

Dilation of a non-quasifree dissipative evolution

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Abstract

A semigroup evolution for the $\frac{1}{2}$ -spin which admits a conservative dilation is known to be governed by a Bloch equation in a standard form. Here we construct a conservative dilation directly from the Bloch equation, thus yielding an example of a dilation scheme for an evolution which is not quasifree. Moreover, we show that this conservative evolution is never ergodic in the non-quasifree case.

The following problem arises naturally in quantum statistical mechanics: given a dissipative evolution of a dynamical system satisfying such conditions as are necessary for it to be part of a larger conservative system, *construct* the minimal conservative system of which the former is a part. For dissipative systems with a nonabelian algebra of observables, such a construction has been heretofore available only for *quasifree* systems [2, 4] for which one may use the Sz.-Nagy unitary dilation theorem [5] for semigroups of contractions on a test-function space. We present below a dilation scheme for a specific non-quasifree evolution, namely the Bloch equation for a $\frac{1}{2}$ -spin at finite temperature [3]. As a corollary to our construction we note that, in contrast to the quasifree case, the ergodicity of the conservative dilation cannot, in general, be deduced from corresponding properties of the dissipative subsystem.

In the context of statistical mechanics we consider, as in [3], that an adequate mathematical framework for a ‘dynamical system’ should consist of a triple $\{\mathfrak{M}, \phi, \gamma(\mathbb{R}_+)\}$ where \mathfrak{M} is a von Neumann algebra acting on a separable Hilbert space, ϕ is a faithful normal state of \mathfrak{M} , and for each $t \geq 0$, γ_t is a completely positive linear map from \mathfrak{M} to \mathfrak{M} such that $\gamma_0 = \text{id}$ and $\phi \circ \gamma_t = \phi$, $\gamma_t(1) = 1$ for all $t \geq 0$. If $\gamma(\mathbb{R}^+)$ is the restriction to \mathbb{R}^+ of a group of automorphisms of \mathfrak{M} which leave ϕ invariant, we say that the system is *conservative*; otherwise we say that $\{\mathfrak{M}, \phi, \gamma(\mathbb{R}_+)\}$ is a *dissipative dynamical system*.

Definition. We say that the dynamical system $\{\mathfrak{M}, \phi, \gamma(\mathbb{R}_+)\}$ *admits a conservative dilation* if there exist:

- (i) a conservative dynamical system $\{\mathfrak{N}, \psi, \alpha(\mathbb{R})\}$, and
- (ii) a pair (i, \mathcal{E}) where $i: \mathfrak{M} \rightarrow \mathfrak{N}$ is an injective $*$ -isomorphism and $\mathcal{E}: \mathfrak{N} \rightarrow \mathfrak{M}$ is such that the composite map $i \circ \mathcal{E}$ is a projection of norm one of \mathfrak{N} onto its von Neumann subalgebra $i(\mathfrak{M})$, such that:

(iii) $\phi \circ \mathcal{E} = \psi$, $\psi \circ i = \phi$ and $\mathcal{E} \circ \alpha_t \circ i = \gamma_t$ for all $t \geq 0$.

(\mathcal{E} is called a ‘conditional expectation’ from \mathfrak{N} to \mathfrak{M} : it verifies the equation $\mathcal{E}(i(A) X i(B)) = A \mathcal{E}(X) B$ for any $A, B \in \mathfrak{M}$, $X \in \mathfrak{N}$).

Consider the case where $\mathfrak{M} = \mathfrak{M}_2(\mathbb{C})$ is the algebra of 2×2 matrices, ϕ is a faithful non-tracial state of \mathfrak{M} , and $\gamma(\mathbb{R}^+)$ is a semigroup of completely positive maps. In a previous article [3], G. G. Emch and the author have shown that for this system to admit a conservative dilation, it is necessary that the generator of the semigroup $\gamma(\mathbb{R}^+)$ satisfy a Bloch equation in standard form: we may summarize this condition as follows. Choose an orthonormal basis for \mathbb{C}^2 for which ϕ has a diagonal density matrix, and let \mathbf{a} be an off-diagonal matrix unit for this basis. Then $\eta := \phi(\mathbf{a}^* \mathbf{a})$ satisfies $0 < \eta < 1$ and $\eta \neq \frac{1}{2}$; and γ is determined by $\lambda, \mu, \omega \in \mathbb{R}$, subject to $0 \leq \mu \leq 2\lambda$, through the equations

$$\gamma_t(\mathbf{a}^* \mathbf{a} - \eta 1) = e^{-\mu t} (\mathbf{a}^* \mathbf{a} - \eta 1), \quad \gamma_t(\mathbf{a}) = e^{-(\lambda - i\omega)t} \mathbf{a}. \quad (1)$$

In [3] it was noted that the evolution is quasifree [4] if and only if $\mu = 2\lambda$, and in this case a dilation scheme is available so that \mathfrak{N} is a CAR algebra and ψ and $\alpha(\mathbb{R})$ are quasifree, by a modification of the algorithm developed in [2]. Before proceeding to the general case, we establish some notations.

(i) Let $K := L^2(\mathbb{R})$, and let $\mathcal{A}(K)$ be the CAR algebra over K , and let ψ_0 be the quasifree state of $\mathcal{A}(K)$ determined by $\psi_0(a^*(f) a(g)) := \eta(g | f)$; since ψ_0 is faithful (because of $0 < \eta < 1$), we may regard $\mathcal{A}(K)$ as a concrete C^* -algebra acting on the GNS representation space corresponding to ψ_0 , with bicommutant $\mathcal{A}(K)''$. We put $\mathfrak{N} := \mathcal{A}(K)'' \bar{\otimes} L^\infty(\mathbb{R})$.

(ii) For $t, x \in \mathbb{R}$, let

$$f_1(x) := \frac{\lambda - \mu/2}{\pi((\lambda - \mu/2)^2 + (x + \omega)^2)}, \quad f_2(x) := \sqrt{\frac{\mu/2\pi}{\mu^2/4 + x^2}}, \quad u_t(x) := e^{-itx}.$$

Observe that $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^2(\mathbb{R})$, $u_t \in L^\infty(\mathbb{R})$, and that f_1 is strictly positive if $\mu < 2\lambda$; in this case $g \mapsto \int f_1(x) g(x) dx$ is a faithful normal state of $L^\infty(\mathbb{R})$, which we denote by ψ_1 . In particular, $\psi_1(u_t) = \exp(-(\lambda - i\omega - \mu/2)t)$ for $t \geq 0$. We denote by ψ the product state $\psi := \psi_0 \otimes \psi_1$ of \mathfrak{N} .

(iii) For $f \in K$, $g \in L^\infty(\mathbb{R})$, $t \in \mathbb{R}$, we define

$$\alpha_t(1 \otimes g) := 1 \otimes g, \quad \alpha_t(a(f) \otimes 1) := a(u_t f) \otimes u_t, \quad (2)$$

which extends by linearity, multiplicativity and σ -weak continuity to a one-parameter group of $*$ -automorphisms of \mathfrak{N} . For all N of the form

$$N = a^*(f_1) a^*(f_2) \cdots a^*(f_m) a(g_n) \cdots a(g_2) a(g_1) \otimes h,$$

we have that $\psi \circ \alpha_t(N) = \psi(N)$ since $\eta(u_t g_j | u_t f_i) = \eta(g_j | f_i)$ and since $\delta_{mn} u_{(n-m)t} h = \delta_{mn} h$, for all $t \in \mathbb{R}$; thus ψ is invariant under each α_t , because such N generate a σ -weakly dense subset of \mathfrak{N} .

(iv) Let $i: \mathfrak{M} \rightarrow \mathfrak{N}$ be the embedding defined by $i(\mathbf{a}) = a(f_2) \otimes 1$. To define $\mathcal{E}: \mathfrak{N} \rightarrow \mathfrak{M}$ we recall that by [4, Appendix] there exists a normal conditional expectation $\mathcal{E}_0: \mathcal{A}(K)'' \rightarrow \mathcal{A}(\mathbb{C}) \simeq \mathfrak{M}$, faithful since $0 < \eta < 1$, such that $\phi \circ \mathcal{E}_0 = \psi_0$, where in particular

$$\mathcal{E}_0(a(h)) = (h | f_2) \mathbf{a} \quad \text{and} \quad \mathcal{E}_0(-a^*(g) a(h) + \eta 1) = -(h | f_2)(f_2 | g) \mathbf{a}^* \mathbf{a} + \eta (h | g) 1,$$

for $g, h \in K$. For $A \in \mathcal{A}(K)''$, $f \in L^\infty(\mathbb{R})$, put $\mathcal{E}(A \otimes f) := \mathcal{E}_0(A) \psi_1(f)$; then \mathcal{E} extends to a faithful normal conditional expectation of \mathfrak{N} onto \mathfrak{M} such that $\phi \circ \mathcal{E} = \psi$.

Theorem. *Let ϕ be a faithful non-tracial state of $\mathfrak{M} = \mathfrak{M}_2(\mathbb{C})$ and let $\gamma(\mathbb{R}^+)$ be the evolution of \mathfrak{M} defined via (1); then if $0 < \mu < 2\lambda$, the system $\{\mathfrak{N}, \psi, \alpha(\mathbb{R})\}$ defined above is a minimal conservative dilation of the dissipative system $\{\mathfrak{M}, \phi, \gamma(\mathbb{R}^+)\}$.*

Proof. We must show that $\mathcal{E} \circ \alpha_t \circ i = \gamma_t$ for all $t \geq 0$. Since both sides of this equation are faithful completely positive maps which preserve ϕ , and since \mathfrak{M} is linearly spanned by $\{1, \mathbf{a}, \mathbf{a}^*, \mathbf{a}^* \mathbf{a}\}$, it suffices to check the following two equalities:

$$\begin{aligned} \phi(\mathbf{a}^* \mathcal{E} \circ \alpha_t \circ i(\mathbf{a})) &= \psi(i(\mathbf{a}^*) \alpha_t \circ i(\mathbf{a})) = \psi(a^*(f_2) a(u_t f_2) \otimes u_t) \\ &= \eta (f_2 | u_t f_2) \psi_1(u_t) = \eta e^{-\mu t/2} e^{-(\lambda - i\omega - \mu/2)t} \\ &= \phi(\mathbf{a}^* \gamma_t(\mathbf{a})); \quad \text{and} \\ \phi(\mathbf{a}^* \mathcal{E} \circ \alpha_t \circ i(\mathbf{a}^* \mathbf{a}) \mathbf{a}) &= \psi(a^*(f_2) a^*(u_t f_2 a(u_t f_2)) a(f_2) \otimes u_t^* u_t) \\ &= \eta^2 (1 - |(f_2 | u_t f_2)|^2) = \eta^2 (1 - e^{-\mu t}) \\ &= \phi(\mathbf{a}^* \gamma_t(\mathbf{a}^* \mathbf{a}) \mathbf{a}). \end{aligned}$$

Finally, let $\mathfrak{N}_0 := \{\alpha_t \circ i(\mathfrak{M}) : t \in \mathbb{R}\}''$. Observe that

$$[a^*(u_s f_2) \otimes u_s^*, a(u_t f_2) \otimes u_t]_+ = (f_2 | u_{s-t} f_2) 1 \otimes u_{t-s} = e^{-\mu(s-t)/2} 1 \otimes u_{t-s}$$

and thus $1 \otimes u_{t-s} \in \mathfrak{N}_0$, so that $1 \otimes L^\infty(\mathbb{R}) = \{1 \otimes u_t : t \in \mathbb{R}\}'' \subset \mathfrak{N}_0$. Since $a(u_t f_2) \otimes 1 = (a(u_t f_2) \otimes u_t)(1 \otimes u_{-t})$, we also obtain $\mathcal{A}(K)'' \otimes 1 \subset \mathfrak{N}_0$ and hence $\mathfrak{N} = \mathcal{A}(K)'' \overline{\otimes} L^\infty(\mathbb{R}) = \mathfrak{N}_0$, which establishes the minimality of the dilation. \square

Remarks. (1) For the remaining cases of $0 \leq \mu \leq 2\lambda$, we note that if $0 = \mu = 2\lambda$ the system is already conservative; if $0 < \mu = 2\lambda$ the scheme above may be modified to yield a *quasifree* minimal dilation with $\mathfrak{N} = \mathcal{A}(K)''$, $\psi = \psi_1$; and finally if $0 = \mu < 2\lambda$ we likewise obtain a minimal dilation with $\mathfrak{N} = \mathfrak{M} \overline{\otimes} L^\infty(\mathbb{R})$.

(2) The semigroup property of the evolution is not essential to the existence of a minimal dilation. Indeed [6], if F_1 and F_2 are any pair of positive-type functions on \mathbb{R} which are Fourier transforms of strictly positive integrable functions, then Eqn. (1) with its coefficients replaced by $|F_1(t)|^2$ and $F_1(t) F_2(t)$ respectively, yields a collection of maps γ_t on \mathfrak{M} which admits a conservative dilation.

(3) The dilated algebra \mathfrak{N} is a *factor*, and the evolution $\alpha(\mathbb{R})$ is *ergodic*, if and only if the reduced evolution $\gamma(\mathbb{R}^+)$ is *quasifree* [4]; indeed, if $\gamma(\mathbb{R}^+)$ is not quasifree ($\mu < 2\lambda$), the centre of \mathfrak{N} is $\mathbb{C} 1 \otimes L^\infty(\mathbb{R})$, which is left pointwise invariant (2) by the automorphisms α_t . In the quasifree case [2], the condition that $\tau \circ \gamma_t \rightarrow \phi$ as $t \rightarrow \infty$ for all states τ of \mathfrak{M} is sufficient to guarantee ergodicity (indeed, one gets a K -flow) but our example shows that such is not the case for non-quasifree evolutions; it thus answers negatively a question raised in [1].

- (4) The evolution $\alpha(\mathbb{R})$ can be alternatively defined by writing $a(u_t f) \otimes u_{(\lambda-\mu/2)t}$ on the right-hand side of (2); this procedure suggests that the case of a quasifree heat bath for a $\frac{1}{2}$ -spin may be considered as a limiting case of the more usually observed physical situation for which the inverse relaxation times μ and λ satisfy $\mu < 2\lambda$.

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