Distinguished Hamiltonian theorem for homogeneous symplectic manifolds

José F. Cariñena,[1] José M. Gracia-Bondía,[1] Luis A. Ibort,[2]
Carlos López[2] and Joseph C. Várilly[3]

[1] Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain

[2] Departamento de Física Teórica, Universidad Complutense de Madrid, 28040 Madrid, Spain

[3] Escuela de Matemática, Universidad de Costa Rica, 11501 San José, Costa Rica

Lett. Math. Phys. **23** (1991), 35–44

Abstract

A diffeomorphism of a finite-dimensional flat symplectic manifold which is canonoid with respect to all linear and quadratic Hamiltonians preserves the symplectic structure up to a factor: so runs the "quadratic Hamiltonian theorem". Here we show that the same conclusion holds for much smaller "sufficiency subsets" of quadratic Hamiltonians, and the theorem may thus be extended to homogeneous infinite-dimensional symplectic manifolds. In this way we identify the distinguished Hamiltonians for the Kähler manifold of equivalent quantizations of a Hilbertizable symplectic space.

1 Introduction

Given a symplectic manifold (M, ω), a *conformal symplectomorphism* is a diffeomorphism $\phi\colon M \to M$ preserving the symplectic structure modulo a factor: $\phi^*\omega = \lambda\omega$, $\lambda \in \mathbb{R} \setminus \{0\}$. Denote by $\mathrm{Diff}_{\mathrm{CS}}(M)$ the group of conformal symplectomorphisms. There has been of late a lot of interest in $\mathrm{Diff}_{\mathrm{CS}}(M)$, its subgroup $\mathrm{Diff}_{\mathrm{S}}(M)$ of symplectomorphisms and other subgroups. One of the purposes of this letter is to point out that considerable insight into the structure of $\mathrm{Diff}_{\mathrm{CS}}(M)$ and its subgroups can be gained by well-worn methods of classical mechanics.

From the outset, we shall be dealing with infinite dimensional manifolds (our examples arise mainly from field theory). We suppose, to fix ideas, that (M, ω) is a separable Riemannian manifold. The bilinear form ω_m is continuous, strongly nondegenerate on T_mM for each $m \in M$. An element of the Lie algebra of $\mathrm{Diff}(M)$ belonging to the Lie subalgebra $\mathfrak{G}$ of $\mathrm{Diff}_{\mathrm{S}}(M)$, i.e., a vector field X with the property $\mathcal{L}_X\omega = 0$, is said to be *locally Hamiltonian*. The Cartan identity tells us that if $X \in \mathfrak{G}$, then the 1-form $i(X)\omega$ is closed; if it is exact, we say that X is *Hamiltonian*. We actually have the following exact sequence of Lie algebra homomorphisms:

$$0 \longrightarrow \mathfrak{X}_H \longrightarrow \mathfrak{G} \longrightarrow H^1(M;\mathbb{R}) \longrightarrow 0, \tag{1.1}$$

where the ideal of Hamiltonian vector fields is denoted $\mathfrak{X}_H$. We shall say that f smooth is a *Hamiltonian function* for X_f if $i(X_f)\omega = df$.

Now, let $X \in \mathfrak{G}$ and $\phi \in \mathrm{Diff}(M)$ be given and assume that the transformed vector field $\phi_* X$ remains in $\mathfrak{G}$. Then ϕ is said to be *canonoid* with respect to X. It is easily seen that if ϕ belongs to $\mathrm{Diff}_{\mathrm{CS}}(M)$ then it is canonoid with respect to any element of $\mathfrak{G}$. For finite dimensional manifolds, the converse is an immediate consequence of Lee Hwa-Chung's theorem [1]. At any rate, the following argument extends the validity of this result to strongly symplectic infinite dimensional manifolds. Note that $X_{f^2} = 2f\,X_f$. Assume that α is a nonzero closed 2-form on M such that for every function h on M, $\mathcal{L}_{X_h}\alpha = 0$. Then $d\alpha_h = 0$ where $\alpha_h := i(X_h)\alpha$. If $h = f^2$ we get

$$0 = \mathcal{L}_{X_h}\alpha = 2\,df \wedge \alpha_f.$$

Then necessarily $\alpha_f = c\,df$, where c must be a nonzero constant, and the conclusion follows.

Some twenty years ago, it was remarked [2] that for the simplest kind of symplectic manifolds, the property of being canonoid with respect to a subset of $\mathfrak{G}$ is enough to characterize a conformal symplectomorphism.

Quadratic Hamiltonian Theorem (Currie and Saletan, 1972)**.** *Let us consider $\mathbb{R}^{2n} = \{q^i, p_i\}$ endowed with the standard symplectic structure $\omega := dq^i \wedge dp_i$. A diffeomorphism $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a conformal symplectomorphism if and only if it is canonoid with respect to every Hamiltonian vector field generated by a linear or quadratic Hamiltonian function in the $\{q^i, p_i\}$ coordinates:*

$$\phi^*\omega = \lambda\omega \quad \text{for some nonzero} \quad \lambda \in \mathbb{R} \iff \phi_*\mathfrak{G}^{1,2} \subseteq \mathfrak{G}. \tag{1.2}$$

The set of Hamiltonian functions generating $\mathfrak{G}^{1,2}$ is called a *sufficiency set*. There is no global meaning for linear or quadratic Hamiltonians in a general symplectic manifold. There have been several efforts for developing an analogue of the quadratic Hamiltonian theorem in a more intrinsic context; see [3] and for global considerations [4]. The implausibility of results of this type for general symplectic manifolds transpires.

Now, one can remark that the sufficiency set found by Currie and Saletan generates the action of the affine symplectic group on $\mathbb{R}^{2n}$. One conjectures that something similar can be said about $\mathrm{Diff}_{\mathrm{CS}}(M)$ if M is a *homogeneous* manifold. This belief is substantiated in what follows. For reasons that soon will become apparent, we shall only consider manifolds with an almost complex structure; this is why we employ strong symplectic forms from the beginning.

In our context, the quadratic Hamiltonian theorem transmutes into a fairly general uniqueness result for invariant symplectic structures. It turns out that unitary invariance of ω in the linear case is all that is needed and so we find, as a byproduct, that in the old theorem by Currie and Saletan there was room for improvement.

In Section 2, the linear case is dealt with as preparation. Then, a general result is proved in Section 3 and exemplified in full detail on the infinite dimensional Kähler manifold of equivalent bosonic quantizations of a hilbertizable symplectic space.

2 Sufficiency sets for linear unitarizable symplectics

Lemma 1. *Let H be a complex Hilbert space with inner product $\langle\cdot\,|\,\cdot\rangle = \Re\langle\cdot\,|\,\cdot\rangle + i\Im\langle\cdot\,|\,\cdot\rangle =: d + iS$. Consider H as a symplectic space with the underlying real-linear structure and symplectic form given by S. Suppose that a diffeomorphism $\phi\colon H \to H$ is such that both constant and complex-linear skewadjoint vector fields are still Hamiltonian for ϕ^*S. Then ϕ is a conformal symplectomorphism.*

Remark. There is no difference between Hamiltonian and locally Hamiltonian vector fields in the present context.

Proof. First note that a continuous real-linear operator A on H can be regarded as a vector field, and linear locally Hamiltonian vector fields corresponding to operators skewadjoint with respect to S:

$$A \in \mathfrak{G} \iff \mathcal{L}_A S(u,v) = S(Au,v) + S(u,Av) = 0 \quad \text{for all} \quad u,v \in H. \tag{2.1}$$

In other words, linear elements of $\mathfrak{G}$ form the Lie algebra $\mathfrak{sp}$ of the symplectic group $\mathrm{Sp}(H)$, with Lie brackets of vector fields going over to commutators. Naturalness of the Lie derivative:

$$\phi^*(\mathcal{L}_X \alpha) = \mathcal{L}_{\phi_* X}(\phi^* \alpha) \quad \text{for all forms } \alpha \tag{2.2}$$

gives at once:

$$\mathcal{L}_{\phi_* X} S = 0 \iff \mathcal{L}_X(\phi^* S) = 0. \tag{2.3}$$

The hypothesis thus says that $\mathcal{L}_X(\phi^* S) = 0$ with respect to all *constant* vector fields; it means that $\phi^* S$ is constant in the original chart, so we can think of it as a bilinear form on H, just like S.

Denote by J the operator of multiplication by i on H. This is a linear skewadjoint vector field, so that $\mathcal{L}_J(\phi^* S) = 0$ by hypothesis. That is to say, $\phi^* S(u, Jv) + \phi^* S(Ju, v) = 0$. Define $h(u,v) := \phi^* S(u, Jv) + i\phi^* S(u,v)$. This is a (not necessarily positive-definite) hermitian form in view of (2.1) and our last remark. Moreover, h is invariant under the unitary group $\mathrm{U}(H)$, since $\mathcal{L}_X(\phi^* S) = 0$ for infinitesimally unitary X. Now consider the hermitian operator O defined (via the Riesz theorem) by $\langle \cdot \mid O \cdot \rangle = h(\cdot,\cdot)$. It commutes with the whole unitary group, so we have $O = \lambda 1$, i.e., $h(\cdot,\cdot) = \lambda \langle \cdot \mid \cdot \rangle$ with $\lambda \in \mathbb{R} \setminus \{0\}$. In particular, $\phi^* S = \lambda S$ and ϕ is a conformal symplectomorphism. □

There is nothing intrinsic in our particular identification of J with $i\,1$. Given a real vector space V with a symplectic form S, choices of complex Hilbert embeddings are given by the set (which may be empty in infinite dimensions) of real endomorphisms J such that

$$\begin{cases} J^2 = -1, \\ S(Ju, Jv) = S(u,v) & \text{for all } u, v \in V, \\ S(v, Jv) > 0 & \text{for all nonzero } v \in V. \end{cases} \tag{2.4}$$

In other words, J is a complex structure, a symplectic operator and is such that the *symmetric* bilinear form

$$d_J(u,v) := S(u, Jv) \tag{2.5}$$

satisfies $d_J(v,v) > 0$ for all nonzero v. Therefore the *hermitian* form

$$\langle u \mid v \rangle := S(u, Jv) + i\,S(u,v) = d_J(u,v) + i\,d_J(Ju, v) \tag{2.6}$$

is a positive definite inner product on V, regarded as a complex vector space by defining $(\alpha + i\beta)v := \alpha v + \beta J v$. Note that the set of operators satisfying the first two properties in (2.4) is $\mathrm{Sp}(V) \cap \mathfrak{sp}(V)$.

Given one such J, the conjugate gJg^{-1} by operators $g \in \mathrm{Sp}(V)$ is an operator of the same type; all of them are conjugate in this way, and the topologies associated to inner products corresponding to different J's are all equivalent (see Section 3). We denote by $\mathrm{U}_J(V)$ the subgroup of symplectic operators commuting with J; these are conjugates of the elements of a conventional unitary group.

Corollary 1 (Infinite-dimensional quadratic Hamiltonian theorem)**.** *A sufficiency set on a Hilbertizable symplectic space is the subset of linear Hamiltonians plus the subset of quadratic (in general unbounded) Hamiltonians generating the action of any subgroup of the type* $\mathrm{U}_J(V)$.

Therefore: the straightforward generalization of the Currie–Saletan theorem is *a fortiori* valid for a symplectic space whose skewsymmetric form is the imaginary part of an inner product. In the context – admittedly less general than theirs – of infinite-dimensional Hilbertizable symplectic spaces, our results improve those in [3] and [5] in several respects. By no means has ϕ to be supposed S-skew, nor are the new Hamiltonians assumed to be the pullbacks of the original ones by ϕ.

In particular:

Corollary 2 (Improved finite-dimensional quadratic Hamiltonian theorem)**.** *A sufficiency set on* $\mathbb{R}^{2n}$ *with the standard symplectic structure is the subset of linear Hamiltonians plus the subset of quadratic Hamiltonians generating the action of* $\mathrm{U}(n)$ *or of any one of its conjugate subgroups in* $\mathrm{Sp}(\mathbb{R}^{2n})$.

For instance, identify as usual $\mathbb{R}^{2n}$ in Darboux coordinates to $\mathbb{C}^n$ by $z = q + ip \leftrightarrow \binom{q}{p}$. The real form of $i\,1$ is then $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and corresponding Hamiltonians are obtained as follows. For $A \in \mathfrak{sp}(n, \mathbb{R})$ we have $H_A = \frac{1}{2} v^t J_0 A v$, because

$$X_A(v) = Av\,\frac{\partial}{\partial v} \quad \text{and} \quad S = J_0\, dv \wedge dv$$

as a differential form. If $A \in \mathfrak{u}(n)$, then $A^t J_0 + J_0 A = 0$ and $A^t + A = 0$, so that:

$$A = \begin{pmatrix} M & N \\ -N & M \end{pmatrix} \tag{2.7}$$

with N symmetric, M skewsymmetric and

$$H_A = \tfrac{1}{2}(q^t N q + p^t N p - q^t M p + p^t M q). \tag{2.8}$$

It is well known that, in the infinite dimensional case, the proper choice of J is a nontrivial, delicate matter, since different choices are not quantum-mechanically equivalent in general. Generally one chooses J in order to unitarize the Hamiltonian dynamics one is interested in. We shall give a couple of archetypal examples, in order to exemplify what we have in mind.

As a first example, let V be the space of solutions of the real Klein–Gordon equation in Minkowski spacetime. To be precise, we have $V = H^{1/2} \oplus H^{-1/2}$, where H^s for real s denotes the usual Sobolev spaces, and the strongly symplectic S is given by:

$$S(v_1, v_2) := S\left(\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) = \int (f_1(x) g_2(x) - f_2(x) g_1(x))\, d^3x. \tag{2.9}$$

We have the first-order system

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = A \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.10}$$

Here the domain of the Hamiltonian vector field A is not the whole of V, but it does not really matter. The *unique* complex structure commuting with A is the orthogonal part of the polar decomposition of A:

$$J(A) = A(-A^2)^{-1/2} = \begin{pmatrix} 0 & (m^2 - \Delta)^{-1/2} \\ -(m^2 - \Delta)^{1/2} & 0 \end{pmatrix}. \tag{2.11}$$

This is a bounded operator; second quantization on $H^{1/2} \oplus H^{-1/2}$ with the complex Hilbert space structure given by $J(A)$ yields correctly the neutral scalar quantum field.

As another example, consider the loop group $L\mathbb{T} = \mathrm{Map}(S^1; \mathbb{T})$. Its component of winding number zero, modulo the constant maps, can be identified to the vector space V of real smooth functions on S^1 with vanishing constant term in their Fourier expansions. We enlarge V so as to include all the square-summable functions. On V there is the strong symplectic form [6]:

$$S(f, g) := \frac{1}{2\pi} \int_0^{2\pi} f'(\varphi)\, g(\varphi)\, d\varphi. \tag{2.12}$$

The natural complex structure is the one commuting with (the infinitesimal generator of) the rotations. We obtain it as before:

$$J\left(\frac{d}{d\varphi}\right) = \frac{d}{d\varphi}\left(-\frac{d^2}{d\varphi^2}\right)^{-1/2}. \tag{2.13}$$

Explicitly, if we write:

$$f(\varphi) = \sum_{n \neq 0} \widehat{f}(n)\, e^{in\varphi}, \tag{2.14}$$

then

$$Jf(\varphi) = \sum_{n \neq 0} i \,\mathrm{sign}(n) \widehat{f}(n)\, e^{in\varphi}. \tag{2.15}$$

J is thus the classical operator giving the conjugate series [7, Chap. 12]; $(1 - iJ)V$ is the polarization consisting of holomorphic functions with vanishing constant term inside $V^{\mathbb{C}} = L^2(S^1; \mathbb{C})$.

We summarize our method in a form suitable for generalization in the next section. On $(V, \langle \cdot | \cdot \rangle)$ supporting a unitary irreducible representation of a group G, there is only one G-invariant symplectic structure $\omega = \Im\langle \cdot \mid \cdot \rangle$ compatible with the complex structure. There might be more, not compatible with J. This is avoided, as in the case of G being the unitary group, if the group representation contains in its centre the complex structure; under this assumption, G-invariance implies J-compatibility.

3 Sufficiency sets for homogeneous complex manifolds

Theorem 1. *Consider a connected Lie group G acting transitively on (M, ω) by symplectomorphisms and let G_{m*} denote the representation of the isotropy subgroup G_m on T_mM (given by restriction of the tangent action of G) at a point $m \in M$. Suppose there is an element j in the center of G_m such that $J_m := G_{m*}j$ is a complex structure on T_mM. If G_{m*} is irreducible, then ω is the unique, up to a constant factor, G-invariant 2-form on M. Consequently, if $\phi \in \mathrm{Diff}(M)$ is canonoid with respect to the fundamental vector fields of the action, then ϕ is a conformal symplectomorphism.*

Proof. Let us employ J_m to define an almost complex structure J on M, invariant under the action of G, in the obvious way. This J is compatible with ω by hypothesis. Then there are Hilbert space structures in $T_{m'}M$ given pointwise at any $m' \in M$ by:

$$\langle u \mid v \rangle_{m'} := \omega_{m'}(u, J_{m'}v) + i\omega_{m'}(u, v), \qquad u, v \in T_{m'}M. \tag{3.1}$$

and the representation $G_{m'}$ on $T_{m'}M$ is made complex. A Schur-lemma argument identical to the one employed in Lemma 1 tells us that $\omega_{m'}$ and its nonzero multiples are the only $G_{m'}$-invariant symplectic forms on $T_{m'}M$. By homogeneity, every invariant form on M is determined by its value at a point, proving unicity. Taking $\widehat{\omega} = \phi^*(\omega)$, the hypothesis of ϕ being canonoid gives G-invariance of $\widehat{\omega}$, and, therefore, $\widehat{\omega}$ is a multiple of ω. □

Remark. If M is a suitable symmetric space, then one can prove that J is integrable and M is actually Kähler [8, Chap. VIII]. The above arguments apply to many classical bounded symmetric domains – they trivially show, for instance, that the usual area form on S^2 (and metric) are the unique $\mathrm{SO}(3)$-compatible ones – and their infinite-dimensional generalizations.

We now proceed to examine in detail a very interesting infinite-dimensional Kähler manifold, related to the linear problem: the manifold of equivalent – in the sense of quantum field theory – complex structures on a Hilbertizable symplectic space. We explicitly compute the *distinguished Hamiltonians*, i.e., those belonging to the sufficiency set.

In order to see clearly what is involved, it is convenient to adopt a real notation and language. Let us choose and fix J_0, satisfying (2.4), on the symplectic space (V, S). If $A \in \mathrm{GL}_{\mathbb{R}}(V)$, we use the notation $A^\sharp := (A^{-1})^t$ for convenience, where the transpose map is defined with respect to d_{J_0} defined as in (2.5). If $A \in \mathrm{End}_{\mathbb{R}} V$, its decomposition into J_0-linear and J_0-antilinear parts is given by

$$a(A) := \tfrac{1}{2}(A - J_0 A J_0), \qquad b(A) := \tfrac{1}{2}(A + J_0 A J_0). \tag{3.2}$$

For $g \in \mathrm{GL}_{\mathbb{R}}(V)$, we have $g \in \mathrm{Sp} := \mathrm{Sp}(V, S) \iff J_0 g = g^\sharp J_0$. Then it is readily seen that a is invertible. Now define $T(g) := b(g)a(g)^{-1}$ for $g \in \mathrm{Sp}$. [For the next computations, we abbreviate often $a := a(g)$, $T := T(g)$.] We can parametrize $g \in \mathrm{Sp}$ by the pair (a, T). From the definitions, $g = (1 + T)a$ and one can prove the following proposition.

Proposition. *For $g \in \mathrm{Sp}$, $T = T(g)$ is J_0-antilinear and symmetric, and $1 - T^2 > 0$ (i.e., is positive-definite with respect to d_{J_0}). Also, a is J_0-linear and one has $a^t(1 - T^2)a = 1$. Conversely, given a pair (a, T) of real endomorphisms of V satisfying these conditions, the endomorphism $g := (1+T)a$ belongs to* Sp. □

We define $\mathcal{D}(V) := \{\, X \in \mathrm{End}_{\mathbb{R}} V : XJ_0 = -J_0 X,\ X^t = X,\ 1 - X^2 > 0 \,\}$, which we may call the *open Cartan–Siegel disk* of (V, S).

Now, we want to consider the set of all positive symplectic complex structures on V. Define $\Sigma(V)$ as the orbit of J_0 in $\mathfrak{sp}$ under the adjoint action of Sp. Computation reveals that this is a Cayley transform:

$$\begin{aligned} J(g) &:= gJ_0g^{-1} = (1 + T(g))a(g)J_0a(g)^{-1}(1 + T(g))^{-1} \\ &= J_0(1 - T(g))(1 + T(g))^{-1}. \end{aligned} \tag{3.3}$$

The formula establishes a continuous bijection between $\mathcal{D}(V)$ and $\Sigma(V)$. The space $\Sigma(V)$ generalizes the Lorentz hyperboloid in $\mathbb{R}^3$.

So we have constructed two homogeneous spaces for the symplectic group. On $\mathcal{D}(V)$ the symplectic group acts by linear fractional transformations $T \mapsto (aT+b)(bT+a)^{-1}$. The stabilizer of J_0 in $\Sigma(V)$ and 0 in $\mathcal{D}(V)$ is $\mathrm{U}_{J_0}(V)$, which is also the *Cartan subgroup* for the involution $g \mapsto -J_0 g J_0$.

Denote by $\mathcal{I}_2$ the ideal of Hilbert–Schmidt operators on (V, d_{J_0}). In view of Shale's theorem [9] on bosonic quantization, it is natural to consider the restricted sets:

$$\begin{aligned} \mathcal{D}'(V) &:= \{\, T \in \Sigma(V) : T \in \mathcal{I}_2 \,\}, \\ \Sigma'(V) &:= \{\, J \in \Sigma(V) : J - J_0 \in \mathcal{I}_2 \,\}. \end{aligned} \tag{3.4}$$

Clearly, $T(g) \in \mathcal{D}'(V) \iff J(g) \in \Sigma'(V)$. These are the orbits of 0 and J_0, respectively, under the action of the subgroup Sp' of restricted linear canonical transformations:

$$\mathrm{Sp}'(V) := \{\, g \in \mathrm{Sp} : T(g) \in \mathcal{I}_2 \,\}. \tag{3.5}$$

It is useful to note that the projections $\mathrm{Sp}' \to \Sigma'(V) \colon g \mapsto J(g)$ and $\mathrm{Sp}' \to \mathcal{D}'(V) \colon g \mapsto T(g)$ have nice global sections. To identify the latter, let $T \in \mathcal{D}(V)$ and define

$$h_T := (1+T)(1-T^2)^{-1/2}. \tag{3.6}$$

This is a symplectic operator. It is immediate that $T(h_T) = T$, so $T \mapsto h_T$ provides a global section of the principal bundle $\mathrm{Sp}(V) \to \mathcal{D}(V)$ with fibre $\mathrm{U}_{J_0}(V)$. Restriction to $\mathcal{D}'(V)$ gives a global section of $\mathrm{Sp}' \to \mathcal{D}'(V)$. For $\Sigma'(V)$, note that $h_T J_0 = J_0 h_T^\sharp = J_0 h_T^{-1}$, and so $h_T J_0 h_T^{-1} = J_0(1-T)(1+T)^{-1} =: J_T$. Therefore the map $J_T \mapsto h_T$ is a global section of $\mathrm{Sp} \to \Sigma(V)$, which restricts to a global section of $\mathrm{Sp}' \to \Sigma'(V)$.

We may avoid the detour through $\mathcal{D}(V)$ by noting that $h_T^2 = -J_T J_0$; indeed, h_T is the positive square root of $-J_T J_0$, so we have established that if $J \in \Sigma(V)$, then $J \mapsto (-JJ_0)^{1/2}$ provides a global section of the bundle $\mathrm{Sp} \to \Sigma(V)$; and finally, there is an "intertwining formula"

$$(-JJ_0)^{1/2} J_0 (-J_0 J)^{1/2} = J \tag{3.7}$$

between two positive symplectic complex structures on (V, S). This shows also that the topologies associated to any two different complex structures are equivalent. From now on we shall work only in $\Sigma'(V)$ and leave the corresponding statements for $\mathcal{D}'(V)$ to the reader.

The sets $\mathcal{D}'(V)$ and $\Sigma'(V)$ are Riemannian manifolds: the Lie algebra of $\mathrm{Sp}'(V)$ is $(\mathfrak{p} \cap \mathcal{I}_2) \oplus \mathfrak{k}$, where $\mathfrak{p} \oplus \mathfrak{k}$ is the Cartan splitting of $\mathfrak{sp}$ and the tangent space, at J_0 for example, is a copy of $\mathfrak{p} \cap \mathcal{I}_2$; the homogeneity of $\Sigma'(V)$ shows that we can identify the tangent space $T_J\Sigma'(V)$ with $\{\, X \in \mathcal{I}_2 : XJ = -JX,\ S(Xu, v) = -S(u, Xv) \,\}$. The *Riemannian metric* d is defined on $\Sigma'(V)$ by

$$d_J(X_J, Y_J) := \mathrm{Tr}[X_J Y_J] \quad \text{for} \quad X_J, Y_J \in T_J\Sigma'(V). \tag{3.8}$$

This metric is clearly invariant under the action of Sp'. Write $L_{J_0} := L := (1+J_0)/\sqrt{2}$ and note that $L \in \mathrm{U}_{J_0}(V)$, $L^{-1} = (1-J_0)/\sqrt{2}$ and that $L^2 = J_0$. Also note that $(\mathrm{Ad}\, L)^2 = (\mathrm{Ad}\, J_0) = -1$ on the J_0-antilinear part of $\mathrm{End}_{\mathbb{R}}(V)$ and in particular on $\mathfrak{p} \cap \mathcal{I}_2$; indeed, $LXL^{-1} = L^2 X = J_0 X$ whenever X is J_0-antilinear. We see that $\mathrm{Ad}\, L$ preserves $\Sigma'(V)$. Notice, finally, that $L_J := (-JJ_0)^{1/2} L (-J_0 J)^{1/2} = (1+J)/\sqrt{2}$.

The *almost complex structure* $\mathcal{J}$ on $\Sigma'(V)$ is defined by

$$\mathcal{J}_J X_J := L_J X_J L_J^{-1} = J X_J \quad \text{for} \quad X_J \in T_J\Sigma'(V). \tag{3.9}$$

The invariance of $\mathcal{J}$ is clear, as in the proof of Theorem 1. The integrability condition is the vanishing of the Nijenhuis tensor field:

$$N(\mathcal{X}, \mathcal{Y}) := [\mathcal{X}, \mathcal{Y}] + \mathcal{J}[\mathcal{J}\mathcal{X}, \mathcal{Y}] + \mathcal{J}[\mathcal{X}, \mathcal{J}\mathcal{Y}] - [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}] \tag{3.10}$$

where $\mathcal{X}$, $\mathcal{Y}$ are vector fields on $\Sigma'(V)$. It is enough to evaluate at J_0. We get $[\mathcal{X}_X, \mathcal{X}_Y](J_0) = \mathcal{X}_{[X,Y]}(J_0) = 0$, because $[X, Y]$ commutes with J_0. A straightforward computation shows that $[\mathcal{J}\mathcal{X}_X, \mathcal{X}_Y](J_0) = 0$ and $[\mathcal{J}\mathcal{X}_X, \mathcal{J}\mathcal{X}_Y](J_0) = 0$ also, so that the Nijenhuis tensor field vanishes, and we conclude that $\mathcal{J}$ is integrable.

Note also that

$$d_J(JX_J, JY_J) = \mathrm{Tr}[JX_J JY_J] = \mathrm{Tr}[X_J Y_J] = d_J(X_J, Y_J) \tag{3.11}$$

since X_J and J anticommute, so the metric is a *Hermitian metric* on the complex manifold $(\Sigma'(V), \mathcal{J})$.

The symplectic structure on $\Sigma'(V)$ is now obvious: set

$$\omega_J(X_J, Y_J) := d_J(JX_J, Y_J) = \mathrm{Tr}[JX_J Y_J]. \tag{3.12}$$

The invariance of ω can be immediately read from this formula. We remark that this is in fact a Kirillov–Kostant–Souriau form, the orbits of the coadjoint action being here isomorphic to the orbits of the adjoint action, so that $d\omega = 0$ is automatically satisfied and $(\Sigma'(V), \mathcal{J}, d)$ is a *Kähler* manifold.

Alternatively, $\Sigma'(V)$ is a Kähler manifold since $\mathcal{J}$ is invariant under parallel translation [10]. Indeed, we need only consider parallel translation along the geodesics of $\Sigma'(V)$ generated by the fundamental Hamiltonian vector fields, and this is given by the group action of $\mathrm{Sp}'(V)$, under which $\mathcal{J}$ is invariant.

According to the discussion in this letter, if a diffeomorphism of $\Sigma'(V)$ is canonoid with respect to the fundamental vector fields of Sp', then it belongs to $\mathrm{Diff}_{\mathrm{CS}}(\Sigma'(V))$. It remains to compute the set of distinguished Hamiltonians. These are in fact generic (affine) coordinate functions on $\mathfrak{sp}'$, when restricted to the orbit $\Sigma'(V)$. Given $A \in \mathfrak{sp}'$, the associated fundamental vector field is $\mathcal{X}_A(J) = JA - AJ$, from which

$$\omega_J(\mathcal{X}_A(J), Y_J) = \mathrm{Tr}[J(JA - AJ)Y_J] = \langle dH_A(J), Y_J\rangle.$$

The undetermined constant in the definition of H_A can be employed to obtain

$$H_A(J) \propto \mathrm{Tr}[(J_0 - J)A],$$

this "renormalization" helping to make sense of the expression in the infinite-dimensional case.

The result is of course applicable to the family of all symmetric domains generalized by $\Sigma'(V)$ (this is the "$C\,I$ noncompact" type in the classification of [8, Chap. X]). The "fermionic" type "$D\,III$ compact" can be developed in a completely parallel way.

Acknowledgement

JCV wishes to thank the Departamento de Física Teórica of the Universidad de Zaragoza for its hospitality.

References

[1] H.-C. Lee, "The universal integral invariants of Hamiltonian systems and application to the theory of canonical transformations", Proc. Roy. Soc. Edinburgh A **62** (1948), 237–246.

[2] D. G. Currie and E. J. Saletan, "Canonical transformations and quadratic Hamiltonians", Nuovo Cim. B **9** (1972), 143–153.

[3] G. Marmo, E. J. Saletan, R. Schmid and A. Simoni, "Bi-Hamiltonian dynamical systems and the quadratic Hamiltonian theorem", Nuovo Cim. B **100** (1987), 297–317.

[4] J. F. Cariñena and M. F. Rañada, "Canonoid transformations from a geometric perspective", J. Math. Phys. **29** (1988), 2181–2186.

[5] R. Schmid, "The quadratic Hamiltonian theorem in infinite dimensions", J. Math. Phys. **29** (1988), 2010–2011.

[6] G. B. Segal, "Unitary representation of some infinite dimensional groups", Commun. Math. Phys. **80** (1981), 301–342;
A. Pressley and G. B. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.

[7] R. L. Wheeden and A. Zygmund, *Measure and Integral*, Marcel Dekker, New York, 1977.

[8] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[9] D. Shale, "Linear symmetries of free Boson fields", Trans. Amer. Math. Soc. **103** (1962), 149–167.

[10] M. Nakahara, *Geometry, Topology and Physics*, Adam Hilger, Bristol, 1990.