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SCATTERING OF E POLARIZED WHISPERING-GALLERY MODE FROM CONCAVE BOUNDARY

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Abstract

In this work we present numerical results for the 2D problem of scattering E polarised whispering-gallery mode from concave convex perfectly conducting boundary. The results were obtained by applying the developed method of currants integral equations (CIE) [6,7] for high frequency domain when the size of the scatterer match is greater than the wave length. We have applied the described procedure in order to find numerical solutions of scattering whispering-gallery mode by concave finite convex boundary as a part of a circular cylinder or part of parabolic cylinder. As incident wave we have considered cylindrical waves from line source and Gauss beam [6] with different effective width. It is shown that we have a complicated process of focusing and oscillating of the beam's reflected field, both cylindrical and Gauss beam incident fields. The distortions of reflected field depend on shape of the boundary and parameters of the incident fields.

Keywords: Scattering problem, E polarized whispering-gallery mode, concave boundary, integral equation of the first kind, numerical solution.

Resumen

En este trabajo presentamos resultados numéricos para el problema 2D de modo de galería susurrante E polarizada, de una frontera perfectamente conducente cóncava y convexa. Los resultados fueron obtenidos aplicando el método desarrollado de ecuaciones integrales pasas [6,7] para dominios de alta frecuencia cuando el tamaño de la pareja dispersora es mayor que la longitud de onda. Hemos aplicado el procedimiento descrito con tal de encontrar soluciones numéricas de modo galería susurrante de dispersión por frontera convexa finita cóncava, como parte de un cilindro circular

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o parte de un cilindro parabólico. Como onda de incidencia hemos considerado ondas cilíndricas de fuente de línea y rayo de Gauss [6] con diferentes anchos efectivos. Se muestra que tenemos un proceso complicado de enfoque y oscilación del campo reflajado del rayo, tanto en el campo de incidencia cilíndrico como con el rayo de Gauss. Las distorsiones del campo reflejado depende en la forma de la frontera y los parámetros de los campos de incidencia.

Palabras clave: Problema de dispersión, modo de galería susurrante E polarizada, frontera cóncava, ecuación integral de primer tipo, solución numérica.

Mathematics Subject Classification: 34L25, 74S15, 65R20.

1 Introduction

The problem of scattering whispering-gallery mode from concave convex boundary is known to be under wide scientific discussion within middle part of 20^{th}. It is important to underline that all achieved results were obtained by asymptotic methods only: the method of geometrical theory of diffraction (GTD), its uniform or local modifications, the method of physical optics, the method of parabolic equation or Kirhhoff approximation [1-5] and deal with describing of the field nearby scatterer's surface. All these methods have restrictions and all attempts to improve the results run against huge difficulties. Actually, it is impossible.

In this work we present a strict numerical solution for 2D problem of scattering whispering-gallery mode from concave smooth boundary with finite size. The results were obtained by applying developed method of prolonged boundary conditions for currants integral equations (CIE) [6,7] in high frequency domain ($kD \gg 1, k = 2\pi/\lambda$–wave number,$\lambda$–wave length, D–maximum size of the scatterer surface).

2 Scattering of E polarized wave from concave finite cylindrical structure

Let at first consider the scattering problem for E polarized incident cylindrical wave:

$$U_0(\overrightarrow{r}) = H_0^{(2)}(k \mid \overrightarrow{r} - \overrightarrow{r}_0 \mid) \tag{1}$$

by perfectly conducting finite cylindrical surface S with cross-section function $\rho_0 = \rho(\varphi)$, $\varphi \in [\varphi_B, \varphi_E]$ in cylindrical coordinate system $\{r, \varphi\}$. In (1), point $\overrightarrow{r}_0 = \{R_0, \varphi_0\}$ indicates a position of the source.

The scattering field $U_1(\overrightarrow{r})$ has to satisfy a wave equation outside of S, boundary condition on S :

$$U_0(\overrightarrow{r}) \mid_S + U_1(\overrightarrow{r}) \mid_S = 0 \tag{2}$$

and Sommerfield condition when $\mid \overrightarrow{r} \mid \to \infty$:

$$\frac{\partial U_i(\overrightarrow{r})}{\partial r} = ikU_i(\overrightarrow{r}) = o(\mid \overrightarrow{r} \mid^{-1/2}), \quad \mid \overrightarrow{r} \mid \to \infty.$$

It is known that in this case the scattering problem could be reduced to Dirichlet value boundary problem and following integral equation of the first kind with singular kernel could be obtained:

$$U_0(\overrightarrow{r})_S = \int_S \mu(\overrightarrow{r}_S H_0^{(2)}(k \mid \overrightarrow{r}_S - \overrightarrow{r}_\sigma \mid) d\sigma, \tag{3}$$

In (3) the unknown function $\mu(\overrightarrow{r})$ is a current on surface $\overrightarrow{r}_S \in S$, $\overrightarrow{r_\sigma} \in S$, $H_0^{(2)}(k \mid \overrightarrow{r} - \overrightarrow{r}_\Sigma \mid)$, fundamental solution to Helmholts equation,

$$\mid r - \overrightarrow{r}_S \mid = \sqrt{r^2 + r_S^2 - 2rr_S \cos(\varphi - \varphi_S)}$$

is a distance between points $\overrightarrow{r} = \{r, \varphi\}$ and $\overrightarrow{r}_s = \{r_S, \varphi_S)$ in a cylindrical coordinate system $\{r, \varphi\}$.

An ordinary way to make a numerical solution of integral equation (3) is to extract a singularity in the kernel and use a piece-wise system of basis functions [4] or some other full system (for example, any kind of orthogonal polynoms, splines or Fourier sets). But it is impossible to obtain stable and accurate results for high frequency domain ($kD \gg 1$). That is why we have made another approach, based on two ideas. First one, is that any numerical solution makes with finite accuracy only. Second one, we have used analytical properties for presentation of scattering field in simple potential layer form [7]. It allows us to make displacement for points r_S from S into Σ surface ($\Sigma = S + i\widetilde{\Delta}$, $\widetilde{\Delta} \ll 1$, where S is the original surface, and $\widetilde{\Delta}$ is a value of displacement from original surface). In other words, we have a "small" displacement δ or continuation into complex region for surface's points $\overrightarrow{r}_i$ and condition $\delta \ll 1$ guarantees us that we do not cross any singularities. So, as a result we have an integral equation of the first kind with smooth kernel and Haar wavelet functions (as a system of basis functions) could be effective applied for its stable numerical solution in the region $kD \gg 1$.

It was mentioned above that one of the main problems deals with applying the method CIE for numerical solution of the scattering problems connected with applying the most effective basis function. To solve this problem we have used a wavelet technique [8]. So using the wavelet technique one can obtain the following presentation for current $\mu(\theta)$ in form of a set:

$$\mu(\theta) = c_0\phi_0(\theta) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{jk}\Psi_{jk}(\theta) \tag{4}$$

where

$$\begin{aligned}
\phi_0(\theta) &= \begin{cases} 1, & 0 \le \theta < 2\pi \\ 0, & \theta \notin [0, 2\pi[, \end{cases} \\
\Psi_{jk}(\theta) &= 2^{j/2}\Psi_H(2^j\theta - 2k\pi), \\
\Psi_H(t) &= \begin{cases} 1, & 0 \le t < \pi, \\ -1, & \pi \le t < 2\pi, \\ 0, & t \notin [0, 2\pi[. \end{cases}
\end{aligned}$$

$\Psi_H(t)$ is a Haar function [8] and d_0, d_{jk} are to be found. We would like to emphasize that Haar function are forming an orthogonal and unconditional basis on $0 \leq \theta < 2\pi$ interval and any function from $L^2(\Re)$ could be presented as a set of Haar functions.

Placing (4) into (3) will we get the following expression for (3):

$$c_0 L\phi_0 + \sum_{j=0}^{p} \sum_{k-0}^{2^j-1} d_{jk} L\Psi_{jk} \approx \Psi(\alpha) \tag{5}$$

where is an integral operator in (3), $\Psi(\alpha) = U_0(\overrightarrow{r}_S)$. Then making a procedure of description with functions in form of the delta functions: $\xi_m = \delta(\alpha - \alpha_m)$ (where α_m are the points from the interval $[0, 2\pi]$ on the contour S, $m = 1, \ldots, M$) one could get a system of linear equations. This system could be written in matrix form as

$$[A_{mn}][d_n] = [b_m] \tag{6}$$

where

$$A_{mn} = \langle \xi_m, L\Psi_{ik} \rangle, \quad m = 1, \ldots, M1; \quad b_m = \langle \xi_m, \Psi \rangle. \tag{7}$$

$$j = 0, \ldots, p; \quad k = 0, \ldots, 2^j - 1; \quad n = p + 2^j.$$

If the $\mu(\overrightarrow{r})$ is known, then the scattering pattern can be calculated as follows

$$g(\varphi) = \int_S \exp[ik\rho_0(\theta)\cos(\theta - \varphi)]\mu(\theta)d\theta \tag{8}$$

We estimate the error in the solution of the problem as the residual Δ of the boundary condition on S'.

3 Numerical results

We have applied the described procedure for numerical solution of scattering whispering-gallery mode by concave finite convex boundary as a part of circular cylinder with cross section function $\rho(\theta)$:

$$\rho(\theta) = a, \varphi \in [\varphi_B, \varphi_E], \tag{9}$$

or part of parabolic cylinder with cross section function (parabola):

$$\rho(\theta) = 2p/[1 + \cos(\varphi)])\varphi \in [\varphi_B, \varphi_E] \tag{10}$$

As incident wave we have considered a cylindrical wave (1) and a Gauss beam [6]

$$U_0(x, y) = \exp\{-ikx - k^2(y - Y_0)^2/k^2\sigma^2\}, \tag{11}$$

where σ determines an "effective width" of the beam and Y_0 a position of its center.

The relative amplitude of scattering pattern $g(\varphi)(g(\varphi) \equiv| g(\varphi)/\max\{g(\varphi)\} |$ for Gauss beam (11) with parameters: $k\sigma = 1$, $kY_0 = -98$, or $k\sigma = 5$, $kY_0 = -95$; the surface as a part of cylinder (9) with parameters: $ka = 120$, $\varphi_{B,E} = \pm\pi/2$ are illustrated by Figures

1 and 2. It is seen that the process of beam's interaction with finite reflector (9) is accompanied by a complicate oscillating of scattering field's amplitude.

The relative amplitude of scattering pattern $g(\varphi)$ for cylindrical E polarized cylindrical wave (1) and cylindrical surface (8) presents at Figures 3 and 4. Parameters of surface (9) were $KA = 80$; $\varphi_{B,E} = \pm\pi/2$ and coordinates of the source were $kR_0 = 78$; $\varphi_0 = -\pi/2$.

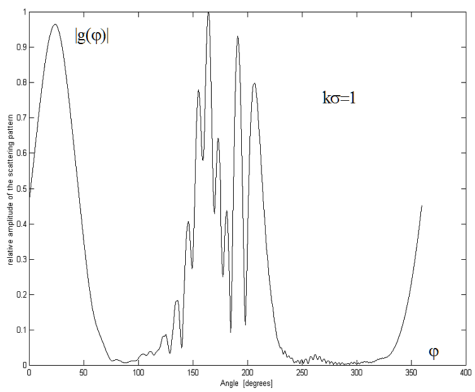


Figura 1.

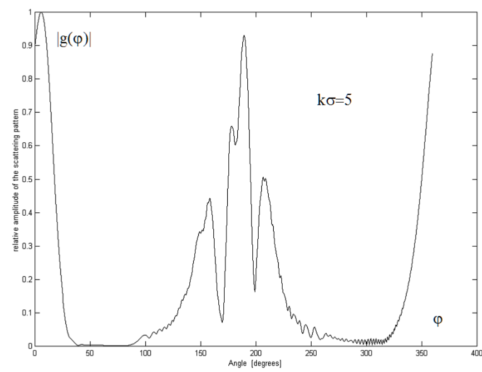


Figura 2.

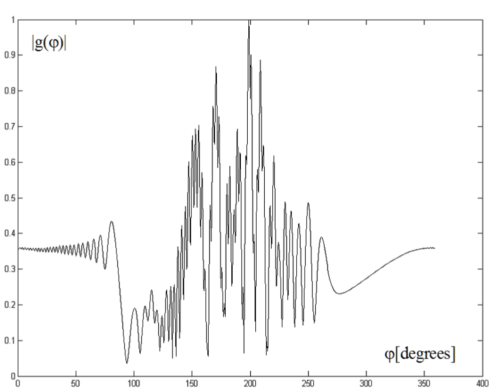
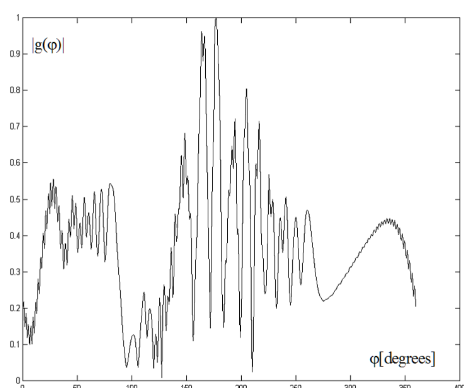


Figura 3.



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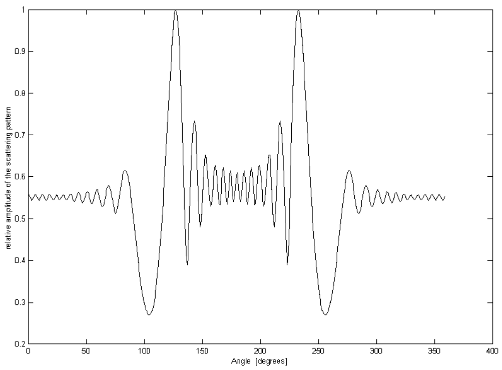

Figura 5.

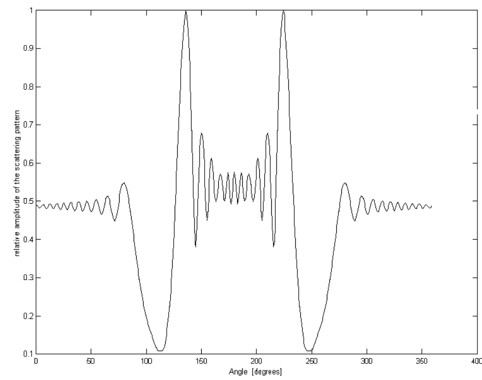

Figura 6.

The relative amplitude of scattering pattern $g(\varphi)$ for cylindrical E polarized cylindrical wave (1) and cylindrical surface (8) presents at Figures 5 and 6. Parameters of surface (9) were $ka = 120$, $\varphi_{B,E} = \pm\pi/2.$, and coordinates of the source were $kR_0 = 115, \varphi_0 = 0$. These figures show that structure of the scattering pattern depends strongly on source's location with respect to scatterer surface.

The relative amplitude of scattering pattern is a parabolic reflector (10) (see Figure 7) depicted in Figures 8–10. Parameters of parabola were: $kp = 60$, $\varphi_{B,E} = \pm\pi/3$, parameters of the source: $kR_0 = 58$, $kR_0 = 54$, $kR_0 = 30$, $\varphi_0 = 0$ (a case of symmetrical location of the sourcer) for Figures 8–10 accordingly.

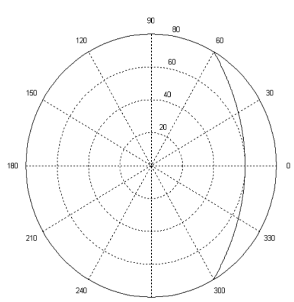

Figura 7.

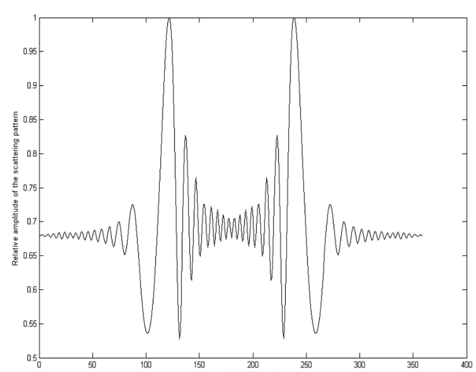

Figura 8.

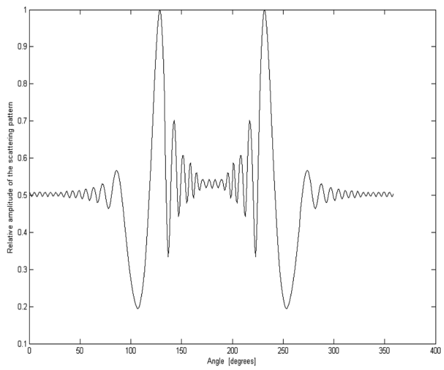

Figura 9.

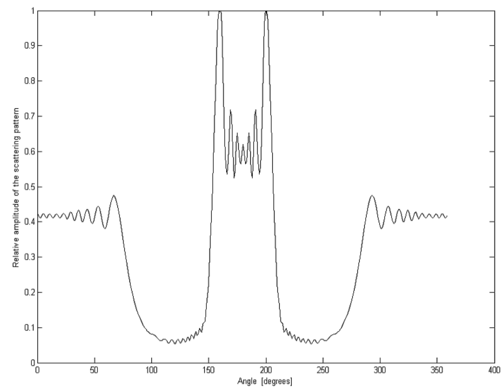

Figura 10.

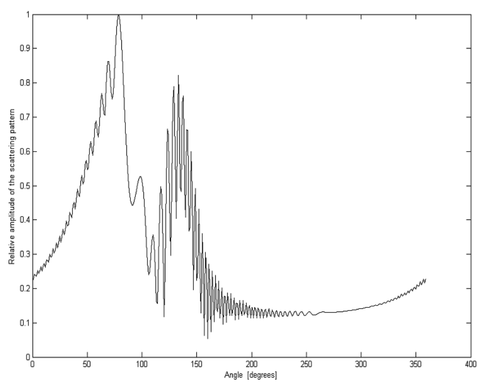

Figura 11.

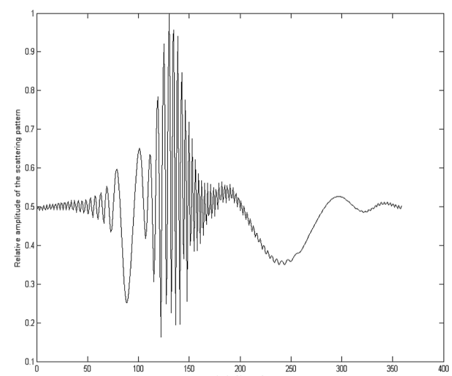

Figura 12.

The case of “nonsymmetrical” location of the source with regard to the scattererb when its parameter were : $kR_0 = 80$, $\varphi_0 = -55^\circ$ or $kR_0 = 74.26$, $\varphi_0 = -55^\circ$ presented at Figures 11 and 12 accordingly.

It is seen that structure of the scattering pattern in this case is closed to a case of cylindrical surface and difference deal with curvature’s changing of the scatterer. It is important to underline that distortions of the scattering pattern depend strongly on location of the source with regard to scatterer’s surface.

Acknowledgements

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