Minimum depth of double cross product extensions

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Resumen

En este artículo exploramos las profundidades mínimas, par e impar, de extensiones de subalgebras en el contexto de productos dobles cruzados de algebras de Hopf de dimensión finita. Comenzamos por definir álgebras de factorización y detallamos como la profundidad de subanillos se relaciona con profundidad de módulos bajo este concepto. A continuación estudiamos la profundidad mínima impar para productos dobles cruzados y determinamos su valor en términos de la profundidad de módulos, concluimos que la profundidad mínima impar de un álgebra de Hopf de dimensión finita H en su doble de Drinfel'd $D(H)$ es 3. Finalmente producimos una condición necesaria y suficiente para que un álgebra de Hopf A tenga profundidad par mínima igual a 2 en un producto doble cruzado $A \bowtie B$. Esta condición suficiente es utilizada luego para producir resultados de profundidad mínima igual a 2 en el caso del doble de Drinfel'd, particularmente en el caso de álgebras de grupo finito. Finalmente producimos fórmulas para el centralizador de una subálgebra normal en un producto doble cruzado

Abstract

In this paper we explore minimum odd and minimum even depth subalgebra pairs in the context of double cross products of finite dimensional Hopf algebras. We start by defining factorization algebras and outline how subring depth in this context relates with the module depth of the regular left module representation of the given subalgebra. Next we study minimum odd depth for double cross product Hopf subalgebras and determine their value in terms of their related module depth, we conclude that minimum odd depth of a Hopf subalgebra H in its Drinfel'd double $D(H)$ is 3. Finally we produce a necessary and sufficient condition for depth 2 of a Hopf subalgebra A in a double cross product $A \bowtie B$. This sufficient condition is then used to prove results regarding minimum depth 2 in Drinfel'd double Hopf subalgebras, particularly in the case of finite Group Hopf algebras. Lastly we provide formulas for the centralizer of a normal Hopf subalgebra in a double cross product scenario.

Key words: Subring depth, Hopf subalgebras, Double cross product Hopf algebras, Drinfel'd double, normality.

Mathematics subject classification: 16S40, 16E99, 16T20

1 Introduction and preliminaries

The study of ring extensions and in particular finite dimensional algebra extensions has been central in the development of abstract algebra for the grater part of the last hundred years. The concept of depth of a ring extension can be traced back to 1968 to Hirata' s work generalizing certain aspects of Morita theory [16]. This work was followed by Sugano in [30], and others throughout the nineteen seventies and the nineteen eighties such as [31], [27] and [29]

In 1972 W. Singer introduced the idea of a matched pair of Hopf algebras in the connected case [28], this was extended by Takeuchi [32] in the early nineteen eighties by considering the non connected case. Both these works set the basis for the study of double cross products of Hopf algebras that was brought forward by S. Majid and others [25], starting from the early nineteen nineties.

More recently in the early two thousands the idea of depth of a ring extension was further studied in the context of Galois coring structures [17], and to characterize structure properties involving self duality, Forbenius extensions and normality such as in [20], [6] and [22]. Moreover, a fair amount of research regarding combinatorial aspects of finite group extensions has been done recently as well, we point out [3] and [2]. Other interesting results may be found in [7], [8], [9] and [10]. Again, in recent years, the study of depth in the context of finite dimensional Hopf algebra extensions has been developed, for example [3], [11], [12], [13], [14], [18], [21] and others. Is in the spirit of the latter that we develop the work presented here, in the context of extensions of finite dimensional Hopf algebras in double cross products of Hopf algebras.

Throughout this paper all rings R and algebras A are associative with unit, all algebras are finite dimensional over a field k of characteristic zero. All modules

M are finite dimensional as well. All subring pairs $S \subseteq R$ satisfy $1_S = 1_R$ and we denote the extension as $S \hookrightarrow R$.

The paper is organized as follows: In Subsection (1.1) preliminaries on the concept of depth will be reviewed. Mainly definitions on subring depth, the concept of module depth in a tensor category and some results that will be of interest further into this study. Other concepts will be introduced when needed.

Section (2) deals with the concept of an algebra extension that factorizes as a tensor product of two subalgebras. We adapt the concept of subring depth to this scenario and prove two preliminary results on depth of an extension of an algebra in a factorization algebra in Theorems (2.1), (2.2) and Corollary (2.3). Example (2.4) reviews the case of the minimum depth of a Hopf algebra H in its smash product with and H-module algebra A, in particular the case of the Heisenberg double $\mathcal{H}(H)$ of a finite dimensional Hopf algebra, which motivates the next two sections.

Section (3) deals with the definitions of double cross products as factorization algebras in Propositions (3.1) and (3.2) and explores minimum odd depth for this cases in Theorems (3.3) and (3.5).

Section (4) contains our main result in the form of Theorem (4.1). Our result establishes a necessary and sufficient condition for minimum even depth to be less or equal to 2 in the case of double cross product extensions of Hopf algebras. This sufficient condition is then used to prove particular cases for Drinfel'd double extensions in the case of finite group algebras in Corollary (4.2) and to provide formulas for the centralizer of a Hopf subalgebra in the case of a depth two double cross product extension in Proposition (4.4) and Corollary (4.5).

1.1 Preliminaries on Depth

Let R be a ring and M and N two left (or right) R-modules. We say M is similar to N as an R module if there are positive integers p and q such that $M|pN$ and $N|qM$, where nV means $\oplus^n V$ for every n and every R module V and $M|pN$ means that M is a direct summand of pN or equivalently that $M \otimes * \cong pN$, in this case we denote the similarity as $M \sim N$. Notice that this similarity is compatible with induction and restriction functors on ${}_R\mathcal{M}$, for if $R \hookrightarrow L$ is an extension of R and K is an right L module then $M \sim N$ as R modules implies $M \otimes_R K \sim N \otimes_R K$ as right L modules. Moreover, if $S \hookrightarrow R$ is a subring then $M \sim N$ as R modules implies $M \sim N$ as S modules.

Consider now a ring extension $B \hookrightarrow A$. Let $n \geq 1$, by $A^{\otimes_B (n)}$ we mean $A \otimes_B A \otimes_B \cdots \otimes_B A$ n times, and define $A^{\otimes_B (0)}$ to be B. Notice that for $n \geq 1$ $A^{\otimes_B (n)}$ has a natural X-Y-bimodule structure where $X, Y \in \{A, B\}$ and for $n = 0$ we get a B-B-bimodule structure.

Definition 1.1. *Let $B \hookrightarrow A$ be a ring extension, we say B has:*

1. ***Minimum odd depth 2n +1**, denoted $d(B,A) = 2n+1$, if $A^{\otimes_B (n+1)} \sim A^{\otimes_B (n)}$ as B-B modules for $n \geq 0$.*
2. ***Minimum even depth 2n**, denoted d(B,A) = 2n, if $A^{\otimes_B (n+1)} \sim A^{\otimes_B (n)}$ as either B-A or A-B modules for for $n \geq 1$.*

Notice that by the observation made above, one has that for all $n \geq 0$ $d(B,A) = 2n$ implies $d(B,A) = 2n+1$ by module restriction, and that for all $m \geq 1$ $d(A,B) = 2m+1$ implies $d(B,A) = 2m+2$ for all m by module induction. Hence we are only interested in the minimum values for which any of these relations is satisfied. In case there is no such minimum value we say the extension has infinite depth.

A third type of subring depth called **H-depth** denoted by $d_h(B,A) = 2n-1$ if $A^{\otimes_B (n+1)} \sim A^{\otimes_B (n)}$ as A-A modules for $n \geq 1$ was introduced by Kadison in [19] as a continuation of the study of H - separable extensions introduced by Hirata, where such extensions are exactly the ones satisfying $d_h(B,A) = 1$. For the purposes of this paper we will restrict our study to minimum odd and even depth only. In particular the cases $d(B,A) \leq 3$ and $d(A,B) \leq 2$.

Let $B \hookrightarrow A$ be a ring extension, $R = A^B$ the centralizer and $T = (A \otimes_B A)^B$ the B central tensor square. It is shown in [17][Section 5] that $d(A,B) \leq 2$ implies a Galois A-coring structure in $A \otimes_R T$ in the sense of [5]. Further more it is also shown in [17] that if the extension $B \hookrightarrow A$ is Hopf Galois for a given finite dimensional Hopf algebra H then $d(B,A) \leq 2$.

Let $R \hookrightarrow H$ be a finite dimensional Hopf algebra extension. Define their quotient module Q as H/R^+H where $R^+ = ker\varepsilon \cap R$ where ε denotes de counit of H. Suppose that R is a normal Hopf subalgebra of H, one can easily show that the extension $R \hookrightarrow H$ is Q-Galois and therefore $d(R,H) \leq 2$. The converse happens to be true as well and the details can be found in [3][Theorem 2.10]. Hence, the following result holds:

Theorem 1.2. *Let $R \hookrightarrow H$ be a finite dimensional Hopf algebra pair. Then R is a normal Hopf subalgebra of H if and only if*

$$d(R,H) \leq 2$$

Now we consider again a k algebra A and an A-module M. Recall that the n-th truncated tensor algebra of M in ${}_A\mathcal{M}$ is defined as

$$T_n(M) = \bigoplus_{i=1}^{n} M^{\otimes(n)} \quad and \quad T_0(M) = k$$

We then define the **module depth** of M in ${}_A\mathcal{M}$ as $d(M, {}_A\mathcal{M}) = n$ if and only if $T_n(M) \sim T_{n+1}(M)$. In case M is an A-module coalgebra (a coalgebra in the category of A modules) then $d(M,_A\mathcal{M}) = n$ if and only if $M^{\otimes(n)} \sim M^{\otimes(n+1)}$ [18], [12].

We point out that an A-module M has module depth n if and only if it satisfies a polynomial equation $p(M) = q(M)$ in the representation ring of A. In this case p and q are polynomials of degree at most $n+1$ with integer coefficients. A brief proof of this can be found in [11]. For this reason we say that a module M has finite module depth in $_A\mathcal{M}$ if and only if it is an algebraic element in the representation ring of A.

Finally, we would like to mention that in the case of Hopf subalgera extensions $R \hookrightarrow H$ there is a way to link subalgebra depth with module depth. The reader will find a proof of the following in [18][Example 5.2]:

Theorem 1.3. *Let $R \hookrightarrow H$ a Hopf subalgebra pair. Consider their quotient module Q, then the minimum depth of the extension satisfies:*

$$2d(Q,_R\mathcal{M}) + 1 \leq d(R,H) \leq 2d(Q,_R\mathcal{M}) + 2$$

2 Depth of factorization algebra extensions

Let A and B be two finite dimensional algebras. Consider the following map:

$$\psi : B \otimes A \longrightarrow A \otimes B \quad ; \quad b \otimes a \longmapsto a_\alpha \otimes b^\alpha$$

such that

$$\psi(1_B \otimes a) = a \otimes 1_B, \quad \psi(b \otimes 1_A) = 1_A \otimes b$$

for all $a \in A$ and $b \in B$. Moreover suppose ψ satisfies the following commutative octagon for all $a, d \in A$, and all $b, c \in B$:

$$(ad_\alpha)_\beta \otimes b^\beta c^\alpha = a_\beta d_\alpha \otimes (b^\beta c)^\alpha \tag{1}$$

We call ψ a factorization of A and B and $A \otimes_\psi B$ a factorization algebra of A and B, a unital associative algebra with product

$$(a \otimes b)(c \otimes d) = a\psi(b \otimes c)d = ac_\alpha \otimes b^\alpha d \tag{2}$$

where $a, c \in A$, $b, d \in B$ and the unit element is $1_A \otimes 1_B$. Besides A and B are $A \otimes_\psi B$ subalgebras via the inclusions $A \hookrightarrow A \otimes_\psi 1_B$ and $B \hookrightarrow 1_A \otimes_\psi B$.

Factorization algebras are ubiquitous: Setting $\psi(b \otimes a) = a \otimes b$ yields the tensor algebra $A \otimes B$. If H is a Hopf algebra and A a left H-module algebra satisfying

$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$, $h \cdot 1_A = \varepsilon(h)1_A$ for all $h \in H$ and $a, b \in A$, define $\psi : H \otimes A; \quad h \otimes a \longmapsto h_1 \cdot a \otimes h_2$ then the product becomes $(a \otimes h)(b \otimes g) = a\psi(h \otimes b)g = a(h_1 \cdot b \otimes h_2)g = ah_1 \cdot b \otimes h_2 g$. It is a routine exercise to verify that $A \otimes_\psi H$ is a factorization algebra and that $A \otimes_\psi H = A\#H$ is the smash product of A and H. Double cross products of Hopf algebras are also examples of factorization algebras, we will study them further in Section (3).

Now let $A \otimes_\psi B$ be a factorization algebra via $\psi : B \otimes A \longmapsto A \otimes B$. For the sake of brevity we will denote it $S_\psi = A \otimes_\psi B$ for the rest of this Section. We point out that due to multiplication in S_ψ and the fact that both A and B are subalgebras of S_ψ we get that for every $n \geq 1$, $S_\psi^{\otimes_B (n)} \in {}_{S_\psi}\mathcal{M}_{S_\psi}$ in the following way:

$$
\begin{aligned}
&(a \otimes_\psi b)(a_1 \otimes b_1 \otimes_B \cdots \otimes_B a_n \otimes b_n)(c \otimes_\psi d) = \\
&= a\psi(b \otimes a_1)b_1 \otimes_B \cdots \otimes_B a_n \psi(b_n \otimes c)d = \\
&= aa_{1\alpha} \otimes b^\alpha b_1 \otimes_B \cdots \otimes_B a_n c_\alpha \otimes b_n^\alpha d
\end{aligned} \tag{3}
$$

The same condition holds for S_ψ as either left or right B module via subalgebra restriction. In this case we can assume $n \geq 0$ and define $S_\psi^{\otimes_B (0)} = B$. This allows us to consider the following isomorphism:

Theorem 2.1. *Let A and B be algebras, $\psi : B \otimes A \longmapsto A \otimes B$ a factorization and S_ψ the corresponding factorization algebra. Then:*

$$S_\psi^{\otimes_B (n)} \cong A^{\otimes (n)} \otimes B \tag{4}$$

as X-Y-bimodules, with X,$Y \in \{S_\psi, B\}$ for $n \geq 1$ and as B-B-bimodules for $n \geq 0$.

Proof. First notice that for $n = 1$, $A \otimes_\psi B \cong A \otimes B$ via $a \otimes_\psi b \longmapsto a \otimes b$, since $A \otimes_\psi B$ is an algebra and multiplication is well defined.

Now, for every $n > 1$, $(A \otimes_\psi B)^{\otimes_B (n)} \cong (A \otimes_\psi B)^{\otimes_B (n-1)} \otimes_B (A \otimes_\psi B)$. By induction on n and using that $B \otimes_B A \cong A$ one gets:

$$
\begin{aligned}
(A \otimes_\psi B)^{\otimes_B (n-1)} \otimes_B A \otimes_\psi B &\cong A^{\otimes_B (n-1)} \otimes B \otimes_B A \otimes B \\
&\cong A^{\otimes (n-1)} \otimes A \otimes B \cong A^{\otimes (n)} \otimes B
\end{aligned} \tag{5}
$$

Finally for $n = 0$ we get $S_\psi^{\otimes_B (0)} = B \cong k \otimes B \cong A^{\otimes (0)} \otimes B$ as B-B bimodules. □

Recall that a Krull-Schmidt category is a generalization of categories where the Krull-Schmidt Theorem holds. They are additive categories such that each object decomposes into a finite direct sum of indecomposable objects having local endomorphism rings, also this decompositions are unique in a categorical sense. For example categories of modules having finite composition length are Krull-Schmidt.

Theorem (2.1) in the context of a Krull-Schmidt category, allows us relate subalgebra depth in a factorization algebra with module depth in the finite tensor category of finite dimensional left B-modules. In turn this will allow us to compute minimum odd depth values in the case of Smash Product algebras and Drinfel'd Double Hopf algebras at the end of this Section as well as in Section (3). The next Theorem and its Corollary provide this connection and they mirror [18][Equation 21] and [12][Equation 21].

Theorem 2.2. *Let $A \otimes_\psi B$ be a factorisation algebra with ${}_B\mathcal{M}_B$ a Krull-Schmidt category, and $A \in_B \mathcal{M}$ Then the minimum odd depth of the extension satisfies:*

$$d(B, S_\psi) \leq 2d(A,_B \mathcal{M}) + 1 \tag{6}$$

Proof. Let $d(A,_B \mathcal{M}_B) = n$. Since ${}_B\mathcal{M}_B$ is a Krull-Schmidt category, standard face and degeneracy functors imply $A^{\otimes_B (m)} | A^{\otimes_B (m+1)}$ for $m \geq 0$. Then $T_n(A) \sim T_{n+1}(A)$ implies $A^{\otimes(n+1)} \sim A^{\otimes(n)}$. Tensoring on the right by $(- \otimes B)$ one gets $A^{\otimes(n+1)} \otimes B \sim A^{\otimes(n)} \otimes B$. By Theorem (2.1) this is equivalent to $(A \otimes_\psi B)^{\otimes_B (n+1)} \sim (A \otimes_\psi B)^{\otimes_B (n)}$. This by definition is $d(B, S_\psi) \leq 2n + 1$. □

Recall that B is a bialgebra if it is both an algebra and a coalgebra such that the coalgebra morphisms are algebra maps, i.e. B is a coalgebra in the category of k algebras. This means that the counit $\varepsilon : B \longrightarrow k$ is an algebra map that splits the coproduct: $(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id$. Via the counit the ground field k becomes a trivial B module via $b \cdot k = \varepsilon(b)k$. Hence, a k vector space V becomes a right B-module via : $V \cong V \otimes k$.

Corollary 2.3. *Let B be a bialgebra. Then the inequality in Theorem* (2.2) *becomes an equality.*

Proof. Let B be a bialgebra, since k becomes a B-module via the counit of B, tensoring by $- \otimes_B k$ or $k \otimes_B -$ is a morphism of B modules. Let $d(B, S_\psi) = 2n+1$, then by definition $S_\psi^{\otimes_B (n)} \sim S_\psi^{\otimes_B (n+1)}$ as B-B bimodules, and by the isomorphism in Theorem (2.1) this implies $A^{\otimes(n)} \otimes B \sim A^{\otimes(n+1)} \otimes B$, then it suffices to tensor on the right by $(- \otimes_B k)$ on both sides of the similarity to get $A^{\otimes n+1} \sim A^{\otimes n}$ which in turn implies $d(A,_B \mathcal{M}) \leq n$. □

Notice that assuming that $A \in_B \mathcal{M}$ makes sense since the factorization algebras we are considering next all depend on this fact to be well defined. On the other hand this result says nothing about *even* depth since by no means one should expect A to be a right or left S_ψ-module.

Example 2.4. *[12, Theorem 6.2] Let H be a Hopf algebra and A an H-module algebra, consider their smash product algebra $A\#H$ and the algebra extension $H \hookrightarrow A\#H$. The extension satisfies:*

$$d(H, A\#H) = d(A,_H \mathcal{M}) + 1$$

Moreover, as a consequence of this one can show the following: Let $dim_k(H) \geq 2$ and consider H^* as a H-module algebra via $h \rightharpoonup f$ and their smash product $H^*\#H$, also known as their Heisenberg double, then the extension $H \hookrightarrow H^*\#H$ satisfies

$$d(H, H^*\#H) = 3$$

This follows since H is a factor $H^*\#H$ subalgebra and the fact that ${}_{H^*}H \cong H^*_{H^*}$ and that minimum depth satisfies $d(H^*, \mathcal{M}_{H^*}) = 1$.

This example motivates the question of whether this result (or an equivalent one) can be attained for a more general class of extensions of Hopf algebras into factorization algebras. The next two Sections deal with this question in the context of the Drinfel'd double $D(H)$ of a Hopf algebra and more generally in the case of the double cross product $A \bowtie B$ of a matched pair of Hopf algebras A and B.

3 Double cross products and minimum odd depth

As we pointed out in the Introduction, the study of double cross products was started in the early seventies by W. Singer with the introduction of matched pairs of Hopf algebras satisfying certain module-comodule factorization conditions in the case of connected module categories, [28]. Later M. Takeuchi [32] furthered the study of matched pairs in the ungraded case, in particular, he aimed at describing natural properties of braided groups. Later S. Majid [25] studied bicrossed products as a means to construct self dual objects in the category of Hopf algebras primarily in the case of non commutative non cocommutative cases, in some sense motivated by the possibility to construct models for quantum gravity. We follow Majid's definition of double cross products as in [24].

Let A and B be two Hopf algebras such that A is a left B-module coalgebra and B a right A-module coalgebra. We say B and A are a matched pair [24][Definition 7.2.1] if there are coalgebra maps

$$\alpha : B \otimes A \longrightarrow B; \quad h \otimes k \longmapsto h \triangleleft k \quad \text{and} \quad \beta : B \otimes A \longrightarrow A; \quad h \otimes k \longmapsto h \triangleright k$$

such that the following compatibility conditions hold:

$$(hg) \triangleleft k = \sum (h \triangleleft (g_1 \triangleright k_1))(g_2 \triangleleft k_2); \quad 1_B \triangleleft k = \varepsilon_A(k) 1_B \tag{7}$$

and
$$h \triangleright (kl) = \sum (h_1 \triangleright k_1)((h_2 \triangleleft k_2) \triangleright l); \quad h \triangleright 1_A = \varepsilon_B(h) 1_A \tag{8}$$

Define a product by

$$(k \bowtie h)(l \bowtie g) = \sum k(h_1 \triangleright l_1) \bowtie (h_2 \triangleleft l_2) g \tag{9}$$

the resulting algebra $B \bowtie A$ is called the **double crossed product** of A and B [24, Theorem 7.2.2], and is a Hopf algebra with coproduct, counit and antipode given by

$$\Delta(k \bowtie h) = k_1 \bowtie h_1 \otimes k_2 \bowtie h_2 \tag{10}$$

$$\varepsilon(k \otimes h) = \varepsilon_K(k)\varepsilon_H(h) \tag{11}$$

$$\begin{aligned} S(k \bowtie h) &= (1_K \bowtie S_H(h))(S_K(k) \bowtie k) \\ &= S_H(h_1) \triangleright S_K(k_1) \bowtie S_H(h_2) \triangleleft S_K(k_2) \end{aligned} \tag{12}$$

respectively.

The following are well known results and are cited here for the sake of completeness, they summarize the fact that Double Cross Products of Hopf algebras are exactly the Hopf algebras that factorize as the product of two Hopf subalgebras. The reader can refer to them in [24] and [25] as well as in [4].

Proposition 3.1. *Double crossed products are factorisation algebras*

The converse is also true:

Proposition 3.2. *[24, Theorem 2.7.3] Suppose H is a Hopf algebra and L and A two sub-Hopf algebras, such that $H \cong A \otimes_\psi L$ is a factorisation, then H is a double crossed product.*

Proof. The multiplication $m : L \otimes A \longrightarrow H$ defined by $a \otimes l \longmapsto al$ is a bijection. This implies $A \bigcap L = k$. Then consider the map:

$$\mu : L \otimes A \longrightarrow A \otimes L; \quad l \otimes a \longmapsto m^{-1}(la)$$

then define

$$\triangleright : L \otimes A \longrightarrow A; \quad l \triangleright a = ((\varepsilon_L \otimes Id) \circ \mu)(l \otimes a)$$

$$\triangleleft : L \otimes A \longrightarrow L; \quad l \triangleleft a = ((Id \otimes \varepsilon_A) \circ \mu)(l \otimes a)$$

□

We wrote the proof of this last Proposition since it allows us to construct examples such as Example (3.4).

Now, let H be any Hopf algebra with bijective antipode S with composition inverse $\overline{S}$. Let S^* be the bijective antipode of H^* and $\overline{S^*}$ its composition inverse, then H is a right H^{*cop}-module coalgebra via

$$h \leftharpoondown\!\!\leftharpoondown f = \sum \overline{S^*}(f_2) \rightharpoonup h \leftharpoonup f_1$$

and H^* is a left H-module coalgebra via

$$h \rightharpoonup\!\!\rightharpoonup f = \sum h_1 \rightharpoonup f \leftharpoonup \overline{S}(h_2)$$

see [26][Chapter 10] for details on this actions. Define the Drinfel'd double of H, $D(H)$ as the double cross product $H^{*cop} \bowtie H$ with product

$$(f \bowtie h)(g \bowtie k) = \sum f(h_1 \rightharpoonup\!\!\rightharpoonup g_2) \bowtie (h_2 \leftharpoondown\!\!\leftharpoondown g_1)k$$

The coproduct, counit and antipode are given by

$$\Delta(f \bowtie h) = \sum (f_2 \bowtie h_1) \otimes (f_1 \bowtie h_2)$$

$$\varepsilon_{D(H)}(f \bowtie h) = \varepsilon_{H^*}(f)\varepsilon_H(h)$$

and

$$S_{D(H)}(f \bowtie h) = \sum (S(h_2) \rightharpoonup S(f_1)) \bowtie (f_2 \leftharpoonup S(h_1))$$

respectively.

Since double crossed products of Hopf algebras are both factorization algebras and Hopf algebras Corollary (2.3) becomes:

Proposition 3.3. *Let H and K be a matched pair of Hopf algebras and consider their double crossed product $H \bowtie K$, then the Hopf algebra extension $H \hookrightarrow H \bowtie K$ satisfies*

$$d(H, H \bowtie K) = 2d(K,_H \mathcal{M}) + 1$$

Example 3.4. *Recall that two Hopf algebras A and B are said to be paired [25][1.4.3] if there is a bilinear map*

$$A \otimes B \longrightarrow k; a \otimes b \longmapsto \langle a, b\rangle$$

Satisfying $\langle ac, b\rangle = \langle a \otimes c, \Delta b\rangle$, $\langle a, 1\rangle = \varepsilon(a)$, $\langle 1, b\rangle = \varepsilon(b)$ and $\langle Sa, b\rangle = \langle a, Sb\rangle$. We also say it is nondegenerate if and only if $\langle a, b\rangle = 0$ for all $b \in B$ implies $a = 0$

and $\langle a,b\rangle = 0$ *for all* $a \in A$ *implies* $b = 0$. *Assume now that* A *and* B *are paired and that* $\langle,\rangle$ *is convolution invertible, define*

$$a \triangleleft b = \sum a_2 \langle a_1, b_1\rangle^{-1}\langle_3, b_2\rangle$$

$$a \triangleright b = \sum b_2 \langle a_, b_1\rangle^{-1}\langle a_2, b_3\rangle$$

With this action we can endow $A^{op} \bowtie B$ *with a double cross product structure. Consider then* H *to be a finite dimensional Hopf algebra and*

$$\langle,\rangle : H \otimes H \longrightarrow k; h \otimes g \longmapsto \varepsilon(a)\varepsilon(b)$$

then $\langle,\rangle$ *satisfies the conditions above, is nodegenerate if and only if* H *is semisimple via Maschke's theorem and is convolution invertible via* $\langle,\rangle\langle,\rangle = \varepsilon$ *Then* $H^{op} \bowtie H$ *is a double cross product isomorphic to the tensor Hopf algebra* $H^{op} \otimes H$, *Proposition* (3.2), *and the minimum odd depth satisfies*

$$d(H, H^{op} \bowtie H) = 3$$

Since $d(H,_{H^{op}} \mathcal{M}) = 1$.

Proposition 3.5. *Let* H *be a finite dimensional Hopf algebra of dimension* $m \geq 2$ *and consider* $D(H) = H^{*cop} \bowtie H$ *its Drinfel'd double. Then the minimum odd depth satisfies:*

$$d(H, D(H)) = 3$$

Proof. The proof is analogous to the one in Example (2.4). □

This result should not come as a surprise: Whenever H is cocommutative it is easy to show that its Drinfel'd double and its Heisenberg double are isomorphic as algebras and since given two isomorphic algebras A and B and an A-module M, module depth satisfies $d(M,_A \mathcal{M}) = d(M,_B \mathcal{M})$, it is immediate that for cocommutative H, minimum odd depth is given by $d(H, D(H)) = d(H, H^*\#H) = 3$. But it is straightforward that depth does not depend on the cocommutativity of the coalgebra structure on H.

4 Depth two

Consider a finite group algebra kG and its dual $(kG)^* = k\langle p_x | x \in G\rangle$ where the $\{p_x\}$ form the dual basis of G satisfying $p_x(y) = \delta_{x,y}$ for all $x, y \in G$. This is an algebra via convolution product and the identity element is $\varepsilon = \sum_{y\in G} p_y$. It is easy to check that $(kG)^*$ has a Hopf algebra structure given by

$$\Delta^* p_x = \sum_{lk=x} p_l \otimes p_k$$

$$\varepsilon^*(p_x) = \delta_{x,1}$$

And antipode S^*.

Consider then $R = kG$ a finite group algebra and $H = D(kG) = (kG)^{*cop} \bowtie kG$ its Drinfel'd double. Multiplication is given by

$$(p_x \bowtie g)(p_y \bowtie k) = p_x p_{gyg^{-1}} \bowtie gk$$

and the antipode is

$$S(p_x \bowtie g) = (\varepsilon \bowtie g^{-1})(S^* p_x \bowtie e) = S^* p_{g^{-1}xg} \bowtie g^{-1}$$

Let $p_x = p_x \bowtie e \in (kG)^*$, and $p_y \bowtie g \in H$. The right adjoint action of H on $(kG)^*$ is given by

$$S(p_y \bowtie g)_1 (p_x \bowtie e)(p_y \bowtie g)_2 = \sum_{lk=y} S(p_l \bowtie g)(p_x \bowtie e)(p_k \bowtie g)$$

a quick calculation and using the formulas above shows that the latter equals

$$\sum_{lk=y} S^*(p_{g^{-1}lg}) p_{g_{-1}xg} p_{g^{-1}kg} \bowtie e$$

A similar calculation shows that the left adjoint action of H on $(kG)^*$ yields

$$(p_y \bowtie g)_1 (p_x \bowtie e) S(p_y \bowtie g)_2 = \sum_{lk=y} p_l p_{gxg^{-1}} p_k \bowtie e$$

and hence $(kg)^*$ is H left and right ad stable and hence normal. As it is shown in Theorem (1.2) this implies then that

$$d((kG)^*, H) \leq 2$$

We point out that this is true since the left coadjoint action of $(kG)^*$ on kG given by $\leftharpoonup$ is trivial on the generators:

$$g \leftharpoonup p_x = g$$

The following theorem tells us that this is in fact a necessary and sufficient condition for depth 2 in the more general case of double cross products:

Theorem 4.1. *Let A, B be a matched pair of Hopf algebras and let $H = A \bowtie B$ be their double cross product. Then $d(A, H) \leq 2$ (Equivalently $d(B, H) \leq 2$)if and only if $B \triangleleft A$ (Equivalently $B \triangleright A$) is trivial.*

Proof. Let $A \bowtie B$ be a double cross product of Hopf algebras. Recall that an extension of finite dimensional Hopf algebras has depth ≤ 2 if and only if the extension is normal. Let $a \bowtie 1_B \in A$ and $h \bowtie g \in A \bowtie B$. Consider the right adjoint action of $A \bowtie B$ on A:

$$S(h_1 \bowtie g_1)(a \bowtie 1_B)(h_2 \bowtie g_2) = (Sg_1 \triangleright Sh_1 \bowtie Sg_1 \triangleleft Sh_1)(a \bowtie 1_B)(h_2 \bowtie g_2)$$
$$= ((Sg_1 \triangleright Sh_1)((Sg_2 \triangleleft Sh_2) \triangleright a_1))(((Sg_3 \triangleleft Sh_3) \triangleleft a_2) \triangleright h_4)$$
$$\bowtie (((Sg_4 \triangleleft Sh_5) \triangleleft a_3) \triangleleft h_6)g_5$$

$\in A$ if and only if

$$(((Sg_4 \triangleleft Sh_5) \triangleleft a_3) \triangleleft h_6)g_5 = \lambda 1_B$$

for some $\lambda \in k$.

Suppose that $B \triangleleft A$ is trivial, then

$$(((Sg_4 \triangleleft Sh_5) \triangleleft a_3) \triangleleft h_6)g_5 = Sg_4\varepsilon(Sh_5)\varepsilon(a_3)\varepsilon(h_5)g_5$$
$$= \varepsilon(g_4)\varepsilon(h_5)\varepsilon(a_3)1_B$$

Take $\lambda = \varepsilon(g_4)\varepsilon(h_5)\varepsilon(a_3)$.

Now assume that

$$(((Sg_4 \triangleleft Sh_5) \triangleleft a_3) \triangleleft h_6)g_5 = \lambda 1_B$$

for some $\lambda \in k$. Without loss of generality we can assume $h = 1_A$ so we obtain

$$(((Sg_4 \triangleleft Sh_5) \triangleleft a_3) \triangleleft h_6)g_5 = (Sg_3 \triangleleft a_3)g_4 = \lambda 1_B$$

apply ε on both sides of the equation to obtain

$$\varepsilon(g_3)\varepsilon(a_3) = \lambda$$

Now let $g \in B$ and $a \in A$ since the antipode is bijective let $g = Sh$, then

$$g \triangleleft a = Sh \triangleleft a = (Sh_1 \triangleleft a)h_2Sh_3 = \varepsilon(Sh_1)\varepsilon(a)Sh_2 = Sh\varepsilon(a) = g\varepsilon(a)$$

Then A is $A \bowtie B$ right ad-stable if and only if $B \triangleleft A$ is trivial.

Consider now the left adjoint action of $A \bowtie B$ on A. Then

$$(h_1 \bowtie g_1)(a \bowtie 1_b)S(h_2 \bowtie g_2) \in A$$

if and only if

$$[(g_3 \triangleleft a_3) \triangleleft (Sg_5 \triangleright Sh_3)](Sg_6 \triangleleft Sh_4) = \lambda 1_B$$

for some $\lambda \in k$. The rest of the proof mirrors what was done above and then A is left $A \bowtie B$ ad stable if and only if $B \triangleleft A$ is trivial, hence the extension is normal if and only if $B \triangleleft A$ is trivial and $d(A, A \bowtie B) \leq 2$ if and only if $B \triangleleft A$ is trivial. The case of the extension $B \hookrightarrow A \bowtie B$ is symmetric. $\square$

Corollary 4.2. *Let G be a finite group and consider $D(kG) = (kG)^{*cop} \bowtie kG$, then*

$$d(kG, D(kG)) \leq 2$$

if and only if G is abelian.

Proof. Let $g, x \in G$. Recall that the left coadjoint action of kG on $(kG)^*$ is given by $g \rightharpoonup p_x = p_{gxg^{-1}}$ which is trivial (i.e $p_{gxg^{-1}} = p_x$ for all $g, x \in G$) if and only if G is abelian. $\square$

Example 4.3. *Consider $H^{op} \bowtie H$ as in Example* (3.4)*, then the minimum depth satisfies*

$$d(H, H^{op} \bowtie H) \leq 2$$

since $h \triangleright g = \sum g_2 \langle h_1, g_1 \rangle^{-1} \langle h_2, g_3 \rangle = g_2 \varepsilon(h_1)\varepsilon(g_1)\varepsilon(h_2)\varepsilon(g_3) = g\varepsilon(h)$ for all $h, g \in H$ and hence $H \triangleright H^{op}$ is trivial.

Now consider the double cross product $H = A \bowtie B$, $Z(A)$, $C_H(A)$ and $N_H(B)$ the center of A, the centralizer of A in H and the normal core of B in H respectively. Then $C_H(A)$ satisfies the following:

Proposition 4.4. *Let $H = A \bowtie B$ be a double cross product such that $d(A, H) \leq 2$. Then*

$$C_H(A) = Z(A) \bowtie N_H(B)$$

as algebras

Proof. Let $f \bowtie k \in C_H(A)$ and $a \bowtie 1_B \in A$. Then $(f \bowtie k)(a \bowtie 1_B) = (a \bowtie 1_B)(f \bowtie k)$. On one hand we have

$$(a \bowtie 1_B)(f \bowtie k) = af_1 \bowtie (1_B \triangleleft f_2)k = af \bowtie k$$

Since depth two implies $A \triangleleft B$ is trivial. On the other hand

$$(f \bowtie k)(a \bowtie 1_B) = f(k_1 \triangleright a) \bowtie k_2$$

Now

$$f(k_1 \triangleright a) \bowtie k_2 = af \bowtie k$$

if and only if $k \triangleright a = \varepsilon(k)a$ and $fa = af$ for all $a \in A$ if and only if $k \in N_H(B)$ and $f \in Z(A)$. $\square$

Corollary 4.5. *Let kG be a finite group algebra and consider $H = D(kG)$ its Drinfel,d double. Then*

$$C_H((kG)^*) = Z((kG)^*) \bowtie Z(kG)$$

as algebras.

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