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L^3: THE GEOMETRY OF PSEUDOQUATERNIONS

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Abstract

We introduce pseudoquaternions as an effective tool to describe the vector analysis in L^3, and we use them to characterize null curves and null cubics in S_1^2.

Keywords: pseudoquaternions, vector analysis, null curves.

Resumen

Introducimos los pseudocuaterniones como una herramienta efectiva para describir el anlisis vectorial de L^3, y los usamos para caracterizar curvas nulas y cbicas nulas en S_1^2.

Palabras-clave: pseudocuaterniones, anlisis vectorial, curvas nulas.

AMS Subject Classification: 14H99

1. Introduction

Let L^3 be the 3-dimensional Lorentzian space with inner product of signature $-, +, +$, which will be denoted by dot.

In this paper we show that pseudoquaternions are an useful and natural tool to study the elementary geometry of L^3 and we have used them to characterize unitary null curves in this space.

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2. Vector analysis in L^3

As a generalization of complex numbers related with the system of quaternions we find the pseudoquaternions [5], given by:

$$z = a + bi + ce + df \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and the complex units hold the following multiplication table:

— i	e	f
i — -1	f	$-e$
e — $-f$	1	$-i$
f — e	i	1

The conjugate pseudoquaternion of $z, (1)$, will be

$$z^* = a - bi - ce - df$$

and its norm or modulus will be

$$N(z) = a^2 + b^2 - c^2 - d^2$$

Trivially,

$$z^{-1} = \frac{z^*}{N(z)}$$

when it is possible, and also, if x and y are two pseudoquaternions we get

$$(x \cdot y)^* = y^* \cdot x^* \text{ and } N(x \cdot y) = N(x) \cdot N(y).$$

We say that a pseudoquaternion $z, (1)$, is **pure** if $a = 0$.

Pure pseudoquaternions verify $z^* = -z$ and $N(z) = -z^2$.

The distance between two pure pseudoquaternions $z_1 = b_1 i + c_1 e + d_1 f$, $z_2 = b_2 i + c_2 e + d_2 f$ is given by

$$d(z_1, z_2) = \sqrt{-(b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2}$$

which coincides with the distance in L^3.

The pseudoquaternions i,e,f are associated to the orthonormal vectors I,E,F.

If we note the inner product by dot, we have

$$I \cdot I = -1, \qquad E \cdot E = 1, \qquad F \cdot F = 1$$

i.e., according to [3], I is timelike vector, E and F are spacelike vectors.

For all above we can identify the vectors of L^3 with pure pseudoquaternions or equivalently, with real linear combination of i, e, f.

We want to define an exterior product in L^3 on the natural way, keeping in mind its analogous in R^3.

Let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ and $C = (c_1, c_2, c_3)$ be vectors in L^3.

Definition 1 *The exterior product of A and B, $A \wedge B$, is the vector of L^3 such that its inner product with C is the determinant of the matrix*

$$\begin{pmatrix} a_1 & -a_2 & -a_3 \\ -b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Equivalently, we say

$$\begin{aligned} A \wedge B &= \det \begin{pmatrix} i & e & f \\ a_1 & -a_2 & -a_3 \\ -b_1 & b_2 & b_3 \end{pmatrix} \\ &= (a_3b_2 - a_2b_3)i - (a_1b_3 - a_3b_1)e + (a_1b_2 - a_2b_1)f \end{aligned} \tag{2}$$

By straightforward computation we can verify

a) $A \wedge A = 0$

b) $A \wedge B = -B \wedge A$

c) $\lambda A \wedge B = A \wedge \lambda B = \lambda(A \wedge B) \qquad \text{si } \lambda > 0$

d) $A \wedge B \cdot B = A \wedge A \cdot A = 0$

e) $(A + B) \wedge C = A \wedge C + B \wedge C$

f) $(A \wedge B) \wedge C = (A \cdot C)B - (B \cdot C)A$

g) If A, B, C are vectors in L^3 and a, b, c its corresponding pure pseudoquaternions, it verifies
$$A \wedge B \cdot C = \frac{1}{2}(abc - cba)$$

h) Let A, B, C be future-pointing timelike vectors in L^3, [1]; A, B, C are on line if and only if
$$|(B - A) \wedge (C - A)| = 0$$

3. Unitary null curves

A curve $q(s)$ verifying $q'(s) \cdot q'(s) = 0$ is called a null curve and if in addition satisfy $q(s) \cdot q(s) = 1$ is called unitary null curve. A null frame in L^3 is an ordered triple of vectors (E^1, E^2, E^3) such that

$$\begin{aligned} &E^1 \cdot E^1 = E^2 \cdot E^2 = 0, \quad E^1 \cdot E^2 = -1, \quad E^3 \cdot E^3 = 1, \\ &E^1 \cdot E^3 = E^2 \cdot E^3 = 0 \quad \text{and } \det \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix} = \pm 1 \end{aligned} \tag{3}$$

Let (E^1, E^2, E^3) be a null frame in L^3. The orthonormal vectors I, E, F are the associated orthonormal frame related to the null frame by

$$I = \frac{1}{2}(E^1 + E^2), \qquad E = \frac{1}{2}(E^1 - E^2), \qquad F = E^3.$$

We take

$$E^1 \wedge E^2 = -E^3, \qquad E^2 \wedge E^3 = E^1 \quad \text{and} \quad E^1 \wedge E^3 = -E^2$$

and we obtain

$$I \wedge E = F, \qquad E \wedge F = \frac{-(I+E)}{2}, \qquad F \wedge I = \frac{(E-I)}{2}$$

and the others vanish.

A rotation in L^3, around the origin, could be defined by the position of a null frame (E^1, E^2, E^3) respect to the initial basis I, E, F.

From the rotation defined by a pseudoquaternion q, the vectors E^i are associated to the pseudoquaternions e^i by

$$e^1 = q^* \ i \ q, \qquad e^2 = q^* \ e \ q, \qquad e^3 = q^* \ f \ q.$$

Explicity, if $q = q_0 + q_1 i + q_2 e + q_3 f$ we know that

$$\begin{array}{rclcrcl} q^* & = & q_0 - q_1 i - q_2 e - q_3 f & \qquad & eq & = & q_0 e - q_1 f + q_2 - q_3 i \\ iq & = & q_0 i - q_1 + q_2 f - q_3 e & & fq & = & q_0 f + q_1 e + q_2 i + q_3 \end{array}$$

and we get

$$\begin{aligned} e^1 &= (q_0^2 + q_1^2 + q_2^2 + q_3^2)i + 2(q_2 q_1 - q_0 q - 3)e + 2(q_0 q_2 + q_3 q_1)f \\ e^2 &= -2(q_0 q_3 + q_1 q_2)i + (q_0^2 - q_1^2 - q_2^2 + q_3^2)e - 2(q_0 q_1 + q_2 q_3)f \\ e^3 &= 2(q_0 q_2 - q_3 q_1)i + 2(q_0 q_1 - q_3 q_2)e + (q_0^2 - q_1^2 + q_2^2 - q_3^2)f \end{aligned}$$

These are the components of the pseudoquaternions e^i as well as components of vectors E^i, $i : 1, 2, 3$.

At every point of an unitary curve se associate the null frame (E^1, E^2, E^3) and following [3] we have the Frenet's equations:

$$\begin{aligned} \frac{dE^1}{ds} &= -k_1(s)E^1 + k_2(s)E^3 \\ \frac{dE^2}{ds} &= -k_1(s)E^2 + k_3(s)E^3 \\ \frac{dE^3}{ds} &= k_3(s)E^1 + k_2(s)E^2 \end{aligned} \tag{4}$$

The "curvatures.ªre

$$k_1 = \frac{-dE^1}{ds} \cdot E^2, \quad k_2 = \frac{dE^1}{ds} \cdot E^3, \quad k_3 = \frac{-dE^3}{ds} \cdot E^2$$

and in terms of the pseudoquaternion q and its derivated

$$\begin{aligned} k_1 &= 2(-q_0' q_3 + q_0 q_3' - q_2 q_1' + q_1 q_2') \\ k_2 &= 2(q_3' q_1 - q_0 q_2' + q_2 q_0' - q_3 q_1') \\ k_3 &= 2(-q_3 q_2' - q_2 q_3' - q_0 q_1' - q_1 q_0') \end{aligned} \tag{5}$$

Also we find that (5) are the relative components (respect to the null frame (E^1, E^2, E^3)) of the instant rotation vector, [4],

$$H = -k_2 E^1 + k_3 E^2 - k_1 E^3$$

since $\frac{dE^i}{ds} = H \wedge E^i$, $i : 1, 2, 3$.

The curve $q = q(s)$ with s no proper time parameter, can be represented by the pseudoquaternion $q = q_0(s) + q_1(s)i + q_2(s)e + q_3(s)f$, with the condition $q \cdot q = 1$ and $q' \cdot q' = 0 \quad (q' = \frac{dq}{ds})$.

We will suposse that the $q_i(s)$ are C^5, as [2].

At every point we can attach a null frame (Q^1, Q^2, Q^3). Without loss of generality we can choose Q^1 as an scalar multiple of q'.

As $Q^i = Q^i(s)$ we can write

$$\frac{dQ^i}{ds} = \sum_j w_j^i \, Q^j$$

with $w_1^1 = w_2^2 = w_2^1 = w_1^2 = w_3^3 = 0$, $w_3^2 = -w_1^3$, $w_3^1 = -w_2^3$.

Now the Frenet's equations are

$$\begin{aligned} \frac{dQ^1}{ds} &= w_3^1 \, Q^3 \\ \frac{dQ^2}{ds} &= w_3^2 \, Q^3 \\ \frac{dQ^3}{ds} &= -w_3^2 \, Q^1 - w_3^1 \, Q^2 \end{aligned} \tag{6}$$

On the natural way, we can consider w_3^1 as curvature and w_3^2 as torsion.

Comparing (4) and (6) we obtain $k_1(s)$ must be zero and from [2], theor. 6.1 the curve is a null straight line. We also obtain $w_3^1 = k_2$ and $w_3^2 = k_3$ and according to [3] (E^1, E^2, E^3) become a Cartan frame and the curve is called a Cartan-framed curve.

In order to know about k_2 and k_3 we study the osculating sphere in L^3, i.e., the sphere passing through four consecutive points of a curve.

Keeping in mind that dot means the inner product of signature $-, +, +$, the equation of this sphere is

$$(x - c) \cdot (x - c) - r^2 = 0$$

where x is a generic point of the sphere, c its center and r its radius.

It is well known that necessary and sufficient condition that the surface $f(s)$ has contact of order n at the point P with the curve is that at P the relation hold:

$$f(s) = f'(s) = ... = f^{(n)}(s) = 0 \quad \text{and} \quad f^{(n+1)}(s) \neq 0.$$

In our case $n = 3$, $f(s) = (x - c) \cdot (x - c) - r^2$ and the relations becomes

$$\begin{aligned} (x-c)\cdot Q^1 &= 0 \\ k_2(x-c)\cdot Q^3 &= 0 \\ (x-c)\cdot(-k_2k_3Q^1 - (k_2)^2Q^2 + k_2'Q^3) &= 0 \end{aligned}$$

We find $(x - c) \cdot (k_2)^2 Q^2 = 0$ then $k_2 = 0$.
The center is $c = x + Q^1$ and the radius is zero.
For all above, we summarize in the following theorem.

Theorem 1 *The curvatures (5) of a null curve is S_1^2 are $k_1 = k_2 = 0$ or equivalently, the null curves in S_1^2 are null straight lines and there not exist osculating sphere of a null spherical curve in L^3.*

At [2], pages 240 and 234, we find that a null cubics is a curve with $k_1 = 1$ and $k_2 = k_3 = 0$, thus

Corollary 1 *There does not exist null cubics in S_1^2.*

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