to generate $K/\mathbb{Q}$ by special Gamma values. We observe this theoretically in the proof of Theorem 1.6. In particular, for the pair (K,F) in that example we construct a choice of generators ϵ_1,ϵ_2 of $\mathcal{O}_F^{\times,+}$ such that $\partial\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}) \neq \varnothing$ for all $\tau \in S_2$, but $C(K/F)=0$ (see e.g. Proposition 2.1).

In Table 1 we display a few pairs of number fields (K,F) satisfying Condition 1.2 with $F=\mathbb{Q}(r)$ cubic of narrow class number 1 and a corresponding choice of generators ϵ_1,ϵ_2 of $\mathcal{O}_F^{\times,+}$ such that $C(K/F)=0$.

F	r	Δ	d_F	d_K	ϵ_1	ϵ_2
x^3-x^2-2x+1	1.8019...	$4r-3$	49	98441	$(r+1)^2$	$r(r+1)$
x^3-x^2-2x+1	−1.2470...	$-4r-3$	49	232897	$(r+1)^2$	$r(r+1)$
x^3-3x-1	−0.3473...	$-3r^2+2r+5$	81	242757	$2r^2-3r-1$	$-r+2$
x^3-x^2-2x+1	−1.2470...	r^2-4r-4	49	271313	$(r-1)^2$	$r(r+1)$
x^3-3x-1	−1.5321...	$-4r-3$	81	347733	$2r^2-3r-1$	$-r+2$
x^3-3x-1	1.8794...	r^2+2r-3	81	373977	$2r^2-3r-1$	$-r+2$
x^3-x^2-2x+1	1.8019...	$5r^2-12$	49	472977	$(r-1)^2$	$r(r+1)$

TABLE 1. List of pairs (K,F) satisfying Condition 1.2 with $C(K/F)=0$.

1.7. **The combinatorial geometry of Shintani sets.** Theorem 1.5 and Section 1.6 reflect the strong connection between the combinatorial geometry of Shintani sets and the algebraicity of special Gamma values. Accordingly, one of our objectives is to undertake an extensive study of these sets.

We first explain how the algebraic numbers in the restricted Shintani sets $\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})$ which determine the special Gamma values $\mathbf{\Gamma}_{K/F,n}$ in Theorem 1.5 are analogous to the rational numbers

$$\widetilde{R_D} := \left\{ \frac{k}{D} \in \frac{1}{D}\mathbb{Z} \;\middle|\; 1 \le k \le D,\ \gcd(k, D) = 1 \right\}$$

at which Euler's Gamma function is evaluated in (1.5). This analogy can be understood from the following group-theoretic perspective. The set of rational numbers defined by

$$R_D := \left\{ \frac{k}{D} \in \frac{1}{D}\mathbb{Z} \;\middle|\; 1 \le k \le D \right\}$$

is a complete set of coset representatives for the quotient group

$$\frac{1}{D}\mathbb{Z} \Big/ \mathbb{Z}$$

inside the standard fundamental domain $(0, 1]$ for the group $\mathbb{R}/\mathbb{Z}$. Similarly, in Proposition 4.1 we will show that the Shintani set $\mathcal{R}^\tau(\mathfrak{D}_{K/F})$ is a complete set of coset representatives for the quotient group

$$G_\tau(\mathfrak{D}_{K/F}^{-1}) := \mathfrak{D}_{K/F}^{-1} \Big/ \bigoplus_{i=1}^{n} \mathbb{Z} f_{\tau,i}$$

whose images under the embedding $\iota : F \hookrightarrow \mathbb{R}^n$ lie inside the standard fundamental parallelotope

$$P_F^\tau = P_F^\tau(\epsilon_1, \ldots, \epsilon_{n-1}) := \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{n} t_i \iota(f_{\tau,i}),\ t_i \in \mathbb{R},\ t_i \in I_{\tau,i} \right\} \subset C_F^\tau$$

for the group

$$\iota(G_\tau(\mathfrak{D}_{K/F}^{-1})) = \mathbb{R}^n \Big/ \bigoplus_{i=1}^{n} \mathbb{Z} \cdot \iota(f_{\tau,i}) .$$

Here we recall that the algebraic numbers $f_{\tau,i}$ are the totally positive units in F defined by (1.11). After removing from the Shintani set $\mathcal{R}^\tau(\mathfrak{D}_{K/F})$ the numbers z for which

$$\chi_{K/F}(\mathfrak{D}_{E/F}\langle z \rangle) = 0$$

by enforcing the condition that $\mathfrak{D}_{K/F}\langle z \rangle$ be coprime to $\mathfrak{D}_{K/F}$, we see that $\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})$ is analogous to $\widetilde{R_D}$.

Now, observe that the size of the set $\widetilde{R_D}$ can be expressed geometrically as a ratio of volumes,

$$\#\widetilde{R_D} = \frac{\mathrm{vol}(\mathbb{R}/\mathbb{Z})}{\mathrm{vol}(\mathbb{Z})} \varphi(D) = \frac{\mathrm{vol}((0, 1])}{\sqrt{d_{\mathbb{Q}}}} \varphi(D) \tag{1.17}$$

where φ is the Euler totient function.

We will prove the following generalization of (1.17) for the size of the restricted Shintani set.

Theorem 1.9. *Assume that $\mathfrak{D}_{K/F}$ is principal. Then*

$$\#\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}) = \frac{\mathrm{vol}(\iota(G_\tau(\mathfrak{D}_{K/F}^{-1}))}{\mathrm{vol}(\iota(\mathcal{O}_F))}\varphi(\mathfrak{D}_{K/F}) = \frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}}\varphi(\mathfrak{D}_{K/F}) = \frac{|\det(A^\tau)|}{\sqrt{d_F}}\varphi(\mathfrak{D}_{K/F}),$$

where

$$\varphi(\mathfrak{D}_{K/F}) := N_{F/\mathbb{Q}}(\mathfrak{D}_{K/F}) \cdot \prod_{\mathfrak{p}|\mathfrak{D}_{K/F}} \Big(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{p})}\Big)$$

is the generalized Euler totient function for number fields.

We will also prove the following orthogonality relations for ray class characters with respect to Shintani sets. This result will be used in the proof of Theorem 1.5.

Theorem 1.10. *Assume that $\mathfrak{D}_{K/F}$ is principal and generated by a totally positive element in F. Let χ be a narrow ray class character modulo $\mathfrak{D}_{K/F}$. Then for any $\tau \in S_{n-1}$, we have*

$$\sum_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} \chi(\mathfrak{D}_{K/F}\langle z\rangle) = \begin{cases} 0, & \text{if } \chi \neq 1, \\ \dfrac{|\det(A^\tau)|}{\sqrt{d_F}}\varphi(\mathfrak{D}_{K/F}), & \text{if } \chi = 1. \end{cases}$$

The proofs of Theorems 1.9 and 1.10 involve a blend of algebraic number theory and discrete geometry. Roughly speaking, we first define a binary operation on $\mathcal{R}^\tau(\mathfrak{D}_{K/F})$ which makes the Shintani set into a finite abelian group. We then use this group structure to prove that

$$\#\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}) = \#(L(P_F^\tau) \cap \mathbb{Z}^n)\varphi(\mathfrak{D}_{K/F}),$$

where L is a certain linear transformation on $\mathbb{R}^n$. Now, a remarkable theorem of Ehrhart (see e.g. [3]) asserts that the number of lattice points in the t-dilation of an n-polytope P is a polynomial

$$E(P,t) := \#(tP \cap \mathbb{Z}^n) = c_n t^n + c_{n-1}t^{n-1} + \cdots + c_0 \in \mathbb{Q}[t]$$

of degree n with rational coefficients whose leading coefficient c_n is given by

$$c_n = \mathrm{vol}(P).$$

This polynomial is called the *Ehrhart polynomial* of P. In particular, the Ehrhart polynomial of the n-parallelotope $L(P_F^\tau)$ takes the form

$$E(L(P_F^\tau), t) = \mathrm{vol}(L(P_F^\tau))t^n + d_{n-1}t^{n-1} + \cdots + d_0$$

for some rational numbers d_i with $i = 0, \ldots, n-1$. On the other hand, we will make crucial use of the structure of $L(P_F^\tau)$ to prove that

$$E(L(P_F^\tau), t) = \#(L(P_F^\tau) \cap \mathbb{Z}^n)t^n.$$

Hence, by comparing leading coefficients we will conclude that

$$\#(L(P_F^\tau) \cap \mathbb{Z}^n) = \mathrm{vol}(L(P_F^\tau)) = \frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}}.$$

Remark 1.11. We emphasize that for computational purposes it is important to have a formula for the size of the Shintani set. For example, since $K/\mathbb{Q}$ is *non-abelian* for $n \geq 2$, the number of elements in the Shintani set will usually be very large. Our formula can be used to choose K so as to control the size the corresponding Shintani set and thus make the computations of examples like Theorem 1.6 more manageable.

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2. Proof of Theorem 1.6

All computations in this section were performed using `SageMath` [16].

Consider the irreducible polynomial

$$p(X) = X^3 - X^2 - 2X + 1 \in \mathbb{Z}[X].$$

The polynomial $p(X)$ has three real roots. We choose the root r which is approximately equal to $-1.2469...$. Then $F = \mathbb{Q}(r)$ is a totally real cubic field with narrow class number 1. Label the three real embeddings of F by $\sigma_1 = \mathrm{id}_F, \sigma_2, \sigma_3$.

Next, define

$$\Delta := r^2 - 2r - 3.$$

Then $K := F(\sqrt{\Delta})$ is a sextic number field such that $\sigma_1(\Delta) > 0$ and $\sigma_i(\Delta) < 0$ for $i = 2, 3$. In particular, the pair of number fields (K, F) satisfies Condition 1.2.

By Theorem 1.5, we have

$$\varepsilon_{K/F,S} = \exp(C(K/F)) \cdot \mathbf{\Gamma}_{K/F,3}.$$

Our objective is to explicitly compute both sides of this identity.

In Section 3 we develop an algorithm to compute the Stark unit $\varepsilon_{K/F,S}$ (see Algorithm 1). Applying this algorithm to the pair (K, F), we find that $\varepsilon_{K/F,S}$ is the root of the monic irreducible polynomial

$$X^6 - 2X^5 + 2X^4 - 3X^3 + 2X^2 - 2X + 1$$

which is approximately equal to 1.6355....

Next, for a particular choice of generators ϵ_1, ϵ_2 of $\mathcal{O}_F^{\times,+}$, we will prove that $C(K/F) = 0$ and compute $\mathbf{\Gamma}_{K/F,3}$.

We have $\Delta \notin \mathcal{O}_F^{\times}(F^{\times})^2$, and thus $v = 0$.

The class numbers of the two fields are $h(K) = h(F) = 1$, the discriminants are $d_K = 31213$ and $d_F = 49$, the relative discriminant $\mathfrak{D}_{K/F} = \langle 2r^2 - 2r + 1\rangle$ is a prime ideal in $\mathcal{O}_F$ lying over 13 of norm 13, and $\varphi(\mathfrak{D}_{K/F}) = 12$. Note that the generator $2r^2 - 2r + 1$ of the ideal $\mathfrak{D}_{K/F}$ is totally positive. Furthermore, we know that K is a ray class field since the narrow ray class group modulo $\mathfrak{D}_{K/F}\mathfrak{p}_\infty^{(1)}\mathfrak{p}_\infty^{(2)}\mathfrak{p}_\infty^{(3)}$ is a group of order 2, where $\mathfrak{p}_\infty^{(i)}$ is the infinite prime of F corresponding to the embedding σ_i.

A set of generators for the group of totally positive units of F is given by

$$\mathcal{O}_F^{\times,+} \cong \langle -r^2 + r + 3\rangle \times \langle r^2 + r\rangle =: \langle \epsilon_1\rangle \times \langle \epsilon_2\rangle.$$

Using the generators ϵ_1, ϵ_2, we compute the weight w_τ for $\tau \in S_2 = \{\mathrm{id}, (12)\}$ as

$$w_{\mathrm{id}} = w_{(12)} = 1.$$

From the preceding data, we find that the constant (1.13) is given by

$$c_{K/F,\tau}(z) = \frac{\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)}{2}. \tag{2.1}$$

Next, using the generators ϵ_1, ϵ_2, we compute the matrices

$$A^{\mathrm{id}} := \begin{pmatrix} 1 & \epsilon_1 & \epsilon_1\epsilon_2 \\ 1 & \sigma_2(\epsilon_1) & \sigma_2(\epsilon_1\epsilon_2) \\ 1 & \sigma_3(\epsilon_1) & \sigma_3(\epsilon_1\epsilon_2) \end{pmatrix} = \begin{pmatrix} 1 & -r^2+r+3 & (r+1)^2 \\ 1 & -r+2 & -3r^2+r+8 \\ 1 & r^2 & 2r^2-3r+1 \end{pmatrix},$$

and

$$A^{(12)} := \begin{pmatrix} 1 & \epsilon_2 & \epsilon_1\epsilon_2 \\ 1 & \sigma_2(\epsilon_2) & \sigma_2(\epsilon_1\epsilon_2) \\ 1 & \sigma_3(\epsilon_2) & \sigma_3(\epsilon_1\epsilon_2) \end{pmatrix} = \begin{pmatrix} 1 & r^2+r & (r+1)^2 \\ 1 & -2r^2+r+5 & -3r^2+r+8 \\ 1 & (r-1)^2 & 2r^2-3r+1 \end{pmatrix}.$$

We have

$$\det(A^{\mathrm{id}}) = 21 \quad \text{and} \quad \det(A^{(12)}) = -7.$$

Then by Theorem 1.9, the sizes of the restricted Shintani sets are given by

$$\#\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F}) = \frac{|\det(A^{\mathrm{id}})|}{\sqrt{d_F}}\varphi(\mathfrak{D}_{K/F}) = \frac{21}{7}\cdot 12 = 36$$

and

$$\#\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F}) = \frac{|\det(A^{(12)})|}{\sqrt{d_F}}\varphi(\mathfrak{D}_{K/F}) = \frac{7}{7}\cdot 12 = 12.$$

In Section 5 we develop an algorithm to compute the restricted Shintani set (see Algorithm 2). Applying this algorithm to the field F, the fractional ideal $\mathfrak{D}_{K/F}$, the permutation $\tau = \mathrm{id}$, and the generators ϵ_1, ϵ_2 yields

$$\mathrm{int}\left(\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})\right) = \{z = t_{z,\mathrm{id},1}\cdot 1 + t_{z,\mathrm{id},2}\cdot\epsilon_1 + t_{z,\mathrm{id},3}\cdot\epsilon_1\epsilon_2 \mid \mathbf{t}_{z,\mathrm{id}} \in Q_{\mathrm{id}}\}$$

and

$$\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F}) = \{z = t_{z,\mathrm{id},1}\cdot 1 + t_{z,\mathrm{id},2}\cdot\epsilon_1 + t_{z,\mathrm{id},3}\cdot\epsilon_1\epsilon_2 \mid \mathbf{t}_{z,\mathrm{id}} \in \partial Q_{\mathrm{id}}\},$$

where

$$Q_{\mathrm{id}} := \left\{ \begin{matrix} \left(\frac{14}{39}, \frac{1}{3}, \frac{22}{39}\right) & \left(\frac{23}{39}, \frac{1}{3}, \frac{25}{39}\right) & \left(\frac{17}{39}, \frac{1}{3}, \frac{10}{39}\right) & \left(\frac{35}{39}, \frac{1}{3}, \frac{16}{39}\right) & \left(\frac{29}{39}, \frac{1}{3}, \frac{1}{39}\right) & \left(\frac{38}{39}, \frac{1}{3}, \frac{4}{39}\right) \\ \left(\frac{1}{39}, \frac{2}{3}, \frac{35}{39}\right) & \left(\frac{10}{39}, \frac{2}{3}, \frac{38}{39}\right) & \left(\frac{4}{39}, \frac{2}{3}, \frac{23}{39}\right) & \left(\frac{22}{39}, \frac{2}{3}, \frac{29}{39}\right) & \left(\frac{16}{39}, \frac{2}{3}, \frac{14}{39}\right) & \left(\frac{25}{39}, \frac{2}{3}, \frac{17}{39}\right) \\ \left(\frac{2}{39}, \frac{1}{3}, \frac{31}{39}\right) & \left(\frac{11}{39}, \frac{1}{3}, \frac{34}{39}\right) & \left(\frac{20}{39}, \frac{1}{3}, \frac{37}{39}\right) & \left(\frac{5}{39}, \frac{1}{3}, \frac{19}{39}\right) & \left(\frac{32}{39}, \frac{1}{3}, \frac{28}{39}\right) & \left(\frac{8}{39}, \frac{1}{3}, \frac{7}{39}\right) \\ \left(\frac{31}{39}, \frac{2}{3}, \frac{32}{39}\right) & \left(\frac{7}{39}, \frac{2}{3}, \frac{11}{39}\right) & \left(\frac{34}{39}, \frac{2}{3}, \frac{20}{39}\right) & \left(\frac{19}{39}, \frac{2}{3}, \frac{2}{39}\right) & \left(\frac{28}{39}, \frac{2}{3}, \frac{5}{39}\right) & \left(\frac{37}{39}, \frac{2}{3}, \frac{8}{39}\right) \end{matrix} \right\}$$

and

$$\partial Q_{\mathrm{id}} := \left\{ \begin{matrix} \left(\frac{1}{13}, 1, \frac{9}{13}\right) & \left(\frac{4}{13}, 1, \frac{10}{13}\right) & \left(\frac{10}{13}, 1, \frac{12}{13}\right) & \left(\frac{3}{13}, 1, \frac{1}{13}\right) & \left(\frac{9}{13}, 1, \frac{3}{13}\right) & \left(\frac{12}{13}, 1, \frac{4}{13}\right) \\ \left(\frac{7}{13}, 1, \frac{11}{13}\right) & \left(\frac{2}{13}, 1, \frac{5}{13}\right) & \left(\frac{5}{13}, 1, \frac{6}{13}\right) & \left(\frac{8}{13}, 1, \frac{7}{13}\right) & \left(\frac{11}{13}, 1, \frac{8}{13}\right) & \left(\frac{6}{13}, 1, \frac{2}{13}\right) \end{matrix} \right\}.$$

Similarly, for the permutation $\tau = (12)$ we compute

$$\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F}) = \partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})$$
$$= \{z = t_{z,(12),1} \cdot 1 + t_{z,(12),2} \cdot \epsilon_2 + t_{z,(12),3} \cdot \epsilon_1\epsilon_2 \mid \mathbf{t}_{z,(12)} \in \partial Q_{(12)}\},$$

where

$$\partial Q_{(12)} := \left\{ \begin{matrix} \left(\frac{1}{13}, 0, \frac{9}{13}\right) & \left(\frac{4}{13}, 0, \frac{10}{13}\right) & \left(\frac{10}{13}, 0, \frac{12}{13}\right) & \left(\frac{3}{13}, 0, \frac{1}{13}\right) & \left(\frac{9}{13}, 0, \frac{3}{13}\right) & \left(\frac{12}{13}, 0, \frac{4}{13}\right) \\ \left(\frac{7}{13}, 0, \frac{11}{13}\right) & \left(\frac{2}{13}, 0, \frac{5}{13}\right) & \left(\frac{5}{13}, 0, \frac{6}{13}\right) & \left(\frac{8}{13}, 0, \frac{7}{13}\right) & \left(\frac{11}{13}, 0, \frac{8}{13}\right) & \left(\frac{6}{13}, 0, \frac{2}{13}\right) \end{matrix} \right\}.$$

Proposition 2.1. *We have $C(K/F) = 0$.*

Proof. Using (1.12) and (2.1) we can write

$$C(K/F) = -\frac{1}{6}\left(S_{\mathrm{id}} + S_{(12)}\right),$$

where the finite sums S_{id} and $S_{(12)}$ are defined by

$$S_{\mathrm{id}} := \sum_{z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})} \sum_{\substack{\mathbf{h}=(h_1,h_2,h_3)\in\mathbb{Z}^3_{\geq 0} \\ \sum_{i=1}^3 h_i = 3}} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) C_{\mathbf{h}}(A^{\mathrm{id}}) \prod_{i=1}^{3} \frac{B_{h_i}(t_{z,\mathrm{id},i})}{h_i!}$$

and

$$S_{(12)} := \sum_{z \in \partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})} \sum_{\substack{\mathbf{h}=(h_1,h_2,h_3)\in\mathbb{Z}^3_{\geq 0} \\ \sum_{i=1}^3 h_i = 3}} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) C_{\mathbf{h}}(A^{(12)}) \prod_{i=1}^{3} \frac{B_{h_i}(t_{z,(12),i})}{h_i!}.$$

We will prove that $S_{\mathrm{id}} + S_{(12)} = 0$.

We first prove that

$$S_{\mathrm{id}} = C_{(2,1,0)}(A^{\mathrm{id}}) B_1(1) \sum_{z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) \frac{B_2(t_{z,\mathrm{id},1})}{2}. \tag{2.2}$$

If $\mathbf{h} = (0,1,2), (0,2,1)$ or $(1,1,1)$, then a direct calculation shows that

$$C_{(0,1,2)}(A^{\mathrm{id}}) = C_{(0,2,1)}(A^{\mathrm{id}}) = C_{(1,1,1)}(A^{\mathrm{id}}) = 0.$$

Now, using our explicit description of the boundary points, one can check that the map

$$\phi : \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F}) \longrightarrow \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})$$

defined by

$$t_{z,\mathrm{id},1} \cdot 1 + 1 \cdot \epsilon_1 + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2 \longmapsto (1 - t_{z,\mathrm{id},1}) \cdot 1 + 1 \cdot \epsilon_1 + (1 - t_{z,\mathrm{id},3}) \cdot \epsilon_1\epsilon_2$$

is an involution. Moreover, this involution ϕ satisfies the following relation with respect to certain values of the quadratic Hecke character $\chi_{K/F}$.

Lemma 2.2. *If $z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})$, then*

$$\chi_{K/F}(\mathfrak{D}_{K/F}\langle \phi(z)\rangle) = \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle).$$

Proof. Since K/F is quadratic, the conductor-discriminant formula (see e.g. [11, §VII.11, (11.9)]) implies that $\chi_{K/F}$ has conductor $\mathfrak{D}_{K/F}$. Also, since F is cubic we have $\chi_{K/F}(\langle -z\rangle) = \chi_{K/F}(\langle z\rangle)$ (see (6.5)). Therefore

$$\begin{aligned}\chi_{K/F}(\mathfrak{D}_{K/F}\langle\phi(z)\rangle) &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle 1-t_{z,\mathrm{id},1}+\epsilon_1+(1-t_{z,\mathrm{id},3})\cdot\epsilon_1\epsilon_2\rangle)\\ &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle -t_{z,\mathrm{id},1}-\epsilon_1-t_{z,\mathrm{id},3}\cdot\epsilon_1\epsilon_2\rangle+\mathfrak{D}_{K/F})\\ &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle -(t_{z,\mathrm{id},1}+\epsilon_1+t_{z,\mathrm{id},3}\cdot\epsilon_1\epsilon_2)\rangle)\\ &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,\mathrm{id},1}+\epsilon_1+t_{z,\mathrm{id},3}\cdot\epsilon_1\epsilon_2\rangle)\\ &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle).\end{aligned}$$

□

Using the involution ϕ, Lemma 2.2, and the relation $B_k(1-x) = (-1)^k B_k(x)$, we have

$$\begin{aligned}&\sum_{z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\prod_{i=1}^{3}\frac{B_{h_i}(t_{z,\mathrm{id},i})}{h_i!}\\ &=\sum_{z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle\phi(z)\rangle)\frac{B_{h_1}(1-t_{z,\mathrm{id},1})B_{h_2}(t_{z,\mathrm{id},2})B_{h_3}(1-t_{z,\mathrm{id},3})}{h_1!\;h_2!\;h_3!}\\ &=\sum_{z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\cdot(-1)^{h_1+h_3}\prod_{i=1}^{3}\frac{B_{h_i}(t_{z,\mathrm{id},i})}{h_i!}.\end{aligned}$$

Therefore, if $\mathbf{h} = (1,0,2),(2,0,1),(1,2,0),(0,0,3)$ or $(3,0,0)$, so that h_1+h_3 is odd, then

$$\sum_{z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\prod_{i=1}^{3}\frac{B_{h_i}(t_{z,\mathrm{id},i})}{h_i!} = -\sum_{z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\prod_{i=1}^{3}\frac{B_{h_i}(t_{z,\mathrm{id},i})}{h_i!},$$

which implies

$$\sum_{z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\prod_{i=1}^{3}\frac{B_{h_i}(t_{z,\mathrm{id},i})}{h_i!} = 0.$$

If $\mathbf{h} = (0,3,0)$ and $z\in\partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})$, then

$$B_{h_2}(t_{z,\mathrm{id},2}) = B_3(1) = 0.$$

Since $B_0(x) = 1$, by combining the preceding analysis we obtain (2.2).

The preceding argument can be repeated to show that

$$S_{(12)} = C_{(2,1,0)}(A^{(12)})B_1(0)\sum_{z\in\partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})}\chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\frac{B_2(t_{z,(12),1})}{2}. \tag{2.3}$$

If $\mathbf{h} = (2,1,0)$, then a direct calculation shows that

$$C_{(2,1,0)}(A^{\mathrm{id}}) = C_{(2,1,0)}(A^{(12)}).$$

Then because $B_1(0) = -B_1(1)$, by (2.2) and (2.3) we have

$$S_{\mathrm{id}} + S_{(12)} = C_{(2,1,0)}(A^{\mathrm{id}})B_1(1) \quad \times$$
$$\left(\sum_{z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\frac{B_2(t_{z,\mathrm{id},1})}{2} - \sum_{z \in \partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\frac{B_2(t_{z,(12),1})}{2} \right).$$

Using our explicit description of the boundary points, one can check that the map

$$\psi : \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F}) \longrightarrow \partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})$$

defined by

$$t_{z,\mathrm{id},1} \cdot 1 + 1 \cdot \epsilon_1 + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2 \longmapsto t_{z,\mathrm{id},1} \cdot 1 + 0 \cdot \epsilon_2 + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2$$

is a bijection. Then using the bijection ψ, we get

$$\sum_{z \in \partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\frac{B_2(t_{z,(12),1})}{2}$$
$$= \sum_{z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,\mathrm{id},1} \cdot 1 + 0 \cdot \epsilon_2 + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2\rangle)\frac{B_2(t_{z,\mathrm{id},1})}{2}.$$

Now, arguing as in the proof of Lemma 2.2 we have

$$\begin{aligned}\chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,\mathrm{id},1} + 1 \cdot \epsilon_1 + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2\rangle) &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,\mathrm{id},1} + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2\rangle + \mathfrak{D}_{K/F}\rangle) \\ &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,\mathrm{id},1} + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2\rangle) \\ &= \chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,(12),1} + 0 \cdot \epsilon_2 + t_{z,(12),3} \cdot \epsilon_1\epsilon_2\rangle).\end{aligned}$$

Then substituting this identity in the preceding sum yields

$$\sum_{z \in \partial\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\frac{B_2(t_{z,(12),1})}{2}$$
$$= \sum_{z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle t_{z,\mathrm{id},1} + 1 \cdot \epsilon_1 + t_{z,\mathrm{id},3} \cdot \epsilon_1\epsilon_2\rangle)\frac{B_2(t_{z,\mathrm{id},1})}{2}$$
$$= \sum_{z \in \partial\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle)\frac{B_2(t_{z,\mathrm{id},1})}{2}.$$

We conclude that $S_{\mathrm{id}} + S_{(12)} = 0$, which completes the proof. □

We now have all of the information we need to compute the special Gamma value $\mathbf{\Gamma}_{K/F,3}$. To express this value in a more compact form, we introduce the following notation. Let $\{x\} = x - [x]$ denote the fractional part of $x \in \mathbb{R}$. Then we can write

$Q_{\mathrm{id}} =$

$$\left\{ \left(\left\{\frac{29}{39}(3m-2)\right\}, \frac{1}{3}, \left\{\frac{3m-2}{39}\right\} \right), \left(\left\{\frac{3m-2}{39}\right\}, \frac{2}{3}, \left\{\frac{35}{39}(3m-2)\right\} \right) : \ 1 \le m \le 13,\ m \ne 5 \right\}$$

and

$$\partial Q_{\mathrm{id}} = \left\{ \left(\left\{ \frac{m}{13} \right\}, 1, \left\{ \frac{9}{13} m \right\} \right) : \ 1 \leq m \leq 12 \right\}.$$

Similarly, we can write

$$Q_{(12)} = \partial Q_{(12)} = \left\{ \left(\left\{ \frac{m}{13} \right\}, 0, \left\{ \frac{9}{13} m \right\} \right) : \ 1 \leq m \leq 12 \right\}.$$

Then by computing the character values $\chi_{K/F}(\mathfrak{D}_{K/F}\langle z \rangle) \in \{\pm 1\}$ for

$$z \in \widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F}) \cup \widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})$$

and pairing according to the values which are 1 and -1, we get

$$\mathbf{\Gamma}_{K/F,3} = \mathbf{\Gamma}^1_{K/F,3} \cdot \mathbf{\Gamma}^2_{K/F,3} \cdot \mathbf{\Gamma}^3_{K/F,3} \cdot \mathbf{\Gamma}^4_{K/F,3},$$

where

$$\mathbf{\Gamma}^1_{K/F,3} := \prod_{i=1}^{3} \prod_{\substack{m=1 \\ m \neq 5}}^{13} \Gamma_3 \left(\left\{ \frac{29}{39}(3m-2) \right\} + \frac{1}{3}\sigma_i(\epsilon_1) + \left\{ \frac{3m-2}{39} \right\} \sigma_i(\epsilon_1\epsilon_2), (1, \sigma_i(\epsilon_1), \sigma_i(\epsilon_1\epsilon_2)) \right)^{\frac{c_1(m)}{2}}$$

$$\mathbf{\Gamma}^2_{K/F,3} := \prod_{i=1}^{3} \prod_{\substack{m=1 \\ m \neq 5}}^{13} \Gamma_3 \left(\left\{ \frac{3m-2}{39} \right\} + \frac{2}{3}\sigma_i(\epsilon_1) + \left\{ \frac{35}{39}(3m-2) \right\} \sigma_i(\epsilon_1\epsilon_2), (1, \sigma_i(\epsilon_1), \sigma_i(\epsilon_1\epsilon_2)) \right)^{\frac{c_1(m)}{2}}$$

$$\mathbf{\Gamma}^3_{K/F,3} := \prod_{i=1}^{3} \prod_{m=1}^{12} \Gamma_3 \left(\left\{ \frac{m}{13} \right\} + \sigma_i(\epsilon_1) + \left\{ \frac{9}{13} m \right\} \sigma_i(\epsilon_1\epsilon_2), (1, \sigma_i(\epsilon_1), \sigma_i(\epsilon_1\epsilon_2)) \right)^{\frac{c_2(m)}{2}}$$

$$\mathbf{\Gamma}^4_{K/F,3} := \prod_{i=1}^{3} \prod_{m=1}^{12} \Gamma_3 \left(\left\{ \frac{m}{13} \right\} + \left\{ \frac{9}{13} m \right\} \sigma_i(\epsilon_1\epsilon_2), (1, \sigma_i(\epsilon_2), \sigma_i(\epsilon_1\epsilon_2)) \right)^{\frac{c_2(m)}{2}}$$

with

$$c_1(m) := \begin{cases} 1, & m \in \{1, 2, 4, 6, 8, 9\} \\ -1, & m \in \{3, 7, 10, 11, 12, 13\} \end{cases}$$

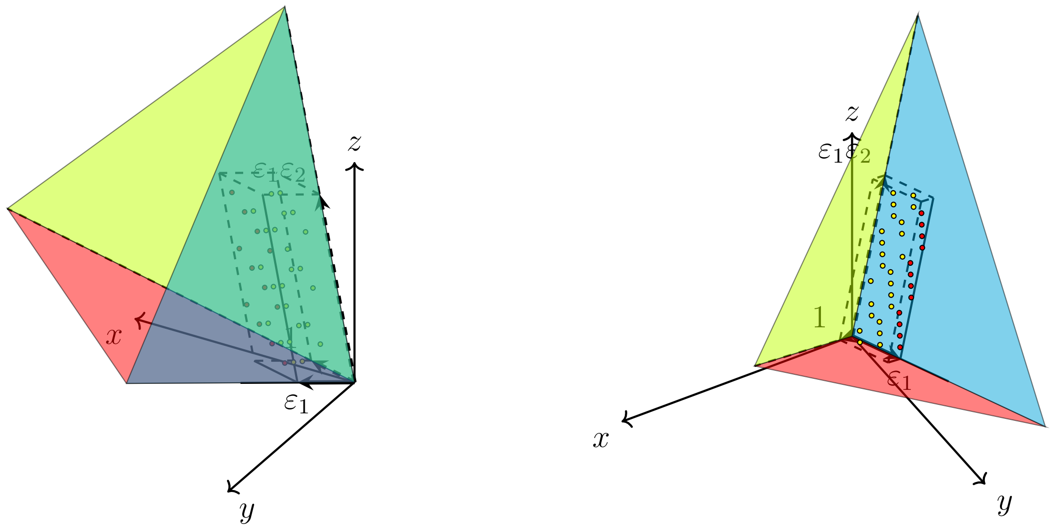
$$c_2(m) := \begin{cases} 1, & m \in \{1, 3, 4, 9, 10, 12\} \\ -1, & m \in \{2, 5, 6, 7, 8, 11\}. \end{cases}$$

Finally, define $\alpha_i := \sigma_i(\epsilon_1)$, $\beta_i := \sigma_i(\epsilon_1\epsilon_2)$ and $\gamma_i := \sigma_i(\epsilon_2)$ for $i = 1, 2, 3$. The values of these real numbers in terms of r are given in the matrices A^{id} and $A^{(12)}$ computed above.

In Figures 1 and 2 we display the embeddings of the restricted Shintani sets

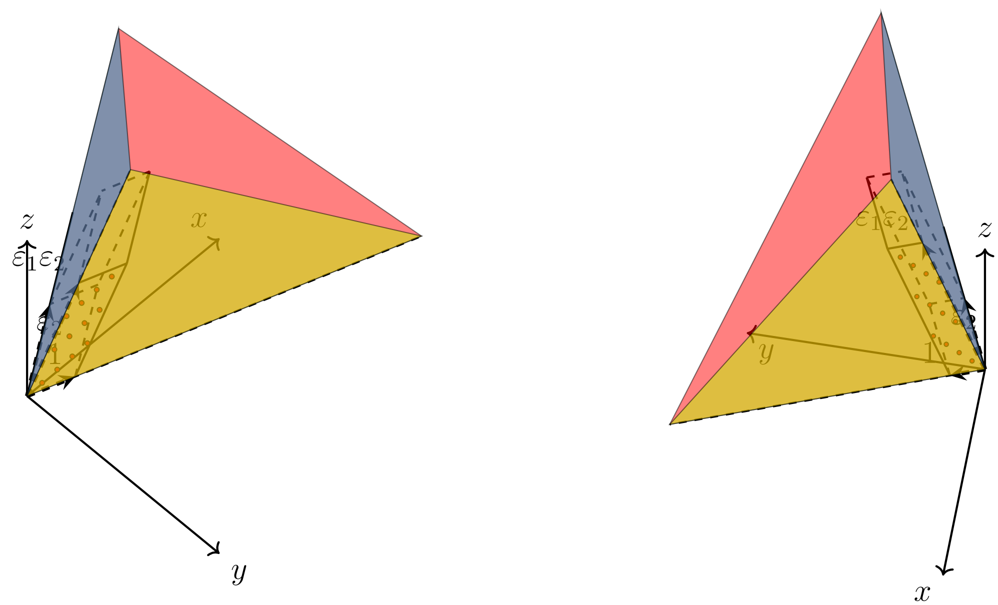
$$\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F}) \quad \text{and} \quad \widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})$$

into the fundamental parallelotopes P_F^{id} and $P_F^{(12)}$ contained in the F-rational cones C_F^{id} and $C_F^{(12)}$, respectively. The interior points are plotted in yellow and the boundary points are plotted in red.



(A) The first view (B) The second view

Figure 1. The embedding of the restricted Shintani set $\widetilde{\mathcal{R}^{\mathrm{id}}}(\mathfrak{D}_{K/F})$ into the parallelotope $P_F^{\mathrm{id}} \subset C_F^{\mathrm{id}}$.



(A) The first view (B) The second view

Figure 2. The embedding of the restricted Shintani set $\widetilde{\mathcal{R}^{(12)}}(\mathfrak{D}_{K/F})$ into the parallelotope $P_F^{(12)} \subset C_F^{(12)}$.

3. An algorithm for computing the Stark unit $\varepsilon_{K/F,S}$

In this section we develop an algorithm to compute the Stark unit $\varepsilon_{K/F,S}$.

Recall that F is a totally real field of degree n over $\mathbb{Q}$ with real embeddings $\sigma_1 := \mathrm{id}_F, \sigma_2, \ldots, \sigma_n$ and $K = F(\sqrt{\Delta})$ is a quadratic extension of F such that $\sigma_1(\Delta) > 0$ and $\sigma_i(\Delta) < 0$ for $i = 2, \ldots, n$.

Write $\mathrm{Gal}(K/F) = \langle \sigma \rangle$, where σ is the embedding of K defined by

$$\sigma : \sqrt{\Delta} \longmapsto -\sqrt{\Delta}.$$

Our algorithm starts with the existence of a unit as in the following proposition.

Proposition 3.1. *There exists a unit* $\epsilon \in \mathcal{O}_K^\times$ *such that*

$$[\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \epsilon \rangle] = \begin{cases} 2 & \text{if } \Delta \in \mathcal{O}_F^\times \cdot (F^\times)^2, \\ 1 & \text{otherwise.} \end{cases}$$

In order to prove Proposition 3.1 we will need the following two lemmas.

Lemma 3.2. *Define the subgroup*

$$N := \{u \in \mathcal{O}_K^\times \mid N_{K/F}(u) = 1\} < \mathcal{O}_K^\times.$$

Then

$$N \cdot \mathcal{O}_F^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^n.$$

Proof. By Dirichlet's unit theorem we have $\mathrm{rank}(\mathcal{O}_K^\times) = n$. Also, since $K \subset \mathbb{R}$ we have $\mathrm{Tor}(\mathcal{O}_K^\times) = \langle \pm 1 \rangle$. Hence $\mathcal{O}_K^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^n$. Therefore, to prove the lemma it suffices to show that the subgroup $N \cdot \mathcal{O}_F^\times < \mathcal{O}_K^\times$ has rank n.

Define the map

$$\varphi : \mathcal{O}_K^\times/(N \cdot \mathcal{O}_F^\times) \longrightarrow N_{K/F}(\mathcal{O}_K^\times)/(\mathcal{O}_F^\times)^2$$
$$\alpha(N \cdot \mathcal{O}_F^\times) \longmapsto N_{K/F}(\alpha)(\mathcal{O}_F^\times)^2.$$

A short calculation shows that φ is a well-defined group homomorphism. The map φ is surjective by definition. To show that φ is injective, let $\alpha \in \mathcal{O}_K^\times$ be such that $\alpha(N \cdot \mathcal{O}_F^\times) \in \ker(\varphi)$. Then $N_{K/F}(\alpha) \in (\mathcal{O}_F^\times)^2$, so there is a unit $\beta \in \mathcal{O}_F^\times$ such that $N_{K/F}(\alpha) = \beta^2$. This implies that $N_{K/F}(\alpha/\beta) = 1$, and hence $\alpha/\beta \in N$, or equivalently, $\alpha \in N \cdot \mathcal{O}_F^\times$. This proves $\ker(\varphi)$ is trivial. Thus φ is an isomorphism.

From the isomorphism φ we have

$$[\mathcal{O}_K^\times : N \cdot \mathcal{O}_F^\times] = [N_{K/F}(\mathcal{O}_K^\times) : (\mathcal{O}_F^\times)^2] \leq [\mathcal{O}_F^\times : (\mathcal{O}_F^\times)^2] < \infty.$$

It follows that $\mathrm{rank}(N \cdot \mathcal{O}_F^\times) = \mathrm{rank}(\mathcal{O}_K^\times) = n$. □

Lemma 3.3. *We have*

$$N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}.$$

Proof. The lemma is equivalent to the statement

$$N/\langle \pm 1 \rangle \cong \mathbb{Z}.$$

Observe that

$$N \cap \mathcal{O}_F^\times = \{u \in \mathcal{O}_F^\times : \ u^2 = 1\} = \langle \pm 1 \rangle.$$

Then by the Second Isomorphism Theorem, we have

$$N/\langle \pm 1 \rangle = N/(N \cap \mathcal{O}_F^\times) \cong (N \cdot \mathcal{O}_F^\times)/\mathcal{O}_F^\times.$$

Now, if G is a finitely generated abelian group with $\mathrm{rank}(G) = n$ and $H < G$ is a subgroup with $\mathrm{rank}(H) = m$, then $m \leq n$ and $\mathrm{rank}(G/H) = n - m$ (see e.g. [9, p. 82]). By Lemma 3.2, we have $\mathrm{rank}(N \cdot \mathcal{O}_F^\times) = n$, and by Dirichlet's unit theorem we have $\mathrm{rank}(\mathcal{O}_F^\times) = n - 1$. Hence $\mathrm{rank}((N \cdot \mathcal{O}_F^\times)/\mathcal{O}_F^\times) = 1$, so that

$$N/\langle \pm 1 \rangle \cong (N \cdot \mathcal{O}_F^\times)/\mathcal{O}_F^\times \cong \mathbb{Z}.$$

□

Proof of Proposition 3.1. Define the subgroup

$$M := \left\{ \frac{u}{\sigma(u)} \;\middle|\; u \in \mathcal{O}_K^\times \right\} < N.$$

Let $\varphi : \mathcal{O}_K^\times \longrightarrow M$ be the map defined by $\varphi(u) = u/\sigma(u)$ for $u \in \mathcal{O}_K^\times$. A short calculation shows that φ is a surjective group homomorphism. Now, observe that

$$u \in \ker(\varphi) \iff \frac{u}{\sigma(u)} = 1 \iff \sigma(u) = u \iff u \in \mathcal{O}_F^\times.$$

Hence $\ker(\varphi) = \mathcal{O}_F^\times$, so that

$$\mathcal{O}_K^\times/\mathcal{O}_F^\times \cong M.$$

By Lemma 3.3 we have $M \leq N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. Since M is infinite, it follows that $M \cong \mathbb{Z}$ if $-1 \notin M$ and $M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ if $-1 \in M$. We have shown that

$$\mathcal{O}_K^\times/\mathcal{O}_F^\times \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} & \text{if } -1 \in M, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Thus $\#\mathrm{Tor}(\mathcal{O}_K^\times/\mathcal{O}_F^\times) = 2$ if $-1 \in M$ and $\#\mathrm{Tor}(\mathcal{O}_K^\times/\mathcal{O}_F^\times) = 1$ otherwise.

Now, let $\epsilon \in \mathcal{O}_K^\times \smallsetminus \mathcal{O}_F^\times$ be a unit such that $\epsilon\mathcal{O}_F^\times$ generates the free part of $\mathcal{O}_K^\times/\mathcal{O}_F^\times$, i.e.,

$$\mathcal{O}_K^\times/\mathcal{O}_F^\times = \mathrm{Tor}(\mathcal{O}_K^\times/\mathcal{O}_F^\times)\langle \epsilon\mathcal{O}_F^\times \rangle.$$

Then by the Third Isomorphism Theorem we have

$$\begin{aligned} \mathcal{O}_K^\times/\langle \mathcal{O}_F^\times, \epsilon \rangle &\cong \left(\mathcal{O}_K^\times/\mathcal{O}_F^\times\right) / \left(\langle \mathcal{O}_F^\times, \epsilon \rangle/\mathcal{O}_F^\times\right) \\ &= \mathrm{Tor}(\mathcal{O}_K^\times/\mathcal{O}_F^\times)\langle \epsilon\mathcal{O}_F^\times \rangle / \left(\langle \mathcal{O}_F^\times, \epsilon \rangle/\mathcal{O}_F^\times\right) \\ &= \mathrm{Tor}(\mathcal{O}_K^\times/\mathcal{O}_F^\times)\langle \epsilon\mathcal{O}_F^\times \rangle / \langle \epsilon\mathcal{O}_F^\times \rangle \\ &\cong \mathrm{Tor}(\mathcal{O}_K^\times/\mathcal{O}_F^\times), \end{aligned}$$

where we used that $\langle \mathcal{O}_F^\times, \epsilon \rangle/\mathcal{O}_F^\times = \langle \epsilon\mathcal{O}_F^\times \rangle$. Hence

$$[\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \epsilon \rangle] = \begin{cases} 2 & \text{if } -1 \in M, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, it remains to prove the following

Claim. $-1 \in M$ if and only if $\Delta \in \mathcal{O}_F^\times \cdot (F^\times)^2$.

We will require the following two facts.

(1) $-1 \in M$ if and only if there exists $u \in \mathcal{O}_K^\times$ such that $\mathrm{Tr}_{K/F}(u) = 0$.
(2) If $\Delta' \in F^\times$ satisfies $K = F(\sqrt{\Delta'})$, then there is a $\beta \in F^\times$ such that $\Delta = \beta^2\Delta'$.

Fact (1) is true since

$$\begin{aligned} -1 \in M &\iff \text{there exists } u \in \mathcal{O}_K^\times \text{ such that } \frac{u}{\sigma(u)} = -1 \\ &\iff \text{there exists } u \in \mathcal{O}_K^\times \text{ such that } \mathrm{Tr}_{K/F}(u) = 0. \end{aligned}$$

Fact (2) is true since $K = F(\sqrt{\Delta})$, and so Δ and Δ' must differ by a square in $F^\times$.

We now prove the Claim. First, suppose that $-1 \in M$. Then by (1), there is a unit $u \in \mathcal{O}_K^\times$ such that $\mathrm{Tr}_{K/F}(u) = 0$. On the other hand, any element $\alpha \in K$ satisfies the equation

$$\alpha^2 - \mathrm{Tr}_{K/F}(\alpha)\alpha + N_{K/F}(\alpha) = 0.$$

Hence $u^2 + N_{K/F}(u) = 0$, or equivalently, $u = \pm\sqrt{|N_{K/F}(u)|}$. Moreover, this unit is not in F, hence

$$K = F(u) = F\left(\sqrt{|N_{K/F}(u)|}\right).$$

Define $\Delta' := |N_{K/F}(u)| \in \mathcal{O}_F^\times$. Then by (2), we have $\Delta \in \mathcal{O}_F^\times \cdot (F^\times)^2$.

Next, suppose that $\Delta \in \mathcal{O}_F^\times \cdot (F^\times)^2$. Then there are elements $\Delta' \in \mathcal{O}_F^\times$ and $\beta \in F^\times$ with $\Delta = \beta^2\Delta'$. Now, we have $K = F(\sqrt{\Delta'})$. Moreover, the minimal polynomial of $\sqrt{\Delta'}$ over F is $x^2 - \Delta'$, and so $\mathrm{Tr}_{K/F}(\sqrt{\Delta'}) = 0$. Finally, we have

$$N_{K/\mathbb{Q}}(\sqrt{\Delta'}) = N_{F/\mathbb{Q}}(N_{K/F}(\sqrt{\Delta'})) = N_{F/\mathbb{Q}}(-\Delta') = \pm 1,$$

hence $\sqrt{\Delta'} \in \mathcal{O}_K^\times$. We have produced a unit $u := \sqrt{\Delta'} \in \mathcal{O}_K^\times$ with $\mathrm{Tr}_{K/F}(u) = 0$. Therefore, by (1) we have $-1 \in M$.

This completes the proof of the Claim, and hence the proposition. □

We can now describe our algorithm to compute the Stark unit. Fix two sets of generators

$$\mathcal{O}_K^\times/\mathrm{Tor}(\mathcal{O}_K^\times) = \langle u_1, \ldots, u_n \rangle \quad \text{and} \quad \mathcal{O}_F^\times/\mathrm{Tor}(\mathcal{O}_F^\times) = \langle \eta_2, \ldots, \eta_n \rangle.$$

We will need the following result which can be deduced from [17, Lemma 4.15] and its proof.

Proposition 3.4. *Let $u \in \mathcal{O}_K^\times$ be such that the units $\{u, \eta_2, \ldots, \eta_n\}$ are multiplicatively independent in $\mathcal{O}_K^\times$. Write*

$$\begin{aligned} u &= \pm u_1^{a_1(u)} \cdots u_n^{a_n(u)} \\ \eta_2 &= \pm u_1^{a_{21}} \cdots u_n^{a_{2n}} \\ &\vdots \\ \eta_n &= \pm u_1^{a_{n1}} \cdots u_n^{a_{nn}} \end{aligned}$$

and define the matrix

$$A(u) := \begin{pmatrix} a_1(u) & \cdots & a_n(u) \\ a_{21} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{Z}).$$

Then

$$|\det(A(u))| = [\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, u \rangle].$$

Let ϵ be a unit as in Proposition 3.1. Since

$$[\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \epsilon \rangle] < \infty,$$

the units $\{\epsilon, \eta_2, \dots \eta_n\}$ are multiplicatively independent in $\mathcal{O}_K^\times$. Then applying Proposition 3.4 with $u = \epsilon$ yields

$$|\det(A(\epsilon))| = [\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \epsilon \rangle] = 2^v.$$

Expanding the determinant along the first row, we have

$$|\det(A(\epsilon))| = \left| \sum_{j=1}^n (-1)^{j+1} a_j(\epsilon) \det(A_{1j}) \right|$$

where $A_{1j} \in M_{n-1}(\mathbb{Z})$ is the matrix obtained from $A(\epsilon)$ by deleting the first row and the j-th column. Define

$$b_j := (-1)^{j+1} \det(A_{1j}) \in \mathbb{Z} \quad \text{for} \quad j = 1, \dots, n.$$

Then $|\det(A(\epsilon))| = 2^v$ if and only if

$$|b_1 a_1(\epsilon) + \cdots + b_n a_n(\epsilon)| = 2^v.$$

In particular, this shows that the linear Diophantine equation

$$b_1 x_1 + \cdots + b_n x_n = 2^v \tag{3.1}$$

has a solution $x(\epsilon) := \pm(a_1(\epsilon), \dots, a_n(\epsilon)) \in \mathbb{Z}^n$.

Since there is at least one solution to (3.1), the Euclidean algorithm provides an effective way of finding infinitely many solutions $c = (c_1, \dots, c_n) \in \mathbb{Z}^n$ to (3.1). Choose any solution

$$c' = (c'_1, \dots, c'_n)$$

to (3.1) constructed as above and define the unit

$$\eta := u_1^{c'_1} \dots u_n^{c'_n} \in \mathcal{O}_K^\times.$$

Write

$$\begin{aligned} \eta &= u_1^{c'_1} \cdots u_n^{c'_n} \\ \eta_2 &= \pm u_1^{a_{21}} \cdots u_n^{a_{2n}} \\ &\vdots \\ \eta_n &= \pm u_1^{a_{n1}} \cdots u_n^{a_{nn}} \end{aligned}$$

and define the matrix

$$A' = \begin{pmatrix} c'_1 & c'_2 & \cdots & c'_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Since c' satisfies (3.1), the calculation with determinants above shows that

$$|\det(A')| = |b_1 c'_1 + \cdots + b_n c'_n| = 2^v.$$

In particular, since $\det(A') \neq 0$, the units $\{\eta, \eta_2, \ldots, \eta_n\}$ are multiplicatively independent in $\mathcal{O}_K^\times$. Then applying Proposition 3.4 with $u = \eta$ yields

$$|\det(A')| = [\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \eta \rangle].$$

Hence

$$[\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \eta \rangle] = 2^v.$$

We have now produced an effectively computable unit

$$\eta = u_1^{c_1'} \ldots u_n^{c_n'} \in \mathcal{O}_K^\times$$

such that

$$[\mathcal{O}_K^\times : \langle \mathcal{O}_F^\times, \eta \rangle] = 2^v.$$

Hence, we have effectively computed the Stark unit

$$\varepsilon_{K/F,S} := \max\left\{ \left|\frac{\eta}{\sigma(\eta)}\right|, \left|\frac{\eta}{\sigma(\eta)}\right|^{-1} \right\}.$$

We summarize the preceding discussion in the following algorithm.

Algorithm 1 Computing the Stark unit $\varepsilon_{K/F,S}$

INPUT: A pair of number fields (K, F) satisfying Condition 1.2.
OUTPUT: The Stark unit $\varepsilon_{K/F,S}$.

1: Compute a set of generators $\{u_1, \ldots, u_n\}$ of $\mathcal{O}_K^\times/\mathrm{Tor}(\mathcal{O}_K^\times)$ and a set of generators $\{\eta_2, \ldots, \eta_n\}$ of $\mathcal{O}_F^\times/\mathrm{Tor}(\mathcal{O}_F^\times)$.

2: Compute integers $a_{ij} \in \mathbb{Z}$ such that

$$\eta_2 = \pm u_1^{a_{21}} \ldots u_n^{a_{2n}}$$
$$\vdots$$
$$\eta_n = \pm u_1^{a_{n1}} \ldots u_n^{a_{nn}}$$

and form the matrix $M = (a_{ij}) \in M_{(n-1)\times n}(\mathbb{Z})$.

3: Compute the integers

$$b_j = (-1)^{j+1} \det(A_{1j}), \quad j = 1, \ldots, n$$

where $A_{1j} \in M_{n-1}(\mathbb{Z})$ is the matrix obtained by deleting the j-th column of M.

4: Find a solution $(c_1', \ldots, c_n') \in \mathbb{Z}^n$ to the linear Diophantine equation

$$b_1 x_1 + \cdots + b_n x_n = 2^v.$$

5: Compute the unit

$$\eta := u_1^{c_1'} \ldots u_n^{c_n'} \in \mathcal{O}_K^\times.$$

6: Compute the Stark unit

$$\varepsilon_{K/F,S} = \max\left\{ \left|\frac{\eta}{\sigma(\eta)}\right|, \left|\frac{\eta}{\sigma(\eta)}\right|^{-1} \right\}.$$

---

4. The combinatorial geometry of Shintani sets

In this section, we investigate the combinatorial geometry of Shintani sets. In particular, we give an explicit formula for the size of Shintani sets, and use this formula to prove that the narrow ray class characters satisfy certain orthogonality relations when evaluated on Shintani sets.

First, we define a binary operation on Shintani sets which gives them the structure of a finite abelian group (see Proposition 4.3). We then use this group structure to define a group homomorphism which allows us to translate the problem of studying the size of Shintani sets to a problem in discrete geometry. More precisely, we will be faced with the problem of counting lattice points inside a lattice polytope in $\mathbb{R}^n$. This problem will be solved by studying the Ehrhart polynomial of the polytope, allowing us (perhaps surprisingly) to find a clean explicit formula for the size of Shintani sets (see Theorem 4.9).

For convenience, we recall the notation and assumptions of Section 1.3, which is adapted from [5]. Let $\mathcal{O}_F^{\times,+}$ denote the group of totally positive units of F. Since F has signature $(n, 0)$, Dirichlet's unit theorem implies that both $\mathcal{O}_F^{\times}$ and $\mathcal{O}_F^{\times,+}$ have rank $n-1$. Fix a choice of generators $\epsilon_1, \dots, \epsilon_{n-1}$ of $\mathcal{O}_F^{\times,+}$.

Let $\iota : F \hookrightarrow \mathbb{R}^n$ be the embedding of F into $\mathbb{R}^n$ given by $\iota(x) := (\sigma_1(x), \dots, \sigma_n(x)) \in \mathbb{R}^n$ for any $x \in F$. For each $1 \leq i \leq n$ and any permutation $\tau \in S_{n-1}$, define the totally positive unit

$$f_{\tau,i} := \epsilon_{\tau(1)}\epsilon_{\tau(2)} \cdots \epsilon_{\tau(i-1)} = \prod_{j=1}^{i-1} \epsilon_{\tau(j)} \in \mathcal{O}_F^{\times,+}.$$

Note that for $i = 1$ this gives the empty product, so $f_{\tau,1} = 1$.

For any $\tau \in S_{n-1}$, define the weight

$$w_\tau := \frac{(-1)^{n-1}\mathrm{sgn}(\tau) \cdot \mathrm{sign}(\det(\sigma_i(f_{\tau,j})))}{\mathrm{sign}(\det(\log|\sigma_i(\epsilon_j)|)_{1\leq i,j\leq n-1})} \in \{0, \pm 1\}.$$

Observe that the matrix appearing in the numerator of w_τ is $n \times n$, whereas the matrix appearing in the denominator is $(n-1) \times (n-1)$. Note that $w_\tau \neq 0$ if and only if $\det(\sigma_i(f_{\tau,j})) \neq 0$. Thus $w_\tau \neq 0$ if and only if the vectors $\iota(f_{\tau,j}) \in \mathbb{R}^n_{>0}$ for $j = 1, \dots, n$ form a basis for $\mathbb{R}^n$. In particular, if $w_\tau \neq 0$, then we can write the n-th standard basis vector in $\mathbb{R}^n$ as

$$e_n := (0, 0, \dots, 1) = \sum_{i=1}^{n} c_i \iota(f_{\tau,i})$$

for some unique real numbers $c_i \in \mathbb{R}$. Define the following intervals in terms of the sign of c_i,

$$I_{\tau,i} := \begin{cases} [0,1) & \text{if } c_i > 0 \\ (0,1] & \text{otherwise.} \end{cases}$$

Similarly, observe that if $w_\tau \neq 0$, then the algebraic numbers $\{f_{\tau,i}\}_{i=1}^n$ form a $\mathbb{Q}$-basis for F. In particular, given a nonzero integral ideal $\mathfrak{f}$ of F, every element $z \in \mathfrak{f}^{-1}$ can be expressed as a linear combination of the form

$$z = \sum_{i=1}^{n} t_{z,\tau,i} f_{\tau,i}$$

for some unique rational numbers $t_{z,\tau,i} \in \mathbb{Q}$. Let

$$\mathbf{t}_{z,\tau} := (t_{z,\tau,1}, \ldots, t_{z,\tau,n}) \in \mathbb{Q}^n$$

be the coordinate vector of z with respect to the $\mathbb{Q}$-basis $\{f_{\tau,i}\}_{i=1}^n$.

For $\tau \in S_{n-1}$ such that $w_\tau \neq 0$ and for $\mathfrak{f}$ a nonzero integral ideal of F, we define the *Shintani set* associated to $\mathfrak{f}$ by

$$\mathcal{R}^\tau(\mathfrak{f}) = \mathcal{R}^\tau(\mathfrak{f}; \epsilon_1, \ldots, \epsilon_{n-1}) := \left\{ z \in \mathfrak{f}^{-1} \;\middle|\; \mathbf{t}_{z,\tau} \in I_{\tau,1} \times \cdots \times I_{\tau,n} \right\}.$$

Similarly, we define the *restricted Shintani set* associated to $\mathfrak{f}$ by

$$\widetilde{\mathcal{R}^\tau}(\mathfrak{f}) = \widetilde{\mathcal{R}^\tau}(\mathfrak{f}; \epsilon_1, \ldots, \epsilon_{n-1}) := \left\{ z \in \mathfrak{f}^{-1} \;\middle|\; \mathbf{t}_{z,\tau} \in I_{\tau,1} \times \cdots \times I_{\tau,n},\ \mathfrak{f}\langle z\rangle \text{ coprime to } \mathfrak{f} \right\}.$$

We next introduce the following modified fractional part function for each interval $I_{\tau,i}$. For $x \in \mathbb{R}$, let $\{x\}_{I_{\tau,i}}$ be the unique element of $I_{\tau,i}$ such that

$$x - \{x\}_{I_{\tau,i}} \in \mathbb{Z}.$$

Thus when $I_{\tau,i} = [0,1)$ we see that $\{x\}_{I_{\tau,i}}$ is just the ordinary fractional part $\{x\}$. On the other hand, when $I_{\tau,i} = (0,1]$, we have

$$\{x\}_{I_{\tau,i}} = \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z} \\ 1 & \text{if } x \in \mathbb{Z}. \end{cases}$$

We begin with the following proposition which we will use to give the Shintani set the structure of a finite abelian group.

Proposition 4.1. *The Shintani set $\mathcal{R}^\tau(\mathfrak{f})$ is a complete set of coset representatives for the quotient group*

$$G_\tau(\mathfrak{f}) = G_\tau(\mathfrak{f}; \epsilon_1, \ldots, \epsilon_{n-1}) := \mathfrak{f}^{-1} \Big/ \bigoplus_{i=1}^n \mathbb{Z} f_{\tau,i}\,.$$

Proof. It suffices to prove that the map

$$\begin{aligned} \mathcal{R}^\tau(\mathfrak{f}) &\longrightarrow G_\tau(\mathfrak{f}) \\ z &\longmapsto [z] \end{aligned}$$

is a bijection. Let $z, \tilde{z} \in \mathcal{R}^\tau(\mathfrak{f})$. Then

$$z = \sum_{i=1}^n t_i f_{\tau,i} \quad \text{and} \quad \widetilde{z} = \sum_{i=1}^n \widetilde{t_i} f_{\tau,i}$$

for some rational numbers $t_i, \widetilde{t_i} \in I_{\tau,i} \cap \mathbb{Q}$. If $[z] = [\tilde{z}]$, then

$$z - \tilde{z} = \sum_{i=1}^n (t_i - \tilde{t_i}) f_{\tau,i} \in \bigoplus_{i=1}^n \mathbb{Z} f_{\tau,i}$$

so that $t_i - \tilde{t_i} \in \mathbb{Z}$. Since $-1 < t_i - \widetilde{t_i} < 1$, it follows that $t_i = \widetilde{t_i}$ for all $i = 1, \ldots, n$, and hence $z = \tilde{z}$. Next, let $[w] \in G_\tau(\mathfrak{f})$. Since $\{f_{\tau,i}\}_{i=1}^n$ is a $\mathbb{Q}$-basis for F, then

$$w = \sum_{i=1}^n r_i f_{\tau,i}$$

for some rational numbers $r_i \in \mathbb{Q}$. Define

$$z_w := \sum_{i=1}^{n} t_i f_{\tau,i}$$

where $t_i := \{r_i\}_{I_{\tau,i}} \in I_{\tau,i} \cap \mathbb{Q}$. Then

$$z_w - w = \sum_{i=1}^{n} (\{r_i\}_{I_{\tau,i}} - r_i) f_{\tau,i} \in \bigoplus_{i=1}^{n} \mathbb{Z} f_{\tau,i} \subset \mathfrak{f}^{-1}.$$

It follows that $z_w \in \mathcal{R}^\tau(\mathfrak{f})$ and $[z_w] = [w]$. □

Remark 4.2. By Proposition 4.1, the Shintani set $\mathcal{R}^\tau(\mathfrak{f}) \neq \varnothing$ since $G_\tau(\mathfrak{f}) \neq \varnothing$.

We can now define a binary operation

$$\oplus : \mathcal{R}^\tau(\mathfrak{f}) \times \mathcal{R}^\tau(\mathfrak{f}) \longrightarrow \mathcal{R}^\tau(\mathfrak{f})$$

by letting $z_1 \oplus z_2$ be the coset representative in $\mathcal{R}^\tau(\mathfrak{f})$ corresponding to the coset

$$z_1 + z_2 + \bigoplus_{i=1}^{n} \mathbb{Z} f_{\tau,i}.$$

We immediately obtain the following proposition.

Proposition 4.3. *The Shintani set $\mathcal{R}^\tau(\mathfrak{f})$ is a finite abelian group with respect to the binary operation $\oplus$.*

Assumption. For the remaining part of this section, we assume that the nonzero integral ideal $\mathfrak{f}$ is *principal* with generator α.

Proposition 4.4. *Define the map*

$$\begin{aligned} \pi_{\alpha,\tau} : \mathcal{R}^\tau(\mathfrak{f}) &\longrightarrow \mathcal{O}_F/\mathfrak{f} \\ z &\longmapsto \alpha z + \mathfrak{f}. \end{aligned}$$

Then $\pi_{\alpha,\tau}$ is a surjective group homomorphism. Moreover, for every coset $w + \mathfrak{f} \in \mathcal{O}_F/\mathfrak{f}$ we have

$$\#\pi_{\alpha,\tau}^{-1}(w + \mathfrak{f}) = \#\ker(\pi_{\alpha,\tau}).$$

Proof. If $z_1, z_2 \in \mathcal{R}^\tau(\mathfrak{f})$ then

$$z_1 + z_2 - z_1 \oplus z_2 = \sum_{i=1}^{n} m_i f_{\tau,i}$$

for some integers $m_i \in \mathbb{Z}$. Hence

$$\begin{aligned} \pi_{\alpha,\tau}(z_1 \oplus z_2) &= \alpha(z_1 \oplus z_2) + \mathfrak{f} \\ &= \alpha z_1 + \alpha z_2 - \alpha \sum_{i=1}^{n} m_i f_{\tau,i} + \mathfrak{f} \\ &= (\alpha z_1 + \mathfrak{f}) + (\alpha z_2 + \mathfrak{f}) \\ &= \pi_{\alpha,\tau}(z_1) + \pi_{\alpha\tau}(z_2) \end{aligned}$$

where we used

$$\alpha \sum_{i=1}^{n} m_i f_{\tau,i} \in \alpha \mathcal{O}_F = \mathfrak{f}. \tag{4.1}$$

Next, let $w + \mathfrak{f} \in \mathcal{O}_F/\mathfrak{f}$. Since $\mathfrak{f}^{-1} = \frac{1}{\alpha}\mathcal{O}_F$, then $\frac{w}{\alpha} \in \mathfrak{f}^{-1}$. Let $z \in \mathcal{R}^\tau(\mathfrak{f})$ be the coset representative corresponding to the coset

$$\frac{w}{\alpha} + \bigoplus_{i=1}^{n} \mathbb{Z} f_{\tau,i}.$$

Then there are integers $n_i \in \mathbb{Z}$ such that

$$z = \frac{w}{\alpha} + \sum_{i=1}^{n} n_i f_{\tau,i}.$$

Hence

$$\pi_{\alpha,\tau}(z) = \alpha z + \mathfrak{f} = \alpha \left(\frac{w}{\alpha} + \sum_{i=1}^{n} n_i f_{\tau,i} \right) + \mathfrak{f} = w + \alpha \sum_{i=1}^{n} n_i f_{\tau,i} + \mathfrak{f} = w + \mathfrak{f}$$

where we again used (4.1). Finally, by the First Isomorphism Theorem we have

$$\mathcal{R}^\tau(\mathfrak{f}) / \ker(\pi_{\alpha,\tau}) \cong \mathcal{O}_F/\mathfrak{f}.$$

It follows immediately that

$$\# \pi_{\alpha,\tau}^{-1}(w + \mathfrak{f}) = \# \ker(\pi_{\alpha,\tau}).$$

□

We now give an initial formula for the size of the Shintani set $\mathcal{R}^\tau(\mathfrak{f})$ and the restricted Shintani set $\widetilde{\mathcal{R}^\tau}(\mathfrak{f})$.

Proposition 4.5. *We have*

$$\# \mathcal{R}^\tau(\mathfrak{f}) = \# \ker(\pi_{\alpha,\tau}) \cdot N_{F/\mathbb{Q}}(\mathfrak{f})$$

and

$$\# \widetilde{\mathcal{R}^\tau}(\mathfrak{f}) = \# \ker(\pi_{\alpha,\tau}) \cdot \varphi(\mathfrak{f}),$$

where

$$\varphi(\mathfrak{f}) := \# \left(\mathcal{O}_F/\mathfrak{f} \right)^\times = N_{F/\mathbb{Q}}(\mathfrak{f}) \cdot \prod_{\mathfrak{p} | \mathfrak{f}} \left(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{p})} \right)$$

is the generalized Euler totient function for number fields.

Proof. We can write the Shintani set $\mathcal{R}^\tau(\mathfrak{f})$ as a disjoint union of the fibers of $\pi_{\alpha,\tau}$,

$$\mathcal{R}^\tau(\mathfrak{f}) = \bigsqcup_{w + \mathfrak{f} \in \mathcal{O}_F/\mathfrak{f}} \pi_{\alpha,\tau}^{-1}(w + \mathfrak{f}). \tag{4.2}$$

Hence by (4.2) and Proposition 4.4 we have

$$\# \mathcal{R}^\tau(\mathfrak{f}) = \sum_{w + \mathfrak{f} \in \mathcal{O}_F/\mathfrak{f}} \# \pi_{\alpha,\tau}^{-1}(w + \mathfrak{f}) = \sum_{w + \mathfrak{f} \in \mathcal{O}_F/\mathfrak{f}} \# \ker(\pi_{\alpha,\tau}) = \# \ker(\pi_{\alpha,\tau}) \cdot N_{F/\mathbb{Q}}(\mathfrak{f}).$$

For the restricted Shintani set $\widetilde{\mathcal{R}^\tau}(\mathfrak{f})$, we simply replace $\mathcal{O}_F/\mathfrak{f}$ with $(\mathcal{O}_F/\mathfrak{f})^\times$ in the preceding argument and observe that it is a classical result that

$$\varphi(\mathfrak{f}) := \#(\mathcal{O}_F/\mathfrak{f})^\times = N_{F/\mathbb{Q}}(\mathfrak{f}) \cdot \prod_{\mathfrak{p}|\mathfrak{f}} \Big(1 - \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{p})}\Big).$$

For example, see Hecke's book [8, §27], in particular the last sentence on p. 88 and Theorem 80 on p. 89. □

We now proceed to determine $\#\ker(\pi_{\alpha,\tau})$ by first expressing it as the number of lattice points in a certain lattice parallelotope in $\mathbb{R}^n$ and then solving the resulting lattice point counting problem by employing techniques from discrete geometry. As we will see, the number $\#\ker(\pi_{\alpha,\tau})$ does not depend on α, but only depends on the totally real number field F, on the permutation $\tau \in S_{n-1}$, and on the choice of independent totally positive units $\epsilon_1, \dots, \epsilon_{n-1}$.

First, we define

$$P_F^\tau = P_F^\tau(\epsilon_1, \dots, \epsilon_{n-1}) := \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^n t_i \iota(f_{\tau,i}),\ t_i \in \mathbb{R},\ t_i \in I_{\tau,i} \right\}.$$

The set P_F^τ is a fundamental parallelotope for the full rank lattice

$$\bigoplus_{i=1}^n \mathbb{Z} \cdot \iota(f_{\tau,i}) \subset \mathbb{R}^n.$$

Observe that the volume of P_F^τ is given by

$$\mathrm{vol}(P_F^\tau) = |\det(A^\tau)|,$$

where A^τ is the matrix defined by

$$A^\tau := (\sigma_i(f_{\tau,j})) \in M_n(\mathbb{R}_{>0}).$$

Lemma 4.6. *We have*

$$\ker(\pi_{\alpha,\tau}) = \mathcal{R}^\tau(\mathfrak{f}) \cap \mathcal{O}_F.$$

Therefore

$$\#\ker(\pi_{\alpha,\tau}) = \#(P_F^\tau \cap \iota(\mathcal{O}_F)).$$

Proof. First recall that $\mathfrak{f} = \alpha\mathcal{O}_F$. Then note that an element $z \in \mathcal{R}^\tau(\mathfrak{f})$ satisfies

$$z \in \ker(\pi_{\alpha,\tau}) \iff \alpha z + \mathfrak{f} = \mathfrak{f} \iff \alpha z \in \mathfrak{f} \iff \alpha z \in \alpha\mathcal{O}_F \iff z \in \mathcal{O}_F.$$

Thus $\ker(\pi_{\alpha,\tau}) = \mathcal{R}^\tau(\mathfrak{f}) \cap \mathcal{O}_F$. The second part of the lemma now follows immediately from the definition of P_F^τ and the identity just proved after taking the embedding $\iota(\mathcal{R}^\tau(\mathfrak{f})) \subset \mathbb{R}^n$. □

In the proof of the following proposition we determine $\#(P_F^\tau \cap \iota(\mathcal{O}_F))$ by using techniques from discrete geometry and thus obtain the final formula for $\#\ker(\pi_{\alpha,\tau})$.

Proposition 4.7. *We have*

$$\#\ker(\pi_{\alpha,\tau}) = \frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}} = \frac{|\det(A^\tau)|}{\sqrt{d_F}},$$

where

$$\mathrm{vol}(P_F^\tau) = \int_{\mathbb{R}^n} \chi_{P_F^\tau}(x)\, dx$$

and $\chi_{P_F^\tau}(x)$ *denotes the characteristic function of* P_F^τ.

Proof. By Lemma 4.6 we have

$$\#\ker(\pi_{\alpha,\tau}) = \#(P_F^\tau \cap \iota(\mathcal{O}_F)).$$

Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation mapping the full rank lattice $\iota(\mathcal{O}_F)$ onto $\mathbb{Z}^n$. Since the vertices of the parallelotope P_F^τ all lie in $\iota(\mathcal{O}_F)$, this implies that the transformed parallelotope $L(P_F^\tau)$ is a *lattice parallelotope* in $\mathbb{Z}^n$, i.e., all the vertices of $L(P_F^\tau)$ lie in $\mathbb{Z}^n$. Moreover $L(P_F^\tau)$ is a fundamental parallelotope for the lattice

$$\Lambda := \bigoplus_{i=1}^{n} \mathbb{Z} \cdot L(\iota(f_{\tau,i})) \subset \mathbb{R}^n.$$

Since the linear transformation L maps interior (resp. boundary) points to interior (resp. boundary) points, we see that

$$\#(P_F^\tau \cap \iota(\mathcal{O}_F)) = \#(L(P_F^\tau) \cap \mathbb{Z}^n).$$

Now, for a positive integer $t \in \mathbb{Z}_{\geq 1}$, we let

$$tL(P_F^\tau) = \{tx \mid x \in L(P_F^\tau)\}$$

be the t-dilation of $L(P_F^\tau)$. More explicitly, we have

$$tL(P_F^\tau) = \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{n} s_i L(\iota(f_{\tau,i})),\ s_i \in \mathbb{R},\ s_i \in tI_{\tau,i} \right\}.$$

Then for each $i = 1, \ldots, n$ we can write

$$tI_{\tau,i} = \bigsqcup_{\substack{m_i \in \mathbb{Z} \\ 0 \leq m_i \leq t-1}} (m_i + I_{\tau,i}),$$

as shown in Figure 3.

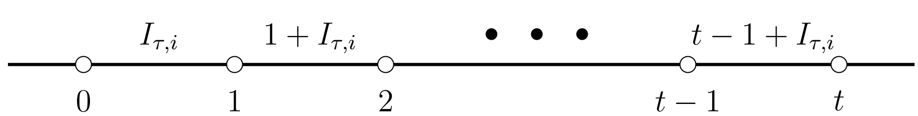


FIGURE 3. The decomposition of $tI_{\tau,i}$ into a disjoint union of integer translates of $I_{\tau,i}$.

It follows that each $s_i \in tI_{\tau,i}$ can be written uniquely as $s_i = m_i + t_i$ for some $m_i \in \mathbb{Z}$ with $0 \leq m_i \leq t-1$ and some $t_i \in I_{\tau,i}$. Hence every element

$$x = \sum_{i=1}^{n} s_i L(\iota(f_{\tau,i}))$$

of the t-dilation $tL(P_F^\tau)$ can be decomposed in a unique way as

$$x = \sum_{i=1}^n s_i L(\iota(f_{\tau,i})) = \sum_{i=1}^n m_i L(\iota(f_{\tau,i})) + \sum_{i=1}^n t_i L(\iota(f_{\tau,i})) \tag{4.3}$$

where

$$\sum_{i=1}^n m_i L(\iota(f_{\tau,i})) \in \bigoplus_{i=1}^n \mathbb{Z} \cdot L(\iota(f_{\tau,i}))$$

and

$$\sum_{i=1}^n t_i L(\iota(f_{\tau,i})) \in L(P_F^\tau).$$

Let

$$\mathcal{B}^\tau = \mathcal{B}^\tau(\epsilon_1, \dots, \epsilon_{n-1}) := \{L(\iota(f_{\tau,i})) \mid i = 1, \dots, n\}.$$

Then $\mathcal{B}^\tau$ is a basis for $\mathbb{R}^n$, so if we denote the coordinates of a vector $x \in \mathbb{R}^n$ with respect to the basis $\mathcal{B}^\tau$ by $[x]_{\mathcal{B}^\tau}$, we find that

$$\begin{aligned} &\left\{ \gamma = \sum_{i=1}^n m_i L(\iota(f_{\tau,i})) \;\middle|\; (m_1, \dots, m_n) \in \mathbb{Z}^n,\ 0 \le m_i \le t-1 \right\} \\ &= \left\{ \gamma \in \bigoplus_{i=1}^n \mathbb{Z} \cdot L(\iota(f_{\tau,i})) \;\middle|\; [\gamma]_{\mathcal{B}^\tau} \in [0, t-1]^n \right\}. \end{aligned} \tag{4.4}$$

Therefore, combining the decomposition (4.3) with (4.4), we have

$$tL(P_F^\tau) = \bigsqcup_{\substack{\gamma \in \bigoplus_{i=1}^n \mathbb{Z} \cdot L(\iota(f_{\tau,i})) \\ [\gamma]_{\mathcal{B}^\tau} \in [0,t-1]^n}} \left(\gamma + L(P_F^\tau)\right), \tag{4.5}$$

where the union is disjoint because $L(P_F^\tau)$ is a fundamental parallelotope for

$$\bigoplus_{i=1}^n \mathbb{Z} \cdot L(\iota(f_{\tau,i}))$$

and each set $\gamma + L(P_F^\tau)$ is just a translate of the fundamental parallelotope $L(P_F^\tau)$ by an element of the lattice.

Since

$$\bigoplus_{i=1}^n \mathbb{Z} \cdot L(\iota(f_{\tau,i})) \subset \mathbb{Z}^n,$$

it follows that for each

$$\gamma \in \bigoplus_{i=1}^n \mathbb{Z} \cdot L(\iota(f_{\tau,i}))$$

we have

$$\#(L(P_F^\tau) \cap \mathbb{Z}^n) = \#((\gamma + L(P_F^\tau)) \cap \mathbb{Z}^n). \tag{4.6}$$

Recalling that $t \in \mathbb{Z}_{\geq 1}$, we find that

$$\#\left\{\gamma \in \bigoplus_{i=1}^{n} \mathbb{Z} \cdot L(\iota(f_{\tau,i})) \,\middle|\, [\gamma]_{\mathcal{B}^\tau} \in [0, t-1]^n\right\} = t^n,$$

and thus by (4.5) and (4.6) we obtain that the number of lattice points in the t-dilation $tL(P_F^\tau)$ is given by

$$\#(tL(P_F^\tau) \cap \mathbb{Z}^n) = t^n \cdot \#(L(P_F^\tau) \cap \mathbb{Z}^n). \tag{4.7}$$

Finally, we consider the Ehrhart polynomial corresponding to the parallelotope $L(P_F^\tau)$,

$$E(L(P_F^\tau), t) := \#(tL(P_F^\tau) \cap \mathbb{Z}^n).$$

As mentioned in the introduction, Ehrhart proved that $E(L(P_F^\tau), t)$ is a polynomial in t of degree n, and moreover, that the leading coefficient is the volume of $L(P_F^\tau)$ (see e.g. [3]),

$$E(L(P_F^\tau), t) = \mathrm{vol}(L(P_F^\tau))t^n + c_{n-1}t^{n-1} + \cdots + c_0.$$

Then after comparing leading coefficients with (4.7), we find that

$$\#(L(P_F^\tau) \cap \mathbb{Z}^n) = \mathrm{vol}(L(P_F^\tau)).$$

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation and $X \subset \mathbb{R}^n$ is a Lebesgue-measurable set, then

$$\mathrm{vol}(T(X)) = |\det T| \mathrm{vol}(X)$$

(see e.g. [6, Theorem 2.44 b]). Applying this result to the linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$, which satisfies $L(\iota(\mathcal{O}_F)) = \mathbb{Z}^n$ and hence is invertible, we get

$$\frac{\mathrm{vol}(L(P_F^\tau))}{\mathrm{vol}(P_F^\tau)} = \frac{\mathrm{vol}(\mathbb{R}^n/L(\iota(\mathcal{O}_F)))}{\mathrm{vol}(\mathbb{R}^n/\iota(\mathcal{O}_F))} = \frac{\mathrm{vol}(\mathbb{R}^n/\mathbb{Z}^n)}{\mathrm{vol}(\mathbb{R}^n/\iota(\mathcal{O}_F))} = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\iota(\mathcal{O}_F))}.$$

Then using that $\mathrm{vol}(\mathbb{R}^n/\iota(\mathcal{O}_F)) = \sqrt{d_F}$ (see e.g. [11, §I.5, Proposition 5.2]), we conclude that

$$\mathrm{vol}(L(P_F^\tau)) = \frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}},$$

which completes the proof. □

Remark 4.8. By Proposition 4.7 we have

$$\frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}} \in \mathbb{Z}^+.$$

By combining Proposition 4.5 and Proposition 4.7, we obtain the following result which immediately implies Theorem 1.9 from the introduction.

Theorem 4.9. *We have*

$$\#\mathcal{R}^\tau(\mathfrak{f}) = \frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}} N_{F/\mathbb{Q}}(\mathfrak{f}) = \frac{|\det(A^\tau)|}{\sqrt{d_F}} N_{F/\mathbb{Q}}(\mathfrak{f})$$

and

$$\#\widetilde{\mathcal{R}^\tau}(\mathfrak{f}) = \frac{\mathrm{vol}(P_F^\tau)}{\sqrt{d_F}} \varphi(\mathfrak{f}) = \frac{|\det(A^\tau)|}{\sqrt{d_F}} \varphi(\mathfrak{f}).$$

We next prove the following orthogonality relations for a narrow ray class character modulo $\mathfrak{f}$, which immediately imply Theorem 1.10 from the introduction.

Theorem 4.10. *Suppose that the generator α of $\mathfrak{f}$ is totally positive, and let χ be a narrow ray class character modulo $\mathfrak{f}$. Then for any $\tau \in S_{n-1}$, we have*

$$\sum_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{f})} \chi(\mathfrak{f}\langle z\rangle) = \begin{cases} 0, & \text{if } \chi \neq 1, \\ \dfrac{|\det(A^\tau)|}{\sqrt{d_F}}\varphi(\mathfrak{f}), & \text{if } \chi = 1. \end{cases}$$

Proof. Write $\mathfrak{f} = \langle\alpha\rangle$ where $\alpha \gg 0$ is totally positive. Since χ is a narrow ray class character modulo $\mathfrak{f}$, on nonzero principal integral ideals it factors as

$$\chi(\langle\beta\rangle) = \chi_{\mathrm{f}}(\beta) N\left(\iota\left(\frac{\beta}{|\beta|}\right)^{\mathbf{p}}\right)$$

where $\mathbf{p} := (p_t) \in \mathbb{Z}^n$ is called an admissible vector, $N(\mathbf{x}) := \prod_t x_t$ for any vector $\mathbf{x} = (x_t) \in \mathbb{R}^n$, $\mathbf{x}^{\mathbf{p}} := (x_t^{p_t})$, and $\chi_{\mathrm{f}} : (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{S}^1$ is a character (see e.g. [11, Chapter 7]). In particular, note that for any nonzero $\beta \in \mathcal{O}_F$ we have

$$N\left(\iota\left(\frac{\beta}{|\beta|}\right)^{\mathbf{p}}\right) = N\left(\left(\sigma_t\left(\frac{\beta}{|\beta|}\right)^{p_t}\right)\right) = \prod_t \operatorname{sign}(\sigma_t(\beta))^{p_t}.$$

Moreover, since $z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{f})$, then the conditions defining $\widetilde{\mathcal{R}^\tau}(\mathfrak{f})$ imply that $\alpha z + \mathfrak{f} \in (\mathcal{O}_F/\mathfrak{f})^\times$ and that $\alpha z \gg 0$. Therefore, for any $z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{f})$ we have

$$\chi(\mathfrak{f}\langle z\rangle) = \chi(\langle\alpha z\rangle) = \chi_{\mathrm{f}}(\alpha z)\prod_t \operatorname{sign}(\sigma_t(\alpha z))^{p_t} = \chi_{\mathrm{f}}(\alpha z),$$

where we used that $\operatorname{sign}(\sigma_t(\alpha z)) = 1$ because $\alpha z \gg 0$. Hence, by Proposition 4.4, Proposition 4.5, Proposition 4.7, and the previous calculation, we get

$$\begin{aligned} \sum_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{f})} \chi(\mathfrak{f}\langle z\rangle) &= \sum_{w \in (\mathcal{O}_F/\mathfrak{f})^\times} \sum_{z \in \pi_{\alpha,\tau}^{-1}(w)} \chi_{\mathrm{f}}(\alpha z) \\ &= \sum_{w \in (\mathcal{O}_F/\mathfrak{f})^\times} \#\pi_{\alpha,\tau}^{-1}(w) \cdot \chi_{\mathrm{f}}(w) \\ &= \#\ker(\pi_{\alpha,\tau}) \sum_{w \in (\mathcal{O}_F/\mathfrak{f})^\times} \chi_{\mathrm{f}}(w) \\ &= \begin{cases} 0, & \text{if } \chi \neq 1, \\ \dfrac{|\det(A^\tau)|}{\sqrt{d_F}}\varphi(\mathfrak{f}), & \text{if } \chi = 1, \end{cases} \end{aligned}$$

where the last equality follows from the orthogonality relations for characters of the group $(\mathcal{O}_F/\mathfrak{f})^\times$. □

5. An algorithm to compute Shintani sets

In this section we give an algorithm to compute Shintani sets.

In order to compute the Shintani set

$$\mathcal{R}^\tau(\mathfrak{f}) = \mathcal{R}^\tau(\mathfrak{f}; \epsilon_1, \ldots, \epsilon_{n-1}) := \left\{ z \in \mathfrak{f}^{-1} \;\middle|\; \mathbf{t}_{z,\tau} \in I_{\tau,1} \times \cdots \times I_{\tau,n} \right\},$$

we must find those $z \in \mathfrak{f}^{-1}$ whose coordinate vector $\mathbf{t}_{z,\tau} = [z]_{\mathcal{B}_f} \in \mathbb{Q}^n$ with respect to the $\mathbb{Q}$-basis $\{f_{\tau,i}\}_{i=1}^n$ for F lies in the half-open hypercube $I_{\tau,1} \times \cdots \times I_{\tau,n}$. We now explain how to translate this problem to the problem of solving an $n \times n$ system of linear inequalities. This will then be summarized as Algorithm 2 below.

First note that $\mathfrak{f}^{-1}$ is a free $\mathbb{Z}$-submodule of F of rank n. Let $\mathcal{B}_\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a $\mathbb{Z}$-basis for $\mathfrak{f}^{-1}$. This $\mathbb{Z}$-basis is also a $\mathbb{Q}$-basis for F. Accordingly, let

$$C = (C_1, \ldots, C_n) \in \mathrm{GL}_n(\mathbb{Q})$$

be the change of basis matrix which changes coordinates from the basis $\mathcal{B}_\alpha$ to the basis $\mathcal{B}_f$. Explicitly, the columns $C_1, \ldots, C_n$ of C are given by

$$C_i = [\alpha_i]_{\mathcal{B}_f} = (c_{1i}, \ldots, c_{ni})^T \in \mathbb{Q}^n$$

for $i = 1, \ldots, n$. Then the equality $[z]_{\mathcal{B}_f} = C[z]_{\mathcal{B}_\alpha}$ holds for any $z \in F$.

Now, write $z \in \mathfrak{f}^{-1}$ as a linear combination of the form

$$z = \sum_{i=1}^n m_i \alpha_i$$

for some unique integers $m_1, \ldots, m_n \in \mathbb{Z}$. Then to find all the elements $z \in \mathfrak{f}^{-1}$ which satisfy $[z]_{\mathcal{B}_f} \in I_{\tau,1} \times \cdots \times I_{\tau,n}$, it is equivalent to find all vectors $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ which satisfy

$$C\mathbf{m}^T \in I_{\tau,1} \times \cdots \times I_{\tau,n},$$

i.e., the set of all integral solutions $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ to the system of inequalities

$$\begin{aligned} c_{11}m_1 + \cdots + c_{1n}m_n &\in I_{\tau,1} \\ c_{21}m_1 + \cdots + c_{2n}m_n &\in I_{\tau,2} \\ &\vdots \\ c_{n1}m_1 + \cdots + c_{nn}m_n &\in I_{\tau,n}. \end{aligned}$$

This system of inequalities has a finite number of solutions, and each solution $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ corresponds to a unique point

$$z_{\mathbf{m}} = \sum_{i=1}^n m_i \alpha_i = \sum_{j=1}^n \left(\sum_{k=1}^n c_{jk} m_k \right) f_{\tau,j} = \sum_{i=1}^n t_{z,\tau,i} f_{\tau,i}$$

in the Shintani set $\mathcal{R}^\tau(\mathfrak{f})$.

We summarize this discussion in the following algorithm.

Algorithm 2 Computing Shintani sets for totally real number fields

INPUT: A quadruple $(F, \mathfrak{f}, \tau, \{\epsilon_1, \dots, \epsilon_{n-1}\})$ consisting of a totally real number field F of degree n, a nonzero integral ideal $\mathfrak{f} \subset \mathcal{O}_F$, a permutation $\tau \in S_{n-1}$ such that $w_\tau \neq 0$, and a system of generators $\{\epsilon_1, \dots, \epsilon_{n-1}\}$ of $\mathcal{O}_F^{\times,+}$.
OUTPUT: The Shintani set $\mathcal{R}^\tau(\mathfrak{f})$ (resp. the restricted Shintani set $\widetilde{\mathcal{R}^\tau}(\mathfrak{f})$).

1: Compute the basis $\mathcal{B}_f = \{f_{\tau,1}, \dots, f_{\tau,n}\}$.
2: Compute the weight w_τ.
3: Determine the half open intervals $I_{\tau,i} = [0,1)$ or $(0,1]$ for $i = 1, \dots, n$.
4: Compute a $\mathbb{Z}$-basis $\{\alpha_1, \dots, \alpha_n\}$ for the fractional ideal $\mathfrak{f}^{-1}$.
5: Compute the change of basis matrix $C = (C_1, \dots, C_n) \in \mathrm{GL}_n(\mathbb{Q})$ with columns given by the coordinate vectors

$$C_i = [\alpha_i]_{\mathcal{B}_f} = (c_{1i}, \dots, c_{ni})^T \in \mathbb{Q}^n.$$

6: Find all integral solutions $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ to the system of inequalities

$$\begin{aligned} c_{11}m_1 + \cdots + c_{1n}m_n &\in I_{\tau,1} \\ c_{21}m_1 + \cdots + c_{2n}m_n &\in I_{\tau,2} \\ &\vdots \\ c_{n1}m_1 + \cdots + c_{nn}m_n &\in I_{\tau,n}. \end{aligned}$$

7: For each solution $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ found in Step 6, compute the algebraic number

$$z_{\mathbf{m}} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij} m_j \right) f_{\tau,i} \in \mathfrak{f}^{-1}.$$

The set of all such numbers $z_{\mathbf{m}}$ is the Shintani set $\mathcal{R}^\tau(\mathfrak{f})$.
8: Discard every number $z_{\mathbf{m}}$ such that $\mathfrak{f}\langle z_{\mathbf{m}} \rangle$ is not coprime to $\mathfrak{f}$. The remaining set of numbers is the restricted Shintani set $\widetilde{\mathcal{R}^\tau}(\mathfrak{f})$.

6. Proof of Theorem 1.5

In this section we prove Theorem 1.5. We refer the reader to the introduction for background and notation.

Given a matrix $A = (a_{ij}) \in M_n(\mathbb{R}_{>0})$ and a non-zero vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0}$, the n-dimensional *Shintani zeta function* is defined by

$$\zeta(s, A, x) := \sum_{m_1, \dots, m_n = 0}^{\infty} \prod_{i=1}^{n} \left\{ \sum_{j=1}^{n} a_{ij}(m_j + x_j) \right\}^{-s}, \quad \mathrm{Re}(s) > 1.$$

The Shintani zeta function has a meromorphic continuation to $\mathbb{C}$ with at most simple poles at $s = 1 - l/n$ for $l \in \mathbb{Z}_{\geq 0}$ and no poles at $s = -k$ for $k \in \mathbb{Z}_{\geq 0}$ (see e.g. Proposition 2.1 of [7]).

Now, by [5, Corollary 3] we have

$$L(\chi_{K/F}, s) = N(\mathfrak{D}_{K/F})^{-s} \sum_{\substack{\tau \in S_{n-1} \\ w_\tau \neq 0}} w_\tau \sum_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z \rangle) \zeta(s, A^\tau, \mathbf{t}_{z,\tau}). \tag{6.1}$$

Since K and F have signatures $(2, n-1)$ and $(n, 0)$, respectively, the Dedekind zeta functions $\zeta_K(s)$ and $\zeta_F(s)$ have zeros of order n and $n-1$ at $s = 0$, respectively. Hence the factorization $\zeta_K(s) = L(\chi_{K/F}, s)\zeta_F(s)$ implies that $L(\chi_{K/F}, s)$ has a simple zero at $s = 0$. A calculation using (6.1) now gives

$$L'(\chi_{K/F}, 0) = \sum_{\substack{\tau \in S_{n-1} \\ w_\tau \neq 0}} w_\tau \sum_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z \rangle)\zeta'(0, A^\tau, \mathbf{t}_{z,\tau}). \tag{6.2}$$

On the other hand, by [13, Proposition 1] we have

$$\zeta'(0, A^\tau, \mathbf{t}_{z,\tau}) = \sum_{i=1}^n \log\left(\frac{\Gamma_n(\langle \mathbf{t}_{z,\tau}, A_i^\tau \rangle, A_i^\tau)}{\rho_n(A_i^\tau)}\right) + \frac{(-1)^n}{n} \sum_{\substack{\mathbf{h}=(h_1,h_2,\ldots,h_n) \in \mathbb{Z}_{\geq 0}^n \\ \sum_{i=1}^n h_i = n}} C_{\mathbf{h}}(A^\tau) \prod_{i=1}^n \frac{B_{h_i}(t_{z,\tau,i})}{h_i!}. \tag{6.3}$$

We evaluate the left hand side of (6.2) using (1.6). We evaluate the right hand side of (6.2) using (6.3) and apply the orthogonality relations in Theorem 4.10 to cancel the residue $\rho_n(A_i^\tau)^{-1}$. After exponentiating, we obtain the following identity for the Stark unit

$$\varepsilon_{K/F,S} = \exp\left(C(K/F)\right) \cdot \mathbf{\Gamma}_{K/F,n},$$

where the constant $C(K/F)$ is defined by

$$C(K/F) := \frac{(-1)^n}{n} \sum_{\tau \in S_{n-1}} \sum_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} c_{K/F,\tau}(z) \sum_{\substack{\mathbf{h}=(h_1,\ldots,h_n) \in \mathbb{Z}_{\geq 0}^n \\ \sum_{i=1}^n h_i = n}} C_{\mathbf{h}}(A^\tau) \prod_{i=1}^n \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} \tag{6.4}$$

with

$$c_{K/F,\tau}(z) := \frac{w_\tau \chi_{K/F}(\mathfrak{D}_{K/F}\langle z \rangle) h(F)}{2^{n-v-2}h(K)},$$

and where the product of special Gamma values is defined by

$$\mathbf{\Gamma}_{K/F,n} := \prod_{\substack{\tau \in S_{n-1} \\ w_\tau \neq 0}} \prod_{z \in \widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})} \prod_{i=1}^n \Gamma_n(\langle \mathbf{t}_{z,\tau}, A_i^\tau \rangle, A_i^\tau)^{c_{K/F,\tau}(z)}.$$

We next show that *only* the boundary points $\partial\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})$ contribute to the sum over z in (6.4). To do this we will show that for any vector

$$\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_{\geq 0}^n$$

such that $\sum_{i=1}^n h_i = n$, we have

$$\sum_{z \in \mathrm{int}(\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}))} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z \rangle) \prod_{i=1}^n \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} = 0.$$

Since (K, F) is a pair of number fields satisfying Condition 1.2 and F has degree n, we have

$$\chi_{K/F}(\langle -z \rangle) = (-1)^{n-1}\chi_{K/F}(\langle z \rangle). \tag{6.5}$$

Also, since $B_k(1-x) = (-1)^k B_k(x)$ we have

$$\prod_{i=1}^{n} \frac{B_{h_i}(1-t_{z,\tau,i})}{h_i!} = (-1)^{\sum_{i=1}^{n} h_i} \prod_{i=1}^{n} \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} = (-1)^n \prod_{i=1}^{n} \frac{B_{h_i}(1-t_{z,\tau,i})}{h_i!}. \tag{6.6}$$

Next, observe that

$$z \in \mathrm{int}(\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})) \quad \Longleftrightarrow \quad 1-z \in \mathrm{int}(\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F})), \tag{6.7}$$

since

$$z = \sum_{i=1}^{n} t_{z,\tau,i} f_{\tau,i}, \quad 1-z = \sum_{i=1}^{n} (1-t_{z,\tau,i}) f_{\tau,i},$$

and

$$t_{z,\tau,i} \notin \{0,1\} \text{ for all } i = 1,\dots,n \quad \Longleftrightarrow \quad 1-t_{z,\tau,i} \notin \{0,1\} \text{ for all } i = 1,\dots,n.$$

Then we get

$$\begin{aligned}
&\sum_{z \in \mathrm{int}(\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}))} \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) \prod_{i=1}^{n} \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} \\
&= \frac{1}{2} \sum_{z \in \mathrm{int}(\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}))} \left\{ \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) \prod_{i=1}^{n} \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} + \chi_{K/F}(\mathfrak{D}_{K/F}\langle 1-z\rangle) \prod_{i=1}^{n} \frac{B_{h_i}(1-t_{z,\tau,i})}{h_i!} \right\} \\
&= \frac{1}{2} \sum_{z \in \mathrm{int}(\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}))} \left\{ \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) \prod_{i=1}^{n} \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} - \chi_{K/F}(\mathfrak{D}_{K/F}\langle z\rangle) \prod_{i=1}^{n} \frac{B_{h_i}(t_{z,\tau,i})}{h_i!} \right\} \\
&= 0
\end{aligned}$$

where the first equality follows from (6.7) and the second equality follows from (6.5), (6.6), and the fact that $\chi_{K/F}$ has conductor $\mathfrak{D}_{K/F}$.

Finally, if $\partial\widetilde{\mathcal{R}^\tau}(\mathfrak{D}_{K/F}) = \varnothing$ for all $\tau \in S_{n-1}$, then by definition the constant $C(K/F) = 0$. Hence the second statement of the theorem follows immediately from (1.15) and (1.8). □

7. Explicit evaluation of the constant $C_{\mathbf{h}}(A^\tau)$

Shintani [13, Remark on p. 206] observed that under certain conditions on the matrix A, the constant $C_{\mathbf{h}}(A)$ can be computed in closed form.

Lemma 7.1. *Let $A = (a_{ij}) \in M_n(\mathbb{R}_{>0})$ and $\mathbf{h} = (h_1,\dots,h_n) \in \mathbb{Z}_{\geq 0}^n$ be a vector such that*

$$\sum_{i=1}^{n} h_i = n.$$

Define the sets

$$P_1(\mathbf{h}) := \{1 \leq i \leq n \mid h_i \geq 1\}$$
$$P_2(\mathbf{h}) := \{1 \leq i \leq n \mid h_i = 0\}.$$

- *If $|P_2(\mathbf{h})| = 0$, then*

$$C_{\mathbf{h}}(A) = 0.$$

- *If* $|P_2(\mathbf{h})| = 1$ *and* $p \in P_2(\mathbf{h})$*, then*

$$C_{\mathbf{h}}(A) = \sum_{1 \le j < \ell \le n} \frac{1}{a_{jp} a_{\ell p}} \prod_{i \in P_1(\mathbf{h})} (a_{ji} a_{\ell p} - a_{jp} a_{\ell i})^{h_i - 1} \log \left(\frac{a_{jp}}{a_{\ell p}} \right).$$

- *If* $|P_2(\mathbf{h})| \ge 2$ *and for all* $p, q \in P_2(\mathbf{h})$ *with* $p \neq q$ *we have*

$$\prod_{1 \le j < \ell \le n} (a_{\ell p} a_{jq} - a_{\ell q} a_{jp}) \neq 0, \tag{7.1}$$

then

$$C_{\mathbf{h}}(A) = \sum_{p \in P_2(\mathbf{h})} \sum_{1 \le j < \ell \le n} \frac{1}{a_{jp} a_{\ell p}} \frac{\prod_{i \in P_1(\mathbf{h})} (a_{ji} a_{\ell p} - a_{jp} a_{\ell i})^{h_i - 1}}{\prod_{q \in P_2(\mathbf{h}), p \neq q} (a_{jq} a_{\ell p} - a_{jp} a_{\ell q})} \log \left(\frac{a_{jp}}{a_{\ell p}} \right).$$

Here we will use Lemma 7.1 to prove that if $F/\mathbb{Q}$ has prime degree, then $C_{\mathbf{h}}(A^\tau)$ can be evaluated in closed form.

Proposition 7.2. *Let* $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_{\ge 0}^n$*. If* F *has prime degree* $[F : \mathbb{Q}] = n$*, then the constant* $C_{\mathbf{h}}(A^\tau)$ *is given as follows.*

- *If* $|P_2(\mathbf{h})| = 0$*, then*

$$C_{\mathbf{h}}(A^\tau) = 0.$$

- *If* $|P_2(\mathbf{h})| = 1$ *and* $p \in P_2(\mathbf{h})$*, then*

$$C_{\mathbf{h}}(A) = \sum_{1 \le j < \ell \le n} \frac{1}{\sigma_j(f_{\tau,p}) \sigma_\ell(f_{\tau,p})} \prod_{i \in P_1(\mathbf{h})} (\sigma_j(f_{\tau,i}) \sigma_\ell(f_{\tau,p}) - \sigma_j(f_{\tau,p}) \sigma_\ell(f_{\tau,i}))^{h_i - 1} \log \left(\frac{\sigma_j(f_{\tau,p})}{\sigma_\ell(f_{\tau,p})} \right).$$

- *If* $|P_2(\mathbf{h})| \ge 2$*, then*

$$C_{\mathbf{h}}(A) =$$
$$\sum_{p \in P_2(\mathbf{h})} \sum_{1 \le j < \ell \le n} \frac{1}{\sigma_j(f_{\tau,p}) \sigma_\ell(f_{\tau,p})} \frac{\prod_{i \in P_1(\mathbf{h})} (\sigma_j(f_{\tau,i}) \sigma_\ell(f_{\tau,p}) - \sigma_j(f_{\tau,p}) \sigma_\ell(f_{\tau,i}))^{h_i - 1}}{\prod_{q \in P_2(\mathbf{h}), p \neq q} (\sigma_j(f_{\tau,q}) \sigma_\ell(f_{\tau,p}) - \sigma_j(f_{\tau,p}) \sigma_\ell(f_{\tau,q}))} \log \left(\frac{\sigma_j(f_{\tau,p})}{\sigma_\ell(f_{\tau,p})} \right).$$

Proposition 7.2 is an immediate consequence of Lemma 7.1 and the following result.

Lemma 7.3. *Let* $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{Z}_{\ge 0}^n$ *be any vector such that* $\sum_{i=1}^n h_i = n$ *and* $|P_2(\mathbf{h})| \ge 2$*. If* F *has prime degree* $[F : \mathbb{Q}] = n$*, then the matrix* $A^\tau = (\sigma_i(f_{\tau,j})) \in M_n(\mathbb{R}_{>0})$ *satisfies condition* (7.1) *of Lemma 7.1 for each* $\tau \in S_{n-1}$*.*

Proof. To verify the condition (7.1) for A^τ, we must show that for all $p, q \in P_2(\mathbf{h})$ with $p \neq q$, we have

$$\prod_{1 \le j < \ell \le n} (\sigma_\ell(f_{\tau,p}) \sigma_j(f_{\tau,q}) - \sigma_\ell(f_{\tau,q}) \sigma_j(f_{\tau,p})) \neq 0 \tag{7.2}$$

where

$$f_{\tau,i} := \epsilon_{\tau(1)} \epsilon_{\tau(2)} \cdots \epsilon_{\tau(i-1)} = \prod_{j=1}^{i-1} \epsilon_{\tau(j)} \in \mathcal{O}_F^{\times,+}.$$

Without loss of generality, we may suppose that $p < q$. Then (7.2) is equivalent to

$$\prod_{1\le j<\ell\le n} (\sigma_\ell(f_{\tau,p})\sigma_j(f_{\tau,q}) - \sigma_\ell(f_{\tau,q})\sigma_j(f_{\tau,p}))$$
$$= \prod_{1\le j<\ell\le n} \sigma_\ell(f_{\tau,p})\sigma_j(f_{\tau,p}) \left(\sigma_j(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)}) - \sigma_\ell(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)})\right) \neq 0.$$

Since $\sigma_\ell(f_{\tau,p})\sigma_j(f_{\tau,p}) \neq 0$ for each j,ℓ, it suffices to show that

$$\prod_{1\le j<\ell\le n} \left(\sigma_j(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)}) - \sigma_\ell(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)})\right) \neq 0.$$

However, if

$$\sigma_j(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)}) - \sigma_\ell(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)}) = 0$$

for some $j < \ell$, then σ_j, σ_ℓ are distinct embeddings of F which are equal after restriction to the subfield $\mathbb{Q}(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)})$. It follows that $\mathbb{Q}(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)})$ is a *proper* subfield of F. However, since $F/\mathbb{Q}$ has prime degree, we must have $\mathbb{Q}(\epsilon_{\tau(p)}\dots\epsilon_{\tau(q-1)}) = \mathbb{Q}$, a contradiction. □

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