Clifford Geometry: A Seminar

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1 Vector bundles and their classification

The procedure commonly known as *geometric quantization* associates a Hilbert space to certain symplectic manifolds, and thereby allows a bridge to be built from the algebra of functions on the manifold to an operator algebra. It has been used, with considerable success, to reconstruct unitary representations of Lie groups from their symplectic homogeneous spaces; this is known as the *Kirillov orbit method*. An alternative method of constructing these representations, at least for compact groups, arises from the study of Dirac operators on these same homogeneous manifolds. The purpose of these notes is to clarify the rôle of the Dirac operator. Hopefully, this will enable us to build a bridge between geometric quantization and *noncommutative geometry*, in which the key concept is a generalization of the Dirac operator to "noncommutative manifolds".

The main geometric objects with which we shall be concerned are *vector bundles* with some extra algebraic structure, such as a metric or a symplectic structure (or both). Roughly speaking, a vector bundle consists of the disjoint union E of a collection of isomorphic vector spaces E_x, indexed by the points of a differential manifold M, together with some smoothness conditions relating E and M; an example is the tangent bundle of the manifold M. The algebraic structure of the "fibres" E_x transfers to give an interesting algebraic structure to the manifold E. The set of all possible vector bundles of a given species on a manifold M, up to a suitable equivalence relation, holds important information about the topology of the manifold M (in fact, it holds all the topological information we need to know for the purposes of noncommutative geometry) and we begin our study with this classification.

We have gathered in Appendix A a compendium of definitions and basic facts on differential manifolds, vector fields and differential forms, with which we assume the reader to be acquainted. We refer to it for the notations used below.

1.1 Generalities on manifolds

Recall that a differentiable manifold of (real) dimension n is a (paracompact) topological space M provided with an atlas, i.e., a collection of charts (U_j, ϕ_j), where the domains U_j form a (locally finite) open covering $\mathcal{U}$ of M, each $\phi_j \colon U_j \to \mathbb{R}^m$ is a homeomorphism, and the "transition maps" $\phi_i \circ \phi_j^{-1}$ are smooth functions on each $\phi_j(U_i \cap U_j)$. If $n = 2m$, we can regard $\mathbb{R}^{2m}$ as $\mathbb{C}^m$; we say that M is a *complex manifold* if the transition maps are holomorphic (as multi-variable vector-valued complex functions); it suffices that they satisfy the Cauchy–Riemann equations.

We shall be mainly interested in the case where M is compact; then its differentiable structure is defined by a finite atlas.

A useful property of the chart domains U_j is that they be *contractible*, that is, that there exist $x_0 \in U_j$ and a smooth function $f \colon [0,1] \times U_j \to U_j$ with $f(0,x) = x_0$, $f(1,x) = x$; informally, U_j may be "deformed" to a point $\{x_0\}$. The intersection of two such domains need not be contractible: just think of the sphere $\mathbb{S}^2$ covered by two large polar caps which overlap at the equator. It is good to know, however, that by using another (equivalent) atlas, we can avoid this difficulty. What we need is that the chart domains form a "good covering" of M [12].

Definition 1.1. An open covering $\mathfrak{U} := \{U_j\}$ of a topological space is a **good covering**[1] if every nonempty finite intersection $U_{j_1} \cap \cdots \cap U_{j_r}$ is contractible. A differentiable manifold M always has a good covering: take a Euclidean metric on M and cover M by an atlas for which each U_j is "geodesically convex" (that is, any two of its points can be connected by a minimal geodesic lying within U_j).[2] All finite intersections are also geodesically convex, and hence are contractible (since one can deform to a point x_0 by retreating along geodesics emanating from that point). So we can replace the original atlas of M by a (possibly larger) atlas whose domains form a good covering. From now on, we will assume that this has been done.

Exercise 1.1. Show that the following recipe defines a good covering $\{U_1, U_2, U_3, U_4\}$ of the sphere $\mathbb{S}^2$. Take four points on $\mathbb{S}^2$ that are vertices of a regular tetrahedron (e.g., if $a = 1/\sqrt{3}$, take $(\pm a, \pm a, \pm a)$ with either one or three positive signs). Connect these points by six great-circle arcs, which determine four spherical triangles F_j. Finally, let $U_j := \{\, x \in \mathbb{S}^2 : d(x, F_j) < \epsilon \,\}$ for some small ϵ (say, $\epsilon = 1/4$).

1.2 Principal bundles and vector bundles

Recall that a *fibre bundle* $E \xrightarrow{\pi} M$ or simply $E \longrightarrow M$ (see Appendix A for notation) is a triple (E, M, π), where the manifolds E and M are its *total space* and *base space* respectively, and $\pi\colon E \to M$ is a surjective submersion, subject to two conditions: (a) that each *fibre* $E_x := \pi^{-1}(\{x\})$ is diffeomorphic to a fixed manifold F (the "typical fibre"), and (b) for some atlas $\{(U_j, \phi_j)\}$ of M there is a family of local trivializations of E, i.e., diffeomorphisms

$$\psi_j\colon \pi^{-1}(U_j) \to U_j \times F \tag{1.1}$$

such that $\pi(\psi_j^{-1}(x, v)) = x$ for all $x \in U_j$, $v \in F$.

Definition 1.2. Let G be a Lie group. A **principal G-bundle** $P \xrightarrow{\eta} M$ is a fibre bundle whose fibres are diffeomorphic to G, together with a free *right action* of G on the total space P, whose orbits are the fibres $P_x = \eta^{-1}(x)$. We use the notation $\chi_j\colon \eta^{-1}(U_j) \to U_j \times G$ for the local trivializations.

Exercise 1.2. Suppose that G is a Lie group and that H is a closed subgroup of G (so that H is also a Lie group); then H acts freely on G by right multiplication $g \cdot h := gh$. If $\eta\colon G \to G/H$ is the quotient map, check that $G \xrightarrow{\eta} G/H$ is a principal H-bundle.

Before examining where principal bundles come from, let us describe a general recipe for manufacturing new fibre bundles using a given principal bundle. Indeed, this recipe is the main reason why principal bundles are used at all.

[1] This is also known as a "contractible covering" [38], or a "Leray covering" [58].

[2] It is a basic proposition of Riemannian geometry that each point has a geodesically convex neighbourhood; see [32] or [36] for the proof.

Definition 1.3. Let $P \xrightarrow{\eta} M$ be principal G-bundle, and let F be a manifold on which the Lie group G acts *on the left*. The product manifold $P \times F$ carries a *right* action of G, given by

$$(p, v) \cdot g := (p \cdot g, g^{-1} \cdot v). \tag{1.2}$$

Let $E := P \times_G F$ be the set of orbits of this action; we denote the orbit of (p, v) by $[p, v]$. Notice that $[p \cdot g, v] = [p, g \cdot v]$ on account of (1.2).

Define $\pi \colon E \to M : [p, v] \mapsto \eta(p)$. For each $p \in P$, the map $v \mapsto [p, v]$ is a diffeomorphism from F to $E_{\eta(p)}$, and $E \xrightarrow{\pi} M$ is a fibre bundle with typical fibre F, which is said to be **associated** to the given principal G-bundle.

Definition 1.4. A **vector bundle** $E \xrightarrow{\pi} M$ is a fibre bundle whose "typical fibre" is a (real or complex) vector space V, such that each fibre E_x is a vector space of dimension $\dim V$, and such that for each $x \in U_j$, the map $v \mapsto \psi_j^{-1}(x, v)$ is a *linear isomorphism* from V onto E_x.

A real vector bundle with typical fibre $V = \mathbb{R}$ is called a **real line bundle**; a complex vector bundle with typical fibre $V = \mathbb{C}$ is called a **complex line bundle**. When we say simply "line bundle", we shall usually mean a complex line bundle.

Examples of (real) vector bundles over M are the tangent bundle $TM \longrightarrow M$, the cotangent bundle $T^*M \longrightarrow M$, and its exterior powers $\Lambda^r T^*M \longrightarrow M$: see Appendix A.

Given a representation $\rho \colon G \to GL(V)$ of a Lie group, and a principal G-bundle $P \xrightarrow{\eta} M$, we can form the associated bundle by taking $g \cdot v \equiv \rho(g)v$; thus

$$[p \cdot g, v] = [p, \rho(g)v] \qquad \text{for} \quad p \in P,\ v \in V,\ g \in G. \tag{1.3}$$

The fibres E_x become vector spaces by taking $[p, u] + [p, v] := [p, u + v]$, and $\lambda[p, v] := [p, \lambda v]$; and the linearity of each $\rho(g)$ shows that $E \xrightarrow{\pi} M$ is a vector bundle with typical fibre V.

We can now reverse the recipe of Definition 1.3 in order to associate a principal bundle to a given vector bundle:

Definition 1.5. Let $E \xrightarrow{\pi} M$ be a vector bundle[3] with typical fibre V. Let P_x, for $x \in M$, denote the set of linear isomorphisms $p \colon V \to E_x$ (each such p is called a *frame* for E_x). If $g \in GL(V)$, then $p \circ g$ is again a frame, so that $GL(V)$ acts on the right (freely and transitively) on P_x by composition $p \mapsto p \circ g$. The disjoint union $P = \biguplus_{x \in M} P_x$, with $\eta(p) := x$ for $p \in P_x$, defines a principal $GL(V)$-bundle $P \xrightarrow{\eta} M$, which is called the **frame bundle** of $E \xrightarrow{\pi} M$.

Exercise 1.3. Using the identity representation of the group $GL(V)$ on V, check that the vector bundle associated to the frame bundle $P \xrightarrow{\eta} M$ by the recipe (1.2) is the original vector bundle $E \xrightarrow{\pi} M$.

[3]We use a notation which covers the real and complex cases simultaneously; thus $GL(V)$ denotes either $GL_{\mathbb{R}}(V)$ or $GL_{\mathbb{C}}(V)$, according as V is a real or complex vector space.

1.3 Hermitian vector bundles

If V carries some extra structure (e.g., an inner product or an orientation), we can restrict to the subgroup $G \leq GL(V)$ which preserves that structure. We need to impose a corresponding structure on the fibres E_x and consider only those frames $p\colon V \to E_x$ which are structure-preserving. Under $p \mapsto p \circ g$, these comprise a principal G-bundle associated to $E \xrightarrow{\pi} M$.

More generally, if we are given a Lie group G with a representation $\rho\colon G \to GL(V)$, we define a right action of G on frames by $p \cdot g := p \circ \rho(g)$. If, and sometimes this is "a big if", we can select a subset of frames for each E_x for which this action is transitive and free (due to some extra structure given on the vector bundle), these will form the fibres Q_x of a principal G-bundle $Q \xrightarrow{\eta'} M$. Since $[p \cdot g, v] = [p, \rho(g)v]$ for $p \in Q$, $v \in V$, the vector bundle associated, via ρ, to the new principal bundle is still $E \xrightarrow{\pi} M$.

For instance, we could ask that a real vector bundle $E \longrightarrow M$ be *Euclidean*, i.e., that each fibre E_x carry a real inner product $g_x(\cdot,\cdot)$, which depends smoothly on x. If V is a real vector space with a positive-definite inner product $q(\cdot,\cdot)$, we consider "orthogonal frames" p satisfying $g_x(p(u), p(v)) = q(u,v)$ for all $u, v \in V$. It is clear that such frames form a principal $O(n)$-bundle associated to $E \xrightarrow{\pi} M$. Such an isomorphism p is determined by choosing an orthonormal basis in (E_x, g_x) (as the image under p of a fixed orthonormal basis in (V, q)), so that this bundle is often called the "orthonormal frame bundle".

Alternatively, if $E \xrightarrow{\pi} M$ is a complex vector bundle, we could ask that it be *Hermitian*, i.e., that each fibre E_x carry a (sesquilinear) inner product $h_x(\cdot,\cdot)$, depending smoothly on x. Then if V is a complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$, we consider "unitary frames" p satisfying $h_x(p(u), p(v)) = \langle u \mid v \rangle$ for $u, v \in V$. Such frames form a principal $U(n)$-bundle associated to $E \xrightarrow{\pi} M$.

1.4 Operations on vector bundles

Definition 1.6. Given any (real or complex) vector bundle $E \longrightarrow M$, one can form a **dual vector bundle** $E^* \longrightarrow M$ whose fibre E^*_x is the dual vector space[4] of E_x. Fix a basis $\{v_1, \dots, v_n\}$ for the typical fibre V. A frame $p\colon V \to E_x$ is defined by selecting a basis $\{e_1, \dots, e_n\}$ for E_x and setting $p(v_j) := e_j$ for each j; matching the dual bases $\{v'_1, \dots, v'_n\}$, $\{e'_1, \dots, e'_n\}$ gives a frame $p'\colon V^* \to E^*_x$ for the dual bundle. Notice that $(p \circ g)' = p' \circ g^{-t}$ by change-of-basis formulae, where $g^{-t} := (g^{-1})^t$ is the contragredient matrix to $g \in GL(V)$.

Exercise 1.4. Verify that $E^* \longrightarrow M$ is the vector bundle associated to the frame bundle of $E \longrightarrow M$, via the representation $\rho(g) := g^{-t}$ of $GL(V)$ on V^*.

Exercise 1.5. Check that the cotangent bundle $T^*M \longrightarrow M$ is the dual bundle to the tangent bundle $TM \longrightarrow M$.

Definition 1.7. Given any two vector bundles $E \longrightarrow M$, $E' \longrightarrow M$ over the same base space, we can form two new vector bundles over M: their *Whitney sum* $E \oplus E' \longrightarrow M$ and their

[4]The dual space of V is $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ or $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, the space of $\mathbb{R}$-linear or $\mathbb{C}$-linear forms on V, according as V is a real or complex vector space.

tensor product $E \otimes E' \longrightarrow M$, whose fibres at $x \in M$ are respectively the direct sum $E_x \oplus E'_x$ and the algebraic tensor product $E_x \otimes E'_x$.

The k-th *exterior power* of $E \longrightarrow M$ is the vector bundle $\Lambda^k E \longrightarrow M$ whose fibre at x is $\Lambda^k E_x$. (For $k = 0$, we take $\Lambda^0 E := M \times \mathbb{R}$.)

These operations can be combined; for instance, the *exterior algebra bundle* $\Lambda^\bullet E \longrightarrow M$ is the Whitney sum of all exterior powers, for $k = 0, 1, \ldots, r$, where $r = \dim E_x$ is the rank of $E \longrightarrow M$.

Definition 1.8. The **complexification** of a real vector bundle $E \longrightarrow M$ is the complex vector bundle $E_\mathbb{C} \longrightarrow M$ with $E_\mathbb{C} := E \otimes_\mathbb{R} \mathbb{C}$, i.e., $(E_x)_\mathbb{C} := E_x \otimes_\mathbb{R} \mathbb{C} = E_x \oplus i\, E_x$ for each $x \in M$.

We write $T_\mathbb{C} M$ and $T^*_\mathbb{C} M$ for the total spaces of the complexified tangent and cotangent bundles of M.

1.5 Equivalent bundles

Definition 1.9. A *morphism* of two fibre bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a pair of smooth maps (τ, σ), with $\tau \colon E \to E'$ and $\sigma \colon M \to M'$, such that $\pi' \circ \tau = \sigma \circ \pi$, i.e., such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\sigma} & M' \end{array}$$

and, in particular, $\tau(E_x) \subseteq E'_{\sigma(x)}$ for each $x \in M$.

A *morphism of vector bundles* is a bundle morphism for which $\tau \colon E_x \to E'_{\sigma(x)}$ is linear. When $M' = M$, we usually take $\sigma = \mathrm{id}_M$.

Definition 1.10. Let $E \xrightarrow{\pi} M$ be a vector bundle and let $\phi \colon N \to M$ be a smooth map. Write $\phi^* E := \{\, (u, y) \in E \times N : \pi(u) = \phi(y) \,\}$ and define $\bar\pi \colon \phi^* E \to N$ and $\tilde\phi \colon \phi^* E \to E$ by $\bar\pi(u, y) := y$ and $\tilde\phi(u, y) := u$. Then $\bar\pi^{-1}(y) = E_{\phi(y)}$, and so $\phi^* E \xrightarrow{\bar\pi} N$ is a vector bundle, called the **pullback bundle** of $E \xrightarrow{\pi} M$ via ϕ. Moreover, $(\tilde\phi, \phi)$ is a bundle morphism; in other words, we have a commutative diagram of smooth maps:

$$\begin{array}{ccc} \phi^* E & \xrightarrow{\tilde\phi} & E \\ {\scriptstyle\bar\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ N & \xrightarrow{\phi} & M \end{array}$$

Exercise 1.6. Show that the pullback bundle has the following universal property: if $E' \xrightarrow{\pi'} N$ is a vector bundle over N and if $\rho \colon E' \to E$ is a map such that (ρ, ϕ) is a bundle morphism from this bundle to $E \xrightarrow{\pi} M$, then there is a unique bundle morphism (τ, id_N), with $\tau \colon E' \to \phi^* E$, so that $\rho = \tilde\phi \circ \tau$. (In other words, any bundle morphism with base map ϕ factors through the pullback bundle.)

Definition 1.11. Let $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} M$ be two vector bundles over the same base space. A **vector bundle equivalence** between them is an invertible vector bundle morphism (τ, id), which is given by a diffeomorphism $\tau\colon E \to E'$ satisfying $\pi' \circ \tau = \pi$ and such that $\tau\colon E_x \to E'_x$ is a linear isomorphism for each x.

We shall denote by $[E]$ the equivalence class of the vector bundle $E \xrightarrow{\pi} M$; for the set of equivalence classes with typical fibre V, we write $\mathrm{Vect}(M; V)$.

A vector bundle $E \longrightarrow M$ is *trivial* if it is equivalent to the product bundle $M \times F \xrightarrow{\mathrm{pr}_1} M$.

Note, from Exercise 1.6, that the pullback bundle is unique up to equivalence, and so defines a unique $[\phi^* E] \in \mathrm{Vect}(N; V)$.

Exercise 1.7. Show that the tangent bundle $T\mathbb{S}^1$ of the unit circle $\mathbb{S}^1$ is trivial, by producing a pair of local trivializations which together form a cylinder.

Exercise 1.8. Let $E \longrightarrow M$ be an arbitrary vector bundle over M; write $E_r = M \times V$ with $\dim V = r$, so that $E_r \longrightarrow M$ denotes the trivial vector bundle of rank r. Show that $E \otimes E_r \longrightarrow M$ is equivalent to the Whitney sum $E \oplus \cdots \oplus E \longrightarrow M$ of r copies of $E \longrightarrow M$.

Definition 1.12. Let $P \xrightarrow{\eta} M$, $P' \xrightarrow{\eta'} M$ be two principal G-bundles over the same base space. An *equivalence* between them is an invertible bundle morphism (χ, id) which is G-*equivariant*, that is, $\chi(p \cdot g) = \chi(p) \cdot g$ for any $p \in P$. Note that $\chi\colon P \to P'$ is a diffeomorphism.

We shall denote by $[P]$ the equivalence class of $P \xrightarrow{\eta} M$; for the set of such equivalence classes we write $\mathrm{Prin}(M; G)$.

Proposition 1.1. *Two vector bundles over M with the same typical fibre V are equivalent if and only if their frame bundles are equivalent as principal $GL(V)$-bundles. The association recipe thus yields a bijection $[E] \leftrightarrow [P]$ between* $\mathrm{Vect}(M; V)$ *and* $\mathrm{Prin}(M; GL(V))$. □

Exercise 1.9. Prove this, using the defining relation (1.3) of an associated vector bundle.

1.6 Sections of vector bundles

Definition 1.13. A **smooth section** of a vector bundle $E \xrightarrow{\pi} M$ is a smooth map $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$, i.e., $s(x) \in E_x$ for each $x \in M$. The totality of smooth sections will be denoted by $\Gamma(E)$, or by $\Gamma(M, E)$ if it is necessary to specify the base space M. Notice that $\Gamma(E)$ is a *module* for the commutative algebra of functions $C^\infty(M)$; the action of $C^\infty(M)$ is just scalar multiplication in each fibre:[5]

$$(fs)(x) := f(x)s(x),$$

for $s \in \Gamma(E)$, $f \in C^\infty(M)$.

[5]It would perhaps be more convenient, in view of an eventual translation to the language of noncommutative geometry, to write the multiplication *on the right*: $(sf)(x) = s(x)f(x)$; but as this conflicts with traditional habits, and could be confusing in the case that s is a vector field, we will for the moment retain the usual notation of multiplying by scalars $f(x)$ on the left.

"Global" smooth sections $s\colon M \to E$ can sometimes be hard to find. For instance, a line bundle admits a *nonvanishing* global section only if it is *trivial*. To see this, notice that any $v \in E_x$ is of the form $\lambda s(x)$, for a unique $\lambda \in \mathbb{C}$, since $s(x) \neq 0$; so $\lambda s(x) \mapsto (x, \lambda)$ is a vector bundle equivalence between E and the trivial line bundle $M \times \mathbb{C}$.[6]

Lemma 1.2. *If $L \longrightarrow M$ is a line bundle, then its tensor product with its dual line bundle, namely $L \otimes L^* \longrightarrow M$, is a trivial line bundle.*

Proof. Indeed, $L_x \otimes L_x^* \simeq \operatorname{End}(L_x)$, so the tensor product bundle has an obvious nonvanishing section s_0, whose value at each x is the identity operator on the line L_x. Since each $\operatorname{End}(L_x)$ is a one-dimensional vector space, any $L \otimes L^* \longrightarrow M$ is a line bundle, with a nonvanishing global section. □

Corollary 1.3. *The set of equivalence classes of line bundles over M has the structure of an abelian group.*

Proof. Let $L \longrightarrow M$, $L' \longrightarrow M$ be any two line bundles over M, and define:

$$[L]\,[L'] := [L \otimes L'], \qquad [L]^{-1} := [L^*]. \tag{1.4}$$

Let $L_0 := M \times \mathbb{C}$ so that $[L_0]$ is the trivial bundle class,[7] and note that $[L \otimes L_0] = [L]$ by Exercise 1.8. Since $[L \otimes L^*] = [L_0]$ by the previous Lemma, the inverse of $[L]$ is $[L^*]$. Moreover, the flip map $u \otimes v \mapsto v \otimes u$ from $L_x \otimes L'_x$ to $L'_x \otimes L_x$ determines a bundle equivalence between $L \otimes L'$ and $L' \otimes L$, so that the product (1.4) is commutative. □

We shall soon identify this group with a cohomology group of M.

1.7 Local sections and transition functions

The question that now arises is how to deal with bundles that are not trivial, and how to give an effective description of such bundles. Since there are no nonvanishing global sections, we must make use of nonvanishing *local* sections $s_j \in \Gamma(U_j, E)$, where $\mathfrak{U} := \{U_j\}$ is an open covering of M by chart domains which admit local trivializations ψ_j as in (1.1). Thus

$$\psi_j(s_j(x)) \equiv (x, f_j(x)) \tag{1.5}$$

where $f_j\colon U_j \to V \setminus \{0\}$ is a smooth nonvanishing function. (Indeed, to say that s_j is smooth is the same as saying that its local representative f_j is a smooth function.)

A global section $s \in \Gamma(M, E)$ is determined, via (1.5), by such a family of local representatives $f_j\colon U_j \to V$ (which may now take zero values).

Let $r = \dim V$ be the rank of the vector bundle $E \xrightarrow{\pi} M$. Suppose we can find a set $\boldsymbol{s}_j = (s_{j1}, \dots, s_{jr})$ with each $s_{jk} \in \Gamma(U_j, E)$ so that $\{s_{j1}(x), \dots, s_{jr}(x)\}$ is a basis for the

[6] Here we are falling into the sloppy habit of referring to a fibre bundle by naming its total space only. If the base space M is fixed, this does no harm.

[7] We use complex line bundles only to be specific; the argument for real line bundles is identical.

fibre E_x at each $x \in U_j$. Then if $t_j \in \Gamma(U_j, E)$ is *any* smooth local section, we can find smooth functions $h_j^1, \dots, h_j^r \in C^\infty(U_j)$ such that $t_j = \sum_{k=1}^r h_j^k s_{jk}$ on U_j. Thus $\Gamma(U_j, E)$ is determined by the set of local sections $\boldsymbol{s}_j$; and globally, the module $\Gamma(M, E)$ is determined by a family $\{(U_j, \boldsymbol{s}_j)\}$ of such sets, one for each local chart of M. Such a family is sometimes called a *local system* of sections [38] for the vector bundle $E \longrightarrow M$.

Note that $\boldsymbol{s}_j$ is a *local frame* over U_j and may be regarded as a local section of the frame bundle $P \xrightarrow{\eta} M$. To be precise, choose and fix a basis $\{v_1, \dots, v_r\}$ for V, and let $p_j \in P_x$, for $x \in U_j$, be the linear isomorphism from V to E_x determined by $p_j(v_k) := s_{jk}(x)$ for $k = 1, \dots, r$. Then $p_j \in \Gamma(U_j, P)$. Conversely, any such local section p_j determines the local frame $\boldsymbol{s}_j$.

Suppose that $(U_i, \boldsymbol{s}_i)$ is another local frame and that the chart domains U_i and U_j overlap: $U_i \cap U_j \neq \emptyset$. Then for $x \in U_i \cap U_j$, the isomorphisms p_i, p_j from V to E_x determined by $p_i(v_k) := s_{ik}(x)$, $p_j(v_k) := s_{jk}(x)$ are related by $p_j = p_i \circ g_{ij}(x)$ for some $g_{ij}(x) \in GL(V)$. In the notation of associated bundles, we have

$$[p_j, v] = [p_i \circ g_{ij}(x), v] = [p_i, g_{ij}(x)v] \tag{1.6}$$

for $v \in V$; or equivalently,

$$\psi_i \circ \psi_j^{-1}(x, v) = (x, g_{ij}(x)v), \tag{1.7}$$

i.e., g_{ij} is the expression in local coordinates of the transition between the local trivializations ψ_i, ψ_j of the vector bundle. Thus each $g_{ij} \colon U_i \cap U_j \to GL(V)$ is a smooth function.

Exercise 1.10. Show that a principal bundle $P \xrightarrow{\eta} M$ is trivial if and only if it admits a global smooth section $q \colon M \to P$.

Definition 1.14. Let $E \longrightarrow M$ be a vector bundle, with typical fibre V, for which $\{\, (U_j, \boldsymbol{s}_j) : j \in J \,\}$ is a local system of pointwise linearly independent local sections. The family of smooth functions $g_{ij} \colon U_i \cap U_j \to GL(V)$, defined whenever $U_i \cap U_j \neq \emptyset$, such that $\boldsymbol{s}_i = g_{ij} \cdot \boldsymbol{s}_j$ on $U_i \cap U_j$, satisfies the consistency conditions

$$g_{ii} = \mathrm{id} \ \text{ on } U_i, \qquad g_{ij} g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k. \tag{1.8}$$

(The notation $g_{ij} \cdot \boldsymbol{s}_j$ denotes the natural action of $GL(V)$ on E, as expressed by (1.6) or (1.7).) The set $\{g_{ij}\}$ is called a family of **transition functions** for the vector bundle $E \longrightarrow M$.

Suppose now that the vector bundle $E \longrightarrow M$ carries extra structure, for instance a Hermitian metric. Then we can assume that the basis $\boldsymbol{s}_j(x)$ for E_x respects this structure (continuing the Hermitian-metric example, we may take it to be an orthonormal basis, with $s_{jk}(x) = p_j(v_k)$ for a fixed orthonormal basis of V). Then the transition functions g_{ij} map $U_i \cap U_j$ into the subgroup G of $GL(V)$ which preserves the appropriate structure (in our example, the unitary group of V). In summary, *every $g_{ij}(x)$ belongs to the structure group G of the principal bundle $P \xrightarrow{\eta} M$* to which the vector bundle is associated.

This suggests that we should study vector bundles by first classifying the corresponding frame bundles, and then invoking Proposition 1.1 to pass the classification to vector bundles.

This procedure works because the frame bundle is entirely determined by the transition functions. Specifically, we have the following "patching-together" construction.

Lemma 1.4. *Let M be a manifold with an atlas of local charts $\{ (U_j, \phi_j) : j \in J \}$; let G be a Lie group and suppose we are given a family of smooth functions $g_{ij} \colon U_i \cap U_j \to G$, defined for $U_i \cap U_j \neq \emptyset$, that satisfies (1.8). Then there exists a principal G-bundle $P \xrightarrow{\eta} M$ and a family of local sections $p_j \in \Gamma(U_j, P)$ such that $p_i(x) \cdot g_{ij}(x) = p_j(x)$ whenever $x \in U_i \cap U_j$.*

Proof. Let Q denote the disjoint union $\biguplus_{j \in J} U_j \times G$ and let P be the quotient space of Q formed by identifying $(x, h)_i \in U_i \times G$ with $(x, g_{ij}(x)\, h)_j \in U_j \times G$ whenever $x \in U_i \cap U_j$. The condition (1.8) simply says that $(x, h)_i \sim (x, g_{ij}(x)\, h)_j$ is an equivalence relation on Q, so the quotient space is well defined. Write $\eta[(x, h)_i] := x$; then it may be checked that P inherits from Q the structure of a differential manifold, of dimension $\dim M + \dim G$, such that $\eta \colon P \to M$ is a submersion. It remains to check that G acts freely and transitively on the right on each fibre $\eta^{-1}(x)$; but it is obvious that the right action of G on Q given by

$$(x, h)_i \cdot g := (x, hg)_i$$

preserves equivalence classes and drops to a right action on P whose orbits are the fibres of η. $\square$

If we are also given a representation $\rho \colon G \to GL(V)$, we may then create a vector bundle with transition functions $\{g_{ij}\}$ by association, using (1.3). Conclusion: one can always patch together a vector bundle with base M and structure group G from a set of transition functions satisfying (1.8) and a representation of G.

The remaining question is how can one describe the equivalence of vector bundles in terms of the transition functions, in order to obtain a manageable classification. The answer is provided by the theory of Čech cohomology.

1.8 Čech cocycles

A family of transition functions forms what is called a "Čech 1-cocycle" with values in the structure group G. Some difficulties arise in the cohomology theory of these objects for noncommutative structure groups, so we shall assume for the present that this group is *abelian*. This still covers many cases of interest, such as the groups $\mathbb{C}$, $\mathbb{C}^\times$, $\mathbb{R}$, $U(1)$, $\mathbb{Z}$, and $\mathbb{Z}_2$.

Definition 1.15. Let $\mathfrak{U} = \{ U_j : j \in J \}$ be an open covering of a manifold M and let A be an abelian group; we will write the group operation additively. For $r \in \mathbb{N}$, a **Čech *r*-cochain** over $\mathfrak{U}$ with coefficients in A is a family of elements $c_{j_0 j_1 \dots j_r} \in A$, indexed by the collections of $(r+1)$ sets $\{U_{j_0}, U_{j_1}, \dots, U_{j_r}\} \subset \mathfrak{U}$ for which $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r} \neq \emptyset$. The set of all r-cochains is denoted $C^r(\mathfrak{U}, A)$; it is an abelian group.

One often needs a broader definition, where A is replaced by a family of abelian groups $\underline{A} := \{A_{j_0 j_1 \dots j_r}\}$ satisfying certain compatibility relations.[8] We will always take $A_{j_0 j_1 \dots j_r}$ to be

[8] To be precise, $\underline{A}$ should be a *sheaf* of abelian groups over M. The interested reader may consult [14] or [58] for the full story.

the collection of smooth functions from $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r}$ into a fixed abelian group A. For $r \in \mathbb{N}$, a *Čech r-cochain* over $\mathfrak{U}$ *with coefficients in* $\underline{A}$ is then a family of smooth functions $f_{j_0 j_1 \dots j_r} \colon U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r} \to A$, defined whenever $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r} \neq \emptyset$. The set of all such families is denoted $C^r(\mathfrak{U}, \underline{A})$.

For instance, the chart maps ϕ_j of a manifold M form a 0-cochain in $C^0(\mathfrak{U}, \underline{\mathbb{R}^n})$ (or in $C^0(\mathfrak{U}, \underline{\mathbb{C}^m})$, if M is a complex manifold). The transition functions of a complex line bundle g_{ij} form a 1-cochain in $C^1(\mathfrak{U}, \underline{\mathbb{C}^\times})$; the transition functions of a Hermitian line bundle form a 1-cochain in $C^1(\mathfrak{U}, \underline{U(1)})$.

Definition 1.16. The **Čech complex** is the cochain complex[9] $(C^\bullet(\mathfrak{U}, \underline{A}), \delta)$ is defined as follows. The *coboundary* operator $\delta_r \colon C^r(\mathfrak{U}, \underline{A}) \to C^{r+1}(\mathfrak{U}, \underline{A})$ is given by

$$(\delta \boldsymbol{a})_{ij} := a_i - a_j, \qquad (\delta \boldsymbol{b})_{ijk} := b_{ij} - b_{ik} + b_{jk}$$

for $\boldsymbol{a} \in C^0(\mathfrak{U}, \underline{A})$, $\boldsymbol{b} \in C^1(\mathfrak{U}, \underline{A})$, and $(\delta \boldsymbol{c})_{j_0 \dots j_r} := \sum_{k=0}^{r} (-1)^k c_{j_0 \dots j_{r-k-1} j_{r-k+1} \dots j_r}$ in general. It is immediate that $\delta^2 = 0$ by cancellation of terms, so we have a complex

$$C^0(\mathfrak{U}, \underline{A}) \xrightarrow{\delta} C^1(\mathfrak{U}, \underline{A}) \xrightarrow{\delta} C^2(\mathfrak{U}, \underline{A}) \xrightarrow{\delta} \cdots$$

which gives rise to cohomology groups in the standard way: $Z^r(\mathfrak{U}, \underline{A}) := \{\, \boldsymbol{c} \in C^r(\mathfrak{U}, A) : \delta \boldsymbol{c} = 0 \,\}$ are the *Čech cocycles*, $B^r(\mathfrak{U}, \underline{A}) := \{\, \delta \boldsymbol{b} : \boldsymbol{b} \in C^{r-1}(\mathfrak{U}, A) \,\}$ are the *Čech coboundaries*, and the quotient group $H^r(\mathfrak{U}, \underline{A}) := Z^r(\mathfrak{U}, \underline{A}) / B^r(\mathfrak{U}, \underline{A})$ is the rth **Čech cohomology group** of the covering $\mathfrak{U}$ with coefficients in $\underline{A}$.

Open coverings of M form a directed set (under refinement); we can eliminate $\mathfrak{U}$ by taking a direct limit: the rth Čech cohomology group *of the manifold* M (with coefficients in $\underline{A}$) is defined as[10]

$$\check{H}^r(M, \underline{A}) := \varinjlim_{\mathfrak{U}} H^r(\mathfrak{U}, \underline{A}).$$

An essential result from topology [14] is that this limit is already attained when $\mathfrak{U}$ is a good covering: $\check{H}^r(M, \underline{A}) = H^r(\mathfrak{U}, \underline{A})$ in this case.

1.9 The Čech cohomology of $\mathbb{S}^2$

Let us compute the Čech cohomology (with real coefficients) of the sphere $\mathbb{S}^2$. We know (Exercise 1.1) that it has a good covering $\mathfrak{U} = \{U_1, U_2, U_3, U_4\}$ by open neighbourhoods of the four spherical triangles F_j obtained by projecting an inscribed regular tetrahedron outward from the centre. Each $U_i \cap U_j$ is a neighbourhood of the edge $F_i \cap F_j$, and each $U_i \cap U_j \cap U_k$ is a neighbourhood of the vertex $F_i \cap F_j \cap F_k$; all are contractible. Thus 0-cochains are labelled by the faces of the tetrahedron, 1-cochains are labelled by its edges, and 2-cochains are labelled by its vertices. Therefore

$$C^0(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}^4, \quad C^1(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}^6, \quad C^2(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}^4,$$

[9] See Appendix A for generalities on cochain complexes and their cohomology groups.

[10] We use the notation $\check{H}$ to distinguish Čech cohomology from singular or de Rham cohomology; although we shall see that this is often unnecessary.

and since $U_1 \cap U_2 \cap U_3 \cap U_4 = \emptyset$, we have $C^r(\mathfrak{U}, \mathbb{R}) = 0$ for $r \geq 3$. Thus the Čech complex reduces to

$$C^0(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta_0} C^1(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta_1} C^2(\mathfrak{U}, \mathbb{R}) \xrightarrow{\delta_2} 0. \tag{1.9}$$

Now $\boldsymbol{a} \in \ker \delta_0$ iff $a_1 = a_2 = a_3 = a_4$, so $\check{H}^0(\mathbb{S}^2, \mathbb{R}) = Z^0(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}$; and (by linearity of δ_0) $B^1(\mathfrak{U}, \mathbb{R}) = \operatorname{im} \delta_0 \simeq \mathbb{R}^3$.

The elements $\boldsymbol{b} \in \ker \delta_1 = Z^1(\mathfrak{U}, \mathbb{R})$ satisfy $b_{ij} - b_{ik} + b_{jk} = 0$ for $\{i, j, k\} \subset \{1, 2, 3, 4\}$; these 4 linear equations for the six b_{ij} form a system of rank 3; and hence $Z^1(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}^3$. Thus the sequence (1.9) is exact at $C^1(\mathfrak{U}, \mathbb{R})$, and so $\check{H}^1(\mathbb{S}^2, \mathbb{R}) = 0$.

Finally, $B^2(\mathfrak{U}, \mathbb{R}) = \operatorname{im} \delta_1 \simeq \mathbb{R}^6 / \ker \delta_1 \simeq \mathbb{R}^3$. Since $Z^2(\mathfrak{U}, \mathbb{R}) = C^2(\mathfrak{U}, \mathbb{R}) \simeq \mathbb{R}^4$, we conclude that $\check{H}^2(\mathbb{S}^2, \mathbb{R}) \simeq \mathbb{R}$.

Exercise 1.11. Compute the de Rham cohomology (Appendix A) of $\mathbb{S}^2$. Show that closed 0-forms on $\mathbb{S}^2$ are constant functions, that the volume form $\Omega = \sin\theta \, d\theta \, d\phi$ is a closed 2-form with nonzero integral, and that if β is another 2-form with $\int_{\mathbb{S}^2} \beta = \int_{\mathbb{S}^2} \Omega$ then $\beta - \Omega$ is exact. If $\alpha = f(\theta, \phi) \, d\theta + g(\theta, \phi) \sin\theta \, d\phi$ is a closed 1-form, use the Poincaré lemma to show that, away from either the north or the south pole, α is of the form $d(\sin\theta \, h(\theta, \phi))$, and hence show that α is exact. Conclude that $\check{H}^r(\mathbb{S}^2, \mathbb{R}) \simeq H^r_{\mathrm{dR}}(\mathbb{S}^2)$ for every r. Is that just a coincidence?

1.10 Line bundle classification

Let $L \longrightarrow M$ be a complex line bundle with a local system $\{(U_j, s_j)\}$ of nonvanishing local sections and with transition functions $g_{ij} \colon U_i \cap U_j \to \mathbb{C}^\times$ such that $s_i = g_{ij} s_j$ on $U_i \cap U_j$. (Since the fibres are one-dimensional, we get[11] $G = GL(1, \mathbb{C}) = \mathbb{C}^\times$ and $s_j \mapsto g_{ij} s_j$ is just the module action of functions on sections; this simplifies the notation considerably.) We may and shall assume that $\mathfrak{U} = \{U_j\}$ is a good covering. The fundamental result we need is the following [57].

Proposition 1.5. *The family of transition functions $\boldsymbol{g} := \{g_{ij}\}$ is a Čech 1-cocycle for the good covering $\mathfrak{U}$; its cohomology class $[\boldsymbol{g}] \in \check{H}^1(M, \underline{\mathbb{C}}^\times)$ is independent of the local system of sections s_j, and depends only on the equivalence class $[L]$ of the complex line bundle $L \longrightarrow M$. Moreover, the correspondence $[L] \mapsto [\boldsymbol{g}] \in \check{H}^1(M, \underline{\mathbb{C}}^\times)$ is an isomorphism of abelian groups.*

Proof. From its definition, $\boldsymbol{g}$ is clearly a Čech 1-cochain in $C^1(\mathfrak{U}, \underline{\mathbb{C}}^\times)$. The consistency condition (1.8) says that $(\delta\boldsymbol{g})_{ijk} := g_{ij} g_{jk} / g_{ik} \equiv 1$ on $U_i \cap U_j \cap U_k$, which means that $\boldsymbol{g}$ is a Čech 1-*cocycle*. If a different local system $\{(U_j, t_j)\}$ is chosen, then $t_i = h_i s_i$, where $h_i \colon U_i \to \mathbb{C}^\times$ is smooth, i.e., $\boldsymbol{h} \in C^0(\mathfrak{U}, \underline{\mathbb{C}}^\times)$. Clearly $t_i = (h_i/h_j) g_{ij} t_j$, so that the transition functions for the new local system form the 1-cocycle $\boldsymbol{g} + \delta\boldsymbol{h}$ (in additive notation). Thus the line bundle determines the class $[\boldsymbol{g}]$ in $\check{H}^1(M, \underline{\mathbb{C}}^\times)$.

The frame bundle $P \longrightarrow M$ of a line bundle is formed simply by deleting the zero section from L, i.e., by taking $P_x = L_x \setminus \{0\}$ for each $x \in M$. A nonvanishing local section s_j of $L \longrightarrow M$ may thus also be regarded as a section of the frame bundle $P \longrightarrow M$. From Exercise 1.4, we see that the transition functions for the dual line bundle $L^* \longrightarrow M$ are

[11] The notation $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers; we revert to multiplicative notation when working with this group.

$1/g_{ij}$. On passing to additive notation, we conclude that the dual bundle determines the class $-[\boldsymbol{g}] \in \check{H}^1(M, \underline{\mathbb{C}}^\times)$.

Similarly, if g'_{ij} are transition functions for another line bundle $L' \longrightarrow M$, then the tensor product bundle $L \otimes L' \longrightarrow M$ has transition functions $g_{ij}g'_{ij}$, and so determines the class $[\boldsymbol{g}] + [\boldsymbol{g}'] \in \check{H}^1(M, \underline{\mathbb{C}}^\times)$. Therefore $[L] \mapsto [\boldsymbol{g}]$ is a *homomorphism* from the group of line bundle classes to the group $\check{H}^1(M, \underline{\mathbb{C}}^\times)$.

If $[\boldsymbol{g}] = 0$ in $\check{H}^1(M, \underline{\mathbb{C}}^\times)$, then $\boldsymbol{g}$ is a coboundary $\delta\boldsymbol{f}$, i.e., $g_{ij} = f_i/f_j$ where each $f_i \colon U_i \to \mathbb{C}^\times$ is a smooth function. But then $f_i^{-1}s_i = f_j^{-1}s_j$ whenever $U_i \cap U_j \neq \emptyset$, and so there is a *global* nonvanishing section $s \in \Gamma(L)$ given by $s := f_j^{-1}s_j$ on U_j; and therefore $L \longrightarrow M$ is a trivial line bundle. Hence the kernel of homomorphism $[L] \mapsto [\boldsymbol{g}]$ is zero. From Lemma 1.4, a line bundle can always be patched together from a given Čech 1-cocycle in $C^1(\mathcal{U}, \underline{\mathbb{C}}^\times)$, having this cocycle as its set of transition functions; this shows that $[L] \mapsto [\boldsymbol{g}]$ is surjective. □

1.11 The second cohomology group

Consider the short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{\epsilon} \mathbb{C}^\times \longrightarrow 0, \tag{1.10}$$

where ι is inclusion and $\epsilon(z) := e^{2\pi i z}$. One may form Čech cochains with values in any of these groups. If $\boldsymbol{c} \in Z^r(\mathcal{U}, \underline{\mathbb{C}}^\times)$, then $\boldsymbol{c} = \epsilon(\boldsymbol{b})$ with $\boldsymbol{b} \in C^r(\mathcal{U}, \underline{\mathbb{C}})$; since $\epsilon(\delta\boldsymbol{b}) = \delta\boldsymbol{c} = 0$, we can find $\boldsymbol{a} \in C^{r+1}(\mathbb{Z})$ with $\iota(\boldsymbol{a}) = \delta\boldsymbol{b}$; since $\iota(\delta\boldsymbol{a}) = \delta(\iota\boldsymbol{a}) = 0$, we have $\delta\boldsymbol{a} = 0$ and so $\boldsymbol{a} \in Z^{r+1}(\mathcal{U}, \mathbb{Z})$. One checks that $[\boldsymbol{c}] \mapsto [\boldsymbol{a}]$ is a well-defined homomorphism[12] $\partial_r \colon \check{H}^r(M, \underline{\mathbb{C}}^\times) \to \check{H}^{r+1}(M, \mathbb{Z})$, called the *Bockstein homomorphism* [23, 58]. We therefore get a long exact sequence in cohomology:

$$\cdots \longrightarrow \check{H}^1(M, \underline{\mathbb{C}}) \longrightarrow \check{H}^1(M, \underline{\mathbb{C}}^\times) \xrightarrow{\partial} \check{H}^2(M, \mathbb{Z}) \longrightarrow \check{H}^2(M, \underline{\mathbb{C}}) \longrightarrow \cdots$$

Notice that the discrete group $\mathbb{Z}$ need not be underlined: if $\mathcal{U}$ is a good covering, an element of $C^r(\mathcal{U}, \underline{\mathbb{Z}})$ is a family of $\mathbb{Z}$-valued smooth functions with connected domains, i.e., a family of constant functions; thus $C^r(\mathcal{U}, \underline{\mathbb{Z}}) = C^r(\mathcal{U}, \mathbb{Z})$ for all r, and so $\check{H}^r(M, \underline{\mathbb{Z}}) = \check{H}^r(M, \mathbb{Z})$.

Exercise 1.12. Verify that ∂ does not depend on the choices of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ within their cohomology classes, and that $\ker \partial_r = \operatorname{im} H^r\epsilon$ and $\operatorname{im} \partial_r = \ker H^{r+1}\iota$.

Exercise 1.13. If $\mathcal{U}$ is a good covering of M and $\{\psi_j\}$ is a partition of unity subordinate to $\mathcal{U}$, define, for $\boldsymbol{c} \in Z^{r+1}(\mathcal{U}, \underline{\mathbb{C}})$, an element $\boldsymbol{b} \in C^r(\mathcal{U}, \underline{\mathbb{C}})$ by $b_{j_0 \dots j_{r-1}} := \sum_j c_{j_0 \dots j_{r-1} j}\psi_j$ (this is a locally finite sum). Show that $\delta\boldsymbol{b} = \boldsymbol{c}$ and conclude that $\check{H}^k(M, \underline{\mathbb{C}}) = 0$ for $k > 0$.

Proposition 1.6. *The Bockstein homomorphism ∂ is an isomorphism of abelian groups between $\check{H}^1(M, \underline{\mathbb{C}}^\times)$ and $\check{H}^2(M, \mathbb{Z})$.*

Proof. This follows from the preceding exercise, but it is instructive to produce the isomorphism explicitly. Let $\boldsymbol{g} \in Z^1(\mathcal{U}, \underline{\mathbb{C}}^\times)$; since each $U_i \cap U_j$ is contractible, we can find a smooth

[12] This is just the standard construction of the "connecting homomorphism" in homological algebra.

function $f_{ij}\colon U_i \cap U_j \to \mathbb{C}$ such that $\epsilon \circ f_{ij} = g_{ij}$; we could write $f_{ij} = (2\pi i)^{-1} \log g_{ij}$, but this is not quite correct since the complex logarithm is "multi-valued". Define

$$a_{ijk} := f_{ij} - f_{ik} + f_{jk} : U_i \cap U_j \cap U_k \to \mathbb{C} \tag{1.11}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Then $\exp(2\pi i a_{ijk}) = g_{ij} g_{jk}/g_{ik} \equiv 1$, and so a_{ijk} is $\mathbb{Z}$-valued. Thus $\boldsymbol{a}$ is an element of $C^2(\mathfrak{U}, \mathbb{Z})$. Now (1.11) says that $\boldsymbol{a} = \delta \boldsymbol{f}$ in $C^2(\mathfrak{U}, \underline{\mathbb{C}})$, and so $(\delta\boldsymbol{a})_{ijkl} := a_{ijk} - a_{ikl} + a_{ijl} - a_{jkl} = 0$ on $U_i \cap U_j \cap U_k \cap U_l$; these are algebraic relations among $\mathbb{Z}$-valued functions, and so $\delta\boldsymbol{a} = 0$ in $C^3(\mathfrak{U}, \mathbb{Z})$; hence $\boldsymbol{a} \in Z^2(\mathfrak{U}, \mathbb{Z})$.

If we take $g'_{ij} := (h_i/h_j) g_{ij}$ with $\boldsymbol{h} \in C^0(\mathfrak{U}, \underline{\mathbb{C}}^\times)$, we can find a smooth function $k_i\colon U_i \to \mathbb{C}$ such that $\epsilon \circ k_i = h_i$; then $\epsilon \circ (f_{ij} + k_i - k_j) = g'_{ij}$. In other words, we modify $\boldsymbol{f}$ to $\boldsymbol{f} + \delta\boldsymbol{k}$ in $C^1(\mathfrak{U}, \underline{\mathbb{C}})$, and $\boldsymbol{a} = \delta\boldsymbol{f}$ is unchanged. Therefore we obtain a well-defined homomorphism $\partial\colon [\boldsymbol{g}] \mapsto [\boldsymbol{a}]$ from $\check{H}^1(M, \underline{\mathbb{C}}^\times)$ to $\check{H}^2(M, \mathbb{Z})$.

To see that this is an isomorphism, let $\{\psi_j\}$ be a smooth partition of unity subordinate to $\mathfrak{U}$. Suppose $\boldsymbol{a} \in Z^2(\mathfrak{U}, \mathbb{Z})$ is any Čech 2-cocycle; define $\boldsymbol{f} \in C^1(\mathfrak{U}, \underline{\mathbb{C}})$ by $f_{ij} := \sum_r a_{ijr}\psi_r$, and define $\boldsymbol{g} \in C^1(\mathfrak{U}, \underline{\mathbb{C}}^\times)$ by $g_{ij} := \exp(2\pi i f_{ij})$. Then

$$f_{ij} - f_{ik} + f_{jk} = \sum_r (a_{ijr} - a_{ikr} + a_{jkr})\psi_r = \sum_r a_{ijk}\psi_r = a_{ijk} \tag{1.12}$$

on using $\delta\boldsymbol{a} = 0$; hence $\delta\boldsymbol{f} = \boldsymbol{a}$. This shows that ∂ is onto.

If $\partial[\boldsymbol{g}] = 0$ in $\check{H}^2(\mathfrak{U}, \mathbb{Z})$, we can arrange, by suitably choosing a representative $\boldsymbol{g}$ in its class, that $f_{ij} - f_{ik} + f_{jk} \equiv 0$ on each $U_i \cap U_j \cap U_k$. Define $\boldsymbol{k} \in C^0(\mathfrak{U}, \underline{\mathbb{C}})$ by $k_i := \sum_r f_{ir}\psi_r$; then $k_i - k_j = \sum_r (f_{ir} - f_{jr})\psi_r = \sum_r f_{ij}\psi_r = f_{ij}$, and so $\boldsymbol{f} = \delta\boldsymbol{k}$. If $h_i := \exp(2\pi i k_i)$, then $\boldsymbol{g} = \delta\boldsymbol{h}$ in $C^1(\mathfrak{U}, \mathbb{C}^\times)$, and so $[\boldsymbol{g}] = 0$ in $\check{H}^2(\mathfrak{U}, \underline{\mathbb{C}}^\times)$. Hence ∂ is one-to-one, which establishes that $\partial\colon \check{H}^1(M, \underline{\mathbb{C}}^\times) \to \check{H}^2(M, \mathbb{Z})$ is an isomorphism. □

The upshot is that complex line bundles over M are classified by the *integral* Čech cohomology group $\check{H}^2(M, \mathbb{Z})$, at the very small price of going up to the second level in cohomology. This is a first taste of "quantization", that is, an unexpected discreteness which appears in a seemingly continuous family of objects. The culprit here is the exact sequence (1.10), due to the periodicity of the exponential function.

1.12 Classification of Hermitian line bundles

We are particularly interested in *Hermitian* line bundles $L \longrightarrow M$, which have an inner product in each fibre (and therefore the local sections s_j can be chosen so that $s_j(x) \in L_x$ is a unit vector). These have structure group $U(1)$, and the previous arguments show that the group of equivalence classes of Hermitian line bundles is isomorphic to the Čech cohomology group $\check{H}^1(M, \underline{U(1)})$.

Exercise 1.14. Construct this isomorphism in detail.

We have a short exact sequence of abelian groups:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{R} \xrightarrow{\epsilon} U(1) \longrightarrow 0,$$

where $\epsilon(t) := e^{2\pi it}$ for $t \in \mathbb{R}$. The corresponding long exact sequence is

$$\cdots \longrightarrow \check{H}^1(M,\underline{\mathbb{R}}) \longrightarrow \check{H}^1(M,\underline{U(1)}) \xrightarrow{\partial} \check{H}^2(M,\mathbb{Z}) \longrightarrow \check{H}^2(M,\underline{\mathbb{R}}) \longrightarrow \cdots$$

and again $\check{H}^1(M,\underline{\mathbb{R}}) = \check{H}^2(M,\underline{\mathbb{R}}) = 0$ by the partition-of-unity construction. Here also, the Bockstein homomorphism is an isomorphism between $\check{H}^1(M,\underline{U(1)})$ and $\check{H}^2(M,\mathbb{Z})$. In this case, the image of each g_{ij} is not the whole of the unit circle $U(1)$, since $U_i \cap U_j$ is contractible; by passing a half-line from the origin through an omitted point of $U(1)$, we can choose a branch of the logarithm for which $f_{ij} := (2\pi i)^{-1} \log g_{ij}$ is well-defined. We then have

$$a_{ijk} := \frac{1}{2\pi i}(\log g_{ij} - \log g_{ik} + \log g_{jk}) \in \mathbb{Z} \tag{1.13}$$

since $\exp(2\pi i a_{ijk}) \equiv 1$ (note that the three branches of the logarithm on the right hand side of (1.13) need not be the same). Once again, we get $\boldsymbol{a} \in C^2(\mathfrak{U},\mathbb{Z})$ with $\delta\boldsymbol{a} = 0$, so $\boldsymbol{a} \in Z^2(\mathfrak{U},\mathbb{Z})$. This leads to the following result.

Proposition 1.7. *The Bockstein homomorphism ∂ is an isomorphism of abelian groups between $\check{H}^1(M,\underline{U(1)})$ and $\check{H}^2(M,\mathbb{Z})$.* □

Exercise 1.15. Write out the proof, using the arguments of Proposition 1.6.

The Čech cohomology groups obviously depend only on the topology of M, and are computed by elementary topological arguments. However, it is desirable to replace them by de Rham cohomology groups, so that one can work with differential forms. This we do in Section 3.

1.13 Classification of vector bundles

In order to classify vector bundles of rank higher than 1 (up to equivalence), we face the obstacle that the group structure on the line bundles does not extend to the higher-rank case. In fact, it turns out that the most useful classification of vector bundles relies on a weaker equivalence relation, called *stable equivalence*, which does not distinguish between a vector bundle $E \longrightarrow M$ and the Whitney sum $E \oplus E_0 \longrightarrow M$ whenever the second summand $E_0 \longrightarrow M$ is a *trivial* bundle. This notion arises from the following fundamental property of vector bundles.

Proposition 1.8. *Let $E \longrightarrow M$ be a vector bundle over a compact manifold M. Then we can find another vector bundle $E' \longrightarrow M$ such that the Whitney sum $E \oplus E' \longrightarrow M$ is a trivial vector bundle.*

Proof. Let $\mathfrak{U} = \{U_1, \dots, U_m\}$ be a finite open covering of M by chart domains, and let $\{s_j^1, \dots, s_j^k\}$ be linearly independent sections in $\Gamma(U_j, E)$ (where k is the rank of E). Let $\{\psi_j\}_{1\leq j\leq m}$ be a smooth partition of unity subordinate to $\mathfrak{U}$. Let $\sigma_j^r \in \Gamma(E)$ be defined as $\psi_j s_j^r$ on U_j and as 0 on the complement of U_j; notice that the vectors $\sigma_j^r(x)$ span the fibre E_x, for any $x \in M$.

Write $n = km$, and define $f\colon M \times \mathbb{C}^n \to E$ (in the case of complex fibres; the real-fibre case is analogous) by $f(x,t) := \sum_{j,r} t_{jr}\sigma_j^r(x)$; then f is a surjective bundle map, i.e., (f, id_M) is a bundle morphism. Write $E'_x := \{\, (x,t) : t \in \mathbb{C}^n,\ f(x,t) = 0 \,\}$. Choose some hermitian (or Riemannian) metric on M, and let F_x be the orthogonal complement of E'_x in $\{x\} \times \mathbb{C}^n$; one checks that the F_x form the fibres of a vector bundle $F \longrightarrow M$, and that (f, id_M) is a bundle equivalence between this bundle and $E \longrightarrow M$. Since $F \oplus E' = M \times \mathbb{C}^n$, we thereby obtain an invertible bundle map from $M \times \mathbb{C}^n$ to $E \oplus E'$. $\square$

Definition 1.17. Let M be a compact manifold. We say that two vector bundles $E \longrightarrow M$ and $F \longrightarrow M$ are **stably equivalent** if there exists a *trivial* bundle $E_0 \longrightarrow M$ such that $E \oplus E_0$ and $F \oplus E_0$ are equivalent. We denote by $[\![E]\!]$ the stable equivalence class of E.

Exercise 1.16. Show that $E \longrightarrow M$ and $F \longrightarrow M$ are stably equivalent iff $[E \oplus G] = [F \oplus G]$ for some third vector bundle $G \longrightarrow M$ (which need not be trivial).

The equivalence classes of vector bundles over M form an *abelian semigroup* with identity, under the operation $[E] + [F] := [E \oplus F]$. One would like to embed this in an abelian *group* in some canonical way. In fact there is a standard construction of such a group, by abstract nonsense. For any abelian semigroup A with identity, let $K(A)$ be the abelian group with the following universal property: there is a unital semigroup homomorphism $\theta\colon A \to K(A)$ such that, whenever G is a group and $\gamma\colon A \to G$ is a unital semigroup homomorphism, there is a unique group homomorphism $\kappa\colon K(A) \to G$ for which $\gamma = \kappa \circ \theta$. Clearly, $K(A)$ is unique up to isomorphism, and it is called the *Grothendieck group* of A.

Exercise 1.17. Check that a group with the desired universal property is given by the following construction. Define an equivalence relation on $A \times A$ by $(a,b) \sim (a',b')$ iff $a + b' + c = a' + b + c$ for some $c \in A$, and let $K(A)$ be the set of equivalence classes with the obvious sum operation, where the class of (b,a) is inverse to the class of (a,b); let $\theta(a)$ be the class of $(a,0)$. Check also that each element of $K(A)$ is of the form $\theta(a) - \theta(b)$ for some elements $a, b \in A$.

Exercise 1.18. What is $K(\mathbb{N})$? What is $K(A)$ if A is the multiplicative semigroup $\mathbb{Z} \setminus \{0\}$?

An abelian semigroup A is said to "allow cancellation" if $a + c = b + c$ implies $a = b$, for any $c \in A$. Regrettably, the semigroup $\mathrm{Vect}(M)$ of (ordinary) equivalence classes of vector bundles over M does not usually allow cancellation. For example, the tangent bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ is not trivial, whereas the normal bundle (of lines from the origin of $\mathbb{R}^3$ through the sphere $\mathbb{S}^2$) is trivial, and their Whitney sum is the trivial bundle $\mathbb{S}^2 \times \mathbb{R}^3 \to \mathbb{S}^2$; thus $[T\mathbb{S}^2] + [\mathbb{S}^2 \times \mathbb{R}] = [\mathbb{S}^2 \times \mathbb{R}^2] + [\mathbb{S}^2 \times \mathbb{R}]$, so that cancellation is not allowed. In this case, the homomorphism θ is not injective.

Definition 1.18. The ***K*-theory** of a compact manifold M is the Grothendieck group $K^0(M) := K(\mathrm{Vect}(M))$. Two vector bundles $E \longrightarrow M$ and $F \longrightarrow M$ have the same image in $K^0(M)$ iff they are stably equivalent; thus any element of $K^0(M)$ can be written as a difference of two stable equivalence classes, $[\![E]\!] - [\![F]\!]$.

Exercise 1.19. Prove the assertion that $\theta([E]) = \theta([F])$ iff $[\![E]\!] = [\![F]\!]$. Show also that any element of $K^0(M)$ can be written as $[E] - [O_k]$ for some k, where $O_k \longrightarrow M$ denotes the trivial bundle of rank k.

It turns out that $K^0(M)$ carries important topological information about the manifold M; in that context, vector bundles may be viewed as auxiliary tools to study topological spaces. There is a companion group, called $K^1(M)$, which together with $K^0(M)$ forms a cohomology theory distinct from the Čech and de Rham cohomologies. This theory is fully developed in the monograph of Atiyah [3].

2 Complex projective spaces

We illustrate the general theory with a look at a particular class of manifolds, namely the complex projective spaces. These are complex manifolds, that is, differentiable manifolds whose transition maps are holomorphic; thus we may use complex local coordinates to describe them. They possess three important features: (a) a Hermitian metric; (b) a distinguished nondegenerate closed 2-form, which is known as a "symplectic structure"; (c) a "complex structure", which at each point specifies a linear automorphism of the tangent space whose square is -1. Moreover, any two of these three structures determine the third; a complex manifold with such a triple structure is called a Kähler manifold. The complex projective spaces are perhaps the simplest examples of compact Kähler manifolds.

2.1 Complex manifolds

Definition 2.1. A **complex manifold** of complex dimension m is a differentiable manifold of real dimension $n = 2m$ which has an atlas of local charts (U_j, ϕ_j), with $\phi_j \colon U_j \to \mathbb{C}^m$, such that the transition maps $\phi_i \circ \phi_j^{-1}$ are holomorphic. If $(x^1, \dots, x^m, y^1, \dots, y^m)$ are local (real) coordinates on U_j, write $z^k := x^k + iy^k$, $\bar{z}^k := x^k - iy^k$; then $(z^1, \dots, z^m, \bar{z}^1, \dots, \bar{z}^m)$ is an alternative system of local (real) coordinates on U_j.

Complex-valued 1-forms[1] are locally generated by $dz^1, \dots, dz^m, d\bar{z}^1, \dots, d\bar{z}^m$, where we write $dz^k := dx^k + i\,dy^k$, $d\bar{z}^k := dx^k - i\,dy^k$. Complex-valued vector fields in $\mathfrak{X}(M, \mathbb{C})$ are locally generated by $\partial/\partial z^1, \dots, \partial/\partial z^m, \partial/\partial \bar{z}^1, \dots, \partial/\partial \bar{z}^m$, where $\partial/\partial z^k := \frac{1}{2}(\partial/\partial x^k - i\partial/\partial y^k)$ and $\partial/\partial \bar{z}^k := \frac{1}{2}(\partial/\partial x^k + i\partial/\partial y^k)$; more precisely, $\mathfrak{X}(U_j, \mathbb{C})$ is a $C^\infty(U_j, \mathbb{C})$-module with these generators. (The notation is chosen so that $dz^j(\partial/\partial z^k) = \delta^j_k$ and $dz^j(\partial/\partial \bar{z}^k) = 0$.)

In particular, we may write the differential of $f \in C^\infty(M, \mathbb{C})$ as

$$df = \frac{\partial f}{\partial z^k}\, dz^k + \frac{\partial f}{\partial \bar{z}^k}\, d\bar{z}^k \equiv \partial f + \bar{\partial} f, \tag{2.1}$$

where, here and in the future, we use the Einstein summation convention of summing over repeated upper and lower indices.[2]

[1] In this section, all vector fields and differential forms will be taken complex-valued unless stated otherwise.

[2] To avoid possible misunderstandings, we emphasize that we do *not* sum over repeated upper or repeated lower indices, unless a summation sign appears explicitly.

Exercise 2.1. Check that the decomposition $df = \partial f + \bar\partial f$ does not depend on the local coordinate system, on account of the holomorphicity of the transition maps.

Exercise 2.2. A smooth map $f\colon M \to N$ is called *holomorphic* iff all its local expressions $\psi_i \circ f \circ \phi_j^{-1}$ have holomorphic Cartesian components. Verify that $f \in C^\infty(M,\mathbb{C})$ is holomorphic iff $\bar\partial f = 0$.

Definition 2.2. In view of (2.1), we have a splitting $\mathcal{A}^1(M) = \mathcal{A}^{1,0}(M) \oplus \mathcal{A}^{0,1}(M)$ of $C^\infty(M,\mathbb{C})$-modules, and more generally,

$$\mathcal{A}^r(M) = \bigoplus_{p+q=r} \mathcal{A}^{p,q}(M),$$

where each $\mathcal{A}^{p,q}(M)$ is locally spanned by r-forms of the type

$$f(z^1,\dots,z^m,\bar z^1,\dots,\bar z^m)\,dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar z^{j_1}\wedge\cdots\wedge d\bar z^{j_q}.$$

Thus the algebra of differential forms is *bigraded*: $\mathcal{A}^\bullet(M) = \bigoplus_{p,q}\mathcal{A}^{p,q}(M)$. Again by (2.1), the exterior derivative splits as $d = \partial + \bar\partial$, where $\partial\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q}(M)$, $\bar\partial\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)$. The identity $d^2 = 0$ yields the three identities

$$\partial^2 = 0, \qquad \partial\bar\partial = -\bar\partial\partial, \qquad \bar\partial^2 = 0, \tag{2.2}$$

on taking account of the grading degrees. (These identities say that the spaces $\mathcal{A}^{p,q}(M)$ form the vertices of a "double complex".)

Exercise 2.3. Verify that the conjugation $\omega \mapsto \bar\omega$ on $\mathcal{A}^\bullet(M,\mathbb{C}) = \mathcal{A}^\bullet(M,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$ is such that $\overline{\partial\omega} = \bar\partial\bar\omega$ and that $\bar\omega \in \mathcal{A}^{q,p}(M)$ whenever $\omega \in \mathcal{A}^{p,q}(M)$.

2.2 Local charts for complex projective spaces

Definition 2.3. The m-dimensional *complex projective space* $\mathbb{CP}^m$ is the set of complex lines through the origin in $\mathbb{C}^{m+1}$, i.e., the one-dimensional complex subspaces of $\mathbb{C}^{m+1}$. For any nonzero $v \in \mathbb{C}^{m+1}$, the line $\langle v\rangle \equiv \mathbb{C}v$ lies in $\mathbb{CP}^m$, and $\eta\colon \mathbb{C}^{m+1}\setminus\{0\} \to \mathbb{CP}^m : v \mapsto \langle v\rangle$ is a quotient map.[3] If $(z^0, z^1,\dots,z^m)$ denotes coordinates in $\mathbb{C}^{m+1}$ (with respect to the standard orthonormal basis), we may regard each z^j as a linear form on $\mathbb{C}^{m+1}$; these cannot vanish simultaneously on $\mathbb{C}^{m+1}\setminus\{0\}$, so we get the following chart domains for $\mathbb{CP}^m$:

$$U_j := \{\,\langle v\rangle \in \mathbb{CP}^m : z^j(v) \neq 0\,\}, \qquad j = 0,1,\dots,m.$$

Since $\langle v\rangle \notin U_j$ iff $\langle v\rangle \subset \ker z^j$, we can identify the complement of U_j with the set of lines in the hyperplane $\ker z^j$, which is homeomorphic to $\mathbb{CP}^{m-1}$. In particular, each U_j is open and dense in $\mathbb{CP}^m$, since its complement is a lower-dimensional submanifold.

[3]This quotient map actually defines the topology of $\mathbb{CP}^m$.

For $\langle v\rangle \in U_j$ and $k \neq j$, we define

$$w_j^k(\langle v\rangle) := \frac{z^k(v)}{z^j(v)}. \tag{2.3}$$

This is well-defined since the fraction is unchanged under $v \mapsto \lambda v$ with $\lambda \in \mathbb{C}^\times$). Now[4]

$$\phi(\langle v\rangle) := (w_j^0, \overset{j}{\ldots\vee\ldots}, w_j^m) \tag{2.4}$$

is a homeomorphism from U_j onto $\mathbb{C}^m$, and $(w_j^0, \overset{j}{\ldots\vee\ldots}, w_j^m)$ is a system of *local coordinates* for the chart (U_j, ϕ_j).

The w_j^k are Cartesian coordinates corresponding to the "homogeneous" coordinates z^k, that is, $(w_j^0, w_j^1, \ldots, w_j^m) = [z^0 : z^1 : \cdots : z^m]$ in the standard notations.

On the overlap $U_i \cap U_j$, we have

$$w_i^k(\langle v\rangle) = w_j^k(\langle v\rangle)\frac{z^j(v)}{z^i(v)} = \frac{w_j^k(\langle v\rangle)}{w_j^i(\langle v\rangle)}$$

for $k \notin \{i,j\}$, so the transition map $\phi_i \circ \phi_j^{-1}$, when written as $u \mapsto v$, is given by rational functions: $v^j = 1/u^i$ and $v^k = u^k/u^i$ for other k, and hence is holomorphic. Thus the atlas $\{\,(U_j, \phi_j) : j = 0, 1, \ldots, m\,\}$ makes $\mathbb{CP}^m$ a complex manifold.

Since $\mathbb{CP}^m \setminus U_m \approx \mathbb{CP}^{m-1}$ and $U_m \approx \mathbb{C}^m \approx \mathbb{R}^{2m}$, we have (by induction) that $\mathbb{CP}^m$ is topologically a *cell complex* with one k-dimensional cell in every even dimension $k = 0, 2, \ldots, 2m$ and no odd-dimensional cells. This allows us to compute the singular homology groups of the complex projective spaces by standard topological techniques [23]: we get $H_k(\mathbb{CP}^m, \mathbb{Z}) = \mathbb{Z}$ if $k = 0, 2, \ldots, 2m$; $H_k(\mathbb{CP}^m, \mathbb{Z}) = 0$ for all other k. Thus $\mathbb{CP}^m$ is a "torsion-free" space, i.e., all its integral homology groups are free; it is known that then the singular cohomology groups with integer coefficients can be computed by duality:

$$H^k(\mathbb{CP}^m, \mathbb{Z}) = \operatorname{Hom}(H^k(\mathbb{CP}^m, \mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, \ldots, 2m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

2.3 The Kähler form

Definition 2.4. For $v = (z^0, z^1, \ldots, z^m) \in \mathbb{C}^{m+1}$, we write $\|v\|^2 = \sum_k |z^k|^2 = \sum_k z^k \bar{z}^k$, and define

$$\begin{aligned}\Phi_0(v) &:= i\,\partial\bar\partial \log \|v\|^2 = i\,\partial\bigl(\|v\|^{-2}{\textstyle\sum_k} z^k\, d\bar{z}^k\bigr) \\ &= \frac{i}{\|v\|^4}\Biggl(\|v\|^2 \sum_k dz^k \wedge d\bar{z}^k - \sum_{r,s\neq j} \bar{z}^r z^s\, dz^r \wedge d\bar{z}^s\Biggr),\end{aligned}$$

[4]The notation $\overset{j}{\vee}$ indicates that the index j is omitted from the sequence $0, 1, \ldots, m$.

which is a 2-form in $\mathcal{A}^{1,1}(\mathbb{C}^{m+1} \setminus \{0\})$. It is clear that Φ_0 is homogeneous of degree 0, i.e., $\Phi_0(\lambda v) = \Phi_0(v)$ for $\lambda \in \mathbb{C}^\times$. This means that there is a 2-form $\Phi \in \mathcal{A}^{1,1}(\mathbb{CP}^m)$ such that $\Phi_0 = \eta^*\Phi$.

To find an expression for Φ in local coordinates on U_j, we may identify U_j with a subset of the hyperplane $z^j = 1$ in $\mathbb{C}^{m+1}$, by using (2.3) with denominator 1. With this convention, the norm squared of a vector v in this hyperplane defines a positive function Q_j on U_j (which tends to infinity at the boundary); in fact,

$$\|v\|^2 = Q_j(\langle v \rangle) := 1 + \sum_{k \neq j} w_j^k \bar{w}_j^k \tag{2.6}$$

for $\langle v \rangle \in U_j$ with local coordinates (2.4). Thus

$$\Phi = i\,\partial\bar{\partial}\log Q_j = \frac{i}{Q_j^2}\biggl(Q_j \sum_{k \neq j} dw_j^k \wedge d\bar{w}_j^k - \sum_{r,s \neq j} \bar{w}_j^r w_j^s \, dw_j^r \wedge d\bar{w}_j^s\biggr). \tag{2.7}$$

The 2-form Φ is in fact real-valued, as is evident from its local expression (2.7), or alternatively by noting that $\overline{\Phi} = -i\,\bar{\partial}\partial\log Q_j = \Phi$ on account of (2.2); this $\Phi \in \mathcal{A}^2(\mathbb{CP}^m, \mathbb{R})$ is called the **Kähler form** on $\mathbb{CP}^m$.

It is important to note that $\Phi = -\partial\bar{\partial}Q_j$ is a *closed* 2-form. This follows at once from (2.2), since $d\Phi = (\partial + \bar{\partial})\Phi = -\partial^2(\bar{\partial}Q_j) + \bar{\partial}^2(\partial Q_j) = 0$.

Exercise 2.4. A real-valued 2-form $\beta \in \mathcal{A}^2(M)$ is called *nondegenerate* if each $\beta_x \in \Lambda^2 T_x^* M$ is nondegenerate as an alternating $\mathbb{R}$-bilinear form, i.e., $\beta_x(u, v) = 0$ for all $v \in T_x M$ implies $u = 0$ in $T_x M$. Show that β is nondegenerate iff for any local expression $\beta\big|_U = b_{rs}\, dx^r \wedge dx^s$, the matrix of local coefficients $[b_{rs}]$ is nonsingular at each point of U. Verify that the Kähler form Φ is nondegenerate on $\mathbb{CP}^m$ by showing that the matrix A with entries $Q_j\delta^{rs} - \bar{w}_j^r w_j^s$ is positive definite (apply the Schwarz inequality to show that $\bar{z}^t A z > 0$ for all $z \in \mathbb{C}^{m+1} \setminus \{0\}$).

Exercise 2.5. Show that $\Phi^{\wedge m}$, the m-fold exterior power of Φ, is a *volume form* on $\mathbb{CP}^m$, i.e., a $2m$-form which is nonzero at each point of $\mathbb{CP}^m$ when regarded as a section of the line bundle $\Lambda^{2m} T^*\mathbb{CP}^m \to \mathbb{CP}^m$.

2.4 The Fubini–Study metric

Definition 2.5. Let M be a differential manifold of even (real) dimension $n = 2m$. An **almost complex structure** on M is an operator $\mathbb{J} \colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ which is $C^\infty(M, \mathbb{R})$-linear,[5] and which satisfies $\mathbb{J}^2 = -\operatorname{id}$. Thus $(\mathbb{J}X)_x = J_x(X_x)$ where $J_x \in \operatorname{End}_{\mathbb{R}}(T_x M)$ with $(J_x)^2 = -\operatorname{id}$ for each $x \in M$.[6] Almost complex structures need not exist; if one does exist, we say that $(M, \mathbb{J})$ is an *almost complex manifold*.

[5]In other words, $\mathbb{J}$ can be identified with a tensor on M, of bidegree $(1, 1)$, given by $(X, \alpha) \mapsto \alpha(\mathbb{J}(X))$. One also says that $\mathbb{J}$ is a "tensorial operator".

[6]An almost complex structure may therefore also be defined as a bundle automorphism (J, id) of the tangent bundle $TM \longrightarrow M$ for which $J^2 = -\operatorname{id}$.

It turns out that M need not be a complex manifold in order to possess an almost complex structure; a known example is the sphere $\mathbb{S}^6$ which has an almost complex structure related to the Cayley numbers. See [15, 58] for some discussion of this. Here we will stick to complex manifolds.

A complex structure can always be defined locally if $\dim M$ is even; it suffices to take local coordinates $(x^1, \dots, x^m, y^1, \dots, y^m)$ on a chart domain U and let

$$\mathbb{J}\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \quad \mathbb{J}\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}. \tag{2.8}$$

Now $\mathbb{J}$ may be amplified to a $C^\infty(U, \mathbb{C})$-linear operator on $\mathfrak{X}(U, \mathbb{C})$ in the natural way; from (2.8) we derive

$$\mathbb{J}\left(\frac{\partial}{\partial z^k}\right) = i\frac{\partial}{\partial z^k}, \quad \mathbb{J}\left(\frac{\partial}{\partial \bar{z}^k}\right) = -i\frac{\partial}{\partial \bar{z}^k}.$$

On a complex manifold, these local definitions patch together to give a global definition of $\mathbb{J}$. For instance, on $\mathbb{CP}^m$ we have

$$\mathbb{J}\left(\frac{\partial}{\partial w_j^k}\right) = i\frac{\partial}{\partial w_j^k}, \quad \mathbb{J}\left(\frac{\partial}{\partial \bar{w}_j^k}\right) = -i\frac{\partial}{\partial \bar{w}_j^k}, \tag{2.9}$$

which is independent of the chart (U_j, ϕ_j).

Exercise 2.6. Verify this independence on an overlap $U_i \cap U_j$ of charts of $\mathbb{CP}^m$.

Lemma 2.1. *Let M be a manifold with an almost complex structure $\mathbb{J}$, and let $\Phi \in \mathcal{A}^2(M)$ be a real-valued nondegenerate 2-form on M which is invariant under $\mathbb{J}$, i.e., $\Phi(\mathbb{J}X, \mathbb{J}Y) = \Phi(X, Y)$ for $X, Y \in \mathfrak{X}(M)$. Then the recipe $g(X, Y) := \Phi(X, \mathbb{J}Y)$ defines a symmetric tensor on M which is also $\mathbb{J}$-invariant.*

Proof. Clearly g is a tensor of bidegree $(2, 0)$; symmetry follows from invariance, since $g(Y, X) = \Phi(Y, \mathbb{J}X) = -\Phi(\mathbb{J}X, Y) = \Phi(\mathbb{J}X, \mathbb{J}^2Y) = \Phi(X, \mathbb{J}Y) = g(X, Y)$. □

When $M = \mathbb{CP}^m$, Φ is the Kähler form (2.7), and $\mathbb{J}$ is given by (2.9), we obtain on U_j:

$$g = \frac{2}{Q_j^2}\left(Q_j \sum_{k \neq j} dw_j^k \cdot d\bar{w}_j^k - \sum_{r,s \neq j} \bar{w}_j^r w_j^s \, dw_j^r \cdot d\bar{w}_j^s\right). \tag{2.10}$$

Exercise 2.7. Check that g is positive definite, and thus defines a Riemannian metric on $\mathbb{CP}^m$. (Recall that the matrix with entries $Q_j\delta^{rs} - \bar{w}_j^r w_j^s$ is positive definite, by Exercise 2.4.)

Definition 2.6. The metric (2.10) is the **Fubini–Study metric** on $\mathbb{CP}^m$.

2.5 The Riemann sphere

When $m = 1$, the complex projective space $\mathbb{CP}^1$ is identified with the *Riemann sphere* $\mathbb{C}_\infty = \mathbb{C} \uplus \{\infty\}$, by identifying $[z^0 \colon z^1]$ with $z = z^1/z^0 \in \mathbb{C}$ if $z^0 \neq 0$, and $[1\colon 0]$ with ∞. Notice that $z \equiv w_0^1$ is the local complex coordinate in $U_0 = \mathbb{C}_\infty \setminus \{\infty\}$. By writing $z = x + iy$, we can also identify $\mathbb{C}_\infty$ with the two-sphere $\mathbb{S}^2$, regarding the latter as a submanifold of $\mathbb{R}^3$, via the *stereographic projection*

$$f(z) := \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right) \tag{2.11}$$

and $f([1\colon 0]) := (0,0,1)$.

The Kähler form on $\mathbb{C}_\infty$ is easily found. We have $Q_0(\langle v \rangle) = 1 + |z|^2$ for $v = (z^0, z^1) \in \mathbb{C}^2$, so (2.7) reduces to

$$\Phi = \frac{i\,dz \wedge d\bar{z}}{(1+|z|^2)^2} = -\frac{2\,dx \wedge dy}{(1+x^2+y^2)^2}. \tag{2.12}$$

Exercise 2.8. Write (u^1, u^2, u^3) to denote the right hand side of (2.11). Check that $\Phi = f^*\Omega$, where $\Omega \in \mathcal{A}^2(\mathbb{R}^3)$ is given by $\Omega = (du^1 \wedge du^2)/2u^3$. The usual spherical coordinates on $\mathbb{S}^2$ are defined by the map $h\colon \mathbb{R}^2 \to \mathbb{S}^2$ where $h(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Check that $h^*\Omega = -\frac{1}{2}\sin\theta\,d\theta \wedge d\phi$.

The Riemannian metric (2.10) reduces to $g = 2(1+|z|^2)^{-2}\,dz \cdot d\bar{z}$ or equivalently $g = 2(1+x^2+y^2)^{-2}\,(dx^2+dy^2)$, where dx^2 denotes the symmetric product $dx \cdot dx$.

Exercise 2.9. Show that $g = f^*(\frac{1}{2}G)$ where $G = (du^1)^2 + (du^2)^2 + (du^3)^2$ is the standard Riemannian metric on $\mathbb{R}^3$, and that $h^*G = d\theta^2 + \sin^2\theta\,d\phi^2$.

We may regard (θ, ϕ) as local coordinates on the Riemann sphere. Thus we write simply:

$$\Phi = -\tfrac{1}{2}(\sin\theta\,d\theta \wedge d\phi), \qquad g = \tfrac{1}{2}(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{2.13}$$

It may prove useful to have available some relations between the local coordinate systems $(z, \bar{z})$ and (θ, ϕ). We list a few identities that follow from the previous formulae:

$$z = \frac{\sin\theta}{1-\cos\theta}e^{i\phi}, \quad dz = \frac{e^{i\phi}}{1-\cos\theta}(-d\theta + i\sin\theta\,d\phi), \quad Q_0 = \frac{2}{1-\cos\theta}. \tag{2.14}$$

Exercise 2.10. Verify the formulae (2.14) and derive (2.13) directly.

Exercise 2.11. Show that the complex structure $\mathbb{J}$ on $\mathbb{S}^2$ satisfying $g(X,Y) = \Phi(X, \mathbb{J}Y)$ is given by $\mathbb{J}(\partial/\partial\theta) = -\csc\theta\,\partial/\partial\phi$ and $\mathbb{J}(\csc\theta\,\partial/\partial\phi) = \partial/\partial\theta$ in spherical coordinates.

3 The de Rham complex and Hodge duality

3.1 The de Rham complex

Definition 3.1. If M is an n-dimensional differential manifold, its **de Rham complex** is the cochain complex

$$\mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \to \cdots \to \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \to \cdots \to \mathcal{A}^n(M)$$

where $\mathcal{A}^k(M) = \mathcal{A}^k(M,\mathbb{R})$ denotes the real-valued differential forms on M, and d is the exterior derivation. We recall (see Appendix A) that its k-cocycles are the *closed* differential k-forms $Z^k_{\mathrm{dR}}(M) := \{\, \omega \in \mathcal{A}^k(M) : d\omega = 0 \,\}$, and its k-coboundaries are the *exact* differential k-forms $B^k_{\mathrm{dR}}(M) := \{\, d\beta : \beta \in \mathcal{A}^{k-1}(M) \,\}$. The k-th de Rham cohomology group $H^k_{\mathrm{dR}}(M) := H^k(\mathcal{A}^\bullet(M), d)$ is a real vector space.

The 0-cocycles in $Z^0_{\mathrm{dR}}(M)$ are locally constant smooth functions, so $H^0_{\mathrm{dR}}(M) = \mathbb{R}^m$ if M has exactly m connected components. In particular, $H^0_{\mathrm{dR}}(M) \neq 0$. If U is a contractible manifold, then by the Poincaré lemma,[1] $H^k_{\mathrm{dR}}(U) = 0$ for each $k = 1, 2, \dots, n$.

If M is an *orientable* compact manifold (without boundary), with volume form $\nu \in \mathcal{A}^n(M)$, then $d\nu = 0$ and ν is not exact since $\int_M \nu \neq 0$. (If $\mu \in \mathcal{A}^n(M)$ is exact, with $\mu = d\beta$, say, then $\int_M \mu = \int_M d\beta = 0$ by Stokes' theorem, since $\partial M = \emptyset$; thus, a volume form cannot be exact.) Therefore $H^n_{\mathrm{dR}}(M) \neq 0$. Moreover, since the integral vanishes on exact $(n-1)$-forms, again by Stokes' theorem, we see that $[\omega] \mapsto \int_M \omega$ is a well-defined linear form on $H^n_{\mathrm{dR}}(M)$.

3.2 The Riemannian volume form

Definition 3.2. Let g be a Riemannian metric on the compact oriented manifold M. Then $g \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is a symmetric $C^\infty(M)$-bilinear form such that each $g_x \colon T_xM \times T_xM \to \mathbb{R}$ is positive definite. The pair (M, g) is called a **Riemannian manifold**.

If $U \subseteq M$ is a chart domain with local coordinates $x^1, \dots, x^n$, let $g_{ij} := g(\partial/\partial x^i, \partial/\partial x^j) \in C^\infty(U)$; then $[g_{ij}]$ is a symmetric matrix, whose determinant we write as $\det g$ (by a slight abuse of notation, since this determinant is coordinate-dependent), and we have $g = g_{ij}\, dx^i \cdot dx^j$ on U.

For each $x \in M$, there is a vector space isomorphism $\hat{g}_x \colon T_xM \to T^*_xM$ given by $\hat{g}_x(u) := [v \mapsto g_x(u,v)]$; these determine a diffeomorphism $\hat{g} \colon TM \to T^*M$ such that $(\hat{g}, \mathrm{id}_M)$ is an equivalence between the tangent and cotangent bundles. They also determine tensorial operators $X \mapsto X^\flat : \mathfrak{X}(M) \to \mathcal{A}^1(M)$, given by $X^\flat(Y) := g(X,Y)$, and its inverse $\alpha \mapsto \alpha^\sharp : \mathcal{A}^1(M) \to \mathfrak{X}(M)$, given by $\alpha(Y) =: g(\alpha^\sharp, Y)$. These "musical isomorphisms" [8] allow us to identify vector fields and 1-forms and to use them interchangeably, as the occasion demands.

The metric g defines thus defines bilinear pairings, not only of vector fields but also of 1-forms and indeed of differential forms of any degree; we will denote all these pairings by $(\cdot \mid \cdot)$ whenever a fixed g is given. Thus $(X \mid Y) := g(X,Y)$ for $X, Y \in \mathfrak{X}(M)$; $(\alpha \mid \beta) := g(\alpha^\sharp, \beta^\sharp)$ for $\alpha, \beta \in \mathcal{A}^1(M)$; and the pairing on $\mathcal{A}^k(M)$ (for $k > 1$) is determined by

$$(\alpha^1 \wedge \cdots \wedge \alpha^k \mid \beta^1 \wedge \cdots \wedge \beta^k) := \det\bigl[(\alpha^i \mid \beta^j)\bigr]$$

for $\alpha^1, \dots, \alpha^k, \beta^1, \dots, \beta^k \in \mathcal{A}^1(M)$.

Definition 3.3. Let (M, g) be an oriented Riemannian manifold. Choose a local orthonormal frame $X_1, \dots, X_n$ for $\mathfrak{X}(U)$, i.e., vector fields such that $g(X_r, X_s) = \delta_{rs}$, which is *oriented*,

[1] A complex $(C^\bullet, d)$ is called "acyclic" if $H^k(C^\bullet, d) = 0$ for $k > 0$. Thus the Poincaré lemma says that the de Rham complex of a contractible manifold is acyclic.

that is, $\nu(X_1, \dots, X_n) > 0$ where ν is a volume form on M which defines the orientation. Let $\hat{\theta}^k := X_k^\flat \in \mathcal{A}^1(U)$, so that $\hat{\theta}^1, \dots, \hat{\theta}^n$ is a local (oriented) orthonormal frame for $\mathcal{A}^1(U)$. Then

$$\Omega := \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \cdots \wedge \hat{\theta}^n \in \mathcal{A}^n(M) \tag{3.1}$$

is a volume form on M, since $(\Omega \mid \Omega) = 1$ and in particular $\Omega_x \neq 0$ in $\Lambda^n T_x^* M$ for all $x \in M$. This Ω is called the **Riemannian volume form** on (M, g).

To see that Ω is in fact independent of the choice of oriented orthonormal frame, we argue as follows. If $Y_1, \dots, Y_n$ is another oriented orthonormal frame for $\mathfrak{X}(V)$, where $U \cap V \neq \emptyset$, then $Y_i := a_i^j X_j$ where $a = [a_i^j]$ is a smooth function on $U \cap V$ with values in $SO(n)$, the group of orthogonal $n \times n$ matrices of determinant 1. (That is to say, a is a local section of the principal $SO(n)$-bundle of oriented orthonormal frames on M.) Now $\hat{\vartheta}^k := Y_k^\flat$ gives the corresponding oriented orthonormal basis for $\mathcal{A}^1(V)$, and thus $\hat{\vartheta}^k = a_l^k \hat{\theta}_l$, from which it follows that $\hat{\vartheta}^1 \wedge \cdots \wedge \hat{\vartheta}^n = (\det a)\, \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^n = \Omega$ on $U \cap V$. Therefore, Ω is defined globally by (3.1).

One can express Ω in terms of local coordinates $x^1, \dots, x^n$ on U for which the $\partial/\partial x^j$ need not be orthonormal, i.e., $g_{ij} \neq \delta_{ij}$ in general. The local coordinates should, however, be compatible with the orientation, which means that $\det g > 0$. If $y^1, \dots, y^n$ are other local coordinates on V, and if $\tilde{g}_{ij} := g(\partial/\partial y^i, \partial/\partial y^j) \in C^\infty(V)$, then $\det \tilde{g} = J^2 \det g$, where $J := \det[\partial y^r / \partial x^j]$ is the Jacobian of the transition function (which is positive). Hence

$$\sqrt{\det \tilde{g}}\, dy^1 \wedge \cdots \wedge dy^n = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n \tag{3.2}$$

on $U \cap V$, and thus the coordinate-independent expression $\sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$ defines a volume on M. In particular, one may choose $y^1, \dots, y^n$ so that $\tilde{g}_{ij} = \delta_{ij}$, so $\det \tilde{g} = 1$ and $dy^1, \dots, dy^n$ is an orthonormal frame: so the volume form (3.2) coincides with Ω of (3.1). In other words,

$$\Omega = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n \tag{3.3}$$

in any oriented local coordinate system.

3.3 The Hodge star operator

Definition 3.4. Let (M, g) be a compact oriented Riemannian manifold of dimension n, and write $m := \lfloor n/2 \rfloor$ (so $n = 2m$ or else $n = 2m - 1$). Define the **Hodge star operator** $\star\colon \mathcal{A}^\bullet(M, \mathbb{C}) \to \mathcal{A}^\bullet(M, \mathbb{C})$ as follows. Choose a local orthonormal frame $X_1, \dots, X_n$ for $\mathfrak{X}(U)$ on some chart domain $U \subset M$, and let $\hat{\theta}^1, \dots, \hat{\theta}^n$ be the corresponding local orthonormal frame for $\mathcal{A}^1(U)$, determined by $\hat{\theta}^i(X_j) := \delta^i_j$. Define $\star$ on $\mathcal{A}^\bullet(U, \mathbb{C})$ by

$$\star := i^m (\epsilon(\hat{\theta}^1) - \iota(X_1)) \dots (\epsilon(\hat{\theta}^n) - \iota(X_n)), \tag{3.4}$$

where $\iota(X)$ denotes contraction with the vector field X and $\epsilon(\alpha)\colon \omega \mapsto \alpha \wedge \omega$ denotes exterior product with the 1-form α. It is readily checked that the right hand side of (3.4) is independent of these local orthonormal frames, and therefore defines an operator on all of $\mathcal{A}^\bullet(M, \mathbb{C})$.

To simplify products of noncommuting quantities as in (3.4), we use the following notation: if $a_{j_1}, a_{j_2}, \dots, a_{j_k}$ are elements of some algebra, *where the indices j_r are in increasing order*, write

$$\prod_{1 \leq r \leq k}^{\rightarrow} a_{j_r} := a_{j_1} a_{j_2} \dots a_{j_k}. \tag{3.5}$$

Thus $\star = i^m \prod^{\rightarrow}_{1 \leq r \leq k} \epsilon(\hat{\theta}^r) - \iota(X_r)$. It is also convenient to write $J := \{j_1, \dots, j_k\}$ and denote the right hand side of (3.5) by $\prod^{\rightarrow}_{j \in J} a_j$, where it is understood that the indices $j \in J$ are arranged in increasing order.

Exercise 3.1. Let V be an n-dimensional oriented real vector space with a positive definite symmetric bilinear form q. For $u \in V$, $\alpha \in V^*$, define operators $\iota(u) \colon \Lambda^k V^* \to \Lambda^{k-1} V^*$ and $\epsilon(\alpha) \colon \Lambda^k V^* \to \Lambda^{k+1} V^*$ by $[\iota(u)\eta](v_1, \dots, v_{k-1}) := \eta(u, v_1, \dots, v_{k-1})$ and $\epsilon(\alpha)\eta := \alpha \wedge \eta$. Let $\{e_1, \dots, e_n\}$, $\{e'_1, \dots, e'_n\}$ be oriented orthonormal bases for (V, q) (so the change-of-basis matrix $[q(e_i, e'_j)]$ has determinant $+1$) and let $\{\zeta^1, \dots, \zeta^n\}$, $\{\zeta'^1, \dots, \zeta'^n\}$ be the respective dual bases for V^*. Show that $\prod^{\rightarrow}_{1 \leq r \leq n} \epsilon(\zeta'^r) - \iota(e'_r) = \prod^{\rightarrow}_{1 \leq r \leq n} \epsilon(\zeta^r) - \iota(e_r)$ as operators on $\Lambda^\bullet V^*$.

Exercise 3.2. Use the previous exercise to show that the right hand side of (3.4) is independent of the given local orthonormal frames $\{X_r\}$ and $\{\hat{\theta}^k\}$.

Notice that when m is even, i.e., when n is of the form $4k$ or $4k+3$ for some integer k, then it is not necessary to use complex-valued forms since $\star$ takes $\mathcal{A}^\bullet(M, \mathbb{R})$ to $\mathcal{A}^\bullet(M, \mathbb{R})$. However, for n of the form $4k+1$ or $4k+2$, we have $i^m = \pm i$ and complex forms are needed. More traditional treatments of Hodge duality [1, 28, 33, 57] use different sign conventions, so that $\star$ operates on $\mathcal{A}^\bullet(M, \mathbb{R})$ in all cases, but the important involutivity property (see Lemma 3.3 below) is then lost.

Lemma 3.1. *If $J := \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$ and if $J' := \{1, \dots, n\} \setminus J = \{i_1, \dots, i_{n-k}\}$ is its complement, with indices written in increasing order in both cases, let $\eta(J, J') = \pm 1$ denote the sign of the shuffle permutation which reorders $(1, 2, \dots, n)$ as $(j_1, \dots, j_k, i_1, \dots, i_{n-k})$. Write, for brevity, $\hat{\theta}^J := \hat{\theta}^{j_1} \wedge \cdots \wedge \hat{\theta}^{j_k}$, where $\hat{\theta}^1, \dots, \hat{\theta}^n$ is a local orthonormal basis of $\mathcal{A}^1(M)$. Then the Hodge star operator is given explicitly on this basis by the recipe:*

$$\star \hat{\theta}^J = i^m (-1)^{nk + k(k-1)/2} \eta(J, J') \, \hat{\theta}^{J'}. \tag{3.6}$$

Proof. This identity results from the following calculation:

$$\begin{aligned}
\star\hat{\theta}^J &= i^m \overrightarrow{\prod_{1\le j\le n}} (\epsilon(\hat{\theta}^j) - \iota(X_j))\, \hat{\theta}^{j_1} \wedge \cdots \wedge \hat{\theta}^{j_k} \\
&= i^m (-1)^{k(k-1)/2} \overrightarrow{\prod_{1\le j\le n}} (\epsilon(\hat{\theta}^j) - \iota(X_j))\, \hat{\theta}^{j_k} \wedge \cdots \wedge \hat{\theta}^{j_1} \\
&= i^m (-1)^{k(k-1)/2} \eta(J', J) \overrightarrow{\prod_{i\in J'}} \epsilon(\hat{\theta}^i) \overrightarrow{\prod_{j\in J}} (-\iota(X_j))\, \hat{\theta}^{j_k} \wedge \cdots \wedge \hat{\theta}^{j_1} \\
&= i^m (-1)^{k(k-1)/2} (-1)^k \eta(J', J) \overrightarrow{\prod_{i\in J'}} \epsilon(\hat{\theta}^i)\, 1 \\
&= i^m (-1)^{k(k-1)/2} (-1)^k (-1)^{k(n-k)} \eta(J, J')\, \hat{\theta}^{J'}.
\end{aligned}$$

Here, one first reverses the order of the exterior product of 1-forms $\hat{\theta}^{j_1} \wedge \cdots \wedge \hat{\theta}^{j_k}$, by a permutation of sign $(-1)^{k(k-1)/2}$. Then the operators $\epsilon(\hat{\theta}^j) - \iota(X_j)$ act successively on forms of type $\hat{\theta}^j \wedge \beta^j$, yielding $-\beta^j$ at each stage, until only the constant $(-1)^k$ remains; next, the operators $\epsilon(\hat{\theta}^i) - \iota(X_i)$, with $i \in J'$, act successively on forms of type γ^j, yielding $\hat{\theta}^j \wedge \gamma^j$ at each stage, until $\hat{\theta}^{J'}$ is created. Lastly, the identity $\eta(J, J')\eta(J', J) = (-1)^{k(n-k)}$ is used: this is just the observation that the permutation which interchanges the blocks of indices J and J' —while preserving the order within each block— is a product of $k(n-k)$ transpositions. Now (3.6) follows on noticing that $(-1)^k(-1)^{k(n-k)} = (-1)^{nk}(-1)^{k(k+1)} = (-1)^{nk}$. ☐

Corollary 3.2. *The Hodge star operator maps $\mathcal{A}^k(M, \mathbb{C})$ into $\mathcal{A}^{n-k}(M, \mathbb{C})$.* ☐

The next calculation shows that the presence of the factor i^m is what ensures that $\star$ is an involution, i.e., an operator whose square is the identity.

Lemma 3.3. $\star\star = \mathrm{id}$.

Proof. From (3.6) it is clear that $\star\star\hat{\theta}^J = C_J \hat{\theta}^J$ for some constant C_J. One computes:

$$\begin{aligned}
C_J &= i^{2m} (-1)^{nk + k(k-1)/2} (-1)^{n(n-k) + (n-k)(n-k-1)/2} \eta(J, J')\eta(J', J) \\
&= (-1)^m (-1)^{n^2} (-1)^{(k(k-1) + (n-k)(n-k-1))/2} (-1)^{k(n-k)} \\
&= (-1)^m (-1)^{n^2} (-1)^{(n^2 - n(2k+1) + 2k^2)/2} (-1)^{nk - k^2} \\
&= (-1)^m (-1)^{(3n^2 - n)/2} = (-1)^m (-1)^{(n^2+n)/2} = (-1)^m (-1)^{m(2m\pm1)} = +1.
\end{aligned}$$

This establishes that $\star\star = \mathrm{id}$, which also implies that $\star$ maps $\mathcal{A}^k(M, \mathbb{C})$ *onto* $\mathcal{A}^{n-k}(M, \mathbb{C})$. ☐

Exercise 3.3. If $X \in \mathfrak{X}(M)$, show that $\star\epsilon(X^\flat)\star = (-1)^n \iota(X)$ as operators from $\mathcal{A}^k(M)$ to $\mathcal{A}^{k-1}(M)$. (Reduce to the special case $X = X_j$.)

Consider the case $M = \mathbb{R}^3$; though not compact, it is an oriented Riemannian manifold with the usual euclidean metric; here $\hat{\theta}^k = dx^k$, and $\star$ is determined by

$$\star 1 = -\Omega = -dx^1 \wedge dx^2 \wedge dx^3,$$
$$\star dx^1 = dx^2 \wedge dx^3, \quad \star dx^2 = -dx^1 \wedge dx^3, \quad \star dx^3 = dx^1 \wedge dx^2,$$

on account of (3.6).

When $M = \mathbb{C}$ with local coordinates x, y, take $z = x + iy$. For the usual metric, we have $\hat{\theta}^1 = dx$, $\hat{\theta}^2 = dy$, and so $\star dx = i\,dy$, $\star dy = -i\,dx$. Using the coordinates $z, \bar{z}$, we then have $\star dz = dz$, $\star d\bar{z} = -d\bar{z}$. Thus, $\star df = \star(\partial f + \bar{\partial} f) = \partial f - \bar{\partial} f$, so that $f \in C^\infty(\mathbb{C}, \mathbb{C})$ is holomorphic iff $\bar{\partial} f = 0$ (by the Cauchy–Riemann equations) iff $\star df = df$. More generally, any $\alpha \in \mathcal{A}^\bullet(M, \mathbb{C})$ may be written as $\alpha = \alpha^+ + \alpha^-$ where α^+ is *selfdual*, i.e., $\star\alpha^+ = \alpha^+$, and α^- is *antiselfdual*, i.e., $\star\alpha^- = -\alpha^-$. When $n = 2m$ and $\alpha \in \mathcal{A}^m(M)$, we have $\alpha^+, \alpha^- \in \mathcal{A}^m(M)$ also. When $n = 2$ and M is a compact complex manifold (i.e., a Riemann surface), $f \in C^\infty(M, \mathbb{C})$ is holomorphic iff df is selfdual.

An important example is $M = \mathbb{S}^2$ with the Riemannian metric $g = d\theta^2 + \sin^2\theta\, d\phi^2$. In spherical coordinates, a local orthonormal basis is given by

$$\hat{\theta}^1 = d\theta, \quad \hat{\theta}^2 = \sin\theta\, d\phi.$$

The star operator on $\mathbb{S}^2$ is determined by

$$\star 1 = i\Omega = i \sin\theta\, d\theta \wedge d\phi, \quad \star d\theta = i \sin\theta\, d\phi,$$

and thus $\star\Omega = -i$ and $\star(\sin\theta\, d\phi) = -i\, d\theta$.

For a general system of coordinates, the star operator can be described as follows. On a chart domain U, a k-form $\omega \in \mathcal{A}^k(U)$ can be written as $\omega = \omega_{j_1 \dots j_k}\, dx^{j_1} \wedge \dots \wedge dx^{j_k}$. New coefficients with "raised indices" are defined by $\omega^{r_1 \dots r_k} = g^{r_1 j_1} \dots g^{r_k j_k} \omega_{j_1 \dots j_k}$ where $[g^{rs}]$ denotes the matrix inverse to the matrix $[g_{ij}]$. Let $\epsilon_{t_1 \dots t_n} := 0$ if $t_1, \dots, t_n$ contains a repeated index, and otherwise let $\epsilon_{t_1 \dots t_n} := \pm 1$ be the sign of the permutation $(1, \dots, n) \mapsto (t_1, \dots, t_n)$. Then $\star\omega$ is given by

$$\begin{aligned} &\star(\omega_{j_1 \dots j_k}\, dx^{j_1} \wedge \dots \wedge dx^{j_k}) \\ &\quad = i^m (-1)^{nk + k(k-1)/2} \frac{\sqrt{\det g}}{(n-k)!} \epsilon_{i_1 \dots i_{n-k} r_1 \dots r_n} \omega^{r_1 \dots r_k}\, dx^{i_1} \wedge \dots \wedge dx^{i_{n-k}}. \end{aligned} \tag{3.7}$$

Exercise 3.4. Check the validity of (3.7) by showing that its right hand side is invariant under a general change of (oriented) frame, and that it reduces to (3.6) for an orthonormal frame.

Exercise 3.5. Work out the action of $\star$ for the Fubini–Study metric on $\mathbb{CP}^m$.

The Hodge star operator relates the exterior product of forms to the Riemannian volume form, as follows.

Lemma 3.4. *For $\alpha, \beta \in \mathcal{A}^k(M, \mathbb{C})$,*

$$\alpha \wedge \star\beta = i^m (-1)^{nk+k(k-1)/2} (\alpha \mid \beta)\, \Omega. \tag{3.8}$$

Proof. Since both sides of (3.8) are $C^\infty(M)$-bilinear in (α, β), it suffices to verify equality for a local basis; thus we may take $\alpha = \hat\theta^I$, $\beta = \hat\theta^J$ where I and J are subsets of $\{1, \dots, n\}$.

Now $(\hat\theta^I \mid \hat\theta^J) = 1$ or 0 according as $I = J$ or not. If $I \neq J$ then $\hat\theta^I \wedge \star\hat\theta^J = 0$ from (3.6), since $I \cap J' \neq \emptyset$. Also, $\hat\theta^J \wedge \star\hat\theta^J = i^m(-1)^{nk+k(k-1)/2}\eta(J, J')\hat\theta^J \wedge \hat\theta^{J'} = i^m(-1)^{nk+k(k-1)/2}\Omega$. □

Definition 3.5. The complex vector space $\mathcal{A}^k(M, \mathbb{C})$ becomes a prehilbert space under the positive definite Hermitian form

$$\langle\langle \alpha \mid \beta \rangle\rangle := \int_M (\bar\alpha \mid \beta)\, \Omega,$$

where the integral over M is *normalized* by the requirement that $\int_M \Omega = 1$. Its completion with respect to this "integrated inner product"[2] is a Hilbert space which may be denoted $L^{2,k}(M)$. Notice that $\langle\langle \alpha \mid \beta \rangle\rangle = \langle\langle \bar\beta \mid \bar\alpha \rangle\rangle$, i.e., the conjugation of forms is antiunitary. The inner product may be extended to all of $\mathcal{A}^\bullet(M, \mathbb{C})$ by declaring that k-forms and l-forms be orthogonal for $k \neq l$; the completion of $\mathcal{A}^\bullet(M, \mathbb{C})$ is the Hilbert-space direct sum $L^{2,\bullet}(M) := \bigoplus_{k=0}^n L^{2,k}(M)$.

From (3.8), it follows that

$$\langle\langle \alpha \mid \beta \rangle\rangle = i^{-m}(-1)^{nk+k(k-1)/2} \int_M \bar\alpha \wedge \star\beta.$$

Lemma 3.5. *The Hodge star operator is isometric, i.e., if $\alpha, \beta \in \mathcal{A}^k(M, \mathbb{C})$ then*

$$\langle\langle \star\alpha \mid \star\beta \rangle\rangle = \langle\langle \alpha \mid \beta \rangle\rangle.$$

Proof. From (3.6) it follows that $\star\bar\alpha = (-1)^m \overline{\star\alpha}$ for any $\alpha \in \mathcal{A}^\bullet(M, \mathbb{C})$. Applying (3.8) twice,

$$\begin{aligned}
\langle\langle \star\alpha \mid \star\beta \rangle\rangle &= i^{-m}(-1)^{n(n-k)+(n-k)(n-k-1)/2} \int_M \overline{\star\alpha} \wedge \beta \\
&= i^m (-1)^{n^2+n(n-1)/2+k(k+1)/2} \int_M \star\bar\alpha \wedge \beta \\
&= i^m (-1)^{n(n+1)/2+k(k+1)/2} (-1)^{k(n-k)} \int_M \beta \wedge \star\bar\alpha \\
&= (-1)^m (-1)^{n(n+1)/2} \langle\langle \bar\beta \mid \bar\alpha \rangle\rangle = \langle\langle \alpha \mid \beta \rangle\rangle
\end{aligned}$$

since $\frac{1}{2}n(n+1) = m(2m \pm 1) \equiv m \bmod 2$. □

Since $\star\star = id$ on $L^{2,k}(M)$, this isometry also satisfies $\langle\langle \star\alpha | \gamma \rangle\rangle = \langle\langle \alpha | \star\gamma \rangle\rangle$ for $\alpha \in \mathcal{A}^k(M, \mathbb{C})$, $\gamma \in \mathcal{A}^{n-k}(M, \mathbb{C})$; thus the Hodge star operator extends to a selfadjoint unitary operator on $L^{2,k}(M) \oplus L^{2,n-k}(M)$.

[2] The double brackets are intended to distinguished the integrated inner product from the $C^\infty(M)$-valued form $(\cdot \mid \cdot)$.

3.4 The Hodge Laplacian

Definition 3.6. The **codifferential** $\delta\colon \mathcal{A}^k(M,\mathbb{C}) \to \mathcal{A}^{k-1}(M,\mathbb{C})$ is the adjoint of the exterior derivative with respect to the integrated inner product. In other words, if $\alpha \in \mathcal{A}^k(M,\mathbb{C})$, $\beta \in \mathcal{A}^{k-1}(M,\mathbb{C})$, $\delta\alpha$ is determined by the relation

$$\langle\!\langle \delta\alpha \mid \beta \rangle\!\rangle := \langle\!\langle \alpha \mid d\beta \rangle\!\rangle. \tag{3.9}$$

The Riesz theorem "guarantees" that there is a unique element $\delta\alpha \in L^2_{k-1}(M)$ satisfying (3.9), though it is not immediately clear that $\delta\alpha \in \mathcal{A}^{k-1}(M,\mathbb{C})$. That this is indeed the case, and that δ takes $\mathcal{A}^k(M,\mathbb{R})$ to $\mathcal{A}^{k-1}(M,\mathbb{R})$, is a consequence of the following basic identity.

Lemma 3.6. $\delta = (-1)^{n+1}{\star}d{\star}$.

Proof. If $\alpha \in \mathcal{A}^k(M,\mathbb{C})$ and $\beta \in \mathcal{A}^{k-1}(M,\mathbb{C})$, then $d\bar\beta \wedge \star\alpha + (-1)^{k-1}\bar\beta \wedge d(\star\alpha) = d(\bar\beta \wedge \star\alpha)$ is an exact n-form, whose integral is zero by Stokes' theorem. Thus

$$\begin{aligned}
\langle\!\langle \beta \mid \delta\alpha \rangle\!\rangle = \langle\!\langle d\beta \mid \alpha \rangle\!\rangle &= i^{-m}(-1)^{nk+k(k-1)/2}\int_M d\bar\beta \wedge \star\alpha \\
&= i^{-m}(-1)^{nk+k(k-1)/2}\int_M (-1)^k\bar\beta \wedge d(\star\alpha) \\
&= (-1)^{nk+k(k-1)/2}(-1)^k(-1)^{n(k-1)+(k-1)(k-2)/2}\langle\!\langle \beta \mid \star d\star\alpha \rangle\!\rangle \\
&= (-1)^{n+1}\langle\!\langle \beta \mid \star d\star\alpha \rangle\!\rangle,
\end{aligned}$$

and so $\delta\alpha = (-1)^{n+1}{\star}d{\star}\alpha$ since β is arbitrary. □

Clearly $\delta^2 = 0$, and in particular $(\mathcal{A}^\bullet(M), \delta)$ is a chain complex. We say that $\omega \in \mathcal{A}^k(M)$ is **coclosed** if $\delta\omega = 0$, or **coexact** if $\omega = \delta\zeta$ for some $\zeta \in \mathcal{A}^{k+1}(M)$.

Definition 3.7. The **Hodge Laplacian** is the operator Δ on $\mathcal{A}^\bullet(M)$ defined as

$$\Delta := (d+\delta)^2 = d\delta + \delta d.$$

Notice that Δ takes $\mathcal{A}^k(M)$ to $\mathcal{A}^k(M)$, since d raises and δ lowers degrees by one. [The restriction of Δ to 0-forms is called the *Laplace–Beltrami operator* on $C^\infty(M)$.]

A k-form γ is called **harmonic** if $\Delta\gamma = 0$. We denote the vector space of harmonic k-forms by $\mathrm{Harm}^k(M)$.

Lemma 3.7. *A k-form γ is harmonic iff $d\gamma = \delta\gamma = 0$.*

Proof. If γ is both closed and coclosed, then clearly $\Delta\gamma = d(\delta\gamma) + \delta(d\gamma) = 0$. On the other hand, for any $\omega \in \mathcal{A}^\bullet(M)$,

$$\langle\!\langle \omega \mid \Delta\omega \rangle\!\rangle = \langle\!\langle \omega \mid d\delta\omega \rangle\!\rangle + \langle\!\langle \omega \mid \delta d\omega \rangle\!\rangle = \langle\!\langle \delta\omega \mid \delta\omega \rangle\!\rangle + \langle\!\langle d\omega \mid d\omega \rangle\!\rangle \geq 0,$$

so that $\Delta\gamma = 0$ implies $d\gamma = \delta\gamma = 0$. □

The Laplacian Δ commutes with the operators $\star$, d and δ. Indeed, $\star d\delta = (-1)^{n+1}\star d\star d\star = \delta d\star$ and $\star\delta d = (-1)^{n+1}d\star d = d\delta\star$, so $\star\Delta = \star d\delta + \star\delta d = \delta d\star + d\delta\star = \Delta\star$. Also, $d\Delta = d\delta d = \Delta d$ and $\delta\Delta = \delta d\delta = \Delta\delta$ since $d^2 = \delta^2 = 0$.

On the Hilbert space $L^{2,\bullet}(M)$, $\Delta = d\delta + \delta d = dd^* + d^*d$ is a formally selfadjoint positive operator with domain $\mathcal{A}^\bullet(M,\mathbb{C})$; however, it is generally an unbounded operator. It can be made bounded by defining a larger Hilbert space norm on $\mathcal{A}^\bullet(M,\mathbb{C})$ and completing to obtain a smaller Hilbert space $H^{2n,\bullet}(M)$, called a Sobolev space. We shall not go into this matter here; we refer to [28] for the details. It turns out that Δ is then bounded as an operator from $H^{2n,\bullet}(M)$ to $L^{2,\bullet}(M)$, which is, in fact, an elliptic differential operator.[3] Elliptic operators have two main properties. Firstly, they are *Fredholm* operators. This implies both that $\ker\Delta$ is finite-dimensional, and that Δ has closed range in $L^{2,\bullet}(M)$. Secondly, a generalized solution of an elliptic differential operator is in fact smooth.[4] This means that $\Delta u = 0$ for $u \in H^{2n,\bullet}(M)$ only when u in fact lies in $\mathcal{A}^\bullet(M,\mathbb{C})$. In other words, $\ker\Delta = \mathrm{Harm}^\bullet(M,\mathbb{C})$. The Fredholm property of Δ now implies that $\mathrm{Harm}^\bullet(M,\mathbb{C})$ is a *finite-dimensional* vector space.

The theory of the Hodge Laplacian culminates in the fundamental theorem of Hodge, which states that any k-form on a compact oriented Riemannian manifold can be uniquely decomposed as a sum of a closed k-form, a coclosed k-form, and a harmonic k-form.

Theorem 3.8. ***(Hodge).*** *Let (M,g) be a compact oriented Riemannian manifold. Then for each $k = 0,1,\dots,n$ there is an orthogonal direct sum*

$$\mathcal{A}^k(M) = d\mathcal{A}^{k-1}(M) \oplus \delta\mathcal{A}^{k+1}(M) \oplus \mathrm{Harm}^k(M). \tag{3.10}$$

That the summands are orthogonal follows from the identities $\langle\!\langle d\alpha \,|\, \delta\beta\rangle\!\rangle = \langle\!\langle d^2\alpha \,|\, \beta\rangle\!\rangle = 0$, $\langle\!\langle d\alpha \mid \gamma\rangle\!\rangle = \langle\!\langle \alpha \mid \delta\gamma\rangle\!\rangle = 0$, $\langle\!\langle \delta\beta \mid \gamma\rangle\!\rangle = \langle\!\langle \beta \mid d\gamma\rangle\!\rangle = 0$, for $\alpha \in \mathcal{A}^{k-1}(M)$, $\beta \in \mathcal{A}^{k+1}(M)$, $\gamma \in \mathrm{Harm}^k(M)$. If P denotes the orthogonal projector on the Hilbert space $L^{2,k}(M)$ whose range is $\mathrm{Harm}^k(M)$, then if $\omega \in \mathcal{A}^k(M)$ the form $\omega - P\omega$ lies in the range of the selfadjoint operator Δ; it turns out that this range is closed, even if one restricts Δ to its original domain $\mathcal{A}^k(M)$. Hence, $\omega - P\omega = \Delta\eta$ for some $\eta \in \mathcal{A}^k(M)$; and therefore $\omega = d(\delta\eta) + \delta(d\eta) + P\omega$. $\square$

The full proof of Hodge's theorem depends on verifying that Δ is elliptic and identifying carefully the range of Δ. For the original treatment of Hodge, one can examine his book [33]; for more modern proofs, in the somewhat more general setting of an "elliptic complex", one may consult [23, 28]. The inverse operator Δ^{-1}, defined on the orthogonal complement of $\mathrm{Harm}^k(M)$, is an integral operator, namely the "Green operator" for the partial differential equation $\Delta\alpha = 0$.

Corollary 3.9. *Each de Rham class $[\omega] \in H_{\mathrm{dR}}(M)$ contains exactly one harmonic form; thus $\gamma \mapsto [\gamma]$ is an isomorphism of $\mathrm{Harm}^k(M,\mathbb{R})$ onto $H^k_{\mathrm{dR}}(M)$, and therefore every de Rham cohomology space of M is finite dimensional.*

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[3] This means that the "principal symbol" of Δ, which is a certain matrix-valued function on T^*M, becomes invertible after deleting a neighbourhood of the zero section of the cotangent bundle. In fact, the principal symbol σ_Δ of the Laplacian is given by $\sigma_\Delta(\xi_x) = -(\xi_x|\xi_x) := -g_x(\xi_x^\sharp, \xi_x^\sharp)$ for $\xi_x \in T_xM$.

[4] A differential operator whose generalized solutions are automatically smooth is called "hypoelliptic". Any elliptic operator is hypoelliptic.

Proof. If $\omega = d\alpha + \delta\beta + \gamma$ in $\mathcal{A}^k(M,\mathbb{R})$, with γ harmonic, then ω is closed iff $d\delta\beta = 0$ iff $(\beta \mid d\delta\beta) = (\delta\beta \mid \delta\beta) = 0$ iff $\delta\beta = 0$. For $\omega = d\alpha + \gamma \in Z^k_{\mathrm{dR}}(M)$, we clearly have $[\omega] = [\gamma]$. Moreover, if $\omega' \in Z^k_{\mathrm{dR}}(M)$ with $[\omega'] = [\omega]$, then $\omega' = \omega + d\zeta$ for some $\zeta \in \mathcal{A}^{k-1}(M)$, so $\omega' = d(\alpha + \zeta) + \gamma$; hence ω' and ω have the same harmonic component γ. $\square$

Finally, notice that $\star d = (-1)^{n-1}\delta\star$, so that whenever $\omega = d\alpha + \delta\beta + \gamma$ in $\mathcal{A}^k(M,\mathbb{C})$, we also have $\star\omega = d((-1)^{n-1}\star\beta) + \delta((-1)^{n-1}\star\alpha) + \star\gamma$ in $\mathcal{A}^{n-k}(M,\mathbb{C})$. The uniqueness of the Hodge decomposition says that the star operator is a linear isomorphism of $\mathrm{Harm}^k(M,\mathbb{C})$ onto $\mathrm{Harm}^{n-k}(M,\mathbb{C})$; if desired, one can match *real* harmonic forms by $\gamma \mapsto i^{-m}\star\gamma$. Passing to cohomology, this yields a well-defined $\mathbb{R}$-linear isomorphism

$$[\gamma] \mapsto [i^{-m}\star\gamma] : H^k_{\mathrm{dR}}(M) \xrightarrow{\sim} H^{n-k}_{\mathrm{dR}}(M).$$

This isomorphism is called **Poincaré duality**.[5]

4 The Hodge Laplacian on the 2-sphere

In this section, we investigate in detail the Hodge Laplacian on the 2-sphere $\mathbb{S}^2$, both as an illustration of the Hodge theory in general, and in order to introduce a first example of a Dirac operator. The sphere is, of course, an oriented Riemannian manifold, but it is also round, i.e., it is a homogeneous space under the group of rotations of $\mathbb{R}^3$. The rotation invariance of the Laplacian Δ facilitates a complete description of all its eigenvalues and eigenvectors, which in turn leads to a corresponding spectral description of the Dirac operator $\not{D} = d + \delta$.

4.1 The rotation group in three dimensions

Definition 4.1. The **rotation group** $SO(3)$ consists of all 3×3 real matrices A satisfying $A^tA = 1_3$ and $\det A = 1$. Any rotation belongs to a one-parameter subgroup $\{\, \exp tN : t \in \mathbb{R} \,\}$, where N belongs to the Lie algebra $\mathfrak{so}(3)$ of the rotation group, i.e., the 3×3 real matrices satisfying $N^t + N = 0$, $\operatorname{Tr} N = 0$. Write

$$N = \begin{bmatrix} 0 & -n^3 & n^2 \\ n^3 & 0 & -n^1 \\ -n^2 & n^1 & 0 \end{bmatrix}, \tag{4.1}$$

and suppose $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$. Then if $n = (n^1, n^2, n^3) \in \mathbb{R}^3$, the matrix identities $N^2 = nn^t - 1_3$, $N^3 = -N$ hold, and it follows that $\exp tN = I + N\sin t + N^2(1 - \cos t) \in SO(3)$. Now the rotation action of $SO(3)$ on $\mathbb{R}^3$ (or on $\mathbb{S}^2$) is given explicitly by

$$\begin{aligned} (\exp tN)\, x &= (I + N\sin t + N^2(1 - \cos t))\, x \\ &= x + (n \times x)\sin t + (n(n \cdot x) - x)(1 - \cos t) \\ &= x\cos t + (n \times x)\sin t + n(n \cdot x)(1 - \cos t). \end{aligned} \tag{4.2}$$

[5]For noncompact Riemannian manifolds, one can define "compactly supported de Rham cohomology" $H^\bullet_{\mathrm{dR},c}(M)$ by starting from the cochain complex of differential forms with compact support. Then Poincaré duality is a family of isomorphisms between $H^k_{\mathrm{dR}}(M)$ and $H^{n-k}_{\mathrm{dR},c}(M)$. In the compact case, both cohomologies coincide.

Let L_1, L_2, L_3 be the generators of the rotation group, i.e., the elements of $\mathfrak{so}(3)$ given by replacing n in (4.1) by the standard orthonormal basis vectors; so $N = n^1 L_1 + n^2 L_2 + n^3 L_3$. It is immediate that $[L_1, L_2] = L_3$, $[L_2, L_3] = L_1$ and $[L_3, L_1] = L_2$, or more compactly, $[L_i, L_j] = \epsilon_{ij}{}^k L_k$.

It is very convenient to introduce the matrices $L_\pm := L_1 \pm i\, L_2$ (which belong to the complexified Lie algebra $\mathfrak{so}(3, \mathbb{C})$). Then

$$L_1^2 + L_2^2 + L_3^2 = L_- L_+ + L_3^2 - iL_3 \tag{4.3}$$

as 3×3 complex matrices.[1]

Definition 4.2. The action of $SO(3)$ on $\mathbb{S}^2$ induces an action of $C^\infty(\mathbb{S}^2)$ by $(R \cdot f)(x) := f(R^{-1}x)$; more generally, $SO(3)$ acts on $\mathcal{A}^\bullet(\mathbb{S}^2)$ by $R \cdot \omega := (R^{-1})^* \omega$. For the corresponding action of the Lie algebra $\mathfrak{so}(3)$ on $C^\infty(\mathbb{S}^2)$, the homomorphism property of the group action induces a Leibniz rule, so that $N \in \mathfrak{so}(3)$ acts on $C^\infty(\mathbb{S}^2)$ as a vector field $\widetilde{N}$, called the *fundamental vector field* of N, which is given explicitly by

$$\widetilde{N} f(x) := \frac{d}{dt}\bigg|_{t=0} f(\exp(-tN)\, x).$$

Since the rotation action on $SO(3)$ is a restriction of an action on $\mathbb{R}^3$, the same formula yields fundamental vector fields in $\mathfrak{X}(\mathbb{R}^3)$. In particular, since $\exp(-tL_1)\, x = (x^1, x^2 - tx^3, x^3 + tx^2) + O(t^2)$ and similarly for $\exp(-tL_2)\, x$ and $\exp(-tL_3)\, x$, on account of (4.2), the corresponding fundamental vector fields on $\mathbb{R}^3$ are

$$\widetilde{L}_1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}, \quad \widetilde{L}_2 = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}, \quad \widetilde{L}_3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}. \tag{4.4}$$

On transforming (4.4) to spherical coordinates (r, θ, ϕ), one finds that the $\widetilde{L}_j$ are independent of r and of $\partial/\partial r$ (as expected). Indeed,

$$\widetilde{L}_\pm = e^{\pm i\phi} \left(\mp i \frac{\partial}{\partial \theta} + \cot\theta \frac{\partial}{\partial \phi} \right), \quad \widetilde{L}_3 = -\frac{\partial}{\partial \phi}, \tag{4.5}$$

which may be regarded as vector fields either in $\mathfrak{X}(\mathbb{R}^3 \setminus \{0\})$ or in $\mathfrak{X}(\mathbb{S}^2)$.

Exercise 4.1. Verify the formulae (4.5) for the fundamental vector fields.

The sphere $\mathbb{S}^2$ is a homogeneous space for the rotation group, and may be identified with the coset space $SO(3)/SO(2)$, where $SO(2)$ is the subgroup of east-west rotations which fix the north pole. One may therefore consider a system of local coordinates for $\mathbb{S}^2$ which

[1] Any Lie algebra $\mathfrak{g}$ gives rise to an associative algebra, called its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, whose elements are polynomial combinations of elements of $\mathfrak{g}$, reduced by the commutation relations among such elements. The Poincaré–Birkhoff–Witt theorem [35] proves that the natural map from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$ is injective, so that $\mathfrak{g}$ may be regarded as a subspace of $\mathcal{U}(\mathfrak{g})$. Now (4.3) may be regarded as an identity in $\mathcal{U}(\mathfrak{so}(3, \mathbb{C}))$.

privileges the cartesian coordinate x^3. We adopt the following local coordinates for the remainder of this chapter:

$$\zeta := x^1 + ix^2 = e^{i\phi} \sin\theta, \quad x^3 = \cos\theta.$$

We also write $\bar\zeta := x^1 - ix^2 = e^{-i\phi} \sin\theta$, which supplies a third local coordinate for $\mathbb{R}^3$. Let us abbreviate $\partial_3 := \partial/\partial x^3$, $\partial_\zeta := \partial/\partial\zeta$, $\bar\partial_\zeta := \partial/\partial\bar\zeta$. In these coordinates, the fundamental vector fields are given by

$$\widetilde{L}_+ = -2ix^3\bar\partial_\zeta + i\zeta\partial_3, \quad \widetilde{L}_- = 2ix^3\partial_\zeta - i\bar\zeta\partial_3, \quad \widetilde{L}_3 = i\bar\zeta\bar\partial_\zeta - i\zeta\partial_\zeta. \tag{4.6}$$

A polynomial in the variables $\widetilde{L}_j$ gives a differential operator on $\mathbb{S}^2$. For instance, the operator corresponding to (4.3) satisfies

$$(\widetilde{L}_-\widetilde{L}_+ + \widetilde{L}_3^2 - i\widetilde{L}_3)\zeta = (\widetilde{L}_3 - i)\widetilde{L}_3\zeta = (\widetilde{L}_3 - i)(-i\zeta) = -2\zeta, \tag{4.7}$$

since $\bar\partial_\zeta$, and therefore also $\widetilde{L}_+$, vanishes on holomorphic functions of ζ.

4.2 The Hodge operators on the sphere

Lemma 4.1. *The Hodge star operator on $\mathcal{A}^\bullet(\mathbb{S}^2)$ is determined by the relations*

$$\star(d\zeta \wedge dx^3) = \zeta, \qquad \star(d\zeta) = x^3\, d\zeta - \zeta\, dx^3,$$

in the (ζ, x^3) coordinates.

Proof. First observe that

$$\begin{aligned} d\zeta \wedge dx^3 &= (ie^{i\phi} \sin\theta\, d\phi + e^{i\phi}\cos\theta\, d\theta) \wedge (-\sin\theta\, d\theta) \\ &= ie^{i\phi}\sin^2\theta\, d\theta \wedge d\phi = i\zeta\Omega, \end{aligned}$$

so that $\star 1 = i\Omega = \zeta^{-1}\, d\zeta \wedge dx^3$, and $\star(\zeta) = d\zeta \wedge dx^3$; reciprocally, $\star(d\zeta \wedge dx^3) = \zeta$. Hodge duals of 1-forms are given by

$$\star\, d\zeta = e^{i\phi} \star(i\sin\theta\, d\phi) + e^{i\phi}\cos\theta \star(d\theta) = e^{i\phi}\, d\theta + e^{i\phi}\cos\theta(i\sin\theta\, d\phi),$$

which simplifies to $\star(d\zeta) = x^3\, d\zeta - \zeta\, dx^3$, and reciprocally, $\star(x^3\, d\zeta - \zeta\, dx^3) = d\zeta$. □

The codifferential $\delta = -\star d\star$ is now easily found. For instance, $\delta(d\zeta) = \star d(\zeta\, dx^3 - x^3\, d\zeta) = \star(2d\zeta \wedge dx^3) = 2\zeta$, and since $\delta(\zeta) = 0$, it follows that

$$\Delta(\zeta) = (d\delta + \delta d)\zeta = \delta(d\zeta) = 2\zeta. \tag{4.8}$$

Definition 4.3. Since rotations act on differential forms through $R \cdot \omega := (R^{-1})^* \omega$, their generators act as *Lie derivatives*:

$$\mathcal{L}_j \omega := \frac{d}{dt}\bigg|_{t=0} (\exp(-tL_j)^* \omega) = \mathcal{L}_{\widetilde{L}_j} \omega$$

for $j = 1, 2, 3$. Lie derivatives commute with exterior derivation: $\mathcal{L}_X d = d\mathcal{L}_X$ as operators on $\mathcal{A}^\bullet(\mathbb{S}^2)$, for any $X \in \mathfrak{X}(\mathbb{S}^2)$; and, in particular, $\mathcal{L}_j d = d\mathcal{L}_j$ for $j = 1, 2, 3$.

An operator T on $\mathcal{A}^\bullet(\mathbb{S}^2)$ is *rotation-invariant* if $T(R \cdot \omega) = R \cdot (T\omega)$ for any $R \in SO(3)$. Since the three one-parameter subgroups $\{ \exp(tL_j) : t \in \mathbb{R} \}$ generate $SO(3)$, this holds iff $T\mathcal{L}_j = \mathcal{L}_j T$ for $j = 1, 2, 3$. For instance, the Hodge star operator $\star = i(\epsilon(\hat{\theta}^1) - \iota(\hat{\theta}^{1\sharp}))(\epsilon(\hat{\theta}^2) - \iota(\hat{\theta}^{2\sharp}))$ is unchanged if the local oriented orthonormal frame $\{\hat{\theta}^1, \hat{\theta}^2\}$ is replaced by $\{R \cdot \hat{\theta}^1, R \cdot \hat{\theta}^2\}$; this says that $\star$ is invariant under rotations, and therefore $\star\mathcal{L}_j = \mathcal{L}_j \star$ for $j = 1, 2, 3$.

Since $\delta = -{\star} d {\star}$, $\Delta = d\delta + \delta d$, it follows that $\delta\mathcal{L}_j = \mathcal{L}_j \delta$ and $\Delta\mathcal{L}_j = \mathcal{L}_j \Delta$ for $j = 1, 2, 3$. In particular, the Hodge Laplacian is rotation-invariant.

Exercise 4.2. Use the Cartan identity $\mathcal{L}_X = \iota_X d + d\iota_X$ to show that $\mathcal{L}_j \Omega = 0$ for $j = 1, 2, 3$. What can one conclude from this?

The Laplace–Beltrami operator Δ_0 (the restriction of Δ to $C^\infty(\mathbb{S}^2)$) is thus a second-order differential operator which commutes with rotations. It therefore represents a quadratic element in the centre of the algebra $\mathcal{U}(\mathfrak{so}(3))$. Now it is known [34, 35] that this centre is the polynomial algebra $\mathbb{R}[C]$ generated by the "Casimir element" $C = L_1^2 + L_2^2 + L_3^2$. Thus $\Delta_0 = a(\widetilde{L}_1^2 + \widetilde{L}_2^2 + \widetilde{L}_3^2)$ for some constant a; Δ and each $\mathcal{L}_j$ commutes with d, and so $\Delta = a(\mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_3^2) = a(\mathcal{L}_-\mathcal{L}_+ + \mathcal{L}_3^2 - i\mathcal{L}_3)$. From (4.7) and (4.8) one sees that $a = -1$, and therefore

$$\Delta = -\mathcal{L}_-\mathcal{L}_+ - \mathcal{L}_3^2 + i\mathcal{L}_3. \tag{4.9}$$

Exercise 4.3. Show directly, using only the commutation relations $[L_i, L_j] = \epsilon_{ij}{}^k L_k$, that any quadratic polynomial in L_1, L_2, L_3 commuting with each L_j must be a multiple of $L_1^2 + L_2^2 + L_3^2$.

4.3 Eigenvectors for the Laplacian

Since Δ is invariant under rotations, any subspace of differential forms which is stable under the $SO(3)$ action is mapped by Δ into another such subspace. The search for eigenvectors under Δ should therefore start with the irreducible subspaces of the representation $R \mapsto (R^{-1})^*$ of the compact group $SO(3)$ on $\mathcal{A}^\bullet(\mathbb{S}^2)$. This is the point of view adopted by Folland [26], who obtained a complete spectral decomposition of the Hodge Laplacian on a sphere of any dimension. Here we show how this works for the 2-sphere.

It is useful to recall some facts about representations of compact Lie groups, which may be found in many places, e.g. [13, 37, 53]. Any irreducible unitary representation of a compact group G acts on a finite dimensional Hilbert space, and the Hilbert space of any unitary representation may be written as a (possibly infinite) direct sum of irreducible subrepresentations. Moreover, by the Peter–Weyl theorem, all such irreducible representations already occur in the decomposition of the regular representation of G on $L^2(G)$, and their carrier spaces are in fact subspaces of $C^\infty(G)$.

In the case $G = SO(3)$, the identification $SO(3)/SO(2) \approx \mathbb{S}^2$ goes as follows: elements of $SO(3)$ are parametrized by three local coordinates (ϕ, θ, ψ), called "Euler angles", and $SO(2)$ is regarded as the one-parameter subgroup $\{ \exp \psi L_3 : \psi \in \mathbb{R} \}$; the remaining angles (θ, ϕ) as the spherical coordinates on $\mathbb{S}^2$. Thus functions on $\mathbb{S}^2$ are identified with functions on $SO(3)$ which are constant on $SO(2)$-cosets, i.e., functions which do not depend on the variable ψ; in this way the representation of $SO(3)$ on $C^\infty(\mathbb{S}^2)$ —or on its completion $L^2(\mathbb{S}^2)$— becomes a subrepresentation of the regular representation on $L^2(SO(3))$. Its irreducible subspaces are spanned by the *spherical harmonics* $Y_{lm}(\theta, \phi)$, where $l = 0, 1, 2, \dots$ and $m = -l, \dots, l-1, l$; indeed $\mathcal{H}_l := \operatorname{span}\{ Y_{lm} : m = -l, \dots, l \}$ is an irreducible subspace of dimension $2l + 1$, and the Y_{lm} form an orthonormal basis for $L^2(\mathbb{S}^2)$. It turns out, again by the Peter–Weyl theorem, that $SO(3)$ has (up to equivalence) exactly one irreducible unitary representation of each odd dimension, so the orthonormality of the spherical harmonics shows that in the decomposition of the rotation action on $C^\infty(\mathbb{S}^2)$, each such representation occurs once only, and that *there are no other irreducible subrepresentations.*[2]

There is a general principle for finding irreducible representations of a compact Lie group G, called the "theorem of the highest weight" [13, 35, 37]. One finds a maximal torus[3] in G, that is, a subgroup $T \leq G$ which is a torus, i.e., isomorphic to $\mathbb{T}^k$ for some k, with k maximal; if $G = SO(3)$, then $k = 1$ and $T := SO(2) \simeq \mathbb{T}$ will do. In a representation space for G, one looks for a joint eigenvector for the torus T; when $G = SO(3)$, this is just an eigenvector v for L_3, whose eigenvalue is called a "weight" of T. Within $\mathfrak{g}_\mathbb{C}$ one finds k "raising elements" which annihilate v (since the weight is "highest") and k "lowering elements" which, applied successively to v, generate a basis for an irreducible representation space V; the commutation relations in $\mathfrak{g}_\mathbb{C}$ ensure that applying other generators does not enlarge the space V. When $G = SO(3)$, there is one raising element, namely L_+, and one lowering element, namely L_-.

The upshot of the general theory is this. Within each space of k-forms on the sphere ($k = 0, 1, 2$), we must find forms α satisfying $\mathcal{L}_+\alpha = 0$ and $\mathcal{L}_3\alpha = c\alpha$ for some eigenvalue c, and such that the vector space $\operatorname{span}\{ \mathcal{L}_-^r \alpha : r \in \mathbb{N} \}$ is finite dimensional. The identity (4.9) guarantees that α is also an eigenvector for the Laplacian Δ.

Definition 4.4. From (4.6), the identity $\mathcal{L}_+ f = 0$ is satisfied whenever $f \in C^\infty(\mathbb{S}^2)$ is holomorphic in ζ and independent of x^3. Since $(\mathcal{L}_3 f)(\zeta) = -i\zeta\, f'(\zeta)$, this f is an eigenvector for $\mathcal{L}_3$ iff it is of the form $f(\zeta) = \zeta^l$ for some l; since $\zeta^l = e^{il\phi} \sin^l \theta$, the smoothness of f forces the condition $l \in \mathbb{N}$. Define $\phi_{0l} \in \mathcal{A}^0(\mathbb{S}^2)$ by $\phi_{0l}(\zeta, x^3) := \zeta^l$, for $l = 0, 1, 2, \dots$.

Exercise 4.4. Show that $\mathcal{L}_-^k \phi_{0l}$ is a linear combination of terms $(x^3)^{k-2r} \bar\zeta^r \zeta^{l-k+r}$ and check that $\mathcal{L}_-^k \phi_{0l} \neq 0$ for $k = 0, 1, \dots, 2l$ but $\mathcal{L}_-^{2l+1} \phi_{0l} = 0$. Conclude that the functions $\{ R \cdot \phi_{0l} :$

[2] For other compact groups, the completeness of the decomposition of the natural representation on 0-forms on a homogeneous space may be obtained from a counting argument based on Frobenius reciprocity. For the case of $SO(n)$ and the sphere $\mathbb{S}^{n-1}$, we refer to Folland [26].

[3] Any compact connected Lie group is the union of all its maximal tori, and any two maximal tori are conjugate, by a theorem of Weyl: see [13] for a proof.

$R \in SO(3)\,\}$ span an irreducible representation space for $SO(3)$, of dimension $2l+1$, for each $l \in \mathbb{N}$.

Exercise 4.5. Prove that the functions $\{\, \mathcal{L}_-^r \phi_{0l} : l \in \mathbb{N},\ r = 0, \dots, 2l \,\}$ span a dense subset of $L^2(\mathbb{S}^2)$ by showing that any homogeneous polynomial in the variables $\zeta, \bar\zeta, x^3$ is a linear combination of these.

Exercise 4.6. Check that $\langle\!\langle \phi_{0l} \mid \phi_{0m} \rangle\!\rangle = 0$ for $l \neq m$, and that $\langle\!\langle \alpha \mid \mathcal{L}_- \beta \rangle\!\rangle = \langle\!\langle \mathcal{L}_+ \alpha \mid \beta \rangle\!\rangle$ for $\alpha, \beta \in \mathcal{A}^0(\mathbb{S}^2)$, using (4.6). Show that $[L_+, L_-] = -2iL_3$ and $[L_3, L_-] = iL_-$ and conclude that $\mathcal{L}_+ \mathcal{L}_-^r \phi_{0l} = a_{lr} \mathcal{L}_-^{r-1} \phi_{0l}$ for some constant a_{lr}. Deduce that the functions $\mathcal{L}_-^r \phi_{0l}$, when suitably normalized, yield an orthonormal basis for $L^2(\mathbb{S}^2)$.

We have thus identified a complete set of irreducible subrepresentations of the rotation action on $\mathcal{A}^0(\mathbb{S}^2)$. Note that $\mathcal{L}_3(\zeta^l) = -il\zeta^l$, so the $\mathcal{L}_3$-eigenvalue is $-il$. It is immediate from (4.9) that $\Delta(\zeta^l) = (l^2 + l)\,\zeta^l$. We take stock that

$$\mathcal{L}_+ \phi_{0l} = 0, \qquad \mathcal{L}_3 \phi_{0l} = -il\,\phi_{0l}, \qquad \Delta \phi_{0l} = l(l+1)\,\phi_{0l}. \tag{4.10}$$

It is now clear that the Laplace–Beltrami operator Δ_0 is a formally selfadjoint, positive operator on $L^2(\mathbb{S}^2)$, with spectrum $\operatorname{sp}(\Delta_0) = \{\, l(l+1) : l \in \mathbb{N} \,\}$. Since Δ_0 commutes with $\mathcal{L}_-$, each $\mathcal{L}_-^r \phi_{0l}$ $(r = 0, \dots, 2l)$ is an eigenvector for the eigenvalue $l(l+1)$, which therefore has multiplicity $2l+1$. The set $\operatorname{sp}(\Delta_0)$, with these multiplicities, is sometimes referred to as the spectrum of the Riemannian manifold $\mathbb{S}^2$ [8].

Note, in particular, that $\ker \Delta_0 = \operatorname{Harm}^0(\mathbb{S}^2)$ is the one-dimensional space spanned by ϕ_{00}, i.e., the space of constant functions. Thus, the only harmonic functions on $\mathbb{S}^2$ are the constants.

4.4 Spectrum of the Hodge Laplacian

Definition 4.5. Since Δ commutes with the exterior derivative and the Hodge star operator, we can manufacture more eigenvectors by applying these operators to the ϕ_{0l}. Since $d\phi_{00} = 0$ because ϕ_{00} is constant, $d\phi_{0l}$ is an eigenvector only for $l \geq 1$. Define

$$\begin{aligned} \psi_{1l} &:= l^{-1}\, d\phi_{0l} = \zeta^{l-1}\, d\zeta, \\ \phi_{1l} &:= -\star\psi_{1l} = -\zeta^{l-1} \star d\zeta = \zeta^l\, dx^3 - \zeta^{l-1} x^3\, d\zeta, \end{aligned}$$

for $l = 1, 2, 3, \dots$. Since $\mathcal{L}_+$, $\mathcal{L}_3$ and Δ each commutes with d and $\star$, we obtain from (4.10):

$$\begin{aligned} &\mathcal{L}_+ \psi_{1l} = 0, \qquad \mathcal{L}_3 \psi_{1l} = -il\,\psi_{1l}, \qquad \Delta \psi_{1l} = l(l+1)\,\psi_{1l}, \\ &\mathcal{L}_+ \phi_{1l} = 0, \qquad \mathcal{L}_3 \phi_{1l} = -il\,\phi_{1l}, \qquad \Delta \phi_{1l} = l(l+1)\,\phi_{1l}. \end{aligned}$$

Thus the 1-forms ψ_{1l} and ϕ_{1l} are highest-weight vectors for irreducible representations of $SO(3)$, of dimension $2l+1$; in fact, the representation spaces are spanned respectively by $\{\, \mathcal{L}_-^r \psi_{1l} : r = 0, \dots, 2l \,\}$ and $\{\, \mathcal{L}_-^r \phi_{1l} : r = 0, \dots, 2l \,\}$.

Exercise 4.7. Show that $\langle\!\langle \psi_{1l} \mid \psi_{1m} \rangle\!\rangle = \langle\!\langle \phi_{1l} \mid \phi_{1m} \rangle\!\rangle = 0$ for $l \neq m$, and that $\langle\!\langle \psi_{1l} \mid \phi_{1m} \rangle\!\rangle = 0$ for all $l, m \geq 1$.

Exercise 4.8. Check that $\langle\!\langle \alpha \mid \mathcal{L}_-\beta \rangle\!\rangle = \langle\!\langle \mathcal{L}_+\alpha \mid \beta \rangle\!\rangle$ for $\alpha, \beta \in \mathcal{A}^1(\mathbb{S}^2)$, and conclude that the 1-forms $\{\, \mathcal{L}_-^r \psi_{1l}, \mathcal{L}_-^r \phi_{1l} : l \geq 1,\ r = 0, \dots, 2l \,\}$ are orthogonal.

The 1-forms ψ_{1l} are exact, so applying d to them yields only zero. However, the 1-forms ϕ_{1l} are coexact, since $\phi_{1l} = -\star d\phi_{0l} = \delta(\star\phi_{0l})$, so we may define

$$\psi_{2l} := (l+1)^{-1}\, d\phi_{1l} = \zeta^{l-1}\, dx^3 \wedge d\zeta = i\zeta^l \Omega$$

for $l = 1, 2, 3, \dots$. Since the expression $i\zeta^l\Omega$ also makes sense for $l = 0$, we also define $\psi_{20} := i\Omega$. Then

$$\mathcal{L}_+\psi_{2l} = 0, \qquad \mathcal{L}_3\psi_{2l} = -il\,\psi_{2l}, \qquad \Delta\psi_{2l} = l(l+1)\,\psi_{2l}$$

for $l = 0, 1, 2, \dots$. Furthermore, since $\star 1 = i\Omega$, we have $\star\psi_{2l} = \phi_{0l}$ for $l \in \mathbb{N}$.

We may summarize this information with two commutative diagrams:

$$\begin{array}{ccc} \phi_{0l} & \xleftarrow{\ \star\ } & \psi_{2l} \\ {\scriptstyle l^{-1}d}\Big\downarrow & & \Big\uparrow{\scriptstyle (l+1)^{-1}d} \\ \psi_{1l} & \xrightarrow{\ -\star\ } & \phi_{1l} \end{array} \qquad \begin{array}{ccc} \phi_{0l} & \xrightarrow{\ \star\ } & \psi_{2l} \\ {\scriptstyle (l+1)^{-1}\delta}\Big\uparrow & & \Big\downarrow{\scriptstyle l^{-1}\delta} \\ \psi_{1l} & \xleftarrow{\ -\star\ } & \phi_{1l} \end{array} \tag{4.11}$$

where $l \geq 1$. The first diagram takes stock of the foregoing definitions, and the vertical arrows in the second diagram are formed by composing three arrows from the first, using the identity $\delta = -\star d \star$. From this it is evident that the ψ_{kl} are exact forms and the ϕ_{kl} are coexact forms. A complete circuit of four arrows in either diagram corresponds to applying the operator $(l(l+1))^{-1}(-\star d\star d - d\star d\star) = (l(l+1))^{-1}\Delta$, which acts as the identity at each vertex.

We now have a complete set of eigenforms for Δ. Indeed, we have shown already that the $\mathcal{L}_-^r\phi_{0l}$ are a complete set of eigenforms of degree 0. Since $\star$ is a bijection between 0-forms and 2-forms, the $\mathcal{L}_-^r\psi_{2l}$, for $l \in \mathbb{N}$, $r = 0, \dots, 2l$ densely span $\mathcal{A}^2(\mathbb{S}^2)$. For the 1-forms, we may use the Hodge decomposition (3.10); the diagrams (4.11) show that $d\colon \delta\mathcal{A}^1(\mathbb{S}^2) \to d\mathcal{A}^0(\mathbb{S}^2)$ and $\delta\colon d\mathcal{A}^1(\mathbb{S}^2) \to \delta\mathcal{A}^2(\mathbb{S}^2)$ are bijections, that the ψ_{1l} densely span $d\mathcal{A}^0(\mathbb{S}^2)$, and that the ϕ_{1l} densely span $\delta\mathcal{A}^2(\mathbb{S}^2)$. It remains only to observe that *there are no nonzero harmonic 1-forms*. This can be seen by noting that by writing any 1-form as $\alpha = f(\zeta)\, d\zeta + g(x^3)\, dx^3 + h(\zeta, x^3)(x^3 d\zeta - \zeta\, dx^3)$, since $f(\zeta)\, d\zeta + g(x^3)\, dx^3$ is exact, and $h(x^3 d\zeta - \zeta\, dx^3) = \star(h\, d\zeta)$ is coexact. Thus, $\{\, \mathcal{L}_-^r\psi_{1l}, \mathcal{L}_-^r\phi_{1l} : l \geq 1,\ r = 0, \dots, 2l \,\}$ forms an orthogonal basis for $\mathcal{A}^1(\mathbb{S}^2)$.

The spectrum of the Hodge Laplacian is therefore $\operatorname{sp}(\Delta) = \{\, l(l+1) : l \in \mathbb{N} \,\}$, with multiplicities $4l(l+1)$ for the eigenvalue $l(l+1)$ when $l \geq 1$, and multiplicity 2 for the zero eigenvalue. Indeed, $\ker\Delta = \operatorname{span}\{\phi_{00}, \psi_{20}\} = \operatorname{Harm}^\bullet(\mathbb{S}^2)$. If K denotes the operator on $L^{2,\bullet}(\mathbb{S}^2)$ which is zero on $\operatorname{Harm}^\bullet(\mathbb{S}^2)$ and inverts Δ on its orthogonal complement, then K is a bounded selfadjoint operator and, moreover, is *compact*, since its spectrum $\operatorname{sp}(K) = \{\, (l(l+1))^{-1} : l \in \mathbb{N} \,\}$ accumulates only at 0 and consists of eigenvalues with finite multiplicities. Also, $\Delta K = K\Delta = I - P$, where P is the orthogonal projector of rank 2 with range $\operatorname{Harm}^\bullet(\mathbb{S}^2)$.

Corollary 4.2. *The de Rham cohomology spaces of* $\mathbb{S}^2$ *are given by*

$$H^\bullet_{\mathrm{dR}}(\mathbb{S}^2) \simeq \mathrm{Harm}^\bullet(\mathbb{S}^2) \simeq \mathbb{R} \oplus 0 \oplus \mathbb{R}, \tag{4.12}$$

on decomposing $\ker \Delta$ *by degrees of forms.* □

Notice that this coincides with the Čech cohomology $\check{H}^\bullet(\mathbb{S}^2, \mathbb{R})$, computed in subsection 1.9.

Definition 4.6. An important topological invariant of a manifold M is its *Euler characteristic*[4]

$$\chi(M) := \sum_{k=0}^{\dim M} (-1)^k \dim H^k_{\mathrm{dR}}(M).$$

For the 2-sphere, (4.12) yields $\chi(\mathbb{S}^2) = 1 - 0 + 1 = 2$.

4.5 The Hodge–Dirac operator

Definition 4.7. The **Hodge–Dirac operator** on the 2-sphere $\mathbb{S}^2$ is the operator $D\!\!\!/ := d + \delta$, whose square is the Hodge Laplacian: $D\!\!\!/^2 = \Delta$.

Let $\mathcal{A}^{\mathrm{even}}(\mathbb{S}^2) := \mathcal{A}^0(\mathbb{S}^2) \oplus \mathcal{A}^2(\mathbb{S}^2)$ be the algebra of differential forms of even degree on $\mathbb{S}^2$, and write $\mathcal{A}^{\mathrm{odd}}(\mathbb{S}^2) := \mathcal{A}^1(\mathbb{S}^2)$ to denote the odd-degree forms. Then $D\!\!\!/$ is an *odd operator* in the sense that it interchanges forms of even and odd parities: $D\!\!\!/(\mathcal{A}^{\mathrm{even}}(\mathbb{S}^2)) \subseteq \mathcal{A}^{\mathrm{odd}}(\mathbb{S}^2)$ and $D\!\!\!/(\mathcal{A}^{\mathrm{odd}}(\mathbb{S}^2)) \subseteq \mathcal{A}^{\mathrm{even}}(\mathbb{S}^2)$.

From (4.11), the action of $D\!\!\!/$ is given explicitly by

$$\begin{aligned} D\!\!\!/\phi_{0l} &= l\,\psi_{1l}, & D\!\!\!/\psi_{1l} &= (l+1)\,\phi_{0l}, \\ D\!\!\!/\psi_{2l} &= l\,\phi_{1l}, & D\!\!\!/\phi_{1l} &= (l+1)\,\psi_{2l}, \end{aligned} \tag{4.13}$$

for $l = 0, 1, 2, \dots$ on the left, and $l = 1, 2, 3, \dots$ on the right. It follows that

$$\begin{aligned} D\!\!\!/\big(\sqrt{l+1}\,\phi_{0l} \pm \sqrt{l}\,\psi_{1l}\big) &= \pm\sqrt{l(l+1)}\,\big(\sqrt{l+1}\,\phi_{0l} \pm \sqrt{l}\,\psi_{1l}\big), \\ D\!\!\!/\big(\sqrt{l+1}\,\psi_{2l} \pm \sqrt{l}\,\phi_{1l}\big) &= \pm\sqrt{l(l+1)}\,\big(\sqrt{l+1}\,\psi_{2l} \pm \sqrt{l}\,\phi_{1l}\big). \end{aligned}$$

Since these eigenforms for $D\!\!\!/$ densely span $L^{2,\bullet}(\mathbb{S}^2)$, one concludes that $D\!\!\!/$ is a formally selfadjoint[5] unbounded linear operator on $L^{2,\bullet}(\mathbb{S}^2)$, with spectrum $\mathrm{sp}(D\!\!\!/) = \{\, \pm\sqrt{l(l+1)} : l \in \mathbb{N} \,\}$.

Exercise 4.9. What are the multiplicities of the eigenvalues of $D\!\!\!/$?

[4] The Euler characteristic may be computed as the integral of a certain differential form over M, as we shall see later.

[5] The term "formally selfadjoint" means that $\langle\!\langle D\!\!\!/\omega \mid \eta \rangle\!\rangle = \langle\!\langle \omega \mid D\!\!\!/\eta \rangle\!\rangle$ for $\omega, \eta \in \mathcal{A}^\bullet(\mathbb{S}^2)$; operator theorists would say that $D\!\!\!/$ is "symmetric". With some more work, one can check that $D\!\!\!/$ is *essentially selfadjoint*, which means that it has an extension to a larger domain in the Hilbert space $L^{2,\bullet}(\mathbb{S}^2)$ which is a closed, unbounded selfadjoint operator. See [39] for a proof of this.

It also follows from (4.13) that $\ker \not{D} = \operatorname{span}\{\phi_{00}, \psi_{20}\}$, and that $\not{D}$ has dense range, so that $\operatorname{coker} \not{D} = 0$; moreover, the restriction of $\not{D}$ to $L^{2,\bullet}(\mathbb{S}^2) \ominus \operatorname{Harm}^\bullet(\mathbb{S}^2)$ has a compact inverse. Thus $\not{D}$ is a Fredholm operator.[6] Its index is given by

$$\operatorname{ind} \not{D} := \dim(\ker \not{D}) - \dim(\operatorname{coker} \not{D}) = 2 - 0 = 2.$$

Corollary 4.3. $\operatorname{ind} \not{D} = \chi(\mathbb{S}^2)$. $\square$

This is a first example of an *index theorem*, wherein a certain integer obtained by an integral over the manifold (namely, the Euler characteristic) turns out to be equal to the Fredholm index of an operator (namely, the Hodge–Dirac operator) which is bound up with the geometric structure of the manifold. For a wider discussion of index theorems in modern geometry and topology, we refer to [9, 28, 39, 42].

There is an equivalent method of computing the index from the grading of $\mathcal{A}^\bullet(\mathbb{S}^2)$ into forms of even and odd degree. Write $L^{2,\bullet}(\mathbb{S}^2) = L^{2,\mathrm{even}}(\mathbb{S}^2) \oplus L^{2,\mathrm{odd}}(\mathbb{S}^2)$ where $L^{2,\mathrm{even}}(\mathbb{S}^2)$ and $L^{2,\mathrm{odd}}(\mathbb{S}^2)$ are the respective completions of $\mathcal{A}^{\mathrm{even}}(\mathbb{S}^2)$ and $\mathcal{A}^{\mathrm{odd}}(\mathbb{S}^2)$; this is then a *graded Hilbert space*. The odd operator may be written as

$$\not{D} = \begin{pmatrix} 0 & \not{D}_1 \\ \not{D}_0 & 0 \end{pmatrix},$$

where $\not{D}_0 \colon L^{2,\mathrm{even}}(\mathbb{S}^2) \to L^{2,\mathrm{odd}}(\mathbb{S}^2)$ and $\not{D}_1 \colon L^{2,\mathrm{odd}}(\mathbb{S}^2) \to L^{2,\mathrm{even}}(\mathbb{S}^2)$. The selfadjointness of $\not{D}$ says that $\not{D}_1 = \not{D}_0^\dagger$, and this may be verified directly from (4.13) also. Now $\operatorname{coker} \not{D}_1 \simeq (\ker \not{D}_0)^\perp$ and $\operatorname{coker} \not{D}_0 \simeq (\ker \not{D}_1)^\perp$, so the index of $\not{D}$ equals

$$\operatorname{ind} \not{D} = \dim(\ker \not{D}_0) - \dim(\ker \not{D}_1). \tag{4.14}$$

From (4.13), the right hand side of (4.14) equals $2 - 0 = 2$, as expected.

Exercise 4.10. Show that $l \langle\langle \psi_{1l} \,|\, \psi_{1l} \rangle\rangle = (l+1) \langle\langle \phi_{0l} \,|\, \phi_{0l} \rangle\rangle$ and that $l \langle\langle \phi_{1l} \,|\, \phi_{1l} \rangle\rangle = (l+1) \langle\langle \psi_{2l} \,|\, \psi_{2l} \rangle\rangle$. Deduce from (4.13) that $\not{D}_0$ and $\not{D}_1$ are adjoints of each other.

The Hodge–Dirac operator is one of many operators which are collectively known as *Dirac operators*. Some common properties are: (i) they are unbounded selfadjoint Fredholm operators; (ii) the Hilbert spaces on which they act are graded into "even" and "odd" subspaces, which the Dirac operators interchange; (iii) their squares are "generalized Laplacians" (of which we shall have more to say later) acting on Riemannian manifolds.

Definition 4.8. As a second example, let us redefine the grading of differential forms by splitting $\mathcal{A}^\bullet(\mathbb{S}^2)$ into the (± 1)-eigenspaces for the Hodge star operator: thus $\mathcal{A}^+(\mathbb{S}^2) := \{\, \omega : \star\omega = \omega \,\}$ is the space of *selfdual forms*, and $\mathcal{A}^-(\mathbb{S}^2) := \{\, \eta : \star\eta = -\eta \,\}$ is the set of *antiselfdual forms*. Now $\delta\star = -\star d$ implies that $\not{D}\star = -\star\not{D}$, so the *same* operator $\not{D} = d + \delta$ interchanges selfdual and antiselfdual forms: $\not{D}(\mathcal{A}^\pm(\mathbb{S}^2)) \subseteq \mathcal{A}^\mp(\mathbb{S}^2)$.

[6] Conventionally, Fredholm operators are taken to be bounded, whereas $\not{D}$ and Δ are not. One could remedy this by redefining the norm on the domain space, but then the domain and range would lie in different Hilbert spaces. Instead, we use the alternative definition [22] of a Fredholm operator as a closed, possibly unbounded, operator between Hilbert spaces which has dense domain, finite-dimensional kernel and finite-codimensional (therefore closed) range. In this sense, the operator closures of $\not{D}$ and Δ are unbounded Fredholm operators on $L^{2,\bullet}(\mathbb{S}^2)$.

The diagrams (4.11) show that the forms $\psi_{2l} + \phi_{0l}$ $(l \geq 0)$ and $\phi_{1l} - \psi_{1l}$ $(l \geq 1)$ are selfdual, whereas $\psi_{2l} - \phi_{0l}$ $(l \geq 0)$ and $\phi_{1l} + \psi_{1l}$ $(l \geq 1)$ are antiselfdual. Since $\not{D}$ commutes with $\mathfrak{L}_-$, applying powers to $\mathfrak{L}_-$ to these forms generates a complete set [i.e., a set which spans a dense subspace of $L^{2,\bullet}(\mathbb{S}^2)$]. Thus we may rewrite (4.13) as

$$
\begin{aligned}
\not{D}(\psi_{2l} + \phi_{0l}) &= l\,(\phi_{1l} + \psi_{1l}), & \not{D}(\phi_{1l} - \psi_{1l}) &= (l+1)\,(\psi_{2l} - \phi_{0l}),\\
\not{D}(\phi_{1l} + \psi_{1l}) &= (l+1)\,(\psi_{2l} + \phi_{0l}), & \not{D}(\psi_{2l} - \phi_{0l}) &= l\,(\phi_{1l} - \psi_{1l}),
\end{aligned}
$$

where $l \geq 1$ in all cases; and $\not{D}(\psi_{20} \pm \phi_{00}) = 0$.

Observe that, in the new grading of $L^{2,\bullet}(\mathbb{S}^2)$, the odd operator $\not{D}$ can be written as

$$\not{D} = \begin{pmatrix} 0 & \not{D}_- \\ \not{D}_+ & 0 \end{pmatrix}, \tag{4.15}$$

where $\not{D}_\pm \colon L^{2,\pm}(\mathbb{S}^2) \to L^{2,\mp}(\mathbb{S}^2)$. Now we *define* its index as

$$\operatorname{ind} \not{D} := \dim(\ker \not{D}_+) - \dim(\ker \not{D}_-), \tag{4.16}$$

which equals $1 - 1 = 0$ since the $\ker \not{D}_\pm$ are one-dimensional spaces, spanned by $(\psi_{20} \pm \phi_{00})$.

The corresponding topological quantity [9] arises from the bilinear form $s([\alpha],[\beta]) := \int_{\mathbb{S}^2} \alpha \wedge \beta$ on $H^1_{\mathrm{dR}}(\mathbb{S}^2)$. The integral depends only on the cohomology classes $[\alpha]$ and $[\beta]$, and s is antisymmetric; we assign to s a "signature" of zero.[7] The index theorem for this case is the equality of this zero signature with the zero index for $\not{D}$ as defined by (4.15).

5 Connections on vector bundles

Line bundles are classified by integral Čech 2-cocycles, according to the theory developed in Section 1. In this chapter, we show how to produce, for a given Hermitian line bundle $E \longrightarrow M$, a Čech 2-cocycle which is associated to its class. This 2-cocycle comes from the curvature form of a *connection* on the line bundle. A connection, or covariant derivative, is a general structure which supplements that of a vector bundle with a notion of "parallel displacement" among neighbouring fibres. There are several possible ways to introduce connections; we shall adopt here an algebraic approach, in the spirit of Cartan calculus of vector fields and forms.

5.1 Modules of vector-valued forms

Lemma 5.1. *Suppose that $E \longrightarrow M$ and $E' \longrightarrow M$ are two vector bundles over the (compact) manifold M. If $\tau\colon E \to E'$ is a bundle map, i.e., a smooth map such that (τ, id_M) is a vector bundle morphism, there is an $C^\infty(M)$-linear map $\tau_*\colon \Gamma(E) \to \Gamma(E')$ given by $\tau_* s := \tau \circ s$; and the correspondence $\tau \mapsto \tau_*$ satisfies $(\mathrm{id}_E)_* := \mathrm{id}_{\Gamma(E)}$ and $(\tau \circ \sigma)_* = \tau_* \circ \sigma_*$.*

[7] A less trivial example arises on considering a Riemannian manifold M of dimension $n = 4k$; the corresponding formula yields a *symmetric* bilinear form $q([\alpha],[\beta]) := \int_M \alpha \wedge \beta$ on $H^{2k}_{\mathrm{dR}}(M)$, and the topological invariant is the *signature* of this bilinear form.

Proof. The $C^\infty(M)$-linearity of τ means that $\tau_*(fs) = f\tau_* s$ for $s \in \Gamma(E)$, $f \in C^\infty(M)$, which follows from the linearity of $\tau\colon E_x \to E'_x$ for each $x \in M$, since $\tau_*(fs)(x) := \tau(f(x)s(x)) = f(x)\tau(s(x)) = (f\,\tau_* s)(x)$. The remaining assertions are obvious. □

Let $\mathcal{A} := C^\infty(M)$ be the algebra of smooth functions on M. One way to restate the preceding Lemma is to say that $E \mapsto \Gamma(E)$, $\tau \mapsto \tau_*$ is a covariant *functor* $\Gamma\colon \mathrm{Vect}(M) \to \mathrm{Mod}(\mathcal{A})$ from the category of vector bundles over M to the category of $\mathcal{A}$*-modules*.

Exercise 5.1. Verify that $\Gamma(E^*) = \Gamma(E)^*$, where $E^* \to M$ is the dual vector bundle to $E \longrightarrow M$, and the notation $\mathcal{E}^*$, for an $\mathcal{A}$-module $\mathcal{E}$, denotes the module of $\mathcal{A}$-linear maps from $\mathcal{E}$ to $\mathcal{A}$.

Definition 5.1. If $\mathcal{A}$ is any (real or complex) algebra, a *bimodule* $\mathcal{E}$ over $\mathcal{A}$ is a vector space with bilinear operations $\mathcal{A} \times \mathcal{E} \to \mathcal{E}$ and $\mathcal{E} \times \mathcal{A} \to \mathcal{E}$, usually simply as $(a, s) \mapsto as$ and $(s, a) \mapsto sa$, satisfying $1\,s = s\,1 = s$, $a(a's) = (aa')s$, and $(sa')a = s(a'a)$, for $s \in \mathcal{E}$, $a, a' \in \mathcal{A}$. The **tensor product** $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}'$ of two such bimodules is the bimodule[1] whose elements are finite sums $\sum_j s_j \otimes s'_j$ with $s_j \in \mathcal{E}$ and $s'_j \in \mathcal{E}'$, subject only to the relations $(sa) \otimes s' - s \otimes (as') = 0$, for $s \in \mathcal{E}$, $s' \in \mathcal{E}'$, $a \in \mathcal{A}$. When $\mathcal{A}$ is commutative, the identification $sa = as$ makes each $\mathcal{A}$-module into an $\mathcal{A}$-bimodule; the foregoing recipe defines the *tensor product of* $\mathcal{A}$*-modules* in that case.

Proposition 5.2. *Let $\mathcal{A} := C^\infty(M)$ be the algebra of smooth functions on a compact manifold M, and let $E \longrightarrow M$, $E' \longrightarrow M$ be vector bundles over M. Then there is a canonical isomorphism of $\mathcal{A}$-modules:*

$$\Gamma(E) \otimes_{\mathcal{A}} \Gamma(E') \simeq \Gamma(E \otimes E').$$

Proof. If $s \in \Gamma(E)$, $s' \in \Gamma(E')$, let $s \otimes s'$ denote the section $x \mapsto s(x) \otimes s'(x)$ of the tensor product bundle $E \otimes F \longrightarrow M$. We provisionally denote by $s \otimes_{\mathcal{A}} s'$ the element of $\Gamma(E) \otimes_{\mathcal{A}} \Gamma(E')$ given by the definition of the tensor product of $\mathcal{A}$-modules. Let $\theta\colon \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E') \to \Gamma(E \otimes E')$ be the $\mathcal{A}$-linear map determined by $\theta(s \otimes_{\mathcal{A}} s') := s \otimes s'$; the claim is that θ is an isomorphism.

If U is a chart domain in M over which the bundles $E \longrightarrow M$ and $E' \longrightarrow M$ are trivial, then $E \otimes E' \longrightarrow M$ is also trivial over U. Indeed, we have seen in subsection 1.7 that any local section $t \in \Gamma(U, E)$ is of the form $t = \sum_{k=1}^r h^k s_k$, where $\{s_1, \dots, s_r\}$ is a local system of sections for E over U, and $h_1, \dots, h_r \in C^\infty(U)$; in other words, the sections $\{s_1, \dots, s_r\}$ *generate* $\Gamma(U, E)$ *freely* over $C^\infty(U)$. If $\{s'_1, \dots, s'_l\}$ is a local system of sections for E' over U, they generate $\Gamma(U, E')$ freely as a $C^\infty(U)$-module, and it is clear that $\{\, s_j \otimes s'_k : j = 1, \dots, r;\ k = 1, \dots, l \,\}$ generate $\Gamma(U, E \otimes E')$ as a free $C^\infty(U)$-module, since $\{\, s_j(x) \otimes s'_k(x) : j = 1, \dots, r;\ k = 1, \dots, l \,\}$ is a basis for $E_x \otimes E'_x$ for each $x \in U$. In summary, θ is an isomorphism whenever the bundles $E \longrightarrow M$ and $E' \longrightarrow M$ (and consequently $E \otimes E' \longrightarrow M$) are trivial.

In the general case, there are vector bundles $F \longrightarrow M$ and $F' \longrightarrow M$ such that $E \oplus F \longrightarrow M$ and $E' \oplus F' \longrightarrow M$ are trivial,[2] by Proposition 1.8. Let $\iota\colon E \to E \oplus F$ and

[1] The bimodule operations on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}'$ are, of course, defined by $a(s \otimes s') := (as) \otimes s'$ and $(s \otimes s')a := s \otimes (s'a)$.

[2] The use of Proposition 1.8 (existence of the supplementary bundle) is the only point in this proof where the compactness of M is used. It should be said that, with some work to establish the existence of a finite trivialising open covering of M, the compactness assumption can be dropped: we refer to [17] for a proof.

$\sigma\colon E \oplus F \to E$ be the extension and restriction maps: $\iota(u) := (u, 0)$, $\sigma(u, v) := u$, and let $\iota'\colon E' \to E' \oplus F'$, $\sigma'\colon E' \oplus F' \to E'$ be similarly defined. Then $\sigma \circ \iota = \mathrm{id}_E$, so $\sigma_* \circ \iota_* = \mathrm{id}_{\Gamma(E)}$ and also $\sigma'_* \circ \iota'_* = \mathrm{id}_{\Gamma(E')}$; thus, ι_* and ι'_* are injective, whereas σ_* and σ'_* are surjective.

Now $E_x \otimes E'_x$ is a direct summand of the vector space $(E_x \oplus F_x) \otimes (E'_x \oplus F'_x)$ for each $x \in M$; this yields bundle maps $\iota''\colon E \otimes E' \to (E \oplus F) \otimes (E' \oplus F')$, $\sigma''\colon (E \oplus F) \otimes (E' \oplus F') \to E \otimes E'$ satisfying $\sigma''_* \circ \iota''_* = \mathrm{id}_{\Gamma(E \otimes E')}$. Finally, let

$$\iota_* \otimes \iota'_*\colon \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E') \to \Gamma(E \oplus F) \otimes_{\mathcal{A}} \Gamma(E' \oplus F')$$

be defined by $(\iota_* \otimes \iota'_*)(s \otimes s') := \iota_* s \otimes \iota'_* s'$, and let $\sigma_* \otimes \sigma'_*\colon \Gamma(E \oplus F) \otimes_{\mathcal{A}} \Gamma(E' \oplus F') \to \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E')$ be defined similarly. We thus have two commutative diagrams:

$$\begin{array}{ccc} \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E') & \xrightarrow{\ \theta\ } & \Gamma(E \otimes E') \\ {\scriptstyle \iota_* \otimes \iota'_*}\big\downarrow & & \big\downarrow{\scriptstyle \iota''_*} \\ \Gamma(E \oplus F) \otimes_{\mathcal{A}} \Gamma(E' \oplus F') & \xrightarrow{\ \Theta\ } & \Gamma((E \oplus F) \otimes (E' \oplus F')) \end{array} \tag{5.1}$$

where Θ is the isomorphism of free $\mathcal{A}$-modules already obtained, and

$$\begin{array}{ccc} \Gamma(E) \otimes_{\mathcal{A}} \Gamma(E') & \xrightarrow{\ \theta\ } & \Gamma(E \otimes E') \\ {\scriptstyle \sigma_* \otimes \sigma'_*}\big\uparrow & & \big\uparrow{\scriptstyle \sigma''_*} \\ \Gamma(E \oplus F) \otimes_{\mathcal{A}} \Gamma(E' \oplus F') & \xrightarrow{\ \Theta\ } & \Gamma((E \oplus F) \otimes (E' \oplus F')) \end{array} \tag{5.2}$$

From (5.1), θ is injective, since $\iota_* \otimes \iota'_*$ and ι''_* are injective and Θ is bijective; and (5.2) shows analogously that θ is surjective. Thus θ is an $\mathcal{A}$-linear isomorphism in the general case. □

Corollary 5.3. *Each $\mathcal{A}$-linear map from $\Gamma(E)$ to $\Gamma(E')$ is of the form τ_* for some bundle map $\tau\colon E \to E'$.*

Proof. The maps $\tau\colon E \to E'$ form the total space of the vector bundle $\mathrm{Hom}(E, E') \longrightarrow M$, whose fibres are $\mathrm{Hom}(E_x, E'_x) \simeq E^*_x \otimes E'_x$. Thus τ can be identified with a section of the vector bundle $E^* \otimes E' \longrightarrow M$. On the other hand, an $\mathcal{A}$-linear map from $\Gamma(E)$ to $\Gamma(E')$ belongs to $\mathrm{Hom}_{\mathcal{A}}(\Gamma(E), \Gamma(E')) \simeq \Gamma(E)^* \otimes_{\mathcal{A}} \Gamma(E') \simeq \Gamma(E^*) \otimes_{\mathcal{A}} \Gamma(E')$; it is easily seen that $\tau_* \in \mathrm{Hom}_{\mathcal{A}}(\Gamma(E), \Gamma(E'))$ corresponds to $\theta^{-1}(\tau) \in \Gamma(E^*) \otimes_{\mathcal{A}} \Gamma(E')$ under these identifications. Since θ is bijective, these account for all $\mathcal{A}$-linear maps from $\Gamma(E)$ to $\Gamma(E')$. □

Definition 5.2. Consider the vector bundle $E' = \Lambda^r T^* M$, whose smooth sections are the r-forms over M: $\mathcal{A}^r(M) = \Gamma(\Lambda^r T^* M)$. Define

$$\mathcal{A}^r(M, E) := \Gamma(E) \otimes_{\mathcal{A}} \mathcal{A}^r(M) \simeq \Gamma(E \otimes \Lambda^r T^* M), \tag{5.3}$$

using Proposition 5.2. The elements of the $\mathcal{A}$-module $\mathcal{A}^r(M, E)$ are finite sums of the form $\sum_k s_k \otimes \omega_k$, where $s_k \in \Gamma(E)$, $\omega_k \in \mathcal{A}^r(M)$; we shall refer to them as "*E-valued r-forms over M*".

On a complex manifold, we may also define $\mathcal{A}^{p,q}(M, E) := \Gamma(E) \otimes_{\mathcal{A}} \mathcal{A}^{p,q}(M)$.

5.2 Connections

Definition 5.3. A **connection** on a complex vector bundle $E \longrightarrow M$ is a $\mathbb{C}$-linear[3] map $\nabla \colon \Gamma(E) \to \mathcal{A}^1(M, E)$ such that

$$\nabla(sf) = (\nabla s)f + s \otimes df \qquad \text{for all} \quad s \in \Gamma(E),\ f \in C^\infty(M). \tag{5.4}$$

This *Leibniz rule*[4] shows that the definition is local, i.e., that ∇s is determined by its restrictions to chart domains U_j. To see that, let $\{f_j\}$ be a smooth partition of unity subordinate to $\{U_j\}$, and notice that $\sum_j df_j = d\bigl(\sum_j f_j\bigr) = d(1) = 0$, and thus $\nabla s = \sum_j \nabla(sf_j)$ by (5.4).

This locality is immediately useful in showing that connections exist on any vector bundle. First, if $E \longrightarrow M$ is a trivial bundle, $E \approx M \times \mathbb{C}^r$, so that $\Gamma(E) \simeq \mathcal{A}^r$, the exterior derivative $d \colon \mathcal{A}^r \to \mathcal{A}^1(M)^r : (h^1, \dots, h^r) \mapsto (dh^1, \dots, dh^r)$ is a connection, since $d(h^k f) = dh^k\, f + h^k\, df$ for $k = 1, \dots, r$, i.e., $d(\boldsymbol{h}\, f) = (d\boldsymbol{h})f + \boldsymbol{h} \otimes df$ for $\boldsymbol{h} \in \mathcal{A}^r$. In the general case, over each chart domain U_j of M one can choose a local system of sections $\boldsymbol{s}_j = (s_{j1}, \dots, s_{jr})$ for $E \longrightarrow M$; the expansion $t = \sum_k s_{jk}\, h^k$ gives an isomorphism $\psi_j^* \colon \Gamma(U_j, E) \to C^\infty(U_j)^r : t \mapsto \boldsymbol{h}$. (This is just the pullback to sections of the local trivialization ψ_j of (1.1)). Since $\mathcal{A}^1(U_j, E) \simeq \Gamma(U_j, E) \otimes_{C^\infty(U_j)} \mathcal{A}^1(U_j)$, we get an isomorphism $\psi_j^* \otimes \mathrm{id} \colon \mathcal{A}^1(U_j, E) \to \mathcal{A}^1(U_j)^r$ on tensoring with $\mathcal{A}^1(U_j)$. Now define $\nabla^{(j)} := (\psi_j^* \otimes \mathrm{id})^{-1} \circ d \circ \psi_j^*$, so that $\nabla^{(j)} \colon \Gamma(U_j, E) \to \mathcal{A}^1(U_j, E)$ is a connection on $\pi^{-1}(U_j) \longrightarrow U_j$. Finally, take any smooth partition of unity $\{f_j\}$ subordinate to the cover $\{U_j\}$, and define $\nabla s := \sum_j \nabla^{(j)}(sf_j)$; one checks that ∇ is a connection on the vector bundle $E \longrightarrow M$.

Exercise 5.2. Carry out this check.

The Leibniz rule (5.4) means that ∇ is not itself $\mathcal{A}$-linear. However, if ∇_0 and ∇_1 are *two* connections on $E \longrightarrow M$, it is immediate from (5.4) that $(\nabla_1 - \nabla_0)(sf) = (\nabla_1 s - \nabla_0 s)f$ for $s \in \Gamma(E)$ and $f \in \mathcal{A}$. By Corollary 5.3, $\nabla_1 - \nabla_0 = \alpha_*$ for some $\alpha \in \Gamma(\operatorname{End} E \otimes T^*M) = \mathcal{A}^1(M, \operatorname{End} E)$. Therefore,

$$\nabla_1 s = \nabla_0 s + \alpha \circ s \qquad \text{for} \quad s \in \Gamma(E); \tag{5.5}$$

conversely, given ∇_0, this equation defines a connection ∇_1 when $\alpha \in \mathcal{A}^1(M, \operatorname{End} E)$. This says that the set of all connections on $E \longrightarrow M$ is an *affine space*, based on the vector space $\mathcal{A}^1(M, \operatorname{End} E)$.

Definition 5.4. If $X \in \mathfrak{X}(M)$ is a vector field, the *contraction* $\iota_X \colon \mathcal{A}^1(M) \to \mathcal{A}$ is $\mathcal{A}$-linear, since $\iota_X(f\beta) = f\,\beta(X) = f\,\iota_X\beta$ for $f \in \mathcal{A}$, $\beta \in \mathcal{A}^1(M)$. Thus it extends to an $\mathcal{A}$-linear map from $\mathcal{A}^1(M, E) = \Gamma(E) \otimes_{\mathcal{A}} \mathcal{A}^1(M)$ to $\Gamma(E) \otimes_{\mathcal{A}} \mathcal{A} = \Gamma(E)$ by $\iota_X(s \otimes \beta) := s(\iota_X\beta) = s\,\beta(X)$.

If $\nabla \colon \Gamma(E) \to \mathcal{A}^1(M, E)$ is a connection, we write $\nabla_X := \iota_X \circ \nabla \colon \Gamma(E) \to \Gamma(E)$. This is a $\mathbb{C}$-linear map satisfying the Leibniz rule:

$$\nabla_X(sf) = (\nabla_X s)f + s(Xf). \tag{5.6}$$

[3] For real vector bundles, we require only that ∇ be an $\mathbb{R}$-linear map.

[4] It is convenient to regard $\Gamma(E)$ and $\mathcal{A}^1(M, E)$ as right $\mathcal{A}$-modules; if one wishes to regard them as left $\mathcal{A}$-modules by identifying $fs = sf$, the Leibniz rule can equivalently be written as $\nabla(fs) = f(\nabla s) + df \otimes s$.

Definition 5.5. If $E \longrightarrow M$ is a *Hermitian* vector bundle, the metric h on E defines a $\mathcal{A}$-sesquilinear map $\Gamma(E)\times\Gamma(E) \to \mathcal{A}$ by $(s|t) : x \mapsto h_x(s(x), t(x))$, for $s, t \in \Gamma(E)$. This extends to a sesquilinear form $\Gamma(E) \times \mathcal{A}^k(M, E) \to \mathcal{A}^k(M)$ by tensoring, i.e., $(s \mid t \otimes \omega) := (s \mid t)\, \omega$ when $\omega \in \mathcal{A}^k(M)$.

A connection ∇ on $E \longrightarrow M$ is **compatible with the metric** if

$$(\nabla s \mid t) + (s \mid \nabla t) = d(s \mid t) \qquad \text{for all} \quad s, t \in \Gamma(E). \tag{5.7}$$

Notice that this is yet another variant of the Leibniz rule. If $X \in \mathfrak{X}(M)$, a compatible connection satisfies $(\nabla_X s \mid t) + (s \mid \nabla_X t) = X(s \mid t)$.

Exercise 5.3. In general, for any connection ∇ on a vector bundle $E \longrightarrow M$, one may define a *dual connection* ∇^* on the dual bundle $E^* \longrightarrow M$ by stipulating that the following Leibniz rule should hold: $\langle \nabla^* \xi, s\rangle + \langle \xi, \nabla s\rangle = d\langle \xi, s\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the evaluation map $\Gamma(E^*) \times \Gamma(E) \to \mathcal{A}$. Verify that, for given ∇ and $\xi \in \Gamma(E^*)$, this Leibniz rule determines a well-defined element $\nabla^*\xi$ of $\mathcal{A}^1(M, E^*)$, and that the operator ∇^* thus obtained is a connection.

Exercise 5.4. Suppose that $E \longrightarrow M$ and $E' \longrightarrow M$ are equivalent vector bundles and that $\tau \colon E \to E'$ is an invertible bundle map. Extend τ_* to an operator from $\mathcal{A}^k(M, E)$ to $\mathcal{A}^k(M, E')$ by tensoring: $\tau_*(s \otimes \omega) := \tau_* s \otimes \omega$ for $s \in \Gamma(E)$, $\omega \in \mathcal{A}^k(M)$. If ∇ is a connection on $E \longrightarrow M$, show that $\nabla' := \tau_* \circ \nabla \circ \tau_*^{-1}$ is a connection on $E' \longrightarrow M$.

5.3 Curvature of a connection

Definition 5.6. If $\nabla \colon \Gamma(E) \to \mathcal{A}^1(M, E)$ is a connection on a vector bundle $E \longrightarrow M$, there is a canonical way to extend ∇ to a linear map $\nabla \colon \mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E)$. Since $\mathcal{A}^k(M, E)$ is a vector space generated by elements of the form $s \otimes \omega$, for $s \in \Gamma(E)$, $\omega \in \mathcal{A}^k(M)$, it suffices to define

$$\nabla(s \otimes \omega) := (\nabla s) \wedge \omega + s \otimes d\omega \qquad \text{for} \quad s \in \Gamma(E),\ \omega \in \mathcal{A}^k(M). \tag{5.8}$$

If $\eta \in \mathcal{A}^\bullet(M)$, it follows that

$$\nabla(s \otimes (\omega \wedge \eta)) = (\nabla s) \wedge \omega \wedge \eta + s \otimes d\omega \wedge \eta + (-1)^k\, s \otimes \omega \wedge d\eta,$$

from which one obtains the "graded Leibniz rule":

$$\nabla(\zeta \wedge \eta) = (\nabla \zeta) \wedge \eta + (-1)^k\, \zeta \wedge d\eta \qquad \text{if} \quad \zeta \in \mathcal{A}^k(M, E),\ \eta \in \mathcal{A}^\bullet(M). \tag{5.9}$$

Exercise 5.5. Verify that the extended map $\nabla \colon \mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E)$ is well-defined, i.e., if $\sum_j s_j \otimes \omega_j = \sum_r t_r \otimes \eta_r$, with $s_j, t_r \in \Gamma(E)$ and $\omega_j, \eta_r \in \mathcal{A}^k(M)$, then $\sum_j \nabla(s_j) \otimes \omega_j = \sum_r \nabla(t_r) \otimes \eta_r$.

The iterated map $\nabla^2 \colon \Gamma(E) \to \mathcal{A}^2(M, E)$ satisfies

$$\begin{aligned} \nabla^2(sf) &= \nabla\big((\nabla s) f\big) + \nabla(s \otimes df) \\ &= (\nabla^2 s) f - \nabla s \wedge df + \nabla s \wedge df + s \otimes d(df) = (\nabla^2 s) f, \end{aligned}$$

for $s \in \Gamma(E)$, $f \in \mathcal{A}$. Thus ∇^2 is an $\mathcal{A}$-linear map from $\Gamma(E)$ to $\mathcal{A}^2(M, E) = \Gamma(E \otimes \Lambda^2 T^*M)$; by Corollary 5.3, $\nabla^2 s = \omega s$ with $\omega \in \Gamma(\operatorname{End} E \otimes \Lambda^2 T^*M) = \mathcal{A}^2(M, \operatorname{End} E)$. This "matrix-valued 2-form" ω is called the **curvature** of the connection ∇.

Exercise 5.6. If $\tau\colon E \to E'$ is an invertible bundle map between equivalent vector bundles over M, and if ∇, ∇' are connections on E and E' related by $\nabla' = \tau_* \circ \nabla \circ \tau_*^{-1}$, show that their curvatures are likewise related by $\omega' = \tau_* \circ \omega \circ \tau_*^{-1}$.

On the trivial bundle $E = M \times \mathbb{C}^r$, let d be the connection $(h^1, \dots, h^r) \mapsto (dh^1, \dots, dh^r)$ given by the exterior derivative. From (5.5), we can write $\nabla s = ds + \alpha s$ where $\alpha \in \mathcal{A}^1(M, \operatorname{End} E) = \mathcal{A}^1(M)^{r\times r}$ is a matrix of 1-forms on M. Explicitly, if $s \in \mathcal{A}^r$ has k-th component h^k, then αs has k-th component $\alpha^k_l h^l$, so $\alpha = [\alpha^k_l] \in \mathcal{A}^1(M)^{r\times r}$. In this case, the curvature ω is a matrix of 2-forms:

$$\begin{aligned} \omega s = \nabla^2 s &= d(ds + \alpha s) + \alpha \wedge (ds + \alpha s) \\ &= ((d\alpha)s - \alpha \wedge ds) + (\alpha \wedge ds + (\alpha \wedge \alpha)s) = (d\alpha + \alpha \wedge \alpha)\, s, \end{aligned} \tag{5.10}$$

so that $\omega = d\alpha + \alpha \wedge \alpha$ in $\mathcal{A}^2(M)^{r\times r}$. In components, $\omega^k_l = d\alpha^k_l + \alpha^k_m \wedge \alpha^m_l$.

5.4 A curvature formula

Lemma 5.4. *The curvature ω of a connection ∇ on a line bundle $E \longrightarrow M$ satisfies*

$$\omega(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \tag{5.11}$$

for all $X, Y \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$.

Proof. Since $\omega \in \mathcal{A}^2(M, \operatorname{End} E) = \Gamma(\operatorname{End} E) \otimes_{\mathcal{A}} \mathcal{A}^2(M)$, the evaluation on the pair of vector fields X, Y yields $\omega(X,Y) \in \Gamma(\operatorname{End} E)$, so that $\omega(X,Y)s \in \Gamma(E)$. Let the right hand side of (5.11) be denoted provisionally by $F(X,Y)s$. We claim that $s \mapsto F(X,Y)s$ is $\mathcal{A}$-linear. To see that, denote by $\tilde{f}$ the operator on $\Gamma(E)$ of right multiplication by $f \in \mathcal{A}$; then the Leibniz rule (5.6) can be rewritten as $[\nabla_X, \tilde{f}] = \widetilde{Xf}$. The desired $\mathcal{A}$-linearity of $F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ follows from

$$\begin{aligned} [F(X,Y), \tilde{f}] &= [[\nabla_X, \nabla_Y], \tilde{f}] - [\nabla_{[X,Y]}, \tilde{f}] \\ &= [\nabla_X, [\nabla_Y, \tilde{f}]] + [[\nabla_X, \tilde{f}], \nabla_Y] - [\nabla_{[X,Y]}, \tilde{f}] \\ &= [\nabla_X, \widetilde{Yf}] + [\widetilde{Xf}, \nabla_Y] - ([X,Y]f)^{\sim} \\ &= (X(Yf) - Y(Xf) - [X,Y]f)^{\sim} = 0, \end{aligned}$$

on using the Jacobi identity. Thus, by Corollary 5.3, $F(X,Y)$ lies in $\Gamma(\operatorname{End} E)$, for each $X, Y \in \mathfrak{X}(M)$. Furthermore, the formula $\iota_{hX} = \tilde{h}\iota_X$ for $h \in \mathcal{A}$, which entails $\nabla_{hX} = \tilde{h}\nabla_X$, shows that F is $\mathcal{A}$-bilinear in (X,Y), and so $F \in \mathcal{A}^2(M, \operatorname{End} E)$.

We must now show that ω and F coincide. It is enough to check this locally, since if $\{f_j\}$ is a partition of unity on M, the identities $\omega(X,Y)s = \sum_j \omega(X,Y)sf_j$, $F(X,Y)s = \sum_j F(X,Y)sf_j$ show that it suffices to prove $\omega(X,Y)s = F(X,Y)s$ when X, Y and s vanish outside a chart domain U_j. Thus we may suppose that $E \longrightarrow M$ is a trivial bundle and indeed that $E = M \times \mathbb{C}^r$, whereupon $\omega = d\alpha + \alpha \wedge \alpha$ if $\alpha \in \mathcal{A}^2(M)^{r\times r}$ is the matrix of 1-forms satisfying $\nabla = d + \alpha$.

Let us denote by h the element $(h_i, \dots, h_r)$ of $\mathcal{A}^r = \Gamma(E)$; then $\nabla_X h = \iota_X(dh + \alpha h)$ has k-th component $Xh^k + \alpha^k_l(X)h^l$. Thus $F(X,Y)h$ has k-th component

$$\begin{aligned} X(Yh^k + \alpha^k_l(Y)h^l) &+ \alpha^k_l(X)\,Yh^l + \alpha^k_m(X)\alpha^m_l(Y)h^l - Y(Xh^k + \alpha^k_l(X)h^l) \\ &- \alpha^k_l(Y)\,Xh^l - \alpha^k_m(Y)\alpha^m_l(X)h^l - [X,Y]h^k - \alpha^k_l([X,Y])h^l \\ &= X(\alpha^k_l(Y))h^l - Y(\alpha^k_l(X)h^l) - \alpha^k_l([X,Y])h^l + (\alpha^k_m(X)\alpha^m_l(Y) - \alpha^k_m(Y)\alpha^m_l(X))h^l \\ &= d\alpha^k_l(X,Y)h^l + (\alpha^k_m \wedge \alpha^m_l)(X,Y)h^l, \end{aligned}$$

that is, $F(X,Y)$ is a matrix of 2-forms with components $(d\alpha^k_l + \alpha^k_m \wedge \alpha^m_l)(X,Y)$. Hence $F = d\alpha + \alpha \wedge \alpha = \omega$ in $\mathcal{A}^2(M, \operatorname{End} E)$. □

Exercise 5.7. Verify that $F(gX, hY)s = F(X,Y)s\,gh$ for $g,h \in \mathcal{A}$.

The connection ∇ is determined by local 1-forms $\alpha_j \in \mathcal{A}^1(U_j, \operatorname{End} E)$, and by a local system $\boldsymbol{s}_j$ in $\Gamma(U_j, E)$ with respect to which the identification $h \leftrightarrow s_{jk}h^k$ is made. It should be carefully noted that the local 1-forms α_j do not patch together to give a global 1-form $\alpha \in \mathcal{A}^1(M, \operatorname{End} E)$ unless $E \longrightarrow M$ is a trivial bundle, since there is no global d for which $\nabla = d + \alpha$ unless the vector bundle is trivial.[5] However, the local 2-forms $\omega_j = d\alpha_j + \alpha_j \wedge \alpha_j \in \mathcal{A}^2(U_j, \operatorname{End} E)$ do patch together, since they are restrictions to the U_j of the global 2-form ω.

When $L \longrightarrow M$ is a *line bundle*, there is a simplification: by Lemma 1.2, the line bundle $\operatorname{End} L \longrightarrow M$ is trivial, so the curvature form ω belongs to $\mathcal{A}^2(M)$, since $\mathcal{A}^2(M, \operatorname{End} L) \simeq \mathcal{A}^2(M) \otimes_{\mathcal{A}} \Gamma(\operatorname{End} L) \simeq \mathcal{A}^2(M) \otimes_{\mathcal{A}} \mathcal{A} = \mathcal{A}^2(M)$ via canonical isomorphisms. Furthermore, there are local 1-forms $\alpha_j \in \mathcal{A}^1(M)$ such that $\omega = d\alpha_j$ on each chart domain U_j, so that the curvature ω is locally exact and hence is a *closed* 2-form on M.

The curvature form ω depends on the connection ∇, but its de Rham class $[\omega] \in H^2_{\mathrm{dR}}(M)$ does not. To see that, recall from (5.5) that if ∇_0 and ∇_1 are two connections on L, with respective curvatures ω_0 and ω_1, then $\nabla_1 s - \nabla_0 s = \alpha\, s$ for some $\alpha \in \mathcal{A}^1(M)$, and hence $\omega_1 - \omega_0 = d\alpha$, an exact 2-form. Thus the class $[\omega]$ depends only on the line bundle $L \longrightarrow M$.

Exercise 5.8. Suppose $L \longrightarrow M$ and $L' \longrightarrow M$ are equivalent line bundles and that $\tau\colon L \to L'$ is an invertible bundle map. Show that $\tau_* \in \Gamma(\operatorname{Hom}(L, L'))$ intertwines the canonical isomorphisms $\Gamma(\operatorname{End} L) \simeq \mathcal{A}$ and $\Gamma(\operatorname{End} L') \simeq \mathcal{A}$, and deduce that the connections ∇ and $\tau_* \circ \nabla \circ \tau_*^{-1}$ have the same curvature. Conclude that the class $[\omega]$ depends only on the equivalence class $[L]$ of the line bundle.

5.5 From de Rham cohomology to Cech cohomology

We have now associated to any line bundle $L \longrightarrow M$, by very different procedures, two second-degree cohomology classes, namely the Čech class obtained directly from its transition

[5]The local 1-forms α_j may be globalized by pulling back to the frame bundle $P \xrightarrow{\eta} M$; recall that the local system $\boldsymbol{s}_j$ may be regarded as a local section of the frame bundle. It is possible to construct a global 1-form $\tilde\alpha \in \mathcal{A}^1(P, \operatorname{End} V)$ which incorporates each $\eta^*\alpha_j$, and different connections given rise to different $\tilde\alpha$: this is called the *connection 1-form* for ∇, for which one may consult [9, 39, 52] or any standard text on differential geometry.

functions and the de Rham class of the curvature of an arbitrary connection. We are led to suspect that these two cohomologies are related in some underlying fashion. This is indeed the case: one way of proving the de Rham theorem, which says that the cohomology of real-valued differential forms on the manifold M depends only on the topology of M (i.e., not on its differential structure) is to show that the de Rham cohomology is isomorphic to Čech cohomology with constant real coefficients.[6] For the full proof, we refer to [12] or [17, Appendix E]. We need only the second-degree case of this isomorphism; however, its proof illustrates the general method.

Proposition 5.5. *Let M be a compact manifold. Then $H^2_{\mathrm{dR}}(M) \simeq \check{H}^2(M,\mathbb{R})$ by a canonical isomorphism.*

Proof. Select a finite good covering[7] $\mathfrak{U} = \{U_j\}$ for M. We must show that $H^2_{\mathrm{dR}}(M) \simeq H^2(\mathfrak{U},\mathbb{R})$. We first define maps between $Z^2_{\mathrm{dR}}(M)$ and $Z^2(\mathfrak{U},\mathbb{R})$, which have some ambiguities that can be removed by passing to cohomology, thereby yielding well-defined $\mathbb{R}$-linear maps between $H^2_{\mathrm{dR}}(M)$ and $H^2(\mathfrak{U},\mathbb{R})$.

Start with a closed 2-form ω in $\mathcal{A}^2(M)$; denote by $\underline{\omega} := \{\omega_j\}$ the set of restrictions of ω to each set U_j of the good covering; this is an element of $C^0(\mathfrak{U},\underline{\mathcal{A}}^2)$, the set of Čech 0-cochains with 2-form coefficients. Since $d\omega = 0$ and each U_j is contractible, the Poincaré lemma shows that there is $\alpha_j \in \mathcal{A}^1(U_j)$ with $d\alpha_j = \omega_j$ for each j; write $\underline{\alpha} := \{\alpha_j\} \in C^0(\mathfrak{U},\underline{\mathcal{A}}^1)$. Now on each nonvoid overlap $U_i \cap U_j$ we have $d(\alpha_i - \alpha_j) = \omega_i - \omega_j = 0$, so that $\alpha_i - \alpha_j = df_{ij}$ with $\boldsymbol{f} := \{f_{ij}\} \in C^1(\mathfrak{U},\underline{\mathcal{A}}^0)$. On each nonvoid $U_i \cap U_j \cap U_k$ we have $d(f_{ij} - f_{ik} + f_{jk}) = 0$ by cancellation, so that $a_{ijk} := f_{ij} - f_{ik} + f_{jk}$ is a constant real-valued function (since $U_i \cap U_j \cap U_k$ is connected); since $a_{ijk} - a_{ijl} + a_{ikl} - a_{jkl} = 0$ on $U_i \cap U_j \cap U_k \cap U_l$ by cancellation, we have $\delta\boldsymbol{a} = 0$, i.e., $\boldsymbol{a} \in Z^2(\mathfrak{U},\mathbb{R})$.

There are some ambiguities in the choices of α_j and f_{ij}, so this process does not give a well-defined map $\omega \mapsto \boldsymbol{a}$. Firstly, we could replace each α_j by $\alpha_j + dg_j$, where $\boldsymbol{g} := \{g_j\} \in C^0(\mathfrak{U},\underline{\mathcal{A}}^0)$; then f_{ij} becomes $f_{ij} + g_i - g_j$, which leaves a_{ijk} unchanged. Secondly, we could replace each f_{ij} by $f_{ij} + c_{ij}$, where the c_{ij} are constant functions, i.e., $\boldsymbol{c} := \{c_{ij}\} \in C^1(\mathfrak{U},\mathbb{R})$; this changes $\boldsymbol{a}$ to $\boldsymbol{a} + \delta\boldsymbol{c}$, and $[\boldsymbol{a}] \in H^2(\mathfrak{U},\mathbb{R})$ is left unchanged. Therefore $\omega \mapsto [\boldsymbol{a}]$ is well-defined. Thirdly, we could replace ω by $\omega + d\beta$, adding an exact form; then α_j becomes $\alpha_j + \beta_j$, and $f_{ij} = (\alpha_i + \beta_i) - (\alpha_j + \beta_j)$ is unchanged because β_i and β_j agree on $U_i \cap U_j$. Therefore, $[\boldsymbol{a}]$ depends only on the de Rham class of ω, so $[\omega] \mapsto [\boldsymbol{a}]$ is a well-defined $\mathbb{R}$-linear map from $H^2_{\mathrm{dR}}(M)$ to $H^2(\mathfrak{U},\mathbb{R})$.

To go the other way, start from $\boldsymbol{a} \in Z^2(\mathfrak{U},\mathbb{R})$, and take a smooth partition of unity $\{\psi_j\}$ subordinate to the covering $\mathfrak{U}$. As in the proof of Proposition 1.6, define $\boldsymbol{f} \in C^1(\mathfrak{U},\underline{\mathcal{A}}^0)$ by $f_{ij} := \sum_r a_{ijr}\psi_r$; then $f_{ij} - f_{ik} + f_{jk} = a_{ijk}$ just as in (1.12), since $\delta\boldsymbol{a} = 0$; moreover, $df_{ij} - df_{ik} + df_{jk} = 0$ since the a_{ijk} are constant. Define $\underline{\alpha} \in C^0(\mathfrak{U},\underline{\mathcal{A}}^1)$ by $\alpha_j := \sum_k \psi_k\, df_{jk}$; now $\alpha_i - \alpha_j = \sum_k \psi_k\, df_{ij} = df_{ij}$ on $U_i \cap U_j$, and so $d\alpha_i - d\alpha_j = 0$ there, which says that the

[6] Another way is to establish an isomorphism between de Rham cohomology and singular cohomology. This is thoroughly dealt with in [23].

[7] This is the only point at which compactness is invoked; and the compactness assumption may be removed whenever the existence of a finite good covering can be established independently.

local 2-forms $\omega_j := d\alpha_j$ patch together to give a global 2-form ω on M. Since $d\omega_j = 0$, ω is closed, i.e., $\omega \in Z^2_{\mathrm{dR}}(M)$.

The correspondence $\boldsymbol{a} \mapsto \omega$ depends on the partition of unity $\{\psi_j\}$, but it is clear that $\boldsymbol{a} \mapsto \boldsymbol{f} \mapsto \underline{\alpha} \mapsto \omega$ retraces the earlier path $\omega \mapsto \underline{\alpha} \mapsto \boldsymbol{f} \mapsto \boldsymbol{a}$; thus we have shown that the linear map $[\omega] \mapsto [\boldsymbol{a}]$ is surjective. Moreover, if $[\boldsymbol{a}] = 0$, then $\boldsymbol{a} = \delta\boldsymbol{c}$ with $\boldsymbol{c} \in C^1(\mathfrak{U}, \mathbb{R})$, so that $f_{ij} := \sum_r (c_{ij} - c_{ir} + c_{jr})\psi_r = c_{ij} - h_i + h_j$ where $h_j := \sum_r c_{jr}\psi_r$ and thus $df_{ij} = dh_j - dh_i$; but then $\alpha_j = \sum_k \psi_k(dh_k - dh_j) = \beta_j - dh_j$, where β_j is the restriction to U_j of the 1-form $\beta := \sum_k \psi_k \, dh_k \in \mathcal{A}^1(M)$; thus $\omega = d\beta$ is exact, i.e., $[\omega] = 0$. Hence $[\omega] \mapsto [\boldsymbol{a}]$ is injective, and is therefore an isomorphism (or real vector spaces) between $H^2_{\mathrm{dR}}(M)$ and $H^2(\mathfrak{U}, \mathbb{R})$. □

The foregoing proof can be cast in a more algebraic framework, as follows. The abelian groups (actually, vector spaces) $C^{pq} := C^p(\mathfrak{U}, \underline{\mathcal{A}}^q)$ of q-form-valued Čech p-cochains are related by two coboundary operators, $\delta\colon C^{pq} \to C^{p+1,q}$ and $d\colon C^{pq} \to C^{p,q+1}$, satisfying $\delta d - d\delta = 0$; if we introduce $\partial := (-1)^p d$ on C^{pq}, we get $\delta\partial + \partial\delta = 0$, so we obtain a "double complex". By introducing $D := \delta + \partial\colon C^{pq} \to C^{p+1,q} \oplus C^{p,q+1}$, and $E^m := \bigoplus_{p+q=m} C^{pq}$, we obtain a new cochain complex $(E^\bullet, D)$, sometimes called the Čech–de Rham complex. (For instance, a 1-cochain for this complex is of the form $\underline{\alpha} \oplus \boldsymbol{f}$ where $\underline{\alpha}$ and $\boldsymbol{f}$ are as in the proof of Proposition 5.5, and $D(\underline{\alpha} \oplus \boldsymbol{f}) = d\underline{\alpha} \oplus (\delta\underline{\alpha} - d\boldsymbol{f}) \oplus \delta\boldsymbol{f}$, which evaluates to $\underline{\omega} \oplus 0 \oplus \boldsymbol{a}$. A general theorem [12, 17] asserts that the k-th cohomology groups for the complexes $(\mathcal{A}^\bullet(M), d)$, $(E^\bullet, D)$, and $(C^\bullet(\mathfrak{U}, M), \delta)$ are isomorphic, for any $k \in \mathbb{N}$.

Definition 5.7. The standard inclusion $\iota\colon \mathbb{Z} \to \mathbb{R}$ induces an injection of Čech cohomology groups $\iota_*\colon \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathbb{R})$. We regard $\check{H}^2(M, \mathbb{Z})$ as a subset of $\check{H}^2(M, \mathbb{R})$ by identifying it with its image under ι_*. We say that a *de Rham cohomology class* $[\omega] \in H^2_{\mathrm{dR}}(M)$ is **integral** if the corresponding Čech class $[\boldsymbol{a}]$ lies in $\check{H}^2(M, \mathbb{Z})$.

Theorem 5.6. *Let $L \longrightarrow M$ be a Hermitian line bundle over a compact manifold, and let ∇ be a connection on it, compatible with the metric, with curvature ω. Then $[(2\pi i)^{-1}\omega]$ is an integral de Rham class, corresponding to the class $[L]$ of the line bundle in $\check{H}^2(M, \mathbb{Z})$.*

Proof. Let $\mathfrak{U} = \{U_j\}$ be a covering of M by chart domains, and let $\{s_j\}$ be a local system of nonvanishing sections for the line bundle $L \longrightarrow M$. Then $s_j \in \Gamma(U_j, L)$ for each j. By the locality property of ∇, we have $\nabla s_j \in \mathcal{A}^1(U_j, L) = \Gamma(U_j, L) \otimes_{C^\infty(U_j)} \mathcal{A}^1(U_j)$, so $\nabla s_j = s_j \otimes \alpha_j$ for some $\alpha_j \in \mathcal{A}^1(U_j)$ (because s_j generates $\Gamma(U_j, L)$ as a $C^\infty(U_j)$-module). We may take s_j to be normalized with respect to the metric, i.e., $(s_j \mid s_j) = 1$ on U_j. Since ∇ is compatible, we find that α_j is a purely imaginary 1-form, because $\bar{\alpha}_j + \alpha_j = (\nabla s_j \mid s_j) + (s_j \mid \nabla s_j) = 0$.

By (5.8), $\nabla^2 s_j = (s_j \otimes \alpha_j) \wedge \alpha_j + s \wedge d\alpha_j = s \wedge d\alpha_j$, so that $\omega = d\alpha_j$ on U_j. Let us write $\beta_j := (2\pi i)^{-1}\alpha_j$; then the β_j are real-valued 1-forms, satisfying $d\beta_j = (2\pi i)^{-1}\omega$. Thus $[(2\pi i)^{-1}\omega] \in H^2_{\mathrm{dR}}(M)$ is a real de Rham class.

The local sections s_j are related by $s_i = g_{ij}s_j = s_j g_{ij}$ on $U_i \cap U_j$, where the g_{ij} are the $U(1)$-valued transition functions of the line bundle (see Definition 1.14). Then

$$s_j \otimes g_{ij}\alpha_i = s_i \otimes \alpha_i = \nabla s_i = \nabla(s_j g_{ij}) = (s_j \otimes \alpha_j)g_{ij} + s_j \otimes dg_{ij}$$

on $U_i \cap U_j$; since s_j does not vanish there, we get

$$\alpha_i = g_{ij}^{-1} \alpha_j g_{ij} + g_{ij}^{-1}\, dg_{ij} = \alpha_j + g_{ij}^{-1}\, dg_{ij} \qquad \text{on} \quad U_i \cap U_j.$$

This gives

$$\beta_i - \beta_j = (2\pi i)^{-1} g_{ij}^{-1}\, dg_{ij} = (2\pi i)^{-1}\, d(\log g_{ij})$$

for a suitable branch of the logarithm, and $f_{ij} := (2\pi i)^{-1} \log g_{ij}$ is a real-valued function on $U_i \cap U_j$ satisfying $\beta_i - \beta_j = df_{ij}$. Now $a_{ijk} := f_{ij} - f_{ik} + f_{jk}$ gives the Čech 2-cocycle $\boldsymbol{a} \in C^2(\mathfrak{U}, \mathbb{R})$ such that $[\boldsymbol{a}]$ corresponds to $[(2\pi i)^{-1}\omega]$ under the isomorphism of Proposition 5.5. On the other hand, the proof of Proposition 1.6 constructs this very same Čech class $[\boldsymbol{a}]$ as the element of $\check{H}^2(M, \mathbb{Z})$ which corresponds to $[L]$ under the Bockstein isomorphism between $\check{H}^1(M, \underline{U(1)})$ and $\check{H}^2(M, \mathbb{Z})$. Hence, $[(2\pi i)^{-1}\omega]$ is integral and corresponds to $[L]$. $\square$

In particular, we see that a closed 2-form on M is the curvature of some compatible connection only if it equals $2\pi i$ times an *integral* 2-form.[8]

5.6 Line bundles over $\mathbb{CP}^m$

Definition 5.8. Any element of $\mathbb{CP}^m$ is a line through the origin in $\mathbb{C}^{m+1}$, and two such lines intersect only at the origin. The **tautological line bundle** $L \longrightarrow \mathbb{CP}^m$ is given by taking the disjoint union of these lines; the fibre at any element of $\mathbb{CP}^m$ is the very same line. More explicitly, we can take L to be

$$L := \{\, (\ell, v) \in \mathbb{CP}^m \times \mathbb{C}^{m+1} : v \in \ell \,\}, \tag{5.12}$$

and define $\pi(\ell, v) := \ell$.

Here $L \longrightarrow \mathbb{CP}^m$ is manifestly a subbundle of a trivial vector bundle. Indeed, let $\ell^\perp$ be the subspace of $\mathbb{C}^{m+1}$ orthogonal to ℓ (with respect to the usual inner product on $\mathbb{C}^{m+1}$), and if $E := \{\, (\ell, u) \in \mathbb{CP}^m \times \mathbb{C}^{m+1} : u \in \ell^\perp \,\}$, then the Whitney sum $L \oplus E = \mathbb{CP}^m \times \mathbb{C}^{m+1}$ is trivial.

Exercise 5.9. If $\eta \colon \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}^m$ is the quotient map, $\mathbb{C}^{m+1} \setminus \{0\} \xrightarrow{\eta} \mathbb{CP}^m$ is a principal $\mathbb{C}^\times$-bundle. Show that the tautological line bundle is associated to this principal bundle via the representation ρ of $\mathbb{C}^\times$ on $\mathbb{C}$ given by the multiplication $\rho(\lambda)\mu := \lambda\mu$.

Definition 5.9. The dual of the tautological line bundle on $\mathbb{CP}^m$ is the **hyperplane bundle** $H \longrightarrow \mathbb{CP}^m$, where $H_\ell = \ell^*$ (the one-dimensional space of linear functionals on ℓ).

[8] This observation is the launching point for the theory of *geometric quantization*, which seeks to represent certain functions on a symplectic manifold (M, Ω), i.e., a manifold equipped with a closed nondegenerate 2-form, by operators on a Hilbert space. The elements of this Hilbert space come from sections of a certain Hermitian line bundle over M, equipped with a compatible connection whose curvature is $(2\pi i\hbar)^{-1}\Omega$ (where $\hbar$ is a positive constant, identified with Planck's constant). Theorem 5.6 shows that such a line bundle exists iff $[\hbar^{-1}\Omega]$ is integral; this is a discreteness condition on the original symplectic form Ω, hence the use of the word "quantization".

The tensor product bundles $L^{\otimes k} \longrightarrow \mathbb{CP}^m$ and $H^{\otimes l} \longrightarrow \mathbb{CP}^m$, for $k, l \in \mathbb{N}$, give more examples of line bundles over $\mathbb{CP}^m$. (We take $L^{\otimes 0} = H^{\otimes 0} := \mathbb{CP}^m \times \mathbb{C}$ by convention.) We aim to show that these are distinct, and that any line bundle over $\mathbb{CP}^m$ is equivalent to one on this list. Since $[L]^{-1} = [H]$ in the group of line bundle classes, and since $H^2_{\mathrm{dR}}(\mathbb{CP}^m) = \mathbb{R}$, we have $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}) \simeq \mathbb{Z}$, so it suffices to verify that $[H]$ corresponds to a generator of the infinite cyclic group $\check{H}^2(\mathbb{CP}^m, \mathbb{Z})$.

Definition 5.10. A complex vector bundle $E \longrightarrow M$ on a complex manifold M is a **holomorphic vector bundle** if its transition functions $g_{ij} \colon U_i \cap U_j \to GL(m, \mathbb{C})$ are holomorphic. The space of holomorphic sections of this line bundle is denoted by $\mathcal{O}(M, E)$ or simply by $\mathcal{O}(E)$.

Exercise 5.10. Compute the transition functions for the tautological and hyperplane bundles over $\mathbb{CP}^m$ and hence show that these are holomorphic line bundles.

Consider first the trivial bundle $\mathbb{CP}^m \times \mathbb{C} \longrightarrow \mathbb{CP}^m$; a holomorphic section of this bundle is of the form $s(x) = (x, f(x))$, where $f \colon \mathbb{CP}^m \to \mathbb{C}$ is a holomorphic *function*. If (U, ϕ) is a chart with $\phi(U) = \mathbb{C}^m$, then $f \circ \phi^{-1} \colon \mathbb{C}^m \to \mathbb{C}$ is an entire holomorphic function on $\mathbb{C}^m$, which is bounded since $\mathbb{CP}^m$ is compact and f is continuous; thus, by Liouville's theorem, f is constant. More generally, any holomorphic function on a compact complex manifold is constant.

It turns out that the tautological line bundle has no global holomorphic sections, other than the zero section. The hyperplane bundle $H \longrightarrow \mathbb{CP}^m$, by contrast, has nontrivial global holomorphic sections. For any $f \in (\mathbb{C}^{m+1})^*$, define $s_f \in \Gamma(\mathbb{CP}^m, H)$ by $s_f(\ell) := f\big|_\ell$. Conversely, any holomorphic section of $H \longrightarrow \mathbb{CP}^m$ is of the form $s(\ell) := g\big|_\ell$, where $g \colon \mathbb{C}^{m+1} \to \mathbb{C}$ is a holomorphic function whose restriction to each one-dimensional subspace is linear, i.e., g is homogeneous of degree one. Since only first-degree terms can then occur in the Taylor series of g, the function g is itself linear, i.e., $g \in (\mathbb{C}^{m+1})^*$ and $s = s_g$. Thus, $f \mapsto s_f$ is a linear bijection between $(\mathbb{C}^{m+1})^*$ and $\mathcal{O}(H)$.

In particular, $\mathcal{O}(H)$ is finite-dimensional, with $\dim \mathcal{O}(H) = m + 1$. A basis is given by $\sigma_0, \dots, \sigma_m$, where $\sigma_j = s_{z^j}$ and $z^j \in (\mathbb{C}^{m+1})^*$ be the j-th coordinate function, for $j = 0, 1, \dots, m$.

The standard inner product $\langle\!\langle \cdot \mid \cdot \rangle\!\rangle$ on $\mathbb{C}^{m+1}$ gives metrics on the line bundles L and H. We may write $(u \mid v)_\ell := \langle\!\langle u \mid v \rangle\!\rangle$ for $u, v \in L_\ell$, i.e., $u, v \in \ell \subset \mathbb{C}^{m+1}$. Now if $\phi, \psi \in \ell^*$ are given by $\phi(v) = \langle\!\langle u \mid v \rangle\!\rangle$, $\psi(v) = \langle\!\langle w \mid v \rangle\!\rangle$ for $v \in \ell$, then

$$(\phi \mid \psi)_{\ell^*} := \frac{\overline{\langle\!\langle u \mid v \rangle\!\rangle} \langle\!\langle w \mid v \rangle\!\rangle}{\langle\!\langle v \mid v \rangle\!\rangle} = \frac{\langle\!\langle w \mid v \rangle\!\rangle \langle\!\langle v \mid u \rangle\!\rangle}{\langle\!\langle v \mid v \rangle\!\rangle},$$

which is independent of any nonzero $v \in \ell$. When $\ell \in U_j$, we have the equality

$$\langle\!\langle v \mid v \rangle\!\rangle = z^0 \bar{z}^0 + \cdots + z^m \bar{z}^m = z^j \bar{z}^j \Bigl(1 + \sum_{k \neq j} w_j^k \bar{w}_j^k\Bigr) = |z^j(v)|^2 Q_j(\ell),$$

and in particular

$$(\sigma_j \mid \sigma_j) = Q_j^{-1} = \Bigl(1 + \sum_{k\neq j} w_j^k \bar{w}_j^k\Bigr)^{-1} \qquad \text{on} \quad U_j. \tag{5.13}$$

By continuity, $(\sigma_j \mid \sigma_j) = 0$ and thus σ_j vanishes on the complement of U_j.

5.7 Connections on the hyperplane bundle

Definition 5.11. When M is a complex manifold and ∇ is a connection on a holomorphic vector bundle $E \longrightarrow M$, we say that ∇ is *compatible with the holomorphic structure* if $\nabla s \in \mathcal{A}^{1,0}(E)$ whenever $s \in \mathcal{O}(E)$. If the vector bundle is also Hermitian, we say that ∇ is a **canonical connection** if it is compatible with both the metric and the holomorphic structure.

Proposition 5.7. *On the hyperplane bundle $H \longrightarrow \mathbb{CP}^m$, there is a unique canonical connection.*

Proof. Let $\{U_j, \sigma_j\}$ be the local system of holomorphic sections of the previous Section. Since σ_j is nonvanishing on U_j (e.g., on account of (5.13)), the argument of the proof of Theorem 5.6 shows that any connection ∇ on the hyperplane bundle is given on each U_j by

$$\nabla\sigma_j = \sigma_j \otimes \alpha_j, \qquad \text{for some} \quad \alpha_j \in \mathcal{A}^1(U_j).$$

Thus ∇ is compatible with the holomorphic structure iff each α_j lies in $\mathcal{A}^{1,0}(U_j)$.

We may write $\alpha_j = a_{jk}\, dw_j^k$ with each $a_{jk} \in C^\infty(U_j)$, so that ∇ is also compatible with the metric iff

$$\begin{aligned} Q_j^{-1}\bigl(a_{jk}\, dw_j^k + \bar{a}_{jk}\, d\bar{w}_j^k\bigr) &= (\sigma_j \mid \nabla\sigma_j) + (\nabla\sigma_j \mid \sigma_j) = d(\sigma_j \mid \sigma_j) \\ &= d(Q_j^{-1}) = -Q_j^{-2} d\Bigl(\sum_{k\neq j} w_j^k \bar{w}_j^k\Bigr) \end{aligned}$$

iff $a_{jk} = -Q_j^{-1}\bar{w}_j^k$ for all $k \neq j$. Thus ∇ satisfies

$$\nabla\sigma_j = -\sigma_j \otimes Q_j^{-1} \sum_{k\neq j} \bar{w}_j^k\, dw_j^k,$$

so this calculation establishes existence and uniqueness of the canonical connection. □

Proposition 5.8. *The curvature of the canonical connection on the hyperplane bundle is $-i\Phi$, where Φ is the Kähler form on $\mathbb{CP}^m$.*

Proof. The curvature ω satisfies $\omega = d\alpha_j$ on U_j, so

$$\begin{aligned} \omega = d\alpha_j &= Q_j^{-1} \sum_{k\neq j} dw_j^k \wedge d\bar{w}_j^k + Q_j^{-2}\, dQ_j \wedge \sum_{k\neq j} \bar{w}_j^k\, dw_j^k \\ &= Q_j^{-2}\Bigl(Q_j \sum_{k\neq j} dw_j^k \wedge d\bar{w}_j^k - \sum_{r,s\neq j} \bar{w}_j^r w_j^s\, dw_j^r \wedge d\bar{w}_j^s\Bigr) = -i\Phi, \end{aligned}$$

by comparison with (2.7). □

The element of $H^2_{\mathrm{dR}}(\mathbb{CP}^m)$ which corresponds to the equivalence class of the hyperplane bundle is therefore $[-(2\pi)^{-1}\Phi]$.

5.8 Characteristic classes

We have seen that the curvature of a Hermitian line bundles over M is a closed 2-form ω on M and $[(2\pi i)^{-1}\omega]$ is an integral cohomology class. For vector bundles of higher rank, it is possible to obtain integral cohomology classes of even degree from the matrix-valued curvature ω by taking suitable traces of its exterior powers. These classes are topological invariants of the manifold M.

Definition 5.12. Let $\tilde{q}\colon (\mathbb{C}^{r\times r})^k \to \mathbb{C}$ be a symmetric k-linear map on the vector space of complex $r \times r$ matrices. One says that $\tilde{q}$ is *invariant* (under the adjoint representation $\mathrm{Ad}(g)A := gAg^{-1}$ of $GL(r,\mathbb{C})$) if

$$\tilde{q}(gA_1g^{-1}, \dots, gA_kg^{-1}) = \tilde{q}(A_1, \dots, A_k) \tag{5.14}$$

for all $g \in GL(r,\mathbb{C})$, $A \in \mathbb{C}^{r\times r}$.

The function $q\colon \mathbb{C}^{r\times r} \to \mathbb{C}$ defined by $q(A) := \tilde{q}(A, A, \dots, A)$ is called a *homogeneous polynomial* of degree k; when $\tilde{q}$ is invariant, we call it an **invariant polynomial** on $\mathbb{C}^{r\times r}$. With the "polarization formula" $\tilde{q}(A_1, \dots, A_k) := (k!)^{-1}\sum_{|J|=s}(-1)^{k-s}q(A_{j_1} + \cdots + A_{j_s})$, the map $\tilde{q}$ may be recovered from q.

More generally, an invariant polynomial on $\mathbb{C}^{r\times r}$ is a finite sum of homogeneous invariant polynomials of various degrees. (A constant function is an invariant polynomial of degree zero.)

For instance, in the expansion $\det(t - A) = \sum_{k=0}^{r}(-1)^k q_k(A)t^{r-k}$, each q_k is an invariant homogeneous polynomial; here $q_r(A) = \det A$, $q_1(A) = \operatorname{tr} A$, and $q_2(A) = \sum_{i<j}\lambda_i(A)\lambda_j(A)$, where the $\lambda_j(A)$ are the eigenvalues of A. Notice that $2q_2(A) = (\operatorname{tr} A)^2 - \operatorname{tr}(A^2)$.

Lemma 5.9. *If $q\colon \mathbb{C}^{r\times r} \to \mathbb{C}$ is an invariant polynomial on $\mathbb{C}^{r\times r}$, then for $A_1, \dots, A_k, B \in \mathbb{C}^{r\times r}$ we have $\sum_{j=1}^{k}\tilde{q}(A_1, \dots, [B, A_j], \dots, A_k) = 0$.*

Proof. Set $g = e^{tB}$ in (5.14), and differentiate with respect to t at $t = 0$; the result follows on noting that $(d/dt)\big|_{t=0} e^{tB}Ae^{-tB} = BA - AB = [B, A]$. □

If $E_r = M \times \mathbb{C}^r$, so that $E_r \longrightarrow M$ is the trivial bundle of rank r, then q yields a polynomial map from $\Gamma(\operatorname{End} E_r) = \mathcal{A}^{r\times r} = C^\infty(M, \mathbb{C}^{r\times r})$ to $\mathcal{A} = C^\infty(M)$ by writing $q(A)(x) := q(A(x))$; and $\tilde{q}$ similarly defines an $\mathcal{A}$-valued symmetric k-linear map on $\mathcal{A}^{r\times r}$. The invariance property is $q(gAg^{-1}) := q(A)$, where $g \in C^\infty(M, GL(r,\mathbb{C}))$. Since $\tau_*(v) := gv$ for $v \in \mathcal{A}^r = \Gamma(E^r)$ defines a bundle automorphism (τ, id_M) of $E_r \longrightarrow M$, the invariance condition may be reexpressed as $q \circ \mathrm{Ad}\,\tau = q$ for every invertible bundle map $\tau\colon E_r \to E_r$.

Exercise 5.11. Verify that for any vector bundle $E \longrightarrow M$ of rank r, the recipe $q(A)(x) := q(A(x))$ yields a map $q\colon \Gamma(\operatorname{End} E) \to \mathcal{A}$ (and by polarization, a k-linear map $\tilde{q}\colon \Gamma(\operatorname{End} E)^k \to \mathcal{A}$), such that $q(\tau_* \circ A \circ \tau_*^{-1}) = q(A)$ for any invertible bundle map $\tau\colon E \to E$.

These may be extended to maps q, $\tilde{q}$ on $\mathcal{A}^\bullet(M, \operatorname{End} E)$ with values in $\mathcal{A}^\bullet(M)$ by defining

$$\tilde{q}(A_1 \otimes \eta_1, \dots, A_k \otimes \eta_k) := \tilde{q}(A_1, \dots, A_k)\, \eta_1 \wedge \cdots \wedge \eta_k. \tag{5.15}$$

The invariance property of Lemma 5.9 may be expressed in this context as

$$\sum_{j=1}^{k} (-1)^{s_0(s_1+\cdots+s_{j-1})} \tilde{q}(\omega_1, \dots, [\![\beta, \omega_j]\!], \dots, \omega_k) = 0 \tag{5.16}$$

whenever $\beta \in \mathcal{A}^{s_0}(M, \operatorname{End} E)$ and $\omega_j \in \mathcal{A}^{s_j}(M, \operatorname{End} E)$ for $j = 1, \dots, k$. The bracket on $\mathcal{A}^\bullet(M, \operatorname{End} E)$ is defined as the linear extension of the recipe:

$$[\![A \otimes \eta, B \otimes \zeta]\!] := [A, B]\, \eta \wedge \zeta \tag{5.17}$$

for $A, B \in \Gamma(\operatorname{End} E)$, $\eta, \zeta \in \mathcal{A}^\bullet(M)$.

Exercise 5.12. Check the invariance formula (5.16).

Exercise 5.13. If $\alpha \in \mathcal{A}^k(M, \operatorname{End} E_r)$, $\beta \in \mathcal{A}^l(M, \operatorname{End} E_r)$ are matrix-valued forms, deduce from (5.17) that

$$[\![\alpha, \beta]\!] = \alpha \wedge \beta - (-1)^{kl} \beta \wedge \alpha$$

in $\mathcal{A}^{k+l}(M, \operatorname{End} E_r)$.

Exercise 5.14. By taking the exterior derivative of the right hand side of (5.15), show that

$$d(\tilde{q}(\omega_1, \dots, \omega_k)) = \sum_{j=1}^{k} (-1)^{s_1+\cdots+s_{j-1}} \tilde{q}(\omega_1, \dots, d\omega_j, \dots, \omega_k) \tag{5.18}$$

when each $\omega_j \in \mathcal{A}^{s_j}(M, \operatorname{End} E_r)$ is a matrix-valued form.

It is convenient to introduce the notation $q'(\omega; \theta) := \sum_{j=1}^{k} \tilde{q}(\omega, \dots, \theta, \dots, \omega)$, where θ appears in the j-th place and ω elsewhere; if ω has even degree, it follows from (5.16) that $q'(\omega; [\![\beta, \omega]\!]) = 0$ for any $\beta \in \mathcal{A}^\bullet(M, \operatorname{End} E)$. Also, if $\omega \in \mathcal{A}^{\mathrm{even}}(M, \operatorname{End} E_r)$ is a matrix-valued form of even degree, then (5.18) gives $d(q(\omega)) = q'(\omega; d\omega) \in \mathcal{A}^{\mathrm{odd}}(M, \operatorname{End} E_r)$.

Proposition 5.10. *Let ∇ be a connection on a vector bundle $E \longrightarrow M$, with curvature $\omega \in \mathcal{A}^2(M, \operatorname{End} E)$. If q is an invariant polynomial on $\mathbb{C}^{r\times r}$, then $q(\omega)$ is closed in $\mathcal{A}^{\mathrm{even}}(M)$.*

Proof. To show that $d(q(\omega)) = 0$ on M, it is enough to show that $d(q(\omega)) = 0$ on each chart domain U. Thus we can suppose that $E \longrightarrow M$ is a *trivial* bundle, that $\nabla = d + \alpha$ with $\alpha \in \mathcal{A}^1(M, \operatorname{End} E)$ and that $\omega = d\alpha + \alpha \wedge \alpha$.

Now $\mathcal{A}^k(M, \operatorname{End} E) \simeq (\mathcal{A}^k(M))^{r\times r}$ by the triviality of the bundle, and the exterior derivative $d\colon \mathcal{A}^\bullet(M, \operatorname{End} E) \to \mathcal{A}^\bullet(M, \operatorname{End} E)$ is an antiderivation, so

$$d\omega = d(\alpha \wedge \alpha) = d\alpha \wedge \alpha - \alpha \wedge d\alpha = \omega \wedge \alpha - \alpha \wedge \omega = [\![\omega, \alpha]\!], \tag{5.19}$$

which is known as the *Bianchi identity* for the curvature. Now

$$d(q(\omega)) = q'(\omega; d\omega) = -q'(\omega; [\![\alpha, \omega]\!]) = 0$$

by the invariance of q. $\square$

Exercise 5.15. A connection ∇ on $E \longrightarrow M$, with curvature ω, yields a connection on the vector bundle $\operatorname{End} E \longrightarrow M$, also denoted ∇, by adopting the Leibniz rule $\nabla(as) = (\nabla a)s + a(\nabla s)$ as a definition, i.e., by setting $(\nabla a)s := \nabla(as) - a(\nabla s)$ for $a \in \Gamma(\operatorname{End} E)$, $s \in \Gamma(E)$. Check that $a \mapsto \nabla a$ is indeed a connection. This extends to a linear map $\nabla \colon \mathcal{A}^k(M, \operatorname{End} E) \to \mathcal{A}^{k+1}(M, \operatorname{End} E)$ by (5.8). Use (5.8) and (5.9) to show that $\nabla^2\beta = \omega \wedge \beta$ for any $\beta \in \mathcal{A}^1(M, E)$, and deduce that $(\nabla\omega)s = \nabla(\nabla^2 s) - \nabla^2(\nabla s) = 0$ for any $s \in \Gamma(E)$. Finally, show that the equation $\nabla\omega = 0$ reduces to the Bianchi identity (5.19) locally,[9] when ω is of the form $\omega = d\alpha + \alpha \wedge \alpha$.

Proposition 5.11. *Let ∇ be a connection on a vector bundle $E \longrightarrow M$, with curvature ω, and let q be an invariant polynomial. Then the cohomology class $[q(\omega)]$ is independent of ∇.*

Proof. Let ∇_0, ∇_1 be two connections on $E \longrightarrow M$. Then, by (5.5), $\nabla_1 = \nabla_0 + \beta$ with $\beta \in \mathcal{A}^1(M, \operatorname{End} E)$. Set $\nabla_t := (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t\beta$ for $0 \leq t \leq 1$; this is a connection on $E \longrightarrow M$, whose curvature we denote by ω_t. The relation $[q(\omega_0)] = [q(\omega_1)]$ follows from the *transgression formula*:

$$q(\omega_1) - q(\omega_0) = d\biggl(\int_0^1 q'(\omega_t; \beta)\,dt\biggr).$$

To verify this relation, it suffices to show that $(d/dt)q(\omega_t) = d(q'(\omega_t; \beta))$ for $0 < t < 1$. By k-linearity of $\tilde{q}$, we have

$$\frac{d}{dt} q(\omega_t) = \frac{d}{dt} \tilde{q}(\omega_t, \dots, \omega_t) = \sum_{j=1}^{k} \tilde{q}(\omega_t, \dots, (d/dt)\omega_t, \dots, \omega_t) = q'(\omega_t; (d/dt)\omega_t).$$

To see that $q'(\omega_t; (d/dt)\omega_t) = d(q'(\omega_t; \beta))$, we may compare these two $2k$-forms over any chart domain $U \subset M$; or equivalently, we may assume that the bundle $E \longrightarrow M$ is trivial, that $\nabla_t = d + \alpha_t$ and $\omega_t = d\alpha_t + \alpha_t \wedge \alpha_t$. In that case, $\alpha_t = \alpha_0 + t\beta$, and

$$\begin{aligned} \frac{d}{dt}\omega_t &= \frac{d}{dt}\bigl(d\alpha_0 + t\,d\beta + (\alpha_0 + t\beta) \wedge (\alpha_0 + t\beta)\bigr) \\ &= d\beta + \alpha_0 \wedge \beta + \beta \wedge \alpha_0 + 2t(\beta \wedge \beta) = d\beta + \alpha_t \wedge \beta + \beta \wedge \alpha_t \\ &= d\beta + [\![\alpha_t, \beta]\!]. \end{aligned}$$

[9] The identity $\nabla\omega = 0$ for any connection is therefore called the Bianchi identity.

It remains to compute

$$\begin{aligned}
d(q'(\omega_t;\beta)) &= \sum_{j=1}^{k} d(\tilde q(\omega_t,\dots,\beta,\dots,\omega_t)) \\
&= \sum_{i<j} \tilde q(\omega_t,\dots,d\omega_t,\dots,\beta,\dots,\omega_t) + q'(\omega_t;d\beta) - \sum_{i>j} \tilde q(\omega_t,\dots,\beta,\dots,d\omega_t,\dots,\omega_t) \\
&= -\sum_{i<j} \tilde q(\omega_t,\dots,[\![\alpha_t,\omega_t]\!],\dots,\beta,\dots,\omega_t) + q'(\omega_t;d\beta) \\
&\qquad\qquad + \sum_{i>j} \tilde q(\omega_t,\dots,\beta,\dots,[\![\alpha_t,\omega_t]\!],\dots,\omega_t) \\
&= \sum_{j=1^k} \tilde q(\omega_t,\dots,[\![\alpha_t,\beta]\!],\dots,\omega_t) + q'(\omega_t;d\beta) = q'(\omega_t;d\beta + [\![\alpha_t,\beta]\!]) \\
&= q'(\omega_t;(d/dt)\omega_t),
\end{aligned}$$

(with $d\omega_t$ in the i-th position and β in the j-th position in the double summations). Here we have used the Bianchi identity $d\omega_t + [\![\alpha_t,\omega_t]\!] = 0$ and the invariance property (5.16) applied to the form $\tilde q(\omega_t,\dots,d\omega_t,\dots,\beta,\dots,\omega_t)$ and also to $\tilde q(\omega_t,\dots,\beta,\dots,d\omega_t,\dots,\omega_t)$. $\square$

If $E' \longrightarrow M$ is another vector bundle equivalent to $E \longrightarrow M$, and it $\tau\colon E \to E'$ is an invertible bundle map, then by Exercises 5.4 and 5.6, the recipes $\nabla' := \tau_* \circ \nabla \circ \tau_*^{-1}$ and $\omega' := \tau_* \circ \omega \circ \tau_*^{-1}$ match connections and curvatures on both vector bundles; by the invariance property (5.14) of the polynomial q, we have $q(\omega') = q(\omega)$ in $\mathcal{A}^{2k}(M)$. Thus the cohomology class $[q(\omega)]$ *depends only on the equivalence class of the vector bundle* $E \longrightarrow M$, and we may denote it by $q([E])$, or more simply by $q(E) \in H^{2k}_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathbb{C}$.

If $E \longrightarrow M$ is a *Hermitian* vector bundle, we consider only bundle maps $\tau\colon E \to E'$ which preserve the fibre metrics. Thus we may use polynomials q which are only invariant under the unitary group $U(r)$ rather than $GL(r,\mathbb{C})$; in (5.16) we may only use forms β with either $i\beta$ or β itself real-valued. If a connection ∇ on $E \longrightarrow M$ is compatible with the metric, then $-i\omega$ is real-valued, and the cohomology class $[q(-i\omega)]$ belongs to $H^{2k}_{\mathrm{dR}}(M)$.

5.9 Chern classes and the Chern character

Definition 5.13. Let $E \longrightarrow M$ be a Hermitian vector bundle, together with a compatible connection ∇ whose curvature is $\omega \in \mathcal{A}^2(M, \operatorname{End} E)$. Define a $U(r)$-invariant polynomial on $\mathbb{C}^{r\times r}$ by

$$c(A) := \det\Bigl(1_r - \frac{1}{2\pi i} A\Bigr),$$

which may be written as a sum of homogeneous polynomials

$$c(A) = 1 + c_1(A) + c_2(A) + \cdots + c_r(A) \tag{5.20}$$

where $c_1(A) = (i/2\pi)\operatorname{tr} A$, $c_r(A) = (i/2\pi)^r \det A$; and if $\{i\lambda_1,\dots,i\lambda_r\}$ are the eigenvalues of A, then $c_k(A) = (-1/2\pi)^k \sum \lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_k}$; the invariant homogeneous polynomials c_k are real-valued on the Lie algebra $\mathfrak{u}(r) = i\mathbb{R}^{r\times r}$ of the unitary group $U(r)$.

The class $c_k(E) \in H^{2k}_{\mathrm{dR}}(M)$ is called the ***k*-th Chern class**, and $c(E) \in H^{\mathrm{even}}_{\mathrm{dR}}(M)$ is called the **total Chern class** of the vector bundle.

Exercise 5.16. If $E^* \longrightarrow M$ is the dual vector bundle to $E \longrightarrow M$, and if ∇^* is the dual connection to ∇ (see Exercise 5.3), show that ∇^* has curvature $-\omega^t \in \mathcal{A}^2(M, \mathrm{End}(E^*))$, and conclude that $c_k(E^*) = (-1)^k c_k(E)$.

Exercise 5.17. If $\phi\colon N \to M$ is a smooth map, and if ∇ is a connection on a Hermitian vector bundle $E \longrightarrow M$ with curvature ω, find a connection ∇' on the pullback bundle $\phi^*E \longrightarrow N$ whose curvature is $\phi^*\omega \in \mathcal{A}^2(N, \mathrm{End}(\phi^*E))$. Conclude that $c_k(\phi^*E) = \phi^*c_k(E) \in H^{2k}_{\mathrm{dR}}(N)$.

For Hermitian line bundles ($r = 1$), the total Chern class is just $c(L) := 1 + c_1(L)$. From Theorem 5.6, we know that $c_1(L)$ is an *integral* cohomology class, so $c(L)$ is also integral. This integrality property holds for Chern classes of any Hermitian vector bundle. Indeed, if $E \longrightarrow M$ and $E' \longrightarrow M$ are two Hermitian vector bundles with compatible connections ∇ and ∇' and respective curvatures ω and ω', then $\nabla \oplus \nabla'$ is a connection on $E \oplus E' \longrightarrow M$, with curvature $\omega \oplus \omega' \in \mathcal{A}^2(M, \mathrm{End}(E \oplus E'))$. Clearly

$$c(\omega \oplus \omega') = \det\Bigl(1_r - \frac{\omega}{2\pi i}\Bigr) \wedge \det\Bigl(1_{r'} - \frac{\omega'}{2\pi i}\Bigr) = c(\omega) \wedge c(\omega'), \tag{5.21}$$

on account of (5.15). Now the wedge product of closed forms induces a product of cohomology classes, since $(\eta + d\zeta) \wedge (\eta' + d\zeta') = \eta \wedge \eta' + d(\eta \wedge \zeta' + \zeta \wedge (\eta' + d\zeta))$ if $d\eta = 0$ and $d\eta' = 0$, so $[\eta \wedge \eta']$ is not affected by adding an exact form to either η or η'; in other words, the recipe $[\eta]\,[\eta'] := [\eta \wedge \eta']$ is a well-defined product[10] making $H^\bullet_{\mathrm{dR}}(M)$ into a ring. The even-degree classes form a *commutative* subring $H^{\mathrm{even}}_{\mathrm{dR}}(M)$. On passing to cohomology, (5.20) becomes

$$c(E \oplus E') = c(E)\, c(E') \qquad \text{in} \quad H^{\mathrm{even}}_{\mathrm{dR}}(M). \tag{5.22}$$

The integral 2-forms generate an integral subring $H^{\mathrm{even}}(M, \mathbb{Z})$. By (5.21), $c(E)$ is integral, i.e., lies in $H^{\mathrm{even}}(M, \mathbb{Z})$, whenever $E \longrightarrow M$ is a Whitney sum of Hermitian line bundles. An important *splitting principle* (see [12] for a proof) asserts that some pullback bundle $\phi^*E \longrightarrow N$ can be split (into a sum of line bundles) in such a way that $c(E) \mapsto \phi^*(c(E)) = c(\phi^*E)$ is injective, and therefore $c(E)$ is integral since $c(\phi^*E)$ is.

Definition 5.14. The **Chern character** $\mathrm{ch}(E)$ of the Hermitian vector bundle $E \longrightarrow M$ of rank r is given by the invariant power series

$$\mathrm{ch}(A) := \mathrm{tr}\bigl(\exp((2\pi)^{-1} i\, A)\bigr) = 1 + \mathrm{ch}_1(A) + \mathrm{ch}_2(A) + \cdots, \tag{5.23}$$

or equivalently by the invariant polynomial obtained by discarding the terms ch_k with $2k > \dim M$, since $\omega^{\wedge k} \in \mathcal{A}^{2k}(M, \mathrm{End}\, E)$ and hence $\mathrm{ch}_k(\omega) = 0$ for $2k > \dim M$. This polynomial is found explicitly by writing the eigenvalues of A as $\{2\pi i\mu_1, \dots, 2\pi i\mu_r\}$, expanding the function $e^{-\mu_1} + \cdots + e^{-\mu_r}$ in a Taylor series, and discarding high-degree terms.

[10] This product is usually called the *cup product* in de Rham cohomology.

Exercise 5.18. Check that $\mathrm{ch}_1(E) = c_1(E)$, $\mathrm{ch}_2(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E))$, and $\mathrm{ch}_3(E) = \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + c_3(E))$.

In general, the $\mathrm{ch}_k(E)$ are polynomial combinations of the Chern classes $c_j(E)$ with rational coefficients, on account of the $1/k!$ terms in the Taylor series expansion; thus they might not be integral classes, but rational classes, i.e., elements of $H^{\mathrm{even}}(M,\mathbb{Q}) = H^{\mathrm{even}}(M,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$.

Proposition 5.12. *The Chern character satisfies the homomorphism properties:*

$$\begin{aligned}\mathrm{ch}(E \oplus E') &= \mathrm{ch}(E) + \mathrm{ch}(E'),\\ \mathrm{ch}(E \otimes E') &= \mathrm{ch}(E)\ \mathrm{ch}(E').\end{aligned}$$

Proof. The invariant power series $\mathrm{ch}(A)$ of (5.22) satisfies $\mathrm{ch}(A \oplus A') = \mathrm{ch}(A) + \mathrm{ch}(A')$ and $\mathrm{ch}(A \otimes 1 + 1 \otimes A') = \mathrm{ch}(A)\ \mathrm{ch}(A')$; where $\oplus$ and $\otimes$ denote the usual direct sum and tensor product of matrices. These identities are simple to check for diagonal matrices, therefore hold for diagonalizable matrices by invariance, and hence hold generally, since the set of diagonalizable $r \times r$ matrices is dense in $\mathbb{C}^{r\times r}$.

Given connections ∇, ∇' with curvatures ω, ω' on the respective vector bundles $E \longrightarrow M$ and $E' \longrightarrow M$, the curvatures of $\nabla \oplus \nabla'$ and $\nabla \otimes \nabla'$ are $\omega \oplus \omega' \in \mathcal{A}^2(M, \mathrm{End}(E \oplus E'))$ and $\omega \otimes 1 + 1 \otimes \omega' \in \mathcal{A}^2(M, \mathrm{End}(E \otimes E'))$. On replacing A, A' by ω, ω' in the foregoing matrix identities, bearing in mind (5.15), and on passing to cohomology, one obtains the desired formulae (5.23) for $\mathrm{ch}(E \oplus E')$ and $\mathrm{ch}(E \otimes E')$. $\square$

Exercise 5.19. Explain how the tensor product connection $\nabla \otimes \nabla'$ is defined, and check the given formula for its curvature.

A trivial bundle $E_r = M \times \mathbb{C}^r \longrightarrow M$ has a "flat" connection (i.e., a connection with zero curvature) namely d, and thus $\mathrm{ch}(E_r) = 1$. Thus, if two vector bundles $E \longrightarrow M$ and $F \longrightarrow M$ are stably equivalent, then $\mathrm{ch}(E) - \mathrm{ch}(F)$ is an integral multiple of 1 in $H^{\mathrm{even}}(M,\mathbb{Q})$. If one defines the "reduced K-theory" $\widetilde{K}^0(M)$ of M as the quotient of $K^0(M)$ by $\mathbb{Z}$ (on identifying $r \in \mathbb{N}$ with $[\![E_r]\!] \in K^0(M)$), the tensor product of vector bundles makes $\widetilde{K}^0(M)$ a commutative ring, and thus the Chern character defines a ring homomorphism $\mathrm{ch}\colon \widetilde{K}^0(M) \to H^{\mathrm{even}}(M,\mathbb{Q})$.

For Hermitian line bundles $L \longrightarrow M$ and $L' \longrightarrow M$, Proposition 5.12 yields the identity

$$c_1(L \otimes L') = c_1(L) + c_1(L'), \tag{5.24}$$

by extracting the component in $H^2_{\mathrm{dR}}(M)$ from the formula $\mathrm{ch}(L \otimes L') = \mathrm{ch}(L)\ \mathrm{ch}(L)$, using $\mathrm{ch}_1(L) = c_1(L)$. Thus c_1 determines a homomorphism from the group of line bundle classes to the additive group $H^2(M,\mathbb{Z})$ of integral de Rham classes. Since $c_1(L) = [(i/2\pi)\omega]$ when ω is the curvature of a compatible connection on $L \longrightarrow M$, and $c_1(L^*) = -c_1(L)$, we conclude that $[L] \mapsto c_1(L^*)$ is the isomorphism described in Theorem 5.6.

To show that $[L] \neq [L']$, it is enough to show that $c_1(L) \neq c_1(L')$. Moreover, if $\dim M = 2m$, then $c_1(L)^m = [(i/2\pi)^n \omega^{\wedge m}]$ is an integral $2m$-form, and so $\int_M (i/2\pi)^n \omega^{\wedge m} \in \mathbb{Z}$, since the identification of $H^{2m}_{\mathrm{dR}}(M)$ with $\mathbb{R}$ is precisely the map $[\nu] \mapsto \int_M \nu$.

5.10 Classification of line bundles over $\mathbb{CP}^m$

Proposition 5.13. *The group of classes of line bundles over $\mathbb{CP}^m$ is an infinite cyclic group, generated by the class $[H]$ of the hyperplane bundle.*

Proof. Since we already know that $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}) \equiv \mathbb{Z}$, we need only check that $[H]$ is a generator. Equivalently, we must check that $c_1(H) = [(2\pi)^{-1}\Phi]$ is a generator for $H^2(M, \mathbb{Z})$. For this, it is enough to show that $\int_{\mathbb{CP}^m} (2\pi)^{-m}\Phi^{\wedge m} = 1$.

Since the complement of the chart domain U_0 is a lower-dimensional submanifold (diffeomorphic to $\mathbb{CP}^{m-1}$), we need only show that the integral over U_0 equals 1. We may therefore use the formula (2.7) (with $j = 0$) for the Kähler form Φ. Then

$$\int_{U_0} \Phi^{\wedge m} = \int_{\mathbb{C}^m} \left(iQ_0^{-1} \sum_{k=1}^{m} dw_0^k \wedge d\bar{w}_0^k - iQ_0^{-2} \sum_{k,l=1}^{m} \bar{w}_0^k w_0^l \, dw_0^k \wedge d\bar{w}_0^l \right)^{\wedge m}. \tag{5.25}$$

At the point $(r, 0, \dots, 0) \in \mathbb{C}^m$, Q_0 simplifies to $1 + r^2$, and the integrand on the right hand side of (5.24) becomes

$$\begin{aligned}
&\left(i(1+r^2)^{-2} \, dw_0^1 \wedge d\bar{w}_0^1 + i(1+r^2)^{-1} \sum_{k=2}^{m} dw_0^k \wedge d\bar{w}_0^k \right)^{\wedge m} \\
&\qquad = \frac{i^m \, m!}{(1+r^2)^{m+1}} \bigwedge_{k=1}^{m} dw_0^k \wedge d\bar{w}_0^k = \frac{2^m \, m!}{(1+r^2)^{m+1}} \bigwedge_{k=1}^{m} dx^k \wedge dy^k \\
&\qquad = \frac{2^m \, m!}{(1+r^2)^{m+1}} \lambda,
\end{aligned} \tag{5.26}$$

where λ is Lebesgue measure on $\mathbb{C}^m$, and $w_0^k = x^k + iy^k$ give Cartesian coordinates on $\mathbb{C}^m$. Interpreting r as a polar coordinate on $\mathbb{C}^m$, one can write $\lambda = r^{2m-1} \, dr \, d^{2m-1}\theta$, with $\theta \in \mathbb{S}^{2m-1}$ being the angular part. The right hand side of (5.25) is invariant under the unitary group $U(m)$, as is the Kähler form, so it represents the integrand at all points, not just at $(r, 0, \dots, 0)$.

The volume of the sphere $\mathbb{S}^{2m-1}$ is $\Omega_{2m} = 2\,\pi^m/(m-1)!$ (see [1, 8, 28], for instance), so the desired integral is

$$\begin{aligned}
\int_{\mathbb{CP}^m} (2\pi)^{-m}\Phi^{\wedge m} &= \frac{m!}{\pi^m} \int_{\mathbb{C}^m} \frac{r^{2m-1}}{(1+r^2)^{m+1}} \, dr \, d^{2m-1}\theta \\
&= \frac{m!}{\pi^m} \Omega_{2m} \int_0^\infty \frac{r^{2m-1}}{(1+r^2)^{m+1}} \, dr = \int_0^\infty \frac{2mr^{2m-1}}{(1+r^2)^{m+1}} \, dr \\
&= \int_0^\infty \frac{mt^{m-1}}{(1+t)^{m+1}} \, dt = \int_0^1 mu^{m-1} \, du = 1,
\end{aligned}$$

on substituting $t = r^2$, $u = t/(1+t)$. □

Corollary 5.14. *The hyperplane bundle is not trivial, since $c_1(H) \neq 0$.* □

This completes the classification of Hermitian line bundles over $\mathbb{CP}^m$, since any line bundle $L' \longrightarrow \mathbb{CP}^m$ satisfies $c_1(L') = k\, c_1(H)$ for some $k \in \mathbb{Z}$; thus $L' \sim H^{\otimes k}$ if $k > 0$, $L' \sim L^{\otimes(-k)}$ if $k < 0$, and L' is trivial iff $k = 0$. Furthermore, k is precisely the integral over $\mathbb{CP}^m$ of $(-2\pi i)^{-m}\omega_{L'}^{\wedge m}$ where $\omega_{L'}$ is the curvature of any compatible connection on $L' \longrightarrow \mathbb{CP}^m$.

5.11 The Levi-Civita connection on the tangent bundle

Definition 5.15. Let M be a Riemannian manifold, and let ∇ be a connection on the tangent bundle $TM \longrightarrow M$. The **fundamental 1-form** θ is the unique element of $\mathcal{A}^1(M, TM)$ satisfying $\iota_X\theta = X$ for all $X \in \mathfrak{X}(M) = \Gamma(TM)$. The **torsion** of ∇ is $T := \nabla\theta \in \mathcal{A}^2(M, TM)$.

Exercise 5.20. Show that the contraction map $\iota_X \colon \mathcal{A}^1(M, E) \to \Gamma(E)$ of Definition 5.4 extends to an $\mathcal{A}$-linear map $\iota_X \colon \mathcal{A}^k(M, E) \to \mathcal{A}^{k-1}(E)$ such that $\iota_X(\zeta\wedge\eta) = (\iota_X\zeta)\wedge\eta + (-1)^k\zeta\wedge \iota_X\eta$ for $\zeta \in \mathcal{A}^k(M, E)$, $\eta \in \mathcal{A}^\bullet(M)$, provided we define $\iota_X s := 0$ for $s \in \Gamma(E) = \mathcal{A}^0(M, E)$.

We can then define $T(X, Y) := \iota_Y(\iota_X T) \in \mathfrak{X}(M)$.

Lemma 5.15. *If $X, Y \in \mathfrak{X}(M)$, then $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.*

Proof. It is not hard to show that $\nabla_X\zeta = \nabla(\iota_X\zeta) + \iota_X(\nabla\zeta)$ and $\nabla_X(\iota_Y\zeta) = \iota_Y(\nabla_X\zeta) + \iota_{[X,Y]}\zeta$ for $\zeta \in \mathcal{A}^1(M, TM)$. For the particular case $\zeta = \theta$, these identities give

$$\begin{aligned} T(X, Y) &= \iota_Y(\iota_X\nabla\theta) = \iota_Y(\nabla_X\theta - \nabla(\iota_X\theta)) = \iota_Y(\nabla_X\theta - \nabla X) \\ &= \nabla_X(\iota_Y\theta) - \iota_{[X,Y]}\theta - \nabla_Y X = \nabla_X Y - [X, Y] - \nabla_Y X, \end{aligned} \tag{5.27}$$

as claimed. $\square$

Exercise 5.21. Verify the aforementioned formulae for $\nabla_X\zeta$ and $\nabla_X(\iota_Y\zeta)$ by applying (5.8) and (5.9) (and their analogues for ι_X) in the case $\zeta = Z \otimes \alpha$ with $Z \in \mathfrak{X}(M)$, $\alpha \in \mathcal{A}^1(M)$.

Proposition 5.16. *If M is a Riemannian manifold, there is a unique connection ∇ on the tangent bundle $TM \longrightarrow M$, which is compatible with the Euclidean metric on TM and is torsion-free.*

Proof. Compatibility with the metric demands that $(\nabla X \mid Y) + (X \mid \nabla Y) = d(X \mid Y)$, as in (5.7), where $(X \mid Y) = g(X, Y)$ denotes the *bilinear* pairing on $\mathfrak{X}(M) = \Gamma(TM)$ determined by the Euclidean metric g. By Lemma 5.15, ∇ is torsion-free iff $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in \mathfrak{X}(M)$.

Thus $Z(X \mid Y) = \iota_Z(\nabla X \mid Y) + \iota_Z(X \mid \nabla Y) = (\nabla_Z X \mid Y) + (X \mid \nabla_Z Y)$. A short calculation then shows that

$$\begin{aligned} 2(\nabla_Z X \mid Y) &= X(Y \mid Z) - Y(Z \mid X) + Z(X \mid Y) \\ &\qquad + (X \mid [Y, Z]) + (Y \mid [Z, X]) - (Z \mid [X, Y]), \end{aligned} \tag{5.28}$$

which establishes the uniqueness of ∇. On the other hand, it is easy to show that the right hand side is $\mathcal{A}$-linear in Y and Z, hence is of the form $\iota_Z(DX \mid Y)$, where $D \colon \mathfrak{X}(M) \to \mathcal{A}^1(M, TM)$. By replacing X by hX (with $h \in \mathcal{A}$) on the right hand side of (5.27) and then simplifying, one verifies the Leibniz rule for D, which proves the existence of the desired connection. $\square$

Definition 5.16. The unique metric-compatible torsion-free connection on the tangent bundle of a Riemannian manifold M is called its **Levi-Civita connection**. Its curvature, in $\mathcal{A}^2(M,TM)$, is usually denoted by R, and is also called the **Riemannian curvature tensor**[11] of M.

Exercise 5.22. Verify the following property of the Riemannian curvature tensor:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \qquad \text{for} \quad X,Y,Z \in \mathfrak{X}(M), \tag{5.29}$$

using only Lemma 5.4, Lemma 5.15, and the Jacobi identity.

Exercise 5.23. Verify the following symmetry properties of the Riemannian curvature tensor:

$$(W \mid R(X,Y)Z) = -(R(X,Y)W \mid Z) = (X \mid R(W,Z)Y),$$

for $W,X,Y,Z \in \mathfrak{X}(M)$.

6 Clifford algebras

A Clifford algebra is an associative algebra which is generated by starting with a real vector space and defining a product of vectors in such a way that the square of any vector is a scalar. This can be done consistently if the generating vector space is Euclidean, i.e., if it carries a symmetric bilinear form. For the zero bilinear form, the corresponding algebra is just the exterior algebra on the given vector space; otherwise, it has the same underlying vector space as the exterior algebra, but with a modified product operation. The Clifford algebra has interesting matrix representations, whose representation spaces are called "Clifford modules". All of these spaces are graded into an "even" part and an "odd" part, so we begin with a general discussion of vector spaces and algebras which are $\mathbb{Z}_2$-graded.

6.1 Superspaces and superalgebras

Definition 6.1. A **superspace** is just a vector space with a given $\mathbb{Z}_2$-grading: $V = V^+ \oplus V^-$; here V^+ is called the *even subspace* and V^- is called the *odd subspace*.[1]

A **superalgebra** is an algebra whose underlying vector space is a superspace: $A = A^+ \oplus A^-$, where the product respects the grading,[2] i.e., $A^+ \cdot A^+ \subseteq A^+$, $A^- \cdot A^- \subseteq A^+$, $A^+ \cdot A^- \subseteq A^-$, and $A^- \cdot A^+ \subseteq A^-$.

[11]Since $\mathcal{A}^2(M,TM) = \mathfrak{X}(M) \otimes_{\mathcal{A}} \mathcal{A}^2(M)$, the map $(X,Y,\alpha) \mapsto \alpha(R(X,Y))$ is a tensor of bidegree $(2,1)$ on M.

[1]The unfortunate prefix "super", which is simply a synonym for "$\mathbb{Z}_2$-graded", was introduced by Feliks Berezin [7] about 30 years ago, and has since become fashionable. Berezin wished to extend the calculus of Gaussian integrals by regarding an exterior algebra as a "space of functions of anticommuting variables". This crazy idea works astonishingly well.

[2]If we regard the exponents $+$ and $-$ as the elements of the additive group $\mathbb{Z}_2$, these four inclusions may be collected as the formula $A^i \cdot A^j \subseteq A^{i+j}$. In this way we can define a G-graded algebra for any abelian group G, although only the cases $G = \mathbb{Z}_2$ and $G = \mathbb{Z}$ are commonly used.

Definition 6.2. The **exterior algebra** $\Lambda^\bullet V$ of a vector space V is a $\mathbb{Z}$-graded algebra, whose subspace of degree k is $\Lambda^k(V)$ for $k = 0, 1, \dots, \dim V$ (and for $k < 0$ or $k > \dim V$, one sets $\Lambda^k(V) := \{0\}$), since $\Lambda^k(V) \wedge \Lambda^l(V) \subseteq \Lambda^{k+l}(V)$ for $k, l \in \mathbb{Z}$.

But $\Lambda^\bullet V$ is also a superalgebra, since we may define

$$\Lambda^+(V) := \bigoplus_{k \text{ even}} \Lambda^k(V), \qquad \Lambda^-(V) := \bigoplus_{k \text{ odd}} \Lambda^k(V).$$

Indeed, any $\mathbb{Z}$-graded algebra becomes a superalgebra, by collecting the subspaces of even degree and of odd degree in this manner.

Definition 6.3. If $V = V^+ \oplus V^-$ is a superspace, then $\operatorname{End} V$ is a superalgebra, with

$$\begin{aligned} \operatorname{End}^+ V &:= \operatorname{End}(V^+) \oplus \operatorname{End}(V^-), \\ \operatorname{End}^- V &:= \operatorname{Hom}(V^+, V^-) \oplus \operatorname{Hom}(V^-, V^+), \end{aligned}$$

Exercise 6.1. Guess the definition of a supermodule. Any superspace V is a supermodule for the superalgebra $\operatorname{End} V$.

Definition 6.4. We say an element a of a superalgebra A is *homogeneous* if either $a \in A^+$ or $a \in A^-$; its **parity** $\#a$ is defined as $\#a := 0$ if $a \in A^+$, $\#a := 1$ if $a \in A^-$. Analogously, in a $\mathbb{Z}$-graded algebra, we define the *degree* of a homogeneous element as $\#a := k$ if $a \in A^k$.

There is an important, if somewhat informal, *sign rule* in superalgebra, which says that in any calculation in which the order of multiplication of homogeneous elements is reversed (i.e., ab is changed to ba), a sign factor of $(-1)^{\#a\#b}$ must be inserted. Thus, for example, we say that a superalgebra is "supercommutative" if $ab = (-1)^{\#a\#b} ba$ for homogeneous elements $a, b \in A$; in other words, even elements commute with both even and odd elements, but two odd elements anticommute. Notice that *the exterior algebra $\Lambda^\bullet V$ is supercommutative.*

Definition 6.5. The **superbracket** in a superalgebra is the bilinear operation $A \times A \to A$ defined, for a, b homogeneous, by

$$[\![a, b]\!] := ab - (-1)^{\#a\#b} ba.$$

A superalgebra is **supercommutative** iff all supercommutators $[\![a, b]\!]$ vanish, i.e., if the superbracket is trivial. The superbracket satisfies the properties:

$$\begin{aligned} &[\![a, b]\!] + (-1)^{\#a\#b} [\![b, a]\!] = 0, \\ &[\![a, [\![b, c]\!]]\!] = [\![[\![a, b]\!], c]\!] + (-1)^{\#a\#b} [\![b, [\![a, c]\!]]\!]. \end{aligned} \tag{6.1}$$

A vector space with a bilinear operation (of any kind) which satisfies (6.1) is called a **Lie superalgebra**.

Several bracket notations are in general use. Usually one writes $[a, b] = ab - ba$, and anticommutators $ab + ba$ are denoted $\{a, b\}$ or sometimes $[a, b]_+$; thus $[\![a, b]\!] = [a, b]$ if either a or b is even, and $[\![a, b]\!] = \{a, b\}$ if both a and b are odd. *Warning*: many superalgebraists use $[a, b]$ to denote a supercommutator, even when a and b are odd.

Definition 6.6. A **supertrace** on a superalgebra A is a linear form τ which vanishes on supercommutators, i.e., $\tau([\![a,b]\!]) = 0$ for all $a, b \in A$.

When $A = \operatorname{End} E$ for E a superspace, we may write $a \in A^+$ and $b \in A^-$ as

$$a = \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & b^- \\ b^+ & 0 \end{pmatrix}, \tag{6.2}$$

and we define a supertrace on $\operatorname{End} E$ by

$$\operatorname{Str}(a+b) := \operatorname{Tr}(a^+) - \operatorname{Tr}(a^-). \tag{6.3}$$

Exercise 6.2. Write the supercommutator $[\![a,b]\!]$ for a, b homogeneous elements of $\operatorname{End} E$ in the matrix notation (6.2) —there are four cases— and thus verify that (6.3) defines a supertrace.

The *tensor product* $U \otimes V$ of two superspaces U and V is a superspace, with the grading:

$$\begin{aligned} (U \otimes V)^+ &:= (U^+ \otimes V^+) \oplus (U^- \otimes V^-), \\ (U \otimes V)^- &:= (U^+ \otimes V^-) \oplus (U^- \otimes V^+). \end{aligned} \tag{6.4}$$

The tensor product of two superalgebras A and B, with the standard product recipe $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ is *not*, however, a superalgebra, since this product does not respect the grading (6.4). However, one defines a **graded tensor product**, denoted $A \mathbin{\overline{\otimes}} B$, whose underlying superspace is $A \otimes B$, by using the multiplication rule:

$$(a \otimes b)(a' \otimes b') := (-1)^{\#a' \, \#b} aa' \otimes bb' \tag{6.5}$$

(for a, a', b, b' homogeneous) in view of the passage of a' to the left of b.

Exercise 6.3. Verify that the product (6.5) is compatible with the grading (6.4) on the superspace $A \otimes B$.

Exercise 6.4. If U and V are (ungraded) vector spaces, show that $\Lambda^\bullet(U \oplus V) \simeq \Lambda^\bullet U \mathbin{\overline{\otimes}} \Lambda^\bullet V$ as superalgebras.

Exercise 6.5. If U and V are superspaces, define an action of $\operatorname{End} U \mathbin{\overline{\otimes}} \operatorname{End} V$ on the superspace $U \otimes V$ in such a way that $\operatorname{End} U \mathbin{\overline{\otimes}} \operatorname{End} V \simeq \operatorname{End}(U \otimes V)$ as superalgebras.

Exercise 6.6. Let A be a supercommutative superalgebra, and let V be a superspace. Establish the following identity for homogeneous elements of $A \mathbin{\overline{\otimes}} \operatorname{End} V$:

$$[\![a \otimes T, b \otimes S]\!] = (-1)^{\#T \, \#b} ab \otimes [\![T, S]\!].$$

Show that the linear map $\operatorname{Str}_A \colon A \mathbin{\overline{\otimes}} \operatorname{End} V \to A$ given by $\operatorname{Str}_A(a \otimes T) := a \operatorname{Str}(T)$ is an A-valued supertrace, in the sense that it vanishes on supercommutators.

Definition 6.7. A **superbundle** over a smooth manifold M is a vector bundle $E \longrightarrow M$ which is a Whitney sum of two vector bundles: $E = E^+ \oplus E^-$. The fibres $E_x = E_x^+ \oplus E_x^-$ are superspaces.

The space of sections $\Gamma(E)$ is a $\mathbb{Z}_2$-graded $\mathbb{C}^\infty(M)$-module: $\Gamma(E) = \Gamma(E^+) \oplus \Gamma(E^-)$. The E-valued forms on M may also be graded by degree, and so, in accordance with (6.4), $\mathcal{A}^\bullet(M, E)$ carries the total $\mathbb{Z}_2$-grading:

$$\mathcal{A}^+(M,E) := \mathcal{A}^{\text{even}}(M,E^+) \oplus \mathcal{A}^{\text{odd}}(M,E^-),$$
$$\mathcal{A}^-(M,E) := \mathcal{A}^{\text{even}}(M,E^-) \oplus \mathcal{A}^{\text{odd}}(M,E^+).$$

6.2 Clifford algebras

In this section we treat briefly the algebraic theory of Clifford algebras over Euclidean vector spaces. The presentation follows the treatment in [9], and the appendix to [54]. For a more detailed exposition of the algebraic theory, see [27] or [39]. All these rely on the fundamental paper of Atiyah, Bott and Shapiro [4].

Definition 6.8. Let V be a real vector space, with $\dim_{\mathbb{R}} V = n$, and q a positive definite symmetric bilinear form on V; we shall call the pair (V, q) a *Euclidean vector space.*[3] The **Clifford algebra** $\mathrm{C}\ell(V) = \mathrm{C}\ell(V, q)$ is an associative algebra generated by the elements of V subject (only) to the relation $v^2 = -q(v, v)\,1$. More precisely, one defines $\mathrm{C}\ell(V, q) := \mathcal{T}(V)/I(q)$, where $\mathcal{T}(V)$ is the tensor algebra over V and $I(q)$ is the ideal generated by $\{\, v \otimes v + q(v, v)\,1 : v \in E\,\}$. The canonical mapping of V into $\mathrm{C}\ell(V, q)$ is injective, so V may be regarded as a subspace of $\mathrm{C}\ell(V, q)$; we then have the fundamental relation

$$uv + vu = -2\,q(u, v) \qquad \text{for} \quad u, v \in V. \tag{6.6}$$

Exercise 6.7. Check that (6.6) is a consequence of $v^2 = -q(v, v)$.

Proposition 6.1. *The algebra $\mathrm{C}\ell(V, q)$ satisfies the following universal property: if A is a real algebra with identity and $f\colon V \to A$ is a linear mapping such that $f(v)^2 = -q(v, v)\,1$, then there is a unique algebra homomorphism $\tilde{f}\colon \mathrm{C}\ell(V, q) \to A$ such that $f = \tilde{f}\big|_V$.* □

Exercise 6.8. Give the proof of Proposition 6.1.

A Clifford algebra is $\mathbb{Z}_2$-graded, as follows. By taking $f(v) := -v$ in the previous proposition, with $A = \mathrm{C}\ell(V, q)$, we see that there is a unique automorphism α of $\mathrm{C}\ell(V, q)$ extending f, such that $\alpha^2 = \mathrm{id}$. (From the definition, each element of $\mathrm{C}\ell(V, q)$ is a linear combination of products $v_1 v_2 \dots v_k$ with $v_1, \dots, v_k \in V$; clearly we have $\alpha(v_1 v_2 \dots v_k) = (-1)^k\, v_1 v_2 \dots v_k$.) We write $\mathrm{C}\ell^\pm(V, q)$ to denote the (± 1)-eigenspaces of α. Thus $\mathrm{C}\ell(V, q)$ is a superalgebra. Notice that $V \subset \mathrm{C}\ell^-(V, q)$.

Now take A to be the opposite algebra of $\mathrm{C}\ell(V, q)$ (the same vector space with the product reversed), and $f(v) := v$; we get an involutive *anti*automorphism of $\mathrm{C}\ell(V, q)$ given by $(v_1 \dots v_k)^! := v_k \dots v_1$. We define another antiautomorphism $a \mapsto \bar{a}$, called *conjugation,* by $\bar{a} := \alpha(a)^! = \alpha(a^!)$.

Many properties of the Clifford algebra come from the following simple proposition, due to Chevalley [16].

[3] A Clifford algebra can be defined if q is a symmetric bilinear form of any signature. However, we shall use only forms which are either positive definite or (occasionally) negative definite.

Proposition 6.2. *Let (V,q) and (W,r) be two Euclidean vector spaces, and let $q\oplus r$ be the symmetric bilinear form on $V\oplus W$ given by $(q\oplus r)(v_1+w_1,v_2+w_2) := q(v_1,v_2)+r(w_1,w_2)$. Then there is an isomorphism of superalgebras $\mathrm{C}\ell(V\oplus W,q\oplus r)\simeq \mathrm{C}\ell(V,q)\mathbin{\overline{\otimes}}\mathrm{C}\ell(W,r)$.*

Proof. Define $f\colon V\oplus W\to \mathrm{C}\ell(V,q)\mathbin{\overline{\otimes}}\mathrm{C}\ell(W,r)$ by $f(v+w) := v\otimes 1+1\otimes w$. Using (6.5), we see that $f(v+w)^2 = -(q\oplus r)(v+w,v+w)\,1\otimes 1$; therefore f extends to an algebra homomorphism $\tilde f\colon \mathrm{C}\ell(V\oplus W,q\oplus r)\to \mathrm{C}\ell(V,q)\mathbin{\overline{\otimes}}\mathrm{C}\ell(W,r)$. Since the elements $v\otimes 1+1\otimes w$ generate $\mathrm{C}\ell(V,q)\mathbin{\overline{\otimes}}\mathrm{C}\ell(W,r)$ as an algebra, $\tilde f$ is surjective; to see that it is injective, it suffices to compute $\tilde f$ on a basis for $\mathrm{C}\ell(V\oplus W,q\oplus r)$ generated by bases for V and W. $\square$

If $e_1,\dots,e_n$ is an orthonormal basis of (V,q), write $e_J := e_{j_1}\dots e_{j_k}$ for $J=\{j_1,\dots,j_k\}\subseteq\{1,2,\dots,n\}$ with $1\le j_1<\dots<j_k\le n$; and let $e_\emptyset := 1$. Then $\{\,e_J : J\subseteq\{1,2,\dots,n\}\,\}$ is a basis of $\mathrm{C}\ell(V,q)$. In particular, $\dim\mathrm{C}\ell(V,q)=2^n$ and $\dim\mathrm{C}\ell^+(V,q)=\dim\mathrm{C}\ell^-(V,q)=2^{n-1}$.

Exercise 6.9. Use the previous Proposition to verify this basis, by induction on n.

Suppose $V=\mathbb{R}^n$ and that q_n is the standard Euclidean inner product on $\mathbb{R}^n$; in that case, we abbreviate $\mathrm{C}\ell(n) := \mathrm{C}\ell(\mathbb{R}^n,q_n)$. Since $\mathrm{C}\ell(1)=\operatorname{span}\{1,e_1\}$ with $e_1^2=-1$, we have $\mathrm{C}\ell(1)\simeq\mathbb{C}$ (as real algebras); with $\mathrm{C}\ell^+(1)=\mathbb{R}$, $\mathrm{C}\ell^-(1)=i\mathbb{R}$. Next, $\mathrm{C}\ell(2)=\operatorname{span}\{1,e_1,e_2,e_1e_2\}\simeq\mathbb{H}$, the algebra of quaternions. In both these cases, $a\mapsto\bar a$ denotes the usual conjugation.

Since $\mathrm{C}\ell(\mathbb{R},-q_1)=\operatorname{span}\{1,\hat e_1\}$ with $\hat e_1^2=+1$, we have $\mathrm{C}\ell(\mathbb{R},-q_1)\simeq\mathbb{R}\oplus\mathbb{R}$, so $\mathrm{C}\ell(V,q)$ depends on the signature of the form q. Also, $\mathrm{C}\ell(\mathbb{R}^2,-q_2)\simeq\mathbb{R}^{2\times 2}$, by taking $\hat e_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $\hat e_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. A complete list of the algebras $\mathrm{C}\ell(\mathbb{R}^n,\pm q_n)$ is given in [4], and reproduced in [27, 39]. A basic fact is what is called *Bott periodicity*: $\mathrm{C}\ell(n+8)\simeq\mathrm{C}\ell(n)\otimes\mathrm{C}\ell(8)$ (ungraded tensor product). Since $\mathrm{C}\ell(8)\simeq\mathbb{R}^{16\times 16}$ is a real matrix algebra, this means that $\mathrm{C}\ell(n+8)\simeq\mathrm{C}\ell(n)^{16\times 16}$, so one only need determine $\mathrm{C}\ell(n)$ for $n=1,2,\dots,8$.

Exercise 6.10. Show that $\mathrm{C}\ell(3)\simeq\mathbb{H}\oplus\mathbb{H}$. The remaining cases are $\mathrm{C}\ell(4)\simeq\mathbb{H}^{2\times 2}$, $\mathrm{C}\ell(5)\simeq\mathbb{C}^{4\times 4}$, $\mathrm{C}\ell(6)\simeq\mathbb{R}^{8\times 8}$, $\mathrm{C}\ell(7)\simeq\mathbb{R}^{8\times 8}\oplus\mathbb{R}^{8\times 8}$.

6.3 Clifford actions

Definition 6.9. A **Clifford module** for the Clifford algebra $\mathrm{C}\ell(V,q)$ is a superspace $F=F^+\oplus F^-$ together with an *even* homomorphism (called a *Clifford action*) $c\colon \mathrm{C}\ell(V,q)\to\operatorname{End}F$. In other words, $c(a)F^\pm\subseteq F^\pm$ if $a\in\mathrm{C}\ell^+(V,q)$, and $c(b)F^\pm\subseteq F^\mp$ if $b\in\mathrm{C}\ell^-(V,q)$.

Suppose that F has a (Euclidean or hermitian) inner product $(\cdot|\cdot)$, and denote the adjoint of $A\in\operatorname{End}F$ by $A^\dagger$, that is, $(x\,|\,A^\dagger y) := (Ax\,|\,y))$ for $x,y\in F$. We say that F is a *selfadjoint Clifford module* if $c(a)^\dagger=c(\bar a)$ for all $a\in A$. Notice that is enough to see that $c(v)^\dagger=-c(v)$ for all $v\in V$, i.e., that each $c(v)$ is skewadjoint.

Write $\operatorname{End}_{\mathrm{C}\ell(V,q)}F := \{\,R\in\operatorname{End}F : [\![c(a),R]\!]=0,\text{for } a\in\mathrm{C}\ell(V,q)\,\}$ to denote the subalgebra of $\operatorname{End}F$ consisting of operators which supercommute with the Clifford action.

Exercise 6.11. Check that $[\![RS,c(v)]\!]=R[\![S,c(v)]\!]+(-1)^{\#S}[\![R,c(v)]\!]S$ for R,S homogeneous elements of $\operatorname{End}F$, and conclude that $\operatorname{End}_{\mathrm{C}\ell(V,q)}F$ is indeed an algebra.

Definition 6.10. Let (V, q) be a finite-dimensional real vector space with a nondegenerate bilinear form.[4] The dual space V^* of $\mathbb{R}$-linear forms on V is a real vector space of the same dimension; we may define the so-called *musical isomorphisms* [8], namely $\flat\colon V \to V^*$ and $\sharp\colon V^* \to V$, by

$$u^\flat(v) := q(u, v), \qquad q(\lambda^\sharp, v) := \lambda(v).$$

They are mutually inverse: $(u^\flat)^\sharp = u$, $(\lambda^\sharp)^\flat = \lambda$.

If W is a complex vector space with a hermitian inner product $\langle\!\langle \cdot \mid \cdot \rangle\!\rangle$ (or a Hilbert space,[5] not necessarily finite-dimensional), we define $\flat\colon W \to W^*$ and $\sharp\colon W^* \to W$ similarly by $u^\flat(v) := \langle\!\langle u \mid v \rangle\!\rangle$, $\langle\!\langle \lambda^\sharp \mid v \rangle\!\rangle := \lambda(v)$. In the complex case, the musical isomorphisms are antilinear.

Definition 6.11. Let (V, q) be an Euclidean vector space, and let $\Lambda^\bullet V$ be its exterior algebra. For $v \in V$, define the *exterior multiplication* $\epsilon(v)$ and the **contraction** $\iota(v^\flat)$ in $\operatorname{End}(\Lambda^\bullet V)$ by

$$\begin{aligned} \epsilon(v)(v_1 \wedge \cdots \wedge v_k) &:= v \wedge v_1 \wedge \cdots \wedge v_k, \\ \iota(v^\flat)(v_1 \wedge \cdots \wedge v_k) &:= \sum_{j-1}^{k} (-1)^{j-1} q(v, v_j) v_1 \wedge \cdot \overset{j}{\vee} \cdot \wedge v_k. \end{aligned} \tag{6.7}$$

(The notation $\overset{j}{\vee}$ indicates that the term with index j is missing from the exterior product.) For $k = 0$, we define $\epsilon(v)1 := v$, $\iota(v^\flat)1 := 0$. Note that $\iota(v^\flat)$ is a graded derivation:[6]

$$\iota(v^\flat)(\alpha \wedge \beta) = \iota(v^\flat)\alpha \wedge \beta + (-1)^{\#\alpha} \alpha \wedge \iota(v^\flat)\beta \qquad \text{for} \quad \alpha, \beta \in \Lambda^\bullet V.$$

On $\Lambda^\bullet V$ we use the real inner product

$$(u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n) := \delta_{mn} \det\bigl[q(u_k, v_l)\bigr]$$

It is easily seen that $\epsilon(v)^\dagger = \iota(v^\flat)$ for $v \in V$. Define

$$c(v)\alpha := \epsilon(v)\alpha - \iota(v^\flat)\alpha, \qquad \text{for} \quad \alpha \in \Lambda^\bullet V. \tag{6.8}$$

Exercise 6.12. Check that $\epsilon(v)^\dagger = \iota(v^\flat)$ and that

$$\epsilon(u)\iota(v^\flat) + \iota(v^\flat)\epsilon(u) = q(v, u).$$

Conclude that $c(v)^2 = -q(v, v)\,1$ for $v \in V$.

By Proposition 6.1, c extends to a Clifford action of $C\ell(V, q)$ on $\Lambda^\bullet V$. Since $c(v)^\dagger = -c(v)$, this action is selfadjoint.

[4] This definition applies to both symmetric and antisymmetric bilinear forms.

[5] In the infinite-dimensional case, the bijectivity of the musical isomorphisms is a restatement of the Riesz theorem.

[6] It could have been defined as the unique graded derivation that takes u to $q(v, u)$.

Definition 6.12. The $\mathbb{R}$-linear map $\sigma\colon \mathrm{C}\ell(V,q) \to \Lambda^\bullet V$ given by

$$\sigma(a) := c(a)\,1 \tag{6.9}$$

for $a \in \mathrm{C}\ell(V,q)$, is a vector-space isomorphism (but not an algebra isomorphism) between the Clifford algebra and the exterior algebra, called [9] the **symbol map**. (It is easy to see that σ is surjective; since $\mathrm{C}\ell(V,q)$ and $\Lambda^\bullet V$ have the same dimension 2^n, it is also injective.) Note that $\sigma(1) = 1$ and $\sigma(v) = v$ for $v \in V$.

The inverse isomorphism $Q\colon \Lambda^\bullet V \to \mathrm{C}\ell(V,q)$ may be called [9] a **quantization map**, since it maps a supercommutative algebra to an algebra which is no longer supercommutative.[7]

Exercise 6.13. Show that $\sigma(uv) = u\wedge v - q(u,v)\,1$, so that $Q(u\wedge v) = uv + q(u,v)\,1$. Establish the general formula:

$$Q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\tau \in S_k} (-1)^\tau\, v_{\tau(1)} \dots v_{\tau(k)}, \tag{6.10}$$

where $(-1)^\tau$ denotes the sign of the permutation τ. In particular, $Q(e_{j_1} \wedge \cdots \wedge e_{j_k}) = e_{j_1} \dots e_{j_k}$ if $\{e_1, \dots, e_n\}$ is an orthonormal basis for V and $j_1 < \cdots < j_k$.

Exercise 6.14. The map Q does not preserve the $\mathbb{Z}$-grading of $\Lambda^\bullet V$, but does preserve its filtration. A *filtration* of an algebra A is a system of subspaces A_j with $A_j \subseteq A_{j+1}$ and $A_i A_j \subseteq A_{i+j}$. For $A = \mathrm{C}\ell(V,q)$, take A_j to be the subspace generated by products at most j vectors in V. There is an associated graded algebra, namely $\operatorname{gr} A := \bigoplus_j A_j/A_{j-1}$ and a canonical "symbol map" $\sigma\colon A \to \operatorname{gr} A$. Check that for $A = \mathrm{C}\ell(V,q)$, the associated graded algebra is none other than $\Lambda^\bullet V$ with the symbol map given by (6.9).

6.4 Complex Clifford algebras

Definition 6.13. The *complexification* $\mathbb{C}\ell(V,q) := \mathrm{C}\ell(V,q) \otimes_{\mathbb{R}} \mathbb{C}$ can be regarded as the Clifford algebra over $\mathbb{C}$ of the complexified vector space $V_{\mathbb{C}}$ with the same quadratic form q amplified to $V_{\mathbb{C}}$: any vector in $V_{\mathbb{C}}$ can be written uniquely as $w = u + iv$ with $u, v \in V$, and one defines $q(w,w') := q(u,u') - q(v,v') + iq(u,v') - iq(v,u')$; notice that the amplified bilinear form q is not positive definite on $V_{\mathbb{C}}$. On $V_{\mathbb{C}}$ all nondegenerate symmetric bilinear forms are equivalent, so we may abbreviate $\mathbb{C}\ell(V,q)$ to $\mathbb{C}\ell(V)$; in particular, $\mathbb{C}\ell(V,q) \simeq \mathbb{C}\ell(n) := \mathbb{C}\ell(\mathbb{C}^n, q_n)$ if $\dim_{\mathbb{R}} V = n$.

Exercise 6.15. Prove that $\mathbb{C}\ell(1) \simeq \mathbb{C} \oplus \mathbb{C}$ and that $\mathbb{C}\ell(2) \simeq \mathbb{C}^{2\times 2}$.

[7] Quantum field theory deals with Fermi fields, which are systems of operators $\{\, \phi(v) : v \in V \,\}$ satisfying an "anticommutation relation" akin to (6.6). It helps to imagine an analogous situation in which the anticommutators vanish, i.e., the algebra generated by the Fermi fields is replaced by a supercommutative algebra; restoration of the scalar terms $q(u,v)$ then corresponds to quantizing the latter system.

Exercise 6.16. Prove that $\mathbb{C}\ell(n+2) \simeq \mathbb{C}\ell(n) \otimes \mathbb{C}\ell(2)$ (ungraded tensor product)[8] by showing that the $\mathbb{C}$-linear map $f\colon \mathbb{C}^{n+2} \to \mathbb{C}\ell(n) \otimes \mathbb{C}\ell(2)$ given by $f(e_1) := 1 \otimes e_1$, $f(e_2) := 1 \otimes e_2$, $f(e_j) := i\, e_{j-2} \otimes e_1 e_2$ for $j \geq 3$, satisfies $\{f(e_j), f(e_k)\} = -2\delta_{jk}(1 \otimes 1)$, and hence extends to the desired isomorphism.

From this we find that $\mathbb{C}\ell(2m) \simeq \mathbb{C}^{N\times N}$ with $N = 2^m$; and $\mathbb{C}\ell(2m+1) \simeq \mathbb{C}^{N\times N} \oplus \mathbb{C}^{N\times N}$. In particular, $\mathbb{C}\ell(V, q)$ is a simple matrix algebra iff $\dim_{\mathbb{R}} V$ is even. From now on, we consider only the case that V has **even dimension** $n = 2m$.

Definition 6.14. Any finite-dimensional module for the matrix algebra $A = \mathbb{C}^{N\times N}$ is of the form $F = S_1 \oplus \cdots \oplus S_r$ where each S_k has dimension N and A acts irreducibly on each S_k. Another way to express this is to say that $F = W \otimes S$, where $\dim_{\mathbb{C}} S = N$ and A acts trivially on W, that is, $a \cdot (w \otimes s) = w \otimes (a \cdot s)$. Therefore, if $\dim V$ is even, any Clifford module for $\mathbb{C}\ell(V)$ is of the form $F = W \otimes S$, with $c(a)(w \otimes s) := w \otimes c(a)s$, where S is an *irreducible* Clifford module, unique up to equivalence, called the **spinor module** (or "spinor space") for $\mathbb{C}\ell(V)$. As algebras, $\mathbb{C}\ell(V)$ and $\mathrm{End}_{\mathbb{C}}(S)$ are isomorphic.

Exercise 6.17. Define $\mathrm{Hom}_{\mathbb{C}\ell(V)}(S, F)$ to be the vector space of $\mathbb{C}$-linear maps $T\colon S \to F$ which intertwine the Clifford actions of $\mathbb{C}\ell(V)$ on S and on F, i.e., $c(a)(Ts) := T(c(a)s)$ for all $s \in S$, $a \in \mathbb{C}\ell(V)$. Check that the map $T \otimes s \mapsto Ts$ gives an isomorphism $\mathrm{Hom}_{\mathbb{C}\ell(V)}(S, F) \otimes S \simeq F$, and deduce that $\mathrm{Hom}_{\mathbb{C}\ell(V)}(S, F) \simeq W$ and $\mathrm{End}_{\mathbb{C}\ell(V)} F \simeq \mathrm{End}_{\mathbb{C}} W$.

Definition 6.15. Let (V, q) be an *oriented* Euclidean vector space of even dimension $n = 2m$ (over $\mathbb{R}$), and let $\{e_1, \dots, e_n\}$ be an orthonormal basis for (V, q) which is compatible with the given orientation.[9] We define the **chirality element** of $\mathbb{C}\ell(V)$ as

$$\gamma := i^m\, e_1 e_2 \dots e_n. \tag{6.11}$$

If $\{e'_1, \dots, e'_n\}$ is another oriented orthonormal basis, then $e'_j = \sum_{k=1}^n g_{jk} e_k$ for $g \in SO(n)$ an orthogonal matrix of determinant $+1$, and so $i^m\, e'_1 e'_2 \dots e'_n = \det(g)\gamma = \gamma$: thus the recipe (6.11) is independent of the chosen basis.

The chirality element γ satisfies $\gamma^2 = 1$ (that is why the factor i^m belongs in the definition), and it anticommutes with the copy of V within $\mathbb{C}\ell(V)$. Indeed, $e_j\gamma = -\gamma e_j$ since e_j anticommutes with every e_k except e_j itself; by linearity, $v\gamma = -\gamma v$ for all $v \in V$. Therefore $\gamma v \gamma = -v$ for $v \in V$, so by Proposition 6.1 $\gamma a \gamma = \alpha(a)$ for all $a \in \mathbb{C}\ell(V)$. The $\mathbb{Z}_2$-grading on $\mathbb{C}\ell(V)$ induced by γ (by declaring an element even or odd according as it commutes or anticommutes with γ) coincides, fortunately, with the original grading given by α.

Notice that if $n = 4k$ is a multiple of 4, then γ belongs to the *real* Clifford algebra $C\ell(V, q)$, and it is not necessary to use the complexification $\mathbb{C}\ell(V)$.

[8] This is the "Bott periodicity" identity for complex Clifford algebras.

[9] Recall that an orientation of V is a choice of a positive direction in the real line $\Lambda^n V$; a basis $\{v_1, \dots, v_n\}$ for V is compatible with the orientation iff $v_1 \wedge \cdots \wedge v_n$ is positive.

6.5 The Fock space of spinors

The availability of γ means that any (ordinary) complex module F under the Clifford algebra can be given a $\mathbb{Z}_2$-grading by taking $F^\pm$ to be the (± 1)-eigenspaces of $c(\gamma) \in \operatorname{End}_\mathbb{C} F$. In particular, the spinor module is a superspace $S = S^+ \oplus S^-$, with $\mathbb{C}\ell(V) \simeq \operatorname{End}_\mathbb{C} S$ as superalgebras. We now give an explicit construction of the spinor module.

Definition 6.16. A Euclidean vector space (V, q) of even real dimension $n = 2m$ can be made into a complex Hilbert space by choosing an *orthogonal complex structure* J on (V, q); this is an $\mathbb{R}$-linear operator on V satisfying:

$$J^2 = -1 \qquad \text{and} \qquad q(Ju, Jv) = q(u, v) \quad \text{for } u, v \in V. \tag{6.12}$$

Now we regard V as a complex vector space via the rule $(\alpha + i\beta)v := \alpha v + \beta Jv$ for $\alpha, \beta \in \mathbb{R}$. The hermitian form

$$\langle\!\langle u \mid v \rangle\!\rangle := q(u, v) + iq(Ju, v)$$

is positive definite since q is. We shall denote the complex Hilbert space thus obtained by (V, q, J). Notice that $i\langle\!\langle u \mid v \rangle\!\rangle = \langle\!\langle u \mid Jv \rangle\!\rangle = -\langle\!\langle Ju \mid v \rangle\!\rangle$ for $u, v \in V$.

Exercise 6.18. For $(V, q) = (\mathbb{R}^4, q_4)$, find all $J \in \operatorname{End}_\mathbb{R} V = \mathbb{R}^{4\times 4}$ satisfying (6.12). Show that the set of such J has two connected components, each homeomorphic to a 2-sphere $\mathbb{S}^2$.

Exercise 6.19. If $g \in \operatorname{End}_\mathbb{R}(V)$, check that g is a $\mathbb{C}$-linear map on the Hilbert space (V, q, J) iff $gJ = Jg$. Conclude that $\{\, g \in GL_\mathbb{R}(V) : gJ = Jg \,\} = GL_\mathbb{C}(V, q, J)$. If $O(V, q)$ denotes the **orthogonal group** of those $g \in GL_\mathbb{R}(V)$ for which $q(gu, gv) = q(u, v)$ for all $u, v \in V$, show that $U_J(V) := O(V, q) \cap GL_\mathbb{C}(V, q, J)$ is the **unitary group** of the Hilbert space (V, q, J), i.e., the group of complex automorphisms satisfying $\langle\!\langle gu \mid gv \rangle\!\rangle = \langle\!\langle u \mid v \rangle\!\rangle$ for $u, v \in V$.

Exercise 6.20. Let $\mathcal{J}(V, q)$ denote the set of all orthogonal complex structures on a Euclidean vector space (V, q). Show that $\mathcal{J}(V, q)$ is empty if $\dim_\mathbb{R} V$ is odd, and nonempty if $\dim_\mathbb{R} V$ is even (produce an example). Show that the orthogonal group $O(V, q)$ acts transitively on $\mathcal{J}(V, q)$ by $J \mapsto gJg^{-1}$, and that the isotropy subgroup of any J is the unitary group $U_J(V)$.

Exercise 6.21. Check that $\{u_1, Ju_1, \dots, u_m, Ju_m\}$ is an orthonormal basis for (V, q) when $\{u_1, \dots, u_m\}$ is an orthonormal basis for (V, q, J), and deduce that $O(V, q) \simeq O(\mathbb{R}^{2m}, q_{2m}) \equiv O(2m)$ and that $U_J(V) \simeq U_i(\mathbb{C}^m) \equiv U(m)$. Conclude that $\mathcal{J}(V, q)$ is diffeomorphic to the quotient manifold $O(2m)/U(m)$ (which, as it happens, has two connected components).[10]

Definition 6.17. Choose and fix an orthogonal complex structure J on (V, q). Denote by $\mathcal{F}_J(V)$ or simply by $\mathcal{F}(V)$ the complex exterior algebra $\Lambda^\bullet(V, q, J)$ over the Hilbert space (V, q, J). Then $\mathcal{F}(V)$ is a complex vector space of dimension 2^m, called the **Fock space**[11]

[10]The identity component $SO(V, q)$ of the group $O(V, q)$, which consists of those g with determinant $+1$, satisfies $SO(V, q) \simeq SO(2m)$, so one of the two components of $\mathcal{J}(V, q)$ is the set of "orientation-preserving" J, which is diffeomorphic to $SO(2m)/U(m)$.

[11]This is sometimes called a *fermion* Fock space to distinguish it from the Hilbert space formed from a symmetric algebra over V (suitably completed), which is known as a "boson Fock space"; these spaces describe many-particle states of fermions and bosons in quantum field theory.

over (V, q, J). The Fock space is itself a complex Hilbert space, with the inner product determined by

$$\langle\!\langle u_1 \wedge \cdots \wedge u_m \mid v_1 \wedge \cdots \wedge v_n \rangle\!\rangle := \delta_{mn} \det\bigl[\langle\!\langle u_k \mid v_l \rangle\!\rangle\bigr].$$

We can choose (by induction on $\dim V$) an orthonormal basis $\{e_1, \dots, e_n\}$ for V so that $Je_{2r-1} = e_{2r}$ for $r = 1, \dots, m$.

It is useful to identify (V, q, J) with an m-dimensional complex subspace W of the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$; take $W := \{\, v - iJv \in V_{\mathbb{C}} : v \in V \,\}$. Notice that W is *isotropic* for the amplified bilinear form q on $V_{\mathbb{C}}$, i.e., $q(u - iJu, v - iJv) = 0$ for $u, v \in V$, on account of (6.12); and moreover, $\dim_{\mathbb{C}} W = m = \frac{1}{2} \dim_{\mathbb{C}} V_{\mathbb{C}}$, so W is a maximally isotropic subspace.[12] The conjugate subspace $\overline{W} := \{\, v + iJv \in V_{\mathbb{C}} : v \in V \,\}$ satisfies $W \cap \overline{W} = 0$, $W \oplus \overline{W} = V_{\mathbb{C}}$, and indeed $\overline{W}$ is the orthogonal complement of W under the inner product on $V_{\mathbb{C}}$ given by

$$\langle\!\langle w \mid z \rangle\!\rangle := 2\, q(\bar{w}, z).$$

The operator $P_+ := \frac{1}{2}(I - iJ)$ is then a unitary isomorphism between (V, q, J) and W, so we may identify these Hilbert spaces. The Riesz theorem allows us to identify the dual space V^* with $\overline{W}$, and if $w = P_+ v$, we identify $v^\flat$ with $\bar{w} := P_- v := \frac{1}{2}(I + iJ)v$. An orthonormal basis for W is $\{z_1, \dots, z_m\}$, where $z_k := P_+ e_{2k-1}$; notice that $P_+ e_{2k} = P_+(Je_{2k-1}) = iz_k$.

We may also identify $\mathcal{F}(V)$ with the complex exterior algebra $\Lambda^\bullet W$. Now define $c(v)$ on $\Lambda^\bullet W$, for $v \in V$, by

$$c(v)\alpha := \epsilon(P_+ v)\alpha - \iota(P_- v)\alpha, \tag{6.13}$$

where $\iota(\bar{z})(w_1 \wedge \cdots \wedge w_k) = \sum_{j=1}^k (-1)^{j-1} \langle\!\langle z \mid w_j \rangle\!\rangle\, w_1 \wedge \overset{j}{\cdot\overset{\vee}{\cdot}\cdot} \wedge w_k$, in accordance with (6.7), and extend to $V_{\mathbb{C}}$ by $c(v_1 + iv_2) := c(v_1) + ic(v_2)$.

Exercise 6.22. Show that $\{\epsilon(w), \iota(\bar{z})\} = \langle\!\langle z \mid w \rangle\!\rangle$ for $w, z \in W$.

Exercise 6.23. Show that $c(w) = \epsilon(w)$ and $c(\bar{z}) = -\iota(\bar{z})$ for $w, z \in W$. Conclude that c extends to a selfadjoint Clifford action on $\Lambda^\bullet(W)$.

Proposition 6.3. *The operator $c(\gamma)$ on the Fock space is the grading operator on the superspace $\Lambda^\bullet W$.*

Proof. The assertion is that the (± 1)-eigenspaces of $c(\gamma)$ are the even and odd subspaces $\Lambda^\pm W$; in other words, $c(\gamma)\alpha = (-1)^k \alpha$ for all $\alpha \in \Lambda^k W$.

First of all, notice that $z_k \bar{z}_k - \bar{z}_k z_k = i\, e_{2k-1} e_{2k}$, so that $\gamma = \overrightarrow{\prod}_{1 \le k \le m} (z_k \bar{z}_k - \bar{z}_k z_k)$. Moreover, $c(z_k \bar{z}_k - \bar{z}_k z_k) = [c(z_k), c(\bar{z}_k)] = [\iota(\bar{z}_k), \epsilon(z_k)]$. Now a direct calculation shows that, for $\alpha = z_{j_1} \wedge \cdots \wedge z_{j_k}$, we have $[\iota(\bar{z}_r), \epsilon(z_r)]\alpha = \pm\alpha$, with negative sign iff $r \in \{j_1, \dots, j_k\}$. Since $\Lambda^\bullet W$ is generated (as a complex vector space) by such α, we obtain $c(\gamma)\alpha = (-1)^k \alpha$. □

[12]If $Z \le V_{\mathbb{C}}$ has dimension k, the subspace of vectors w with $q(w, z) = 0$ for all $z \in Z$ has dimension $2m - k$, since q is nondegenerate; thus an isotropic subspace can have dimension at most m. A maximally isotropic subspace is also called a *polarization* for q.

6.6 The Pin and Spin groups

Definition 6.18. Let (V, q) be a Euclidean vector space with q positive definite. The invertible elements of $\mathrm{C}\ell(V, q)$ form a group which includes all nonzero scalars and all nonzero $v \in V$ (since $v^{-1} = -v/q(v, v)$); and the "twisted conjugation" $\phi(a)b := \alpha(a)ba^{-1}$ provides a linear action of this group on $\mathrm{C}\ell(V, q)$. The invertible elements which preserve the subspace V under this action form the *Clifford group* $\Gamma(V, q) := \{\, a : \phi(a)(V) \subset V \,\}$. The nonzero scalars $t \in \mathbb{R}^\times$ lie in $\Gamma(V, q)$, and so do the nonzero vectors $u \in V \setminus \{0\}$, since if $v \in V$,

$$\phi(u)v = -uvu^{-1} = (vu - 2q(u, v))u^{-1} = v - 2\frac{q(u, v)}{q(u, u)}u, \tag{6.14}$$

so $u \in \Gamma(V, q)$. Geometrically, (6.14) is the *reflection* in the hyperplane orthogonal to u, since $\phi(u)u = -u$ and $\phi(u)v = v$ iff $q(u, v) = 0$.

Lemma 6.4. *The kernel of $\phi\colon \Gamma(V, q) \to GL_{\mathbb{R}}(V)$ is the subgroup $\mathbb{R}^\times$ of nonzero scalars.*

Proof. If $a \in \ker \phi$, let $a = a^+ + a^-$ with $a^\pm \in \mathrm{C}\ell^\pm(V, q)$; taking even and odd parts of the equation $\alpha(a)v = va$ gives $a^\pm v = \pm va^\pm$ for $v \in V$. If $\{e_1, \dots, e_n\}$ is an orthonormal basis for V, and if V_1 denotes the hyperplane in V orthogonal to e_1, then $a^+ = b^+ + e_1 b^-$ with $b^\pm \in \mathrm{C}\ell^\pm(V_1, q)$. Now the equation $a^+ e_1 = e_1 a^+$ yields $b^+ e_1 + b^- = e_1 b^+ - b^-$, so $b^- = 0$; hence $a^+ \in \mathrm{C}\ell(V_1, q)$. A similar argument shows that $a^- \in \mathrm{C}\ell(V_1, q)$. An obvious induction argument now shows that a has only a scalar component when expanded in the basis $\{\, e_J : J \subseteq \{1, \dots, n\} \,\}$ of $\mathrm{C}\ell(V, q)$, and so $a \in \mathbb{R}^\times$. $\square$

If $v \in V$, then $v\bar{v} = v\alpha(v) = -v^2 = q(v, v)\,1$, so the quadratic form $q(\cdot, \cdot)$ extends to the map $a \mapsto a\bar{a}$ of $\mathrm{C}\ell(V, q)$ into itself. If $a = \sum_J a_J e_J$ is the expansion of a with respect to the basis $\{\, e_J : J \subseteq \{1, 2, \dots, n\} \,\}$ of $\mathrm{C}\ell(V, q)$, then $a\bar{a}$ and $\bar{a}a$ have the same scalar component $\sum_J a_J^2$. If $a \in \Gamma(V, q)$ and $v \in V$, then $\phi(a)v = u$ implies $u = \alpha(a)^{-1}va$, and so $u = u^! = a^! v\bar{a}^{-1} = \phi(\bar{a})v$; in consequence, $\phi(a\bar{a})v = \phi(a)\phi(\bar{a})v = v$, so that $a\bar{a} \in \ker \phi = \mathbb{R}^\times$. Thus $a \mapsto a\bar{a} = \bar{a}a$ yields a group homomorphism of $\Gamma(V, q)$ into $\mathbb{R}^\times$. Its kernel is $\mathrm{Pin}(V, q) := \{\, a \in \Gamma(V, q) : a\bar{a} = 1 \,\}$.

If $a \in \Gamma(V, q)$ and $v \in V$, then

$$q(\phi(a)v, \phi(a)v) = \alpha(a)va^{-1}\bar{a}^{-1}\bar{v}\alpha(\bar{a}) = \alpha(a\bar{a})(\bar{a}a)^{-1}v\bar{v} = -v^2 = q(v, v),$$

since $a\bar{a} = \bar{a}a$ is a scalar, and so $\phi(a)$ lies in the orthogonal group $O(V, q)$. Restriction to $\mathrm{Pin}(V, q)$ yields a homomorphism $\phi\colon \mathrm{Pin}(V, q) \to O(V, q)$ which is *surjective* since the orthogonal group is generated by the reflections[13] (6.14), and $v \in \mathrm{Pin}(V, q)$ whenever $q(v, v) = 1$.

Definition 6.19. The **Spin group** of the Euclidean vector space (V, q) is defined as the even part of $\mathrm{Pin}(V, q)$, i.e.,

$$\mathrm{Spin}(V, q) := \mathrm{C}\ell^+(V, q) \cap \mathrm{Pin}(V, q) := \{\, a \in \Gamma(V, q) \cap \mathrm{C}\ell^+(V, q) : a\bar{a} = 1 \,\}.$$

[13] This is the Cartan–Dieudonné theorem, which can be proved by writing an orthogonal matrix in normal form as a direct of k plane rotation matrices and possibly a -1 diagonal entry, and by recalling that a plane rotation is a product of two reflections in lines.

The image $\phi(\mathrm{Spin}(V,q))$ is the subgroup $SO(V,q)$ of $O(V,q)$ generated by products of an even number of reflections. This is the *rotation group*, also called the special orthogonal group, since $g \in O(V,q)$ lies in $SO(V,q)$ iff $\det g = 1$.

The homomorphisms $\phi\colon \mathrm{Pin}(V,q) \to O(V,q)$ and $\phi\colon \mathrm{Spin}(V,q) \to SO(V,q)$ have the same kernel $\{\pm 1\} \simeq \mathbb{Z}_2$, by Lemma 6.4. In particular, we have a short exact sequence of Lie groups:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V,q) \xrightarrow{\phi} SO(V,q) \longrightarrow 1,$$

so that $\mathrm{Spin}(V,q)$ is a double covering of the rotation group.

Exercise 6.24. Show that $\mathrm{Pin}(V,q)$ is the set of products $\{v_1 \dots v_k\}$ of unit vectors (i.e., vectors $v_j \in V$ such that $q(v_j,v_j) = 1$ for each j), by showing that these products form a normal subgroup which ϕ maps onto $O(V,q)$. Deduce that $\mathrm{Spin}(V,q)$ is the set of products $\{v_1 \dots v_{2r}\}$ of an even number of unit vectors.

Exercise 6.25. If $u, v \in V$ are unit vectors in V, with $u \neq \pm v$, and if $a(t) := 1 + q(u,v)\sin 2t$ for $0 \leq t \leq \frac{\pi}{2}$, define $w(t) := a(t)^{-1/2}(\cos t\, u + \sin t\, v)$, $z(t) := a(t)^{-1/2}(\cos t\, v + \sin t\, u)$, and $b := (1 - q(u,v)^2)^{-1/2}(uv + q(u,v)\,1)$. Show that $t \mapsto -w(t)w(-t)$ is a continuous path in $\mathrm{Spin}(V,q)$ from 1 to -1 through b, and that $t \mapsto w(t)z(-t)$ is a continuous loop in $\mathrm{Spin}(V,q)$ from uv to b and back. Deduce that $\mathrm{Spin}(V,q)$ is a connected group.[14]

On passing to the complexification, we may regard $\mathrm{Spin}(V,q)$ as a subgroup of the invertible elements of the complex Clifford algebra $\mathbb{C}\ell(V)$; the circle group $U(1)$, regarded as complex scalars is another such subgroup. Now if $\lambda, \mu \in U(1)$ and $a, b \in \mathrm{Spin}(V,q)$, then $\lambda a = \mu b$ in $\mathbb{C}\ell(V)$ iff $(\mu, b) = (\lambda, a)$ or else $(\mu, b) = (-\lambda, -a)$; thus $\mathrm{Spin}(V,q)$ and $U(1)$ generate the subgroup

$$\mathrm{Spin}^c(V) \simeq (\mathrm{Spin}(V,q) \times U(1))/\mathbb{Z}_2,$$

where the quotient map is defined by the relation $(\lambda, a) \sim (-\lambda, -a)$. Define $\phi^c(\lambda a)v := \lambda^2 \phi(a)v$ for $\lambda a \in \mathrm{Spin}^c(V)$; then ϕ^c maps $\mathrm{Spin}^c(V)$ onto $SO(V,q) \times U(1)$ with kernel $\{\pm 1\}$, so there is another short exact sequence of Lie groups:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\phi^c} SO(V,q) \times U(1) \longrightarrow 1.$$

The Lie algebra of the spin group $\mathrm{Spin}(V,q)$ is readily identified as a subspace of the Clifford algebra $\mathrm{C}\ell(V,q)$. Indeed, let A_2^+ be the subspace of $\mathrm{C}\ell^+(V,q)$ with basis $\{1\} \cup \{\, e_i e_j : i < j \,\}$ with $\{e_1, \dots, e_n\}$ an (arbitrary) orthonormal basis of V (compare with Exercise 6.14), and let $C^2(V,q)$ be the image of $\Lambda^2 V$ under the quantization map Q.

Lemma 6.5. $C^2(V,q) = \{\, b \in A_2^+ : \bar{b} = -b \,\}$.

Proof. Since $Q(u \wedge v) = uv + q(u,v)\,1$, it follows that $C^2(V,q) \subseteq A_2^+$. Counting dimensions, $\dim C^2(V,q) = \dim \Lambda^2 V = \frac{1}{2}n(n-1) = \dim A_2^+ - 1$, so $C^2(V,q)$ is a hyperplane in A_2^+. Now

[14] Since the fundamental group of $SO(V,q)$ is $\mathbb{Z}_2$, this shows that the covering map ϕ is nontrivial and hence that $\mathrm{Spin}(V,q)$ is the universal covering group of $SO(V,q)$.

$\overline{e_ie_j} = e_je_i = -e_ie_j$ for $i < j$, so the map $b \mapsto \bar{b} + b$ is a scalar-valued $\mathbb{R}$-linear form on A_2^+, whose kernel is a hyperplane; and since

$$\overline{Q(u \wedge v)} = vu + q(u,v)\,1 = -uv - q(u,v)\,1 = -Q(u \wedge v)$$

for $u, v \in V$, it coincides with $C^2(V,q)$. $\square$

If $b \in C^2(V,q)$ and $w \in V$, then $[\![b,w]\!] = [b,w]$ lies in V also; to see that, it suffices to take $b = uv + q(u,v)\,1$, whereupon

$$\begin{aligned} [b,w] &= uvw - wuv = uvw + uwv - uwv - wuv \\ &= 2q(u,w)v - 2q(v,w)u \in V. \end{aligned} \tag{6.15}$$

The only elements of A_2^+ which commute with V are scalars, so $[b,w]$ vanishes for all $w \in V$ iff $b = 0$ in $C^2(V,q)$. Hence $\tau(b)v := [b,v]$ defines an injective $\mathbb{R}$-linear map $\tau\colon C^2(V,q) \to \operatorname{End} V$.

Proposition 6.6. *The map τ is a Lie algebra isomorphism from $C^2(V,q)$ onto the Lie algebra of antisymmetric operators $\mathfrak{so}(V,q) := \{\, A \in \operatorname{End} V : q(Au,v) \equiv -q(u,Av) \,\}$.*

Proof. First notice that $C^2(V,q)$ is a Lie algebra under the bracket $[b,d] := bd - db$ of $C\ell^+(V,q)$. Indeed, on taking $b = uv + q(u,v)\,1$, $d = wz + q(w,z)\,1$, one finds that

$$[b,d] = [uv,wz] = [uv,w]z + w[uv,z] \in A_2^+$$

using (6.15), and $\overline{[b,d]} = [\bar{d},\bar{b}] = [-d,-b] = -[b,d]$, so $[b,d] \in C^2(V,q)$. Also from (6.15),

$$q([b,w],z) = 2q(u,w)q(v,z) - 2q(v,w)q(u,z) = -q(w,[b,z]),$$

which shows that $\tau(b) \in \mathfrak{so}(V,q)$. Since $\dim \mathfrak{so}(V,q) = \frac{1}{2}n(n-1) = \dim C^2(V,q)$ and τ is injective, its image is all of $\mathfrak{so}(V,q)$. Finally, the Jacobi identity yields

$$\tau([b,d])v = [[b,d],v] = [[b,v],d] + [b,[d,v]] = -\tau(d)\tau(b)v + \tau(b)\tau(d)v,$$

so $\tau([b,d]) = [\tau(b),\tau(d)]$ where the latter bracket is the commutator bracket in $\operatorname{End} V$ (which preserves $\mathfrak{so}(V,q)$); thus, τ is a Lie algebra isomorphism. $\square$

Lemma 6.7. *If $A \in \mathfrak{so}(V,q)$ and $\{e_1,\dots,e_n\}$ is an orthonormal basis for V, then*

$$\tau^{-1}(A) = \frac{1}{2}\sum_{j<k} q(Ae_j,e_k)\,e_je_k = \frac{1}{4}\sum_{j,k=1}^{n} q(Ae_j,e_k)\,e_je_k. \tag{6.16}$$

Proof. The series in (6.16) are equal since A is antisymmetric, and they lie in $C^2(V,q)$ by Lemma 6.5; if b denotes either series, we show that $q(e_r,\tau(b)e_s) = q(e_r,Ae_s)$ for all r,s. Indeed,

$$\begin{aligned} \tau(b)e_s &= \frac{1}{2}\sum_{j<k} q(Ae_j,e_k)\,[e_je_k,e_s] = \frac{1}{2}\sum_{j<k} q(Ae_j,e_k)\,(2\delta_{js}e_k - 2\delta_{ks}e_j) \\ &= \sum_{k>s} q(Ae_s,e_k)\,e_k - \sum_{j<s} q(Ae_j,e_s)\,e_j = \sum_{k=1}^{n} q(Ae_s,e_k)\,e_k, \end{aligned}$$

by antisymmetry of A, so $q(e_r,\tau(b)e_s) = q(Ae_s,e_r) = q(e_r,Ae_s)$ as required. $\square$

The Lie algebra $C^2(V,q)$ gives rise to a *Lie group*, namely, the group of invertible elements of $\mathrm{C}\ell^+(V,q)$ generated by the elements $\exp b := 1 + \sum_{k\geq 1} b^k/k!$, for $b \in C^2(V,q)$, i.e., by the image of $C^2(V,q)$ under the exponential map in the Clifford algebra. Notice that if $a = \exp b$ and $w \in V$, then

$$\begin{aligned} awa^{-1} &= (\exp b)w(\exp(-b)) = \sum_{k,l\geq 0} \frac{1}{k!\,l!} b^k w(-b)^l \\ &= \sum_{r\geq 0} \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} b^k w(-b)^{r-k} = \sum_{r\geq 0} \frac{1}{r!}[b,[b,\dots,[b,w]\dots]] \\ &= \sum_{r\geq 0} \frac{1}{r!} \tau(b)^r w, \end{aligned} \tag{6.17}$$

so that $awa^{-1} \in V$, and hence $a \in \Gamma(V,q)$. From the relation $\bar{b} = -b$ it follows that $a\bar{a} = (\exp b)(\exp(-b)) = 1$, and so $a \in \mathrm{Spin}(V,q)$.

Exercise 6.26. If $b = uv + q(u,v)\,1$ with $u,v \in V$, show that

$$q(\tau(b)w, \tau(b)w) \leq 16\, q(u,u)q(v,v)q(w,w).$$

If Σ denotes the series on the right hand side of (6.17), verify its convergence by obtaining the estimate $q(\Sigma,\Sigma) \leq e^{8\,s(b)} q(w,w)$ where $s(b)^2 = q(u,u)q(v,v)$.

Proposition 6.8. *The exponential map* $\exp\colon C^2(V,q) \to \mathrm{Spin}(V,q)$ *is surjective.*

Proof. The equation (6.17) reads

$$\phi(\exp b) = \exp(\tau(b)), \tag{6.18}$$

where the second exponential map is the matrix exponential in $\operatorname{End} V$. Now any rotation in $SO(V,q)$ is of the form $\exp A$ for some antisymmetric operator $A \in \mathfrak{so}(V,q)$, as is seen by expressing the matrix of the rotation in canonical form. Therefore, $\{\, \phi(\exp b) : b \in C^2(V,q)\,\} = SO(V,q)$.

Also, if u,v are orthogonal unit vectors in V and if $\theta \in \mathbb{R}$, then $\theta uv \in C^2(V,q)$, and $\exp(\theta uv) = \cos\theta + \sin\theta\, uv$. In particular, $-1 = \exp(\pi uv)$ lies in $\exp(C^2(V,q))$. Now if $b = \tau^{-1}(A)$ with $A \in \mathfrak{so}(V,q)$, choose an orthonormal basis for V such that $q(Ae_j, e_k) = 0$ unless $k = j \pm 1$. From (6.16), $b = \sum_{r=1}^m b_r e_{2r-1}e_{2r}$ with $b_r = \frac{1}{2}q(Ae_{2r-1}, e_{2r})$. Thus b commutes with $\pi e_1 e_2$, and so $-\exp b = \exp(\pi e_1 e_2 + b)$. It follows that $\exp(C^2(V,q))$ is a subset of $\mathrm{Spin}(V,q)$ doubly covering $SO(V,q)$, and hence is all of $\mathrm{Spin}(V,q)$. $\square$

Thus we may regard $C^2(V,q)$ as the Lie algebra of $\mathrm{Spin}(V,q)$. It should be noted that $\mathrm{Spin}(V,q)$ and $\mathrm{Pin}(V,q)$ are *compact* Lie groups, since they are closed subsets of the "unit sphere" in $\mathrm{C}\ell(V,q)$ determined by the condition $\sum_J a_J^2 = 1$. Therefore, Proposition 6.8 exemplifies the well-known fact that the exponential mapping from the Lie algebra to a compact connected Lie group is surjective [13].

6.7 The spin representation

Proposition 6.9. *Let $S := \Lambda^\bullet_{\mathbb{C}} W$ be the spinor module for $\mathbb{C}\ell(V)$. Define $c\colon \mathrm{Pin}(V,q) \to \mathrm{End}_{\mathbb{C}} S$ by restriction of the Clifford action $c\colon \mathbb{C}\ell(V) \to \mathrm{End}_{\mathbb{C}} S$ given by (6.13). Then c is an irreducible unitary representation of the Lie group $\mathrm{Pin}(V,q)$.*

Proof. Since $c\colon \mathbb{C}\ell(V) \to \mathrm{End}_{\mathbb{C}} S$ is multiplicative, its restriction to $\mathrm{Pin}(V,q)$ is a representation of this group on S. Since the Clifford action is selfadjoint (see Exercise 6.24), we have $c(a)^\dagger = c(\bar{a}) = c(a^{-1}) = c(a)^{-1}$ for $a \in \mathrm{Pin}(V,q)$, because $a\bar{a} = 1$; thus $c(a)$ is a unitary operator on S.

The irreducibility of c follows from Schur's Lemma: we show that any $T \in \mathrm{End}_{\mathbb{C}} S$ commuting with $\{\, c(a) : a \in \mathrm{Pin}(V,q) \,\}$ is a scalar operator. Indeed, T must commute with $c(w) = \epsilon(w)$ for any $w \in W$, since $V \setminus \{0\} \subset \mathrm{Pin}(V,q)$ and $W \subset V_{\mathbb{C}}$; and T must also commute with $c(\bar{z}) = -\iota(\bar{z})$ for $z \in W$. If Ω is a unit vector in the one-dimensional space $\Lambda^0_{\mathbb{C}} W \simeq \mathbb{C}$, then $\iota(\bar{z})T\Omega = T(\iota(\bar{z})\Omega) = 0$, so $T\Omega = t\Omega \in \Lambda^0_{\mathbb{C}} W$ for some $t \in \mathbb{C}$. Now $T(w_1\wedge\cdots\wedge w_r) = T(\epsilon(w_1)\dots\epsilon(w_r)\Omega) = \epsilon(w_1)\dots\epsilon(w_r)(t\Omega) = t\, w_1\wedge\cdots\wedge w_r$, so $T = t\,\mathrm{id}_S$. □

Definition 6.20. The irreducible representation given by the previous Proposition is called the *pin representation* of the group $\mathrm{Pin}(V,q)$. Its restriction to the subgroup $\mathrm{Spin}(V,q)$ is called the **spin representation**.

Proposition 6.10. *The spin representation is not irreducible, but it is the direct sum of two inequivalent irreducible representations on the subspaces $S^+ := \Lambda^+_{\mathbb{C}} W$ and $S^- := \Lambda^-_{\mathbb{C}} W$.*

Proof. Recall Proposition 6.3: $c(\gamma)$ is the grading operator on $S = S^+ \oplus S^-$. Now the chirality element γ anticommutes with every $v \in V$, and so $\gamma a = a\gamma$ for all $a \in \mathrm{Spin}(V,q)$ by Exercise 6.24. Thus $c(\gamma)$ commutes with $\{\, c(a) : a \in \mathrm{Spin}(V,q) \,\}$, so by Schur's lemma the spin representation cannot be irreducible; in fact, it is the direct sum of two subrepresentations on the ± 1-eigenspaces of $c(\gamma)$, which are S^+ and S^-.

If $T \in \mathrm{End}_{\mathbb{C}} S^\pm$ commutes with $c(a)$ for all $a \in \mathrm{Spin}(V,q)$, then T must commute with $c(w_1)c(w_2) = \epsilon(w_1)\epsilon(w_2)$ and with $c(\bar{z}_1)c(\bar{z}_2) = \iota(\bar{z}_1)\iota(\bar{z}_2)$ for $w_1, w_2, z_1, z_2 \in W$, so T is a scalar operator, by adapting the argument of the proof of Proposition 6.9. Thus both subrepresentations are irreducible.

Notice that $\dim S^+ = \dim S^- = 2^{m-1}$, so these "half-spin representations" have the same dimension. However, they are not equivalent, since if $R \in \mathrm{Hom}(S^+, S^-)$ is an intertwining operator, then $Rc(e_{2j-1}e_{2j}) = c(e_{2j-1}e_{2j})R$ for $j = 1, \dots, m$ and hence $Rc(\gamma) = c(\gamma)R$; but then $R\alpha = Rc(\gamma)\alpha = c(\gamma)R\alpha = -R\alpha$ for all $\alpha \in S^+$, so $R = 0$; inequivalence follows from Schur's lemma. □

Exercise 6.27. Show that $\gamma \in \mathrm{Spin}^c(V)$.

The whole theory of Clifford algebras and spin representations may be extended, with a little care, to infinite-dimensional vector spaces with a symmetric bilinear form. The Clifford algebra must be complete, so its complexification is defined as the unique C^*-algebra with the desired universal property. The orthogonal group must be restricted to the subgroup of operators $g \in \mathrm{End}_R V$ whose antilinear part, with respect to a fixed complex structure, is

Hilbert–Schmidt; this is necessary to guarantee convergence of series similar to (6.16). For an exposition of the infinite-dimensional spin representation which extrapolates from the present treatment, we refer to [2]. An alternative method, wherein the spin representation is defined by permuting certain "Gaussian" elements in the Fock space, is developed in [31], based on ideas of [44].

7 Global Clifford modules

Given a compact Riemannian manifold (M, g), we can form a *Clifford bundle* $\mathbb{C}\ell(M) \longrightarrow M$ by gluing together the complex Clifford algebras $\mathbb{C}\ell(T_xM, g_x)$. The space of smooth sections $\Gamma(\mathbb{C}\ell\, M)$ is an algebra, and its completion to the space of continuous sections $\Gamma_{\mathrm{cont}}(\mathbb{C}\ell\, M)$ is canonically a C^*-algebra. Naturally, we want to consider vector bundles of representation spaces for the algebras $\mathbb{C}\ell(T_xM, g_x)$, whose section spaces are modules over the algebra $\Gamma(\mathbb{C}\ell\, M)$. However, although the Clifford algebra bundle can be built over any Riemannian manifold, topological obstructions can prevent putting together the corresponding Clifford modules. We shall restrict our attention to oriented, even-dimensional Riemannian manifolds and we shall say that (M, g) is a *spin manifold* if a Clifford module corresponding to the irreducible spinor representation exists. Such manifolds may have none, one, or several inequivalent spin structures; our first task is to sort out the possibilities.

7.1 Clifford algebra bundles

Definition 7.1. Let M be a compact manifold, and let $E \longrightarrow M$ be a real vector bundle of rank n with a Euclidean metric g. For any $x \in M$, (E_x, g_x) is a Euclidean vector space, and we can form the Clifford algebra $\mathrm{C}\ell(E_x, g_x)$. These form the fibres of a real vector bundle $\mathrm{C}\ell\, E \longrightarrow M$ of rank 2^n, which extends the vector bundle $E \longrightarrow M$ in the sense that there is an injective bundle map $j\colon E \longrightarrow \mathrm{C}\ell\, E$; since $\mathrm{C}\ell(E_x, g_x) \simeq \Lambda^\bullet E_x$ as vector spaces, just take $\mathrm{C}\ell\, E$ to be the total space of the exterior algebra bundle $\Lambda^\bullet E \longrightarrow M$, and let j be the map which identifies each $v_x \in E_x$ with its canonical image in $\Lambda^1 E_x$. The declaration $j(v_x)^2 := -g_x(v_x, v_x)$ determines, by Proposition 6.1, a bilinear product in each fibre which depends smoothly on $x \in M$, and also determines isomorphisms $(\mathrm{C}\ell\, E)_x \simeq \mathrm{C}\ell(E_x, g_x)$.

The complexification of $\mathrm{C}\ell\, E \longrightarrow M$ is $\mathbb{C}\ell\, E \longrightarrow M$, the complex Clifford algebra bundle; notice that $\mathbb{C}\ell\, E = \mathrm{C}\ell\, E_{\mathbb{C}}$.

The module of smooth sections $\Gamma(\mathbb{C}\ell\, E)$ is thus an algebra under the pointwise Clifford product $(\kappa\xi)(x) := \kappa(x)\xi(x)$. If the rank of $E \longrightarrow M$ is even, the centre of this algebra is just $\mathcal{A} = C^\infty(M)$, by identifying smooth functions on M with scalar-valued sections of the Clifford algebra bundle.[1]

[1]The continuous sections $\Gamma_{\mathrm{cont}}(\mathbb{C}\ell\, E)$ form a C^*-algebra whose centre is the algebra $C(M)$ of continuous complex-valued functions on M. This C^*-algebra is noncommutative, but in a fairly trivial way: it is "Morita-equivalent" to the commutative C^*-algebra $C(M)$. This means [46] that $\Gamma_{\mathrm{cont}}(\mathbb{C}\ell\, E) \otimes \mathcal{K} \simeq C(M) \otimes \mathcal{K}$, where $\mathcal{K}$ is the elementary C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. In particular, since $\mathcal{K}$ is a simple C^*-algebra, there is a bijective correspondence between the irreducible representations of these Morita-equivalent C^*-algebras.

Exercise 7.1. Let $Q \to M$ be the principal $O(n)$-bundle of orthonormal frames for $E \longrightarrow M$. Check that any $g \in O(n)$ extends to an automorphism $\rho(g)$ of the Clifford algebra $\mathrm{C}\ell(n)$, and that $\rho(g)\rho(g') = \rho(gg')$ for $g, g' \in O(n)$. Then show that $\mathrm{C}\ell\, E \longrightarrow M$ is the vector bundle associated to $Q \to M$ via the representation ρ of $O(n)$.

Definition 7.2. The **Clifford bundle** over a Riemannian manifold (M, g) is the bundle of Clifford algebras $\mathrm{C}\ell(M) \longrightarrow M$ obtained by taking the tangent bundle $TM \longrightarrow M$ as the generating vector bundle, with the Riemannian metric g defining the Euclidean structure on TM. It would be more consistent to write $\mathrm{C}\ell\, TM$ rather than $\mathrm{C}\ell(M)$, but this is not usually done, for the following reason. The cotangent bundle $T^*M \longrightarrow M$ is equivalent to the tangent bundle via the bundle map $\hat{g}\colon TM \to T^*M$ induced by the metric (see the discussion after Definition 3.2), and this in turn defines a Euclidean structure g^{-1} on the cotangent bundle by

$$g^{-1}(\alpha, \beta) := g(\alpha^\sharp, \beta^\sharp) \qquad \text{for} \quad \alpha, \beta \in \mathcal{A}^1(M), \tag{7.1}$$

where the vector field $\alpha^\sharp$ is determined by $g(\alpha^\sharp, Y) := \alpha(Y)$. This extends to a bundle equivalence $\hat{g}\colon \mathrm{C}\ell\, TM \to \mathrm{C}\ell\, T^*M$. Thus we may and shall regard sections of $\mathrm{C}\ell(M) \longrightarrow M$ as Clifford products of vector fields or of forms, according to momentary convenience. We shall, as before, write the scalar products in (7.1) as $(\alpha \mid \beta)$ and $(\alpha^\sharp \mid \beta^\sharp)$ respectively.

Exercise 7.2. Verify that the map $\hat{g}\colon TM \to T^*M$ extends to an equivalence of the Clifford algebra bundles $\mathrm{C}\ell\, TM$ and $\mathrm{C}\ell\, T^*M$ by applying Proposition 6.1 on the fibres.

Exercise 7.3. Let $g_{ij} := g(\partial/\partial x^i, \partial/\partial x^j)$ be the matrix entries of the metric g in local coordinates, and let $[g^{rs}]$ be the inverse matrix to $[g_{ij}]$; show that $(dx^r)^\sharp = g^{rj}\partial/\partial x^j$ and conclude that $g^{-1}(dx^r, dx^s) = g^{rs}$, thereby justifying the notation g^{-1} for the metric in T^*M.

In view of Proposition 5.2, the sections of the Clifford algebra bundle may be constructed by an alternative method. Starting from a Euclidean vector bundle $E \longrightarrow M$, one can define the tensor bundle $\mathcal{T}(E) \longrightarrow M$ as the (infinite) Whitney sum of the tensor product bundles $E^{\otimes k} \longrightarrow M$, where $E^{\otimes 0} = M \times \mathbb{R}$, the trivial rank-one bundle. The ideals $I(g_x)$ of $\mathcal{T}(E_x)$ glue together to give a bundle of ideals $I(g) \longrightarrow M$, and $\mathrm{C}\ell\, E \to M$ may be defined as the quotient bundle $\mathcal{T}(E)/I(g) \longrightarrow M$. The module of sections $\Gamma(I(g))$ is an $\mathcal{A}$-submodule of $\Gamma(\mathcal{T}(E))$, and the quotient $\mathcal{A}$-module may be identified with $\Gamma(\mathrm{C}\ell\, E)$. Note, in particular, that $\Gamma(I(g))$ is generated as an $\mathcal{A}$-module by the elements $s \otimes s + g(s, s)$, for $s \in \Gamma(E)$.

Proposition 7.1. *Any connection ∇ on a Euclidean vector bundle $E \longrightarrow M$ which is compatible with the metric on E extends to a connection, also denoted by ∇, on $\mathrm{C}\ell\, E \longrightarrow M$, such that*

$$\nabla_X(\kappa\xi) = (\nabla_X\kappa)\xi + \kappa(\nabla_X\xi) \tag{7.2}$$

for $X \in \mathfrak{X}(M)$, $\kappa, \xi \in \Gamma(\mathrm{C}\ell\, E)$; that is, each ∇_X is a derivation on the algebra $\Gamma(\mathrm{C}\ell\, E)$.

Proof. Compatibility of ∇ with the metric is expressed by (5.7), which is equivalent to the condition that $(\nabla_X s \,|\, t) + (s \,|\, \nabla_X t) = X(s \,|\, t)$ for $s, t \in \Gamma(E)$, $X \in \mathfrak{X}(M)$. The operators ∇_X

on $\Gamma(E)$ extend to $\Gamma(\mathcal{T}(E))$ by setting $\nabla_X(s_1 \otimes \cdots \otimes s_k) := \sum_{j=1}^k s_1 \otimes \cdots \otimes \nabla_X s_j \otimes \cdots \otimes s_k$; it is clear that $X \mapsto \nabla_X(s_1 \otimes \cdots \otimes s_k)$ is a connection on the bundle $E^{\otimes k} \longrightarrow M$ for each k. Moreover, each $\nabla_X : \Gamma(\mathcal{T}(E)) \to \Gamma(\mathcal{T}(E))$ is an $\mathcal{A}$-linear derivation, i.e., it satisfies the Leibniz rule

$$\nabla_X(\sigma \otimes \tau) = (\nabla_X \sigma) \otimes \tau + \sigma \otimes (\nabla_X \tau). \tag{7.3}$$

On applying the extended ∇ to a generator of the module of ideals $\Gamma(I(g))$, we get

$$\begin{aligned}\nabla_X[s \otimes s + (s \mid s)] = (s + \nabla_X s) \otimes (s + \nabla_X s) + (s + \nabla_X s \mid s + \nabla_X s) \\ - s \otimes s - (s \mid s) - \nabla_X s \otimes \nabla_X s - (\nabla_X s \mid \nabla_X s).\end{aligned}$$

Therefore, ∇_X preserves $\Gamma(I(g))$, and so drops to the quotient algebra $\Gamma(\mathrm{C}\ell\, E)$. The derivation property (7.3) survives on passing to the quotient, and (7.2) follows. □

One may eliminate the vector field X in the usual way, by noting that both sides of (7.2) are $\mathcal{A}$-linear in X, and by extending the Clifford product in $\Gamma(\mathrm{C}\ell\, E)$ to an $\mathcal{A}$-bilinear form from $\Gamma(\mathrm{C}\ell\, E) \times \mathcal{A}^1(M, \mathrm{C}\ell\, E)$ to $\mathcal{A}^1(M, \mathrm{C}\ell\, E)$. Thus (7.2) may be rewritten as

$$\nabla(\kappa\xi) = (\nabla\kappa)\xi + \kappa(\nabla\xi) \qquad \text{for} \quad \kappa, \xi \in \Gamma(\mathrm{C}\ell\, E), \tag{7.4}$$

which is an equation in $\mathcal{A}^1(M, \mathrm{C}\ell\, E)$.

Definition 7.3. The **Levi-Civita connection** on the Clifford bundle over M is the derivation $\nabla : \Gamma(\mathrm{C}\ell(M)) \to \mathcal{A}^1(M, \mathrm{C}\ell(M))$ which extends the Levi-Civita connection on the tangent bundle (or equivalently, its dual connection on the cotangent bundle). Whenever convenient, we shall denote any of these three connections by ∇^{LC}.

Definition 7.4. A **Clifford module** over a Riemannian manifold M is a module of sections $\Gamma(F)$ of a superbundle $F \longrightarrow M$, together with a *Clifford action* of the sections of the complexified Clifford bundle $\Gamma(\mathbb{C}\ell(M))$. Such an action is defined as an $\mathcal{A}$-bilinear map $\Gamma(\mathbb{C}\ell(M)) \times \Gamma(F) \to \Gamma(F)$, written $(\kappa, s) \mapsto c(\kappa)s$, such that $c(1)s = s$ and $c(\kappa)c(\xi)s = c(\kappa\xi)s$ for all $s \in \Gamma(F)$ and $\kappa, \xi \in \Gamma(\mathbb{C}\ell(M))$, or else as a homomorphism $c : \Gamma(\mathbb{C}\ell(M)) \to \Gamma(\operatorname{End} F)$, which is *even* in the sense that $c : \Gamma(\mathbb{C}\ell^{\pm}(M)) \to \Gamma(\operatorname{End}^{\pm} F)$.

The evenness of the action is equivalent to the condition that for any $\alpha \in \mathcal{A}^1(M)$, the operator $c(\alpha)$ interchanges $\Gamma(F^+)$ and $\Gamma(F^-)$.

If $F \longrightarrow M$ is a Hermitian vector bundle, we say that the Clifford action is *selfadjoint* if $c(\kappa)^\dagger = c(\bar\kappa)$ for $\kappa \in \mathrm{C}\ell(M)$, or equivalently, if $c(\alpha)^\dagger = -c(\alpha)$ for any (real-valued) $\alpha \in \mathcal{A}^1(M)$. More precisely, this condition reads

$$(c(\alpha)s \mid t) + (s \mid c(\alpha)t) = 0,$$

for $s, t \in \Gamma(F)$, using the scalar product obtained from the hermitian structure on F.

Obvious examples of Clifford modules are $\Gamma(\mathbb{C}\ell(M))$ itself, with the left multiplication $c_l(\kappa)\xi := \kappa\xi$; and $\Gamma(\mathbb{C}\ell(M))$ itself, with conjugated right multiplication $c_r(\kappa)\xi := \xi\bar\kappa$. Another example is the module of complex-valued exterior forms $\mathcal{A}^\bullet(M, \mathbb{C})$, with the Clifford action obtained by applying the recipe (6.8) to each fibre:

$$c(\alpha)\omega := \alpha \wedge \omega - \iota(\alpha^\sharp)\omega.$$

However, these examples are "too large", since the Clifford actions on the fibres are reducible, and it is to be expected that the global module should also be decomposable into a direct sum of several submodules. Therefore we seek *irreducible* Clifford modules, where each fibre is linearly isomorphic to the Fock space $S = \Lambda^\bullet W$ of subsection 6.5 (provided $n = \dim M$ is even). In that case, after choosing an oriented orthonormal basis for $T^*_x M$, the spin representation of the group $\mathrm{Spin}(n) := \mathrm{Spin}(\mathbb{R}^n, q_n)$ yields a homomorphism $c\colon \mathrm{Spin}(n) \to GL_{\mathbb{C}}(F_x)$; all such homomorphisms form a *principal* $\mathrm{Spin}(n)$*-bundle* associated to the vector bundle $F \longrightarrow M$ via the spin representation. There is, however, a topological obstruction to the existence of such a principal bundle. Before we can proceed, we must identify this obstruction, in order to know whether irreducible Clifford modules are available at all.

7.2 Existence of spin structures

Definition 7.5. A real vector bundle $E \longrightarrow M$ is *orientable* if its transition functions $g_{ij}\colon U_i \cap U_j \to GL(r, \mathbb{R})$ can be chosen to satisfy $\det g_{ij} > 0$ for all i, j. A Euclidean vector bundle has transition functions with values in the orthogonal group $O(r)$; thus it is orientable iff one can choose the g_{ij} to satisfy $\det g_{ij} = +1$. (This may be expressed by saying that the structure group of E can be reduced from $O(r)$ to $SO(r)$.)

To see whether a Euclidean vector bundle is orientable or not, take a good covering $\mathfrak{U} = \{U_j\}$ for M and choose transition functions g_{ij} for $\mathfrak{U}$. Then $\det g_{ij}\colon U_i \cap U_j \to \{\pm 1\} = \mathbb{Z}_2$, and since $g_{ij} g_{jk} = g_{ik}$ on each nonempty $U_i \cap U_j \cap U_k$, we conclude that $\det \boldsymbol{g}$ is a Čech 1-cocycle[2] in $C^1(\mathfrak{U}, \mathbb{Z}_2)$. Any set of transition functions is of the form $g'_{ij} = f_i g_{ij} f_j^{-1}$ with $f_j\colon U_j \to O(r)$, as follows from the proof of Proposition 1.5. (If a local system of sections $\boldsymbol{s}_j$ for E transforms as $\boldsymbol{s}_i = g_{ij} \boldsymbol{s}_j$, and an equivalent system of local sections is given by $\boldsymbol{s}'_j := f_j \boldsymbol{s}_j$, the new transition functions satisfy $g'_{ij} f_j = f_i g_{ij}$.) Hence $\det \boldsymbol{f} \in C^0(\mathfrak{U}, \mathbb{Z}_2)$ and $\det \boldsymbol{g}' = \det \boldsymbol{g} + \delta(\det \boldsymbol{f})$. Thus the class $w_1(E) := [\det \boldsymbol{g}] \in \check{H}^1(M, \mathbb{Z}_2)$ depends only on the equivalence class $[E]$ of the vector bundle. It is usually called the **first Stiefel–Whitney class** of E [23, 39]. It is also customary to write $w_1(M) := w_1(TM)$ when (M, g) is a Riemannian manifold. We may summarize the situation as follows.

Lemma 7.2. *A Euclidean vector bundle $E \longrightarrow M$ is orientable iff $w_1(E) = 0$ in $\check{H}^1(M, \mathbb{Z}_2)$. A Riemannian manifold (M, g) is orientable iff $w_1(M) = 0$.*

Proof. The condition $w_1(E) = 0$ means that transition functions g'_{ij} can be chosen to satisfy $\det g'_{ij} = +1$ identically. □

Suppose now that $E \longrightarrow M$ is an *oriented* Euclidean vector bundle (i.e., that $w_1(E) = 0$), with transition functions $g_{ij}\colon U_i \cap U_j \to SO(r)$ for a good covering $\mathfrak{U}$ of M. Since each $U_i \cap U_j$ is contractible, one can lift these maps to the double covering $\mathrm{Spin}(r)$ of $SO(r)$, i.e., we can find smooth functions $h_{ij}\colon U_i \cap U_j \to \mathrm{Spin}(r)$ such that $\phi(h_{ij}) = g_{ij}$ for all i, j. The relation $\phi(h_{ij} h_{jk} h_{ik}^{-1}) = g_{ij} g_{jk} g_{ik}^{-1} = 1$ in $SO(r)$ shows that

$$a_{ijk} := h_{ij} h_{jk} h_{ik}^{-1} = \pm 1$$

[2] We identify $\{\pm 1\}$ with $\mathbb{Z}_2$, in order to use additive notation when combining Čech cocycles.

in Spin(r). Thus $\boldsymbol{a}$ is a Čech 2-cochain in $C^2(\mathfrak{U}, \mathbb{Z}_2)$; indeed, $\boldsymbol{a}$ is a Čech 2-cocycle, as a short calculation shows.

Exercise 7.4. Compute $\delta\boldsymbol{a}$, to check that it is trivial in $C^3(\mathfrak{U}, \mathbb{Z}_2)$.

We can make random changes of signs of some of the h_{ij}, i.e., we can let $h'_{ij} := b_{ij} h_{ij}$ where $b_{ij} \colon U_i \cap U_j \to \{\pm 1\}$ are arbitrary (but constant) sign functions. Then $\boldsymbol{b} \in C^1(\mathfrak{U}, \mathbb{Z}_2)$, and $a'_{ijk} := h'_{ij} h'_{jk} (h'_{ik})^{-1}$ yields $\boldsymbol{a}' \in C^2(\mathfrak{U}, \mathbb{Z}_2)$ with $\boldsymbol{a}' = \boldsymbol{a} + \delta\boldsymbol{b}$. Therefore, the class $w_2(E) := [\boldsymbol{a}] \in \check{H}^2(\mathfrak{U}, \mathbb{Z}_2)$ depends only on the equivalence class $[E]$ of the vector bundle; it is called the **second Stiefel–Whitney class** of E. One writes $w_2(M) := w_2(TM)$ when (M, g) is an oriented Riemannian manifold.

Definition 7.6. A **spin structure** on an oriented Riemannian manifold (M, g) of dimension n is a principal Spin(n)-bundle $P \xrightarrow{\eta} M$ together with a bundle map $\tau \colon P \to Q$ where $Q \xrightarrow{\theta} M$ is the principal $SO(n)$-bundle of oriented orthonormal frames for the tangent bundle $TM \longrightarrow M$, such that $\tau(p \cdot g) = \tau(p) \circ \phi(g)$ for $p \in P$, $g \in \mathrm{Spin}(n)$. We say that M is a **spin manifold** if it admits at least one spin structure.

By Lemma 1.4, such a principal Spin(n)-bundle may be assembled from transition functions $h_{ij} \colon U_i \cap U_j \to \mathrm{Spin}(n)$ satisfying $\phi(h_{ij}) = g_{ij}$, where the g_{ij} are the transition functions of the frame bundle, provided only that $h_{ij} h_{jk} = h_{ik}$ on each nonempty $U_i \cap U_j \cap U_k$. Thus M admits a spin structure iff $w_2(M) = 0$ in $\check{H}^2(M, \mathbb{Z}_2)$.

Since $\tau \colon P \to Q$ reproduces the double covering $\phi \colon \mathrm{Spin}(n) \to SO(n)$ on each fibre, the map τ is two-to-one.

Inequivalent spin structures may arise from different choices of h_{ij} covering the same g_{ij}. The only freedom here comes from the sign changes $b_{ij} = \pm 1$ mentioned above, where $h'_{ij} = b_{ij} h_{ij}$ determine another spin structure on M. Since $h_{ij} h_{jk} = h_{ik}$ and similarly for the h'_{ij}, it follows that $b_{ij} b_{jk} = b_{ik}$ also; thus $\boldsymbol{b}$ is a Čech 1-cocycle. Now if $\boldsymbol{b} = \delta\boldsymbol{c}$ with $\boldsymbol{c} \in C^0(\mathfrak{U}, \mathbb{Z}_2)$, i.e., if $b_{ij} = c_i / c_j$ with all $c_j = \pm 1$, then $h'_{ij} = (c_i / c_j) h_{ij}$ as functions from $U_i \cap U_j$ to the group Spin(n), so the corresponding principal Spin(n)-bundles are equivalent. Conversely, a principal bundle equivalence yields relations $h'_{ij} = (a_i / a_j) h_{ij}$ for some smooth functions $a_j \colon U_j \to \mathrm{Spin}(n)$ satisfying $a_i = \pm a_j$ on overlaps; thus, signs c_j can be chosen so that $c_i a_i = c_j a_j$ on overlaps, and consequently $h'_{ij} = (c_i / c_j) h_{ij}$. In summary, the two spin structures are equivalent iff $\boldsymbol{b} = \delta\boldsymbol{c}$ for some $\boldsymbol{c}$, which gives the following result.

Lemma 7.3. *If M is a Riemannian manifold with $w_1(M) = 0$ and $w_2(M) = 0$, the inequivalent spin structures on M are classified by the Čech cohomology group $\check{H}^1(M, \mathbb{Z}_2)$.* $\square$

The computation of the Stiefel–Whitney classes for particular manifolds involves either a good deal of combinatorial calculation (see [28], for instance) or general theorems from topology [23, 39, 41]. One such general theorem is particularly useful: if $E \longrightarrow M$ is a *complex* vector bundle and $E_{\mathbb{R}} \longrightarrow M$ is the underlying real vector bundle, then $w_1(E_{\mathbb{R}}) = 0$ and $w_2(E_{\mathbb{R}})$ is the image of the first Chern class $c_1(E)$ under the canonical homomorphism from $\check{H}^2(M, \mathbb{Z})$ to $\check{H}^2(M, \mathbb{Z}_2)$ obtained from the standard homomorphism $j \colon \mathbb{Z} \to \mathbb{Z}_2$ (namely, reduction modulo 2).[3] That $w_1(E_\mathbb{R}) = 0$ should be no surprise, because a complex vector bundle is always orientable. To see that, notice firstly that the unitary group $U(r)$ can be regarded as a subgroup of $SO(2r)$, since $U(r)$ permutes orthonormal bases in $\mathbb{C}^r$, and any such basis $\{e_1, \dots, e_r\}$ yields an orthonormal basis $\{e_1, f_1, \dots, e_r, f_r\}$ for $\mathbb{R}^{2r}$, by setting $f_r := ie_r$; and since $U(r)$ is connected, it is therefore contained[4] in the identity component $SO(2r)$ of the orthogonal group of $\mathbb{R}^{2r}$. Now, with respect to any Hermitian metric on E, a local orthonormal basis of sections for $\Gamma(U, E)$ gives a local orthonormal basis of sections for $\Gamma(U, E_\mathbb{R})$, and the $U(r)$-valued transition functions of E may thus be regarded as $SO(2r)$-valued transition functions of $E_\mathbb{R}$.

Since the first Chern class $c_1(H)$ of the hyperplane bundle $H \to \mathbb{CP}^m$ is a generator of $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}) \simeq \mathbb{Z}$, so that $c_1(H) \leftrightarrow 1$ under this group isomorphism, its modulo-2 reduction is not zero: therefore, $w_2(H_\mathbb{R}) \neq 0$ in $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)$. Now it can be shown [28] that the tangent bundle $T\mathbb{CP}^m$ has the following property: if $E_r \longrightarrow \mathbb{CP}^m$ denotes the trivial real bundle of rank r, then $T\mathbb{CP}^m \oplus E_2$ and $H_\mathbb{R} \oplus \cdots \oplus H_\mathbb{R}$ (with $(m+1)$ summands) are equivalent real bundles over $\mathbb{CP}^m$. These facts suffice to decide which of the $\mathbb{CP}^m$ are spin manifolds.

Proposition 7.4. *The complex projective space $\mathbb{CP}^m$ is a spin manifold iff m is odd; and for odd m, the spin structure on $\mathbb{CP}^m$ is unique.*

Proof. The isomorphism $\mathrm{C}\ell(V, q) \mathbin{\overline{\otimes}} \mathrm{C}\ell(W, r) \simeq \mathrm{C}\ell(V \oplus W, q \oplus r)$ of Proposition 6.2, restricted to the spin subgroups, shows that $\mathrm{Spin}(V, q)$ and $\mathrm{Spin}(W, r)$ may be regarded as commuting subgroups of $\mathrm{Spin}(V \oplus W, q \oplus r)$. Thus we may embed the direct product $\mathrm{Spin}(k) \times \mathrm{Spin}(l)$ as a subgroup of $\mathrm{Spin}(k+l)$; and ϕ maps this to the usual embedding of $SO(k) \times SO(l)$ in $SO(k+l)$. Thus, whenever $E \longrightarrow M$ and $F \longrightarrow M$ are oriented Euclidean vector bundles, the transition functions $g_{ij} \oplus g'_{ij}$ of $E \oplus F$ take values in $SO(k) \times SO(l)$, and lift to $h_{ij} \oplus h'_{ij}$ in $\mathrm{Spin}(k) \times \mathrm{Spin}(l)$. The second Stiefel–Whitney class therefore satisfies[5] the relation $w_2(E \oplus F) = w_2(E) + w_2(F)$.

When $M = \mathbb{CP}^m$, we thereby obtain

$$w_2(\mathbb{CP}^m) + 0 = w_2(T\mathbb{CP}^m \oplus E_2) = w_2(H_\mathbb{R} \oplus \cdots \oplus H_\mathbb{R}) = (m+1)\, w_2(H_\mathbb{R}). \tag{7.5}$$

Any nontrivial element of the group $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)$ is of order two,[6] and $w_2(H_\mathbb{R}) \neq 0$, so the right hand side of (7.5) vanishes iff m is odd.

To get uniqueness, we must show that $\check{H}^1(\mathbb{CP}^m, \mathbb{Z}_2) = 0$. This follows from the Bockstein homomorphism construction of subsection 1.11, applied to the exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_2 \longrightarrow 0,$$

[3] See Appendix B of [39] for a proof.

[4] If one chooses an orthonormal basis in $\mathbb{C}^r$ for which $g \in U(r)$ is diagonal, i.e., $g(e_j) = e^{i\alpha_j} e_j$, then $g(e_j) = \cos\alpha_j\, e_j + \sin\alpha_j\, f_j$, $g(f_j) = -\sin\alpha_j\, e_j + \cos\alpha_j\, f_j$, so the image of g in $SO(2r)$ is a direct sum of 2×2 rotation blocks.

[5] When E and F are not orientable, this additivity breaks down; there is a "product formula" of Whitney [41] which yields the relation $w_2(E \oplus F) = w_2(E) + w_1(E)w_1(F) + w_2(F)$.

[6] Indeed, $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)$ is a *vector space* over the field $\mathbb{Z}_2$.

where '2' denotes multiplication by two, and $j\colon \mathbb{Z} \to \mathbb{Z}_2$ is reduction modulo 2. The Bockstein construction yields a long exact sequence in Čech cohomology

$$\cdots \to \check{H}^1(M, \mathbb{Z}_2) \xrightarrow{\partial} \check{H}^2(M, \mathbb{Z}) \xrightarrow{2_*} \check{H}^2(M, \mathbb{Z}) \xrightarrow{j_*} \check{H}^2(M, \mathbb{Z}_2) \xrightarrow{\partial} \check{H}^3(M, \mathbb{Z}) \to \cdots$$

for any compact manifold M. Let us chase this diagram backwards from $\check{H}^3(M, \mathbb{Z})$ in the case $M = \mathbb{CP}^m$. We know that $\check{H}^3(\mathbb{CP}^m, \mathbb{Z}) = 0$ from (2.5), so j_* is surjective.[7] We know also that $\check{H}^2(\mathbb{CP}^m, \mathbb{Z}) = \mathbb{Z}$ (these are the Chern classes found in subsection 5.10), so the image of 2_*, which equals the kernel of j_*, is the subgroup $2\mathbb{Z}$ of "even" Chern classes, and 2_* is just multiplication by 2, which is injective. This forces $\partial\colon \check{H}^1(\mathbb{CP}^m, \mathbb{Z}_2) \to \check{H}^2(\mathbb{CP}^m, \mathbb{Z})$ to be the zero homomorphism, and thus $j_*\colon \check{H}^1(\mathbb{CP}^m, \mathbb{Z}) \to \check{H}^1(\mathbb{CP}^m, \mathbb{Z}_2)$ is onto; but $\check{H}^1(\mathbb{CP}^m, \mathbb{Z}) = 0$, again by (2.5), so we conclude that $\check{H}^1(\mathbb{CP}^m, \mathbb{Z}_2) = 0$. □

7.3 Spin^c structures

What can be done about manifolds like $\mathbb{CP}^2$, which are complex manifolds but do not admit a spin structure? Since our immediate aim is to describe an irreducible module for the *complex* Clifford bundle over such manifolds, we could relax our requirements slightly by replacing the required structure group of the tangent bundle by $\mathrm{Spin}^c(n)$ rather than $\mathrm{Spin}(n)$. (Recall that $\mathrm{Spin}^c(n)$ is the product of the group $\mathrm{Spin}(n)$ and the unitary scalars $U(1)$ within the complex Clifford algebra $\mathbb{C}\ell(V)$.) We say that a (compact, oriented, Riemannian) manifold M admits a *spin^c structure* if there is a principal $\mathrm{Spin}^c(n)$-bundle $P^c \xrightarrow{\eta} M$ and a bundle map $\tau\colon P^c \to Q^c$, satisfying $\tau(p \cdot g) = \tau(p) \circ \phi^c(g)$ for $p \in P$ and $g \in \mathrm{Spin}^c(n)$; here Q^c is a principal $SO(n) \times U(1)$-bundle of the form $Q \times R \longrightarrow M$, where[8] Q is the $SO(n)$-bundle of oriented orthonormal frames of the tangent bundle, as before, and $R \longrightarrow M$ is a principal $U(1)$-bundle which we may choose as we please.

Now a principal $U(1)$-bundle is just the frame bundle of a *complex line bundle* $L \to M$, so (up to equivalence) these are classified by $\check{H}^2(M, \mathbb{Z})$. The modulo-2 reduction $j_*[L]$ is an element of $\check{H}^2(M, \mathbb{Z}_2)$, where the second Stiefel-Whitney class also lives. Suppose that $w_2(M) = j_*[L^*]$ in $H^2(M, \mathbb{Z}_2)$, i.e., that $w_2(M) + j_*[L] = 0$; then one can find local sections $\tilde{h}_{ij}\colon U_i \cap U_j \to \mathrm{Spin}^c(n)$ such that $\phi^c(\tilde{h}_{ij}) = \lambda_{ij} g_{ij}$ are the transition functions of $TM \oplus L \longrightarrow M$, which patch together properly to give the required principal bundle $P^c \longrightarrow M$.

Exercise 7.5. Assume that $w_2(M) + j_*[L] = 0$ and construct $P^c \longrightarrow M$ as indicated.

If a spin^c structure exists, M is called a *spin^c manifold*. The foregoing argument says that this is the case iff $w_2(M) \in j_*(\check{H}^2(M, \mathbb{Z}))$. Moreover, inequivalent spin^c structures are parametrized by the classes of complex line bundles, i.e., by $\check{H}^2(M, \mathbb{Z})$.

[7] Actually, the formula (2.5) refers to singular or de Rham cohomology; but we can establish an isomorphism between de Rham and Čech cohomology in degree 3 by a simple modification of the proof of Proposition 5.5. For the isomorphism between the de Rham–Čech cohomologies in any degree, see [12, 17].

[8] The fibrewise product of principal bundles corresponds, by association, to the Whitney sum of vector bundles; thus, if Q and R are the frame bundles of vector bundles E and F over M, with respective structure groups G and H, then $Q \times R \longrightarrow M$ may be defined as the frame bundle of $E \oplus F \longrightarrow M$, which is a principal $G \times H$-bundle.

Lemma 7.5. *The complex projective space $\mathbb{CP}^m$ is a spin^c manifold for any positive integer m.*

Proof. In the proof of Proposition 7.4, we verified the surjectivity of the homomorphism $j_*\colon \check{H}^2(\mathbb{CP}^m, \mathbb{Z}) \to \check{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)$, without reference to the parity of m. $\square$

In fact, any compact complex manifold carries a spin^c structure. This follows from the relation $w_2(M) = j_*(c_1(T_{\mathrm{hol}}M))$, where $T_{\mathrm{hol}}M \longrightarrow M$ is the holomorphic tangent bundle, i.e., the complex vector bundle whose fibres $T_{x,\mathrm{hol}}M$ are spanned by the tangent vectors $(\partial/\partial z^1)_x, \dots, (\partial/\partial z^m)_x$. To get a more concrete construction, we first regard $U(m)$ as a subgroup of $SO(2m)$ and consider the homomorphisms $\tau\colon U(m) \to SO(2m) \times U(1)$ given by $\tau(g) := (g, \det g)$, and $\sigma\colon U(m) \to \mathrm{Spin}^c(2m)$ defined as follows. For each $g \in U(m)$ there is an orthonormal basis $\{e_1, \dots, e_m\}$ of $\mathbb{C}^m$ which diagonalizes g, i.e., $g(e_j) = e^{i\alpha_j} e_j$; write $f_j := ie_j$ and $a_j := e^{i\alpha_j/2}(\cos \frac{1}{2}\alpha_j + (\sin \frac{1}{2}\alpha_j) e_j f_j) \in \mathbb{C}\ell^+(2m)$; then $\sigma(g) := a_1 a_2 \dots a_m \in \mathrm{Spin}^c(2m)$.

Exercise 7.6. Check that all the a_j commute,[9] that σ is a well-defined homomorphism, and that $\phi^c \circ \sigma = \tau$.

Exercise 7.7. The linear map on $\mathbb{R}^{2m}$ determined by $Je_j := f_j$, $Jf_j := -e_j$ is a complex structure; and $U(m) = \{\, g \in SO(2m) : gJ = Jg \,\}$. Write $k := e_1 f_1 + \dots + e_m f_m \in \mathbb{C}\ell^+(2m)$ and check that $\exp(\pi k) = (-1)^m$ and $\phi(\exp(\frac{\pi}{4} k)) = J$. Define the *metaunitary group* $MU(m)$ as $\{\, a \in \mathrm{Spin}(2m) : ak = ka \,\}$ and verify that $\phi(MU(m)) = U(m)$. Show that the centre of the group $MU(m)$ equals $\{\, \exp(\theta k) : -\pi < \theta \le \pi \,\} \simeq U(1)$ if m is odd, whereas the centre is $\{\, \pm \exp(\theta k) : 0 \le \theta < \pi \,\} \simeq U(1) \times \mathbb{Z}_2$ if m is even.

Now we can exhibit a spin^c structure on a complex manifold M. Take any Hermitian metric on M and let $Q' \longrightarrow M$ be the unitary frame bundle of the holomorphic tangent bundle $T_{\mathrm{hol}}M \longrightarrow M$. Associate to it, firstly, a principal $\mathrm{Spin}^c(2m)$-bundle $P \longrightarrow M$ via the homomorphism σ, as in (1.3); and secondly, a principal $U(1)$-bundle $R \longrightarrow M$ via the homomorphism $\det\colon U(m) \to U(1)$. The latter is just the frame bundle of the complex line bundle $K^* \longrightarrow M$ where $K \longrightarrow M$ is the so-called *canonical line bundle* on M, defined by $\Gamma(K) := \mathcal{A}^{m,0}(M)$. Since $\phi^c \circ \sigma = \tau$, the principal bundle $P \longrightarrow M$ yields the required spin^c structure.

Of course, a spin manifold is automatically a spin^c manifold. It suffices to use the trivial $U(1)$ bundle $R = M \times U(1)$ to build a spin^c structure from a given spin structure.

Exercise 7.8. Complete the following construction of the spin structure for $\mathbb{CP}^m$, with m odd, due to Dabrowski and Trautman [21]. Show that $\mathbb{CP}^m$ is diffeomorphic to the homogeneous space $U(m+1)/(U(1) \times U(m))$ and that the unitary frame bundle is given by $Q' := U(m+$

[9] For a *fixed* orthonormal basis $\{e_1, \dots, e_m\}$ of $\mathbb{C}^m$, the elements $a_1, \dots, a_m$ and the scalars $e^{i\theta}1$ generate a subgroup T of $\mathrm{Spin}^c(2m)$ which is isomorphic to an $(m+1)$-dimensional torus $\mathbb{T}^{m+1}$; moreover, since $\phi^c(a_1 a_2 \dots a_m)$ is a block diagonal matrix over $\mathbb{R}^{2m}$, the centralizer of T in $\mathrm{Spin}^c(2m)$ is T itself, so it is a *maximal torus.* If $b \in \mathrm{Spin}^c(2m)$, then by pulling back $\phi^c(b)$ to $U(m)$ via τ and diagonalizing the resulting unitary matrix, one sees that b lies in some conjugate subgroup aTa^{-1}. This exemplifies the well-known theorem of Weyl [13] that the maximal tori in a compact connected Lie group are conjugate and cover the whole group.

$1)/U(1)$. Let $MU(m)$ be the metaunitary group of Exercise 7.7 and let $P' := MU(m+1)/U(1)$, where the subgroup $U(1)$ of $MU(m+1)$ is given as $\{\exp(\theta k) : 0 \leq \theta < \pi\}$, the identity component of the centre. Check that the inclusion $MU(m) \subset MU(m+1)$ drops to a free right action of $MU(m)$ on P', and that the double covering $\phi\colon MU(m+1) \to U(m+1)$ drops to a double covering $\tau'\colon P' \to Q'$, which intertwines the respective actions of $MU(m)$ and $U(m)$. Finally, show how the inclusions $MU(m) \hookrightarrow \mathrm{Spin}(2m)$ and $U(m) \hookrightarrow SO(2m)$ associate to these new principal bundles P and Q together with a double covering $\tau\colon P \to Q$ which yields the desired spin structure.

Catalogues of spin manifolds and spin^c manifolds are given in several places, e.g., [29, 39]. As well as the classes of manifolds just discussed, it is worth mentioning that any sphere $\mathbb{S}^n$ is spin; any simply connected Lie group is spin; any orientable 2-dimensional manifold is spin (e.g., any Riemann surface); any 3-dimensional manifold is spin; and any orientable 4-dimensional manifold is spin^c. As counterexamples, we note that $\mathbb{CP}^2$ is an orientable 4-dimensional manifold which is not spin; and the homogeneous space $SU(3)/SO(3)$ is an orientable 5-dimensional manifold which is not spin^c.

7.4 The spinor module

Definition 7.7. Let M be a spin manifold of even dimension $n = 2m$, and let $P \longrightarrow M$ be the principal $\mathrm{Spin}(n)$-bundle defining its spin structure.[10] Let $S = \Lambda^\bullet_{\mathbb{C}} W$ be an irreducible Clifford module for $\mathbb{C}\ell(n)$, i.e., a Fock space of complex dimension 2^m, and let $c\colon \mathrm{Spin}(n) \to \mathrm{End}_{\mathbb{C}} S$ denote the spin representation. Let $S(M) \longrightarrow M$ be the complex vector bundle associated to the spin structure P via the spin representation c. It is called the **spinor bundle** over M. Its module of sections $\Gamma(S(M))$ is an *irreducible* Clifford module, and will be called the **spinor module** for the algebra $\Gamma(\mathbb{C}\ell(M))$.

Proposition 7.6. *If M is an even-dimensional spin manifold, any Clifford module $\Gamma(F)$ is of the form $\Gamma(W \otimes S(M))$, where $\Gamma(W)$ is a trivial Clifford module, i.e., the Clifford action is $(\kappa, w) \mapsto w$ for $\kappa \in \Gamma(\mathbb{C}\ell(M))$, $w \in \Gamma(W)$.*

Proof. Let $\Gamma(F)$ be given. If $\mathcal{A} = C^\infty(M)$, any $\mathcal{A}$-linear map from $\Gamma(S(M))$ to $\Gamma(F)$ is of the form τ_* where $\tau\colon S(M) \to F$ is a bundle map, i.e., τ_* is an element of $\Gamma(\mathrm{Hom}(S(M), F))$. Thus any such map which intertwines the two Clifford actions belongs to $\Gamma(W)$, where $W := \mathrm{Hom}_{\mathbb{C}\ell(M)}(S(M), F)$ is the vector bundle over M with fibres $W_x = \mathrm{Hom}_{\mathbb{C}\ell(n)}(S(M)_x, F_x)$.

From Exercise 6.17 it follows that $w_x \otimes \sigma_x \mapsto w_x(\sigma_x)$ gives an vector space isomorphism $W_x \otimes S(M)_x \simeq F_x$. (If $w_x(\sigma_x) = 0$ for any nonzero σ_x, then $0 = c(a)[w_x(\sigma_x)] = w_x(c(a)\sigma_x)$ for all a, so $w_x = 0$ by irreducibility of $S(M)_x$.) The intertwining property $w_x(c(a)\sigma_x) = c(a)[w_x(\sigma_x)]$ shows that $W_x \otimes S(M)_x$ becomes a $\mathbb{C}\ell(n)$-module via the recipe $c(a)[w_x \otimes \sigma_x] := w_x \otimes c(a)\sigma_x$, and the aforementioned isomorphism intertwines this action with the given action on F_x. Globally, we obtain an invertible bundle map from $W \otimes S(M)$ to F and hence an $\mathcal{A}$-linear isomorphism from $\Gamma(W \otimes S(M))$ to $\Gamma(F)$ which intertwines the action

$$c(\kappa)[w \otimes \sigma] := w \otimes c(\kappa)\sigma \tag{7.6}$$

[10] If M carries more than one spin structure, we choose and fix a particular one.

with the given Clifford action on $\Gamma(F)$. $\square$

Exercise 7.9. Prove that $\Gamma(\operatorname{End}_{\mathbb{C}\ell(M)} F) \simeq \Gamma(\operatorname{End}_{\mathbb{C}} W)$; in other words, match $\mathcal{A}$-linear operators commuting with the Clifford action on $\Gamma(F)$ to $\mathcal{A}$-linear operators on $\Gamma(W)$.

Definition 7.8. The passage from the irreducible spinor module $\Gamma(S(M))$ to a Clifford module of the form $\Gamma(W \otimes S(M))$ whose Clifford action is given by (7.6) is called a **twisting** by the vector bundle $W \longrightarrow M$ (which in principle may be any complex vector bundle). We refer to $\Gamma(W \otimes S(M))$ as a **twisted Clifford module**.

Notice, in particular, that any Clifford module $\Gamma(F)$ where $F \longrightarrow M$ has minimal rank 2^m is obtained by twisting the spinor module with a complex line bundle. Clearly, then, the twisting may affect the global topology of F, but for algebraic properties of the Clifford action it is enough to study the spinor module.

7.5 The spin connection

The next task is to show that the spinor module admits a distinguished connection, to be called the "spin connection", which satisfies a "Leibniz rule" with respect to the Clifford action. This is not a trivial matter, as the spinor module is not "tensorial", that is, its existence has topological obstructions not faced by modules of tensors, vector fields, or differential forms. Therefore the algebraic techniques used in Section 5 to construct new connections from old ones are not enough to produce a suitable connection on the spinor bundle.

What is needed is to use the spin representation. This we have defined, in Section 6, as a group representation of $\operatorname{Spin}(V, q)$ as unitary operators on S which does not drop to a representation of the special orthogonal group $SO(V, q)$. It may, however, be regarded as a *projective* (or "double-valued") representation of $SO(V, q)$. The corresponding *infinitesimal* representations of Lie algebras are not troubled by this topological problem,[11] and there is a linear isomorphism $\tau\colon C^2(V, q) \to \mathfrak{so}(V, q)$ between the Lie algebras.

Definition 7.9. Let $\dot{\mu}\colon \mathfrak{so}(V, q) \to \operatorname{End}^+_{\mathbb{C}}(S)$ be the linear map given by

$$\dot{\mu}(A) := c(\tau^{-1}(A)), \tag{7.7}$$

Note that $\dot{\mu}(A)$ is skewhermitian since the Clifford action is selfadjoint, so $c(b)^\dagger = c(\bar{b}) = -c(b)$ for $b \in C^2(V, q)$ by Lemma 6.5. We call $\dot{\mu}$ the **derived spin representation**.

Lemma 7.7. *The derived spin representation satisfies the identity*

$$[\dot{\mu}(A), c(v)] = c(Av) \tag{7.8}$$

for all $A \in \mathfrak{so}(V, q)$, $v \in V$.

[11] This is because V is finite-dimensional. In the infinite-dimensional case, the covering map ϕ has a 1-dimensional kernel (a circle), which gives rise to a "spin anomaly" in the infinitesimal representation of $SO(V, q)$. For details, consult [31].

Proof. It is equivalent to show that $[\tau^{-1}(A), v] = Av$ as elements of the Clifford algebra $\mathbb{C}\ell(V,q)$. However, this follows immediately from the formula of Lemma 6.7 for $\tau^{-1}(A)$, and from (6.15):

$$\begin{aligned}[]
[\tau^{-1}(A), v] &= \frac{1}{4} \sum_{j,k=1}^{n} q(Ae_j, e_k)\,[e_j e_k, v] \\
&= \frac{1}{2} \sum_{j,k=1}^{n} q(Ae_j, e_k)\,\big(q(e_j, v)e_k - q(e_k, v)e_j\big) \\
&= \frac{1}{2} \sum_{j=1}^{n} q(e_j, v)Ae_j + \frac{1}{2} \sum_{k=1}^{n} q(e_k, v)Ae_k = Av,
\end{aligned}$$

on using the antisymmetry of A. $\square$

It would be helpful to have a version of (7.8) which is valid when vectors $v \in V$ are replaced by arbitrary elements $b \in \mathbb{C}\ell(V,q)$. To accomplish this, one needs to extend the operator A on V to a linear operator on $\mathbb{C}\ell(V,q)$. In fact, A extends as a *derivation* of the Clifford algebra: define $\hat{A}(1) := 0$, $\hat{A}(v) := Av$ for $v \in V$, and

$$\hat{A}(v_1 \dots v_k) := \sum_{j=1}^{k} v_1 \dots v_{j-1}(Av_j)v_{j+1} \dots v_k \tag{7.9}$$

for $v_1, \dots, v_k \in V$.

Exercise 7.10. Check that the definition (7.9) is consistent by first verifying that $\hat{A}(uv) + \hat{A}(vu) = 0$ for $u, v \in V$. Then show that (7.8) extends to the identity $[\dot{\mu}(A), c(b)] = c(\hat{A}b)$ for all $A \in \mathfrak{so}(V,q)$, $b \in \mathbb{C}\ell(V,q)$.

Now we can translate these algebraic identities to relations among sections of bundles over a spin manifold M, by applying them at each fibre. Thus, if $A \in \Gamma(\operatorname{End} E)$ where $E \longrightarrow M$ is a Euclidean vector bundle, and if $(As \mid t) = -(s \mid At)$ for $s, t \in \Gamma(E)$, then $A(x) \in \mathfrak{so}(E_x)$ for $x \in M$, so we may write $A \in \Gamma(\mathfrak{so}(E))$. In particular, if $E = TM$ is the tangent bundle, the notation $A \in \Gamma(\mathfrak{so}(TM))$ means that A is an operator on $\Gamma(TM) = \mathfrak{X}(M)$ satisfying $g(AX, Y) = -g(X, AY)$ for $X, Y \in \mathfrak{X}(M)$. We write $\dot{\mu}(A)$ to denote the section $x \mapsto \dot{\mu}(A(x))$ of the spinor bundle $S(M) \longrightarrow M$. This defines a map $\dot{\mu}\colon \Gamma(\mathfrak{so}(TM)) \to \Gamma(S(M))$. Furthermore, by tensoring these $\mathcal{A}$-modules of sections with $\mathcal{A}^k(M)$, we obtain maps $\dot{\mu}\colon \mathcal{A}^k(M, \mathfrak{so}(TM)) \to \mathcal{A}^k(M, S(M))$.

If ∇ is a connection on the tangent bundle compatible with the metric, such as the Levi-Civita connection for instance, then, locally at least, $\nabla = d + \alpha$ where $\alpha \in \mathcal{A}^1(U, \mathfrak{so}(TM))$ in view of the metric compatibility condition (5.7). Thus $\dot{\mu}(\alpha)$ makes sense as an element of $\mathcal{A}^1(U, S(M))$, where U is the domain of α.

Theorem 7.8. *Let M be an even-dimensional spin manifold and let ∇ be the Levi-Civita connection on the complex Clifford bundle $\mathbb{C}\ell(M) \longrightarrow M$. Then there is a connection ∇^S on* *the spinor bundle $S(M) \longrightarrow M$ satisfying the following Leibniz rule:*

$$\nabla^S(c(\kappa)\sigma) = c(\nabla\kappa)\sigma + c(\kappa)(\nabla^S\sigma), \tag{7.10}$$

for all $\kappa \in \Gamma(\mathbb{C}\ell(M))$, $\sigma \in \Gamma(S(M))$.

Proof. Let $\mathcal{U} = \{U_j\}$ be a covering of M by chart domains and let $\{f_j\}$ be a smooth partition of unity subordinate to $\mathcal{U}$, with $f_j(x) > 0$ iff $x \in U_j$. Then $f_j\nabla$ is a connection on U_j compatible with the metric $f_j g$. Suppose for the moment that there exist connections ∇^S_j on each restriction of the spinor bundle $S(M) \longrightarrow M$ to U_j, satisfying $\nabla^S_j(c(\kappa)\sigma) = c(f_j\nabla\kappa)\sigma + c(\kappa)(\nabla^S_j\sigma)$, whenever σ vanishes outside U_j; then the recipe $\nabla^S\tau := \sum_j \nabla^S_j(f_j\tau)$, for any $\tau \in \Gamma(S(M))$, defines a connection satisfying (7.10).

Therefore, we may replace M by any chart domain U_j; or we may equivalently suppose that the vector bundles $\mathbb{C}\ell(M) \longrightarrow M$ and $S(M) \longrightarrow M$ are trivial, and that the Levi-Civita connection may be written as $\nabla = d + \alpha$ with $\alpha \in \mathcal{A}^1(M, \mathfrak{so}(TM))$. The formula $\nabla = d + \alpha$, defined initially on vector fields, is also applicable to Clifford products of vector fields, provided $\alpha\kappa$ is interpreted as the *derivation action* of α on $\kappa \in \Gamma(\mathbb{C}\ell(M))$, in view of the Leibniz rule (7.4).

Now introduce a connection ∇^S on $S(M)$ by *defining*

$$\nabla^S := d + \dot\mu(\alpha). \tag{7.11}$$

From (7.8), extended to Clifford algebra, we get $[\dot\mu(\alpha), c(\kappa)] = c(\alpha\kappa)$ on $\Gamma(S(M))$, and then

$$\begin{aligned}
\nabla^S(c(\kappa)\sigma) &= d(c(\kappa)\sigma) + \dot\mu(\alpha)(c(\kappa)\sigma) \\
&= c(d\kappa)\sigma + c(\kappa)\,d\sigma + [\dot\mu(\alpha), c(\kappa)]\sigma + c(\kappa)\dot\mu(\alpha)\sigma \\
&= c(d\kappa)\sigma + c(\alpha\kappa)\sigma + c(\kappa)\big(d\sigma + \dot\mu(\alpha)\sigma\big) \\
&= c(\nabla\kappa)\sigma + c(\kappa)(\nabla^S\sigma),
\end{aligned}$$

verifying (7.10). □

We have constructed one solution ∇^S to (7.10). If $\widetilde\nabla^S$ is another connection on $S(M)$ satisfying the same module-derivation property, then $\widetilde\nabla^S - \nabla^S$ is given by $\beta \in \mathcal{A}^1(M, \operatorname{End} S(M))$ such that $\beta(c(\kappa)\sigma) \equiv c(\kappa)(\beta\sigma)$. That is to say, β lies in $\mathcal{A}^1(M, \operatorname{End}_{\mathbb{C}\ell(M)} S(M)) \simeq \mathcal{A}^1(M)$ in view of Exercise 7.9 and the irreducibility of $S(M)$. Therefore, ∇^S is unique up to addition of a scalar action by a 1-form on M.

We may apply the same reasoning to any Clifford module, with the following result.

Proposition 7.9. *Let $\Gamma(F) = \Gamma(W \otimes S(M))$ be a Clifford module over a spin manifold M, and let $\widetilde\nabla$ be a connection on F which is a module derivation, i.e., $\widetilde\nabla(c(\kappa)\tau) \equiv c(\nabla\kappa)\tau + c(\kappa)(\widetilde\nabla\tau)$. Then there is a unique connection ∇^W on W such that $\nabla^W \otimes 1 + 1 \otimes \nabla^S = \widetilde\nabla$.*

Proof. Since $\tau \in \Gamma(F)$ is a finite sum of the form $\sum_j w_j \otimes \sigma_j$, we may rewrite the module derivation property of $\widetilde\nabla$ as $\widetilde\nabla(w \otimes c(\kappa)\sigma) = w \otimes c(\nabla\kappa)\sigma + c(\kappa)(\widetilde\nabla(w \otimes \sigma))$. Let ∇^W_0 be an arbitrary connection on W. Then we also get

$$[\nabla^W_0 \otimes 1 + 1 \otimes \nabla^S, 1 \otimes c(\kappa)](w \otimes \sigma) = w \otimes c(\nabla\kappa)\sigma,$$

so $\beta := \widetilde{\nabla} - (\nabla_0^W \otimes 1 + 1 \otimes \nabla^S)$ lies in $\mathcal{A}^1(M, \operatorname{End} F)$ and commutes with the Clifford action, that is, $\beta \in \mathcal{A}^1(M, \operatorname{End}_{\mathbb{C}\ell(M)} F)$. By Exercise 7.9 it is of the form $\beta = \alpha \otimes 1$ for a unique $\alpha \in \mathcal{A}^1(M, \operatorname{End} W)$. Writing $\nabla^W := \nabla_0^W + \alpha$ now gives $\widetilde{\nabla} = \nabla^W \otimes 1 + 1 \otimes \nabla^S$, as desired. □

Lemma 7.10. *The curvature of the spin connection is $\dot{\mu}(R)$, where R denotes the Riemannian curvature, $R \in \mathcal{A}^2(M, \mathfrak{so}(TM))$.*

Proof. It suffices to check this on a chart domain U, where we may write $\nabla^S = d + \dot{\mu}(\alpha)$. The curvature form of the spin connection is given by (5.10) as $\omega^S := \dot{\mu}(d\alpha + \alpha \wedge \alpha) = \dot{\mu}(R)$. □

7.6 Local coordinate formulas

Definition 7.10. On a chart domain U of M with local coordinates $\{x^1, \dots, x^n\}$, we shall write the basic vector fields as $\partial_j \equiv \partial/\partial x^j$. If M is Riemannian, the **Christoffel symbols** Γ^k_{ij} of the Levi-Civita connection on the chart domain U are the functions in $C^\infty(U)$ defined by

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k, \tag{7.12}$$

or equivalently, $\nabla \partial_j = \Gamma^k_{ij}\, dx^i \otimes \partial_k$. In particular, $\Gamma^k_{ij} = dx^k\big(\nabla_{\partial_i} \partial_j\big)$. Notice also that $\Gamma^k_{\bullet j}$ give the matrix components α^k_j of $\alpha \in \mathcal{A}^1(U, \operatorname{End} TM)$.

Exercise 7.11. Check that the Levi-Civita connection on the cotangent bundle is determined locally by $\nabla\, dx^k = -\Gamma^k_{ij}\, dx^i \otimes dx^j$, or equivalently, $\nabla_{\partial_i} dx^k = -\Gamma^k_{ij}\, dx^j$.

Exercise 7.12. From the definition (5.27) of the Levi-Civita connection, show that

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \tag{7.13}$$

using the relations $[\partial_i, \partial_j] = 0$.

Exercise 7.13. Show that the Christoffel symbols on the sphere $\mathbb{S}^2$ are given in spherical coordinates by

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^k_{ij} = 0 \text{ otherwise}, \tag{7.14}$$

by applying (7.13) to the metric $g := d\theta^2 + \sin^2\theta\, d\phi^2$.

Exercise 7.14. The components of the Riemann curvature tensor may be written [52] as $R^i_{jkl} := dx^i(R(\partial_k, \partial_l)\partial_j)$. Verify the "Ricci identities":

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm},$$

by using the curvature formula (5.11).

It is somewhat more common to use the components $R_{ijkl} = g_{im} R^m_{jkl} = (\partial_i \mid R(\partial_k, \partial_l \partial_j))$. One sees from (5.11) that, for k, l fixed, the matrix with (i,j)-entry R_{ijkl} is antisymmetric.

Exercise 7.15. Verify that the curvature of the spin connection is given locally by

$$\omega^S(\partial_k, \partial_l) = -\tfrac{1}{4} R_{ijkl}\, c(dx^i)\, c(dx^j), \tag{7.15}$$

on account of (7.7).

We now express the spin connection itself in local coordinates. Fix a chart domain $U \subset M$, and write $g = g_{ij}\, dx^i \cdot dx^j$ there. Since the matrix $G = [g_{ij}]$ is positive definite, we can find a matrix $H = [h^\alpha_j]$ of functions in $C^\infty(U)$ such that $H^t H = G$. Indeed, if $G^{1/2}$ is the positive definite square root of G, we may take $H = AG^{1/2}$ where A is an orthogonal matrix at each point of U with $A \colon U \to SO(n)$ smooth.[12] Choose and fix such an H, and let $H^{-1} = [\tilde{h}^r_\beta]$ be its inverse matrix. Recall that $g^{-1} = g^{rs}\, \partial_r \cdot \partial_s$ is the metric on the cotangent bundle $T^*M \longrightarrow M$; thus we have

$$h^\alpha_i \delta_{\alpha\beta} h^\beta_j = g_{ij}, \qquad \tilde{h}^i_\alpha \delta^{\alpha\beta} \tilde{h}^j_\beta = g^{ij}.$$

Orthonormal bases for $\mathcal{A}^1(U)$ and $\mathfrak{X}(U)$ are then given by

$$\theta^\alpha := h^\alpha_j\, dx^j, \qquad E_\beta := \tilde{h}^r_\beta \partial_r.$$

(We use the convention of reserving Latin indices for coordinate bases and Greek indices for orthonormal bases.) By construction, we get

$$g(E_\alpha, E_\beta) = \delta_{\alpha\beta}, \qquad g^{-1}(\theta^\alpha, \theta^\beta) = \delta^{\alpha\beta},$$

and also $(\theta^\alpha)^\sharp = E_\alpha$, $(E_\beta)^\flat = \theta^\beta$ on U.[13]

Locally, a smooth section of the spinor bundle looks like a smooth map from an open subset U of M (which can actually be taken as the complement of the closure of an arbitrarily small open set in M [19]) to the Hilbert space S of the spinor representation. Let $\{\, \gamma^\alpha \equiv \gamma_\alpha : \alpha = 1, \dots, n \,\}$ be a fixed set of unitary skewadjoint operators on S with the property

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2\, \delta^{\alpha\beta}.$$

[For instance, take $\gamma^\alpha := c(e_\alpha)$ where $\{e_1, \dots, e_n\}$ is an orthonormal basis for (V, q).] We set

$$c(dx^r) := \tilde{h}^r_\beta \gamma^\beta. \tag{7.16}$$

From (7.7) it is immediate that

$$c(dx^r)c(dx^s) + c(dx^s)c(dx^r) = -2g^{rs} = -2(dx^r \mid dx^s), \tag{7.17}$$

so that (7.16) in fact defines a local Clifford action of $\mathcal{A}^1(U)$ on $\Gamma(U, S)$. Conversely, a *given* Clifford action on the spinor bundle, which necessarily satisfies (7.17), together with a given fixed set of γ^α, defines via (7.16) matrices $[\tilde{h}^r_\beta]$ and $[h^\alpha_j]$ satisfying (7.7).[14]

It is convenient to introduce "mixed" or "orthogonal" Christoffel symbols $\widetilde{\Gamma}^\beta_{i\alpha}$, $\widehat{\Gamma}^\beta_{\epsilon\alpha}$ in $C^\infty(U)$ by

$$\nabla_{\partial_i} E_\alpha =: \widetilde{\Gamma}^\beta_{i\alpha} E_\beta; \qquad \nabla_{E_\epsilon} E_\alpha =: \widehat{\Gamma}^\beta_{\epsilon\alpha} E_\beta.$$

[12] We require $\det A = 1$ so as to preserve the orientation when passing to orthonormal bases in (7.7).

[13] These last relations give rise to the common phrasing: "we need not worry about raising and lowering indices, so long as we deal with orthonormal bases".

[14] The possibility of varying the Clifford action by premultiplying H by arbitrary $SO(n)$-valued functions, provided only that these be compatible with changes of local charts, reflects the noncanonical nature of the spinor bundle.

The $\widetilde{\Gamma}^\beta_{i\alpha}$ and $\widehat{\Gamma}^\beta_{\epsilon\alpha}$ are antisymmetric in the indices α, β; for instance,

$$\widetilde{\Gamma}^\beta_{i\alpha} + \widetilde{\Gamma}^\alpha_{i\beta} = g(\nabla_{\partial_i} E_\alpha, E_\beta) + g(E_\alpha, \nabla_{\partial_i} E_\beta) = \partial_i(\delta_{\alpha\beta}) = 0,$$

on account of the compatibility with the metric: $\alpha \in \mathcal{A}^1(U, \mathfrak{so}(TM))$.

The components of the spin connection are given by the $\operatorname{End} S$-valued functions

$$\omega_i := \tfrac{1}{4}\widetilde{\Gamma}^\beta_{i\alpha}\,\gamma^\alpha\gamma_\beta, \qquad \nu_\epsilon := \tfrac{1}{4}\widehat{\Gamma}^\beta_{\epsilon\alpha}\,\gamma^\alpha\gamma_\beta = \tfrac{1}{4}\tilde{h}^i_\epsilon\widetilde{\Gamma}^\beta_{i\alpha}\,\gamma^\alpha\gamma_\beta. \tag{7.18}$$

Exercise 7.16. Show that $h^\alpha_j\widetilde{\Gamma}^\beta_{i\alpha} = -\partial_i(h^\beta_j) + \Gamma^k_{ij}h^\beta_k$. Deduce the formula

$$\omega_i = \tfrac{1}{4}\big(\Gamma^k_{ij}g_{kl} - \partial_i(h^\alpha_j)\delta_{\alpha\beta}h^\beta_l\big)\, c(dx^j)\, c(dx^l),$$

that expresses the spin connection in a coordinate basis.

Exercise 7.17. Let $X = a^j\partial_j$ be the local expression of a vector field on M. Show that the contraction of X with the spin connection is given locally as an operator on $C^\infty(U, S)$ by:

$$\nabla^S_X\sigma = a^j(\partial_j\sigma + \omega_j)\sigma,$$

and, in particular, that $\nabla^S_{E_\epsilon}\sigma = E_\epsilon\sigma + \nu_\epsilon(x)\sigma$.

Exercise 7.18. Check that ∇^S is a *hermitian* connection on the spinor module.

8 Dirac operators and Laplacians

A Dirac operator is an operator of odd parity on a Clifford module over a spin (or spin^c) manifold, which is a first-order differential operator whose corresponding Leibniz rule involves Clifford multiplication by differentials of functions. Its square is therefore an even-parity second-order differential operator, which gives a far-reaching generalization of the Laplace-Beltrami operator on a Riemannian manifold: we may call this square a "generalized Laplacian", using the terminology of [9]. Moreover, any Dirac operator over a compact manifold is elliptic, which implies that its inverse (off its finite-dimensional kernel) is a compact operator, so it has a discrete spectrum of eigenvalues which give precise information about the geometry of the manifold. Even more importantly, as Connes [18] has pointed out, the Dirac operator determines the Riemannian metric, and therefore serves as a gateway for reformulating the entire corpus of Riemannian geometry in analytic terms, allowing its extension to discrete spaces, fractals, spaces of tilings and group orbits, and many other contexts whose geometric structure seemed only a few short years ago to be hopelessly beyond reach.

8.1 Connections and differential forms

Definition 8.1. Let ∇ be a connection on the cotangent bundle $T^*M \longrightarrow M$ of a compact manifold M; the Leibniz rule $\nabla(\xi \wedge \omega) = (\nabla\xi) \wedge \omega + \xi \wedge (\nabla\omega)$ extends it to a connection on the exterior product bundle $\Lambda^\bullet T^*M \longrightarrow M$, so that ∇ maps $\mathcal{A}^\bullet(M)$ to $\mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^\bullet(M)$.[1] The exterior product defines an $\mathcal{A}$-linear map $\hat\epsilon\colon \mathcal{A}^1(M) \otimes_{\mathcal{A}} \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ by $\hat\epsilon(\omega \otimes \eta) := \omega \wedge \eta$. The composition $\hat\epsilon \circ \nabla$ is an $\mathbb{R}$-linear endomorphism of $\mathcal{A}^\bullet(M)$. In local coordinates $\nabla\omega = dx^j \otimes \nabla_{\partial_j}\omega$, and so $\hat\epsilon(\nabla\omega) = \epsilon(dx^j)\,\nabla_{\partial_j}\omega$ where $\epsilon(dx^j)$ denotes exterior multiplication by dx^j.

We introduce also a contraction map $\hat\iota\colon \mathfrak{X}(M) \otimes_{\mathcal{A}} \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ by $\hat\iota(X \otimes \eta) := \iota_X\eta$. If (M, g) is a Riemannian manifold, we identify $\mathfrak{X}(M)$ with $\mathcal{A}^1(M)$ via the metric g, and write $\hat\iota(\alpha \otimes \eta) := \iota_{\alpha^\sharp}\eta$. The composed map $\hat\iota \circ \nabla$ is another $\mathbb{R}$-linear endomorphism of $\mathcal{A}^\bullet(M)$.

Lemma 8.1. *Let ∇ be a connection on the cotangent bundle $T^*M \longrightarrow M$; and let $T \in \mathcal{A}^2(M, TM)$ be the torsion of the dual connection ∇^* on the tangent bundle. Then for any $\omega \in \mathcal{A}^\bullet(M)$,*

$$\hat\epsilon(\nabla\omega) = d\omega - \iota_T\omega, \tag{8.1}$$

where $\iota_T\colon \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ denotes contraction with the torsion.

Proof. Write $\nabla\omega = \beta^k \otimes \eta_k \in \mathcal{A}^1(M) \otimes \mathcal{A}^\bullet(M)$; then $\nabla_X\omega = \beta^k(X)\,\eta_k$ by contraction. Therefore, if $X, Y \in \mathfrak{X}(M)$ and $\omega \in \mathcal{A}^1(M)$,

$$\begin{aligned}
\hat\epsilon(\nabla\omega)(X, Y) &= (\beta^k \wedge \eta_k)(X, Y) = \beta^k(X)\eta_k(Y) - \beta^k(Y)\eta_k(X) \\
&= (\nabla_X\omega)(Y) - (\nabla_Y\omega)(X) \\
&= X(\omega(Y)) - Y(\omega(X)) - \omega(\nabla^*_X Y - \nabla^*_Y X) \\
&= d\omega(X, Y) - \omega(\nabla^*_X Y - \nabla^*_Y X - [X, Y]) \\
&= d\omega(X, Y) - \omega(T(X, Y)).
\end{aligned} \tag{8.2}$$

This establishes (8.1) for 1-forms. Moreover, $\iota_Y\iota_X(\hat\epsilon(\nabla\omega)) = \iota_Y\iota_X(d\omega) - \iota_{T(X,Y)}\omega$, for a form $\omega \in \mathcal{A}^k(M)$ of any degree, by an easy extension of the above argument. ☐

Exercise 8.1. Generalize (8.2) to higher-degree forms.

Corollary 8.2. *If ∇ is a connection on the cotangent bundle $T^*M \longrightarrow M$ whose dual connection is torsion-free, then $\hat\epsilon \circ \nabla$ equals the exterior derivation, d.* ☐

In particular, $\hat\epsilon \circ \nabla^{LC} = d$ for the Levi-Civita connection ∇^{LC} (on the cotangent bundle).

There is a kind of dual formula relating the Levi-Civita connection with the codifferential δ. Before revealing that, it is useful to consider the divergence of a vector field on a Riemannian manifold.

[1] We find it more convenient in this Section to write $\mathcal{A}^r(M, E) = \mathcal{A}^r(M) \otimes_{\mathcal{A}} \Gamma(E)$, reversing the order of the tensor product given in (5.3). The Leibniz rule (5.4) is adapted accordingly, and the contraction operator $\iota_X : \mathcal{A}^1(M, E) \to \Gamma(E)$ is now given by $\iota_X(\beta \otimes s) := \beta(X)\,s$.

8.2 Divergence of a vector field

Definition 8.2. Let X be a vector field on an oriented manifold M with volume form ν. The **divergence** of X relative to ν is the function $\operatorname{div}_\nu X \in C^\infty(M)$ determined by

$$(\operatorname{div}_\nu X)\nu := \mathcal{L}_X \nu.$$

If $f \in C^\infty(M)$ is never zero, $f\nu$ is another volume form with $(\operatorname{div}_{f\nu} X)\, f\nu = \mathcal{L}_X(f\nu) = (Xf)\nu + f\mathcal{L}_X\nu = (Xf + f \operatorname{div}_\nu X)\nu$, so that

$$\operatorname{div}_{f\nu} X = \operatorname{div}_\nu X + \frac{Xf}{f}. \tag{8.3}$$

If (M, g) is a Riemannian manifold with Riemannian volume form Ω, we write $\operatorname{div} X \equiv \operatorname{div}_\Omega X$.

Since $\mathcal{L}_X\Omega = d(\iota_X\Omega)$ by Cartan's identity, the identity $\int_M (\operatorname{div} X)\Omega = 0$ (the *divergence theorem*) is an immediate consequence of Stokes' theorem.

Over a chart domain U, we may consider the local volume form $\mu = dx^1 \wedge \cdots \wedge dx^n$. If $X = X^j\,\partial_j \in \mathfrak{X}(U)$, then by computing $\mathcal{L}_X(dx^1 \wedge \cdots \wedge dx^n)$ we obtain the standard formula $\operatorname{div}_\mu X = \partial_j X^j$. Since $\Omega = \sqrt{\det g}\,\mu$ on U by (3.3), we derive from (8.3) a local formula for the Riemannian divergence:

$$\operatorname{div} X = \partial_j X^j + X^j\,\partial_j(\ln\sqrt{\det g}) = \frac{1}{\sqrt{\det g}}\,\partial_j(X^j\sqrt{\det g}).$$

Lemma 8.3. *The Riemannian divergence of a vector field is obtained from the Levi-Civita connection on the tangent bundle via the local formula*

$$\operatorname{div} X = dx^j(\nabla^{LC}_{\partial_j} X). \tag{8.4}$$

Proof. The connection ∇^{LC} is determined over a chart domain U by the Christoffel symbols Γ^k_{ij} of (7.12), satisfying $\Gamma^k_{ij} = dx^k\big(\nabla^{LC}_{\partial_i}\partial_j\big)$. The Leibniz rule now yields, for $X = X^j\,\partial_j \in \mathfrak{X}(U)$,

$$dx^j(\nabla^{LC}_{\partial_j} X) = \partial_j X^j + \Gamma^j_{jk} X^k.$$

The verification of (8.4) thus reduces to checking the formula

$$\Gamma^j_{jk} = \partial_k(\ln\sqrt{\det g}).$$

This relation is well known [36], but it is instructive to see how it goes. Recall that the Christoffel symbols are given by (7.13):

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

If $[G^{ij}]$ denotes the adjugate matrix to $[g_{ij}]$ (that is, the matrix of cofactors), then by Cramer's rule $g^{ij} = G^{ij}/\det g$. Now $g^{jl}(\partial_j g_{kl} - \partial_l g_{kj}) = 0$ since the matrix $[g_{ij}]$ is symmetric, and so

$$\begin{aligned} \Gamma^j_{jk} &= \tfrac{1}{2} g^{jl}\,\partial_k g_{jl} = \frac{1}{2\det g} G^{jl}\,\partial_k g_{jl} = \frac{1}{2\det g}\frac{\partial \det g}{\partial g_{jl}}\,\partial_k g_{jl} \\ &= \frac{1}{2\det g}\partial_k(\det g) = \tfrac{1}{2}\partial_k(\ln\det g), \end{aligned}$$

on using the Cramer expansion $\det g = g_{jl}G^{jl}$. $\square$

If we choose local orthonormal bases $\{\theta^\alpha\}$ for $\mathcal{A}^1(U)$ and $\{E_\beta\}$ for $\mathfrak{X}(U)$ with $E_\alpha = (\theta^\alpha)^\sharp$, we may write, as in (7.7), $\theta^\alpha = h^\alpha_j\,dx^j$ and $E_\beta = \tilde{h}^k_\beta \partial_k$. Then $\theta^\alpha(\nabla^{LC}_{E_\alpha} X) = h^\alpha_j\,dx^j(\tilde{h}^k_\alpha \nabla^{LC}_{\partial_k} X)$, so we obtain the alternative formula to (8.4), using local orthonormal bases:

$$\operatorname{div} X = \theta^\alpha(\nabla^{LC}_{E_\alpha} X). \tag{8.5}$$

Yet another divergence formula, for a particular set of vector fields, is the following.

Lemma 8.4. *Let M be an oriented Riemannian manifold. Given $\zeta \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{k-1}(M)$, define $Z \in \mathfrak{X}(M)$ by $\omega(Z) := (\zeta \mid \omega \wedge \eta)$ for $\omega \in \mathcal{A}^1(M)$. The divergence of this vector field is given locally by*

$$\operatorname{div} Z = E_\alpha(\zeta \mid \theta^\alpha \wedge \eta),$$

where $\{\theta^1, \dots, \theta^n\}$ and $\{E_1, \dots, E_n\}$ are local orthonormal bases of 1-forms and vector fields respectively, with $E_\alpha = (\theta^\alpha)^\sharp$.

Proof. Since $\omega \mapsto (\zeta \,|\, \omega \wedge \eta)$ is $\mathcal{A}$-linear on $\mathcal{A}^1(M)$, it indeed defines a vector field $Z \in \mathfrak{X}(M)$. Since $\Omega = \theta^1 \wedge \cdots \wedge \theta^n$ locally, we find that $\iota_Z\Omega = \sum_{\alpha=1}^n (-1)^{\alpha-1}(\zeta \mid \theta^\alpha \wedge \eta)\, \theta^1 \wedge \cdot\overset{\alpha}{\vee}\cdot \wedge \theta^n$, and it follows that $\mathcal{L}_Z\Omega = d(\iota_Z\Omega) = E_\alpha(\zeta \mid \theta^\alpha \wedge \eta)\,\Omega$. $\square$

8.3 The Hodge–Dirac operator revisited

Lemma 8.5. *If ∇^{LC} denotes the Levi-Civita connection on the cotangent bundle of an oriented Riemannian manifold M, and $\delta \colon \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ is the codifferential, then $\hat{\iota} \circ \nabla^{LC} = -\delta$.*

Proof. Choose $\zeta \in \mathcal{A}^k(M)$ and $\eta \in \mathcal{A}^{k-1}(M)$ with $k \geq 1$, and define $Z \in \mathfrak{X}(M)$ by $\omega(Z) := (\zeta \mid \omega \wedge \eta)$. Choose local orthonormal bases $\{\theta^1, \dots, \theta^n\}$ and $\{E_1, \dots, E_n\}$ of 1-forms and vector fields, with $E^\alpha \equiv E_\alpha = (\theta^\alpha)^\sharp$. Since $(\xi \mid \beta \wedge \omega) = (\iota_{\beta^\sharp}\xi \mid \omega)$ for any $\xi, \omega \in \mathcal{A}^\bullet(M)$ and $\beta \in \mathcal{A}^1(M)$, we obtain

$$\begin{aligned} (\zeta \mid d\eta) &= (\zeta \mid \hat{\epsilon}(\nabla^{LC}\eta)) = (\zeta \mid \epsilon(\theta^\alpha)\nabla^{LC}_{E_\alpha}\eta) \\ &= (\iota_{E^\alpha}\zeta \mid \nabla^{LC}_{E_\alpha}\eta) = E_\alpha(\iota_{E^\alpha}\zeta \mid \eta) - (\nabla^{LC}_{E_\alpha}\iota_{E^\alpha}\zeta \mid \eta) \\ &= E_\alpha(\zeta \mid \theta^\alpha \wedge \eta) - (\iota_{E^\alpha}\nabla^{LC}_{E_\alpha}\zeta \mid \eta) \\ &= \operatorname{div} Z - (\hat{\iota}(\nabla^{LC}\zeta) \mid \eta), \end{aligned}$$

where we have used Corollary 8.2, the metric compatibility of ∇^{LC}, and the commutation relation $[\nabla_X, \iota_Y] = \iota_{[X,Y]}$ (see Exercise 5.21). On multiplying both sides by Ω and integrating, the term $\operatorname{div} Z$ is killed by Stokes' theorem, and we obtain $\langle\!\langle \delta\zeta \,|\, \eta \rangle\!\rangle := \langle\!\langle \zeta \,|\, d\eta \rangle\!\rangle = -\langle\!\langle \hat\iota(\nabla^{LC}\zeta) \,|\, \eta \rangle\!\rangle$; since η is arbitrary, this gives $\hat\iota(\nabla^{LC}\zeta) = -\delta\zeta$. □

Corollary 8.6. *If $\hat c\colon \Gamma(\mathbb{C}\ell(M)) \otimes \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ denotes the Clifford action on the de Rham algebra of an oriented Riemannian manifold, and if ∇^{LC} denotes the Levi-Civita connection, then*

$$\hat c \circ \nabla^{LC} = d + \delta. \tag{8.6}$$

Proof. If $\nabla\omega = \beta^k \otimes \eta_k$, then $\hat c(\nabla\omega) = c(\beta^k)\eta_k = \epsilon(\beta^k)\eta_k - \iota_{(\beta^k)^\sharp}\eta_k$, so that $\hat c = \hat\epsilon - \hat\iota$. Thus (8.6) follows immediately from Corollary 8.2 and Lemma 8.5. □

In subsection 4.5, we referred to $d+\delta$ as the "Hodge–Dirac operator", which we obtained, in a somewhat ad-hoc fashion, as a square root of the Hodge Laplacian. The formula (8.6) reveals its true nature as the composition of a Clifford action and a connection which is compatible with this action (recall Proposition 7.1). This opens the way for the general definition of Dirac operators.

8.4 Dirac operators

Definition 8.3. Let $\Gamma(F)$ be a selfadjoint Clifford module over a compact oriented Riemannian manifold M, that is, let $F \longrightarrow M$ be a Hermitian vector bundle and let there be given a selfadjoint Clifford action $\hat c\colon \Gamma(\mathbb{C}\ell(M) \otimes F) \to \Gamma(F)$, where we write $\hat c(\kappa \otimes \psi) := c(\kappa)\psi$. Let ∇ be a Hermitian connection on $F \longrightarrow M$ that satisfies the Leibniz rule $\nabla(c(\kappa)\psi) = c(\nabla^{LC}\kappa)\psi + c(\kappa)(\nabla\psi)$, for $\kappa \in \Gamma(\mathbb{C}\ell(E))$, $\psi \in \Gamma(F)$. The **Dirac operator** associated to the connection ∇ and the Clifford action $\hat c$ is

$$D\!\!\!/ := \hat c \circ \nabla.$$

Via $\Gamma(F) \xrightarrow{\nabla} \mathcal{A}^1_{\mathbb{C}}(M) \otimes_{\mathcal{A}} \Gamma(F) \xrightarrow{\hat c} \Gamma(F)$, this $D\!\!\!/$ is a $\mathbb{C}$-linear endomorphism of $\Gamma(F)$. In local coordinates, $D\!\!\!/\psi = c(dx^j)\nabla_{\partial_j}\psi$.

There are many Dirac operators to be found. First of all, if M is a spin manifold, we may consider the *irreducible* Clifford module $\Gamma(S)$, i.e., the spinor module, with its spin connection ∇^S. Then $D\!\!\!/^S := \hat c \circ \nabla^S$ is the original Dirac operator.[2] Any other Clifford module $\Gamma(S)$ on M is obtained by twisting, that is, $S = W \otimes F$ where $W \longrightarrow M$ is a Hermitian vector bundle carrying the trivial Clifford action, and the compatible connection is determined, via Proposition 7.9, by an arbitrary Hermitian connection on W. The Dirac operator $d+\delta$ on the de Rham algebra arises in this way, since $\Lambda^\bullet T^*M \simeq S \otimes S'$ where $S' \longrightarrow M$ is the superbundle obtained from the spinor bundle by taking the opposite grading: $(S')^\pm := S^\mp$.

[2] $D\!\!\!/^S$ is usually called **the** Dirac operator on the spinor module.

Exercise 8.2. If $c\colon \mathbb{C}\ell(V) \to \operatorname{End}_{\mathbb{C}} \Lambda^\bullet_{\mathbb{C}} W$ is the Clifford action on Fock space, and if $v \in V$ is a unit vector, show that $c'(a) := -c(vav)$ is another selfadjoint Clifford action, whose restriction to $\mathrm{Spin}(V,q)$ switches the two irreducible subrepresentations of the spin representation. Conclude that there is an irreducible Clifford module $S' \longrightarrow M$ on any spin manifold, whose "even" subspace $\Gamma(S'^+)$ is the "odd" subspace $\Gamma(S^-)$ of the spinor module, and viceversa.[3]

Exercise 8.3. Define a Clifford action on the conjugate Fock space $\bar{c}\colon \mathbb{C}\ell(V) \to \operatorname{End}_{\mathbb{C}} \Lambda^\bullet_{\mathbb{C}} \overline{W}$ by setting $\bar{c}(v)\bar{\alpha} := \epsilon(P_- v)\bar{\alpha} - \iota(P_+ v)\bar{\alpha}$ for $v \in V_{\mathbb{C}}$, $\bar{\alpha} \in \overline{W}$ [compare with (6.13)]. Check that $\bar{c}(w) = -\iota(w)$ for $w \in W$ and $\bar{c}(\bar{z}) := \epsilon(\bar{z})$ for $z \in \overline{W}$. If γ denotes, as usual, the chirality element of $\mathbb{C}\ell(V)$, show that $\bar{c}(\gamma)\bar{\alpha} = (-1)^{k+1}\bar{\alpha}$ whenever $\bar{\alpha} \in \Lambda^k_{\mathbb{C}} \overline{W}$ [compare with Proposition 6.3]. If $S = \Lambda^\bullet_{\mathbb{C}} W$ is the spinor module for $\mathbb{C}\ell(V)$, the dual Hilbert space S^* is identified with $\Lambda^\bullet_{\mathbb{C}} \overline{W}$ via

$$\langle \bar{z}_1 \wedge \cdots \wedge \bar{z}_r, w_1 \wedge \cdots \wedge w_s \rangle := \delta_{rs} \det\bigl[2q(\bar{z}_k, w_l)\bigr],$$

that is, the conjugation $C\colon w_1 \wedge \cdots \wedge w_s \mapsto \bar{w}_1 \wedge \cdots \wedge \bar{w}_s$ is the (antilinear) Riesz isomorphism of S with S^*. Check that C intertwines the action $\bar{c}$ on S^* with the action c on S, and conclude that S^* is an irreducible Clifford module for $\mathbb{C}\ell(V)$ whose $\mathbb{Z}_2$-grading is given by $(S^*)^+ = C(S^-)$ and $(S^*)^- = C(S^+)$.[4]

Even if M is not spin, there may be many Clifford modules with compatible connections; the de Rham algebra with the Levi-Civita connection is again the prime example. In any case, since the obstruction to the existence of a spin structure is global, we can always write $\Gamma(U,F) = \Gamma(U, W \otimes S)$ over a chart domain (where W may depend on U) and from any one compatible connection we can manufacture others by altering ∇^W over U only. Alternatively, we may vary the Clifford action by redefining (7.16) (which amounts to an action of the $SO(n)$ frame bundle), as explained in subsection 7.6. Moreover, we have the further freedom of making a smooth change in the metric g and thereby changing the Clifford action c on $\Gamma(F)$ (and also the connection, if necessary, to preserve compatibility). Thus any one Dirac operator gives rise to a large family of "smoothly perturbed" Dirac operators on the same Clifford module.

Any Dirac operator is a first-order differential operator. To avoid any possible confusion of terminology, we make a formal definition.

Definition 8.4. Let $E \longrightarrow M$ be a vector bundle. Any $A \in \Gamma(\operatorname{End} E)$ defines a $\mathbb{C}$-linear operator on $\Gamma(E)$ by left multiplication, that is, $(As)_x := A_x(s_x)$ for $s \in \Gamma(E)$, $x \in M$. Any connection[5] ∇ on $E \longrightarrow M$, when contracted by vector fields $X \in \mathfrak{X}(M)$, provides other $\mathbb{C}$-linear operators ∇_X on $\Gamma(E)$. The **differential operators** on $E \longrightarrow M$ are defined as the elements of the subalgebra $\mathcal{D}(M,E)$ generated by $\Gamma(\operatorname{End} E)$ and by $\{\, \nabla_X : X \in \mathfrak{X}(M) \,\}$.

[3] In more sophisticated terms, the spinor bundles S and S' over a spin manifold are associated to the spin structure $P \longrightarrow M$ via the representations c and c' of $\mathrm{Spin}(2m)$; these vector bundles are inequivalent since the representations c and c' are inequivalent.

[4] We are indebted to William Ugalde for clarification on this point.

[5] In view of (5.5), two such connections differ by the action of an element of $\Gamma(\operatorname{End} E)$, so only a single connection is needed here.

A finite sum of operators of the form $A\nabla_{X_1}\dots\nabla_{X_r}$ with $r \le k$ (and $r = k$ for at least one summand) is called a differential operator *of order* k. By a partition-of-unity argument, a differential operator is of order k if it can be written as such a sum in each chart domain separately. Thus a Dirac operator $D\!\!\!/ = c(dx^j)\nabla_{\partial_j}$ is a differential operator of first order.

As an operator on the superspace $\Gamma(F)$, the Dirac operator $D\!\!\!/$ is *odd*, that is, $D\!\!\!/(\Gamma(F^\pm)) \subseteq \Gamma(F^\mp)$, since $\hat{c}\colon \mathcal{A}^1_{\mathbb{C}}(M) \otimes_{\mathcal{A}} \Gamma(F^\pm) \to \Gamma(F^\mp)$. Thus we may write

$$D\!\!\!/ =: \begin{pmatrix} 0 & D\!\!\!/_- \\ D\!\!\!/_+ & 0 \end{pmatrix}, \tag{8.7}$$

where $D\!\!\!/_\pm\colon \Gamma(F^\pm) \to \Gamma(F^\mp)$. Its square is an *even* operator: $D\!\!\!/^2_\pm\colon \Gamma(F^\pm) \to \Gamma(F^\pm)$; and

$$D\!\!\!/^2 = \begin{pmatrix} D\!\!\!/_- D\!\!\!/_+ & 0 \\ 0 & D\!\!\!/_+ D\!\!\!/_- \end{pmatrix}.$$

Last but not least, a Dirac operator is *essentially selfadjoint*. This means that $D\!\!\!/\colon \Gamma(F) \to \Gamma(F)$ extends uniquely to a selfadjoint operator on the Hilbert space $L^2(F)$ obtained by completing the space of smooth sections $\Gamma(F)$ with respect to the inner product $\langle\langle \phi \mid \psi \rangle\rangle := \int_M (\phi \mid \psi)\,\Omega$. Regrettably, $D\!\!\!/$ is always an unbounded operator, so this selfadjoint extension is still only densely defined. Rather than worry about identifying the precise domain of the extended $D\!\!\!/$, we will stick to the original domain $\Gamma(F)$; a full proof of essential selfadjointness (see [39], for instance) shows that nothing is thereby lost, as the closure of $D\!\!\!/$ on this domain is the full selfadjoint extension.[6] We shall therefore show merely that $D\!\!\!/$ is **formally selfadjoint**, that is,

$$\langle\langle D\!\!\!/\phi \mid \psi \rangle\rangle = \langle\langle \phi \mid D\!\!\!/\psi \rangle\rangle \qquad \text{for} \quad \phi, \psi \in \Gamma(F). \tag{8.8}$$

Proposition 8.7. *The Dirac operator $D\!\!\!/$ is formally selfadjoint.*

Proof. The argument is similar to that of Lemma 8.5. By invoking partitions of unity, we can reduce the problem to verifying (8.8) when ϕ, ψ are smooth sections of $F \longrightarrow M$ which vanish outside some chart domain; thus we write $D\!\!\!/ = c(\theta^\alpha)\nabla_{E^\alpha}$, where $\{\theta^1, \dots, \theta^n\}$ and $\{E_1, \dots, E_n\}$ are local orthonormal bases of 1-forms and vector fields, with $E^\alpha \equiv E_\alpha = (\theta^\alpha)^\sharp$. Now

$$\begin{aligned}
(\phi \mid D\!\!\!/\psi) &= (\phi \mid c(\theta^\alpha)\nabla_{E_\alpha}\psi) = -(c(\theta^\alpha)\phi \mid \nabla_{E_\alpha}\psi) \\
&= -E_\alpha(c(\theta^\alpha)\phi \mid \psi) + (\nabla_{E_\alpha} c(\theta^\alpha)\phi \mid \psi) \\
&= -E_\alpha(c(\theta^\alpha)\phi \mid \psi) + (c(\nabla^{LC}_{E_\alpha}\theta^\alpha)\phi \mid \psi) + (c(\theta^\alpha)\nabla_{E_\alpha}\phi \mid \psi) \\
&= -E_\alpha(c(\theta^\alpha)\phi \mid \psi) + (c(\nabla^{LC}_{E_\alpha}\theta^\alpha)\phi \mid \psi) + (D\!\!\!/\phi \mid \psi),
\end{aligned} \tag{8.9}$$

[6] In other words, $\Gamma(F)$ is a "core" for $D\!\!\!/$ [45]. The compactness of M is not indispensable here: essential selfadjointness can be proven under the weaker assumption that the Riemannian manifold M is complete [39, 53, 61].

where we have used the skewadjointness of $c(\theta^\alpha)$, the hermiticity of the connection ∇, and the compatibility of ∇ with the Levi-Civita connection on the cotangent bundle. We claim that we can find a vector field $Z \in \mathfrak{X}(M)$, depending on ϕ and ψ, such that $\operatorname{div} Z = -E_\alpha(c(\theta^\alpha)\phi|\psi) + (c(\nabla^{LC}_{E_\alpha}\theta^\alpha)\phi|\psi)$. Then integration over M of both sides of (8.9) (multiplied by the Riemannian volume form Ω) yields the desired relation (8.8).

The vector field Z is defined simply by

$$\omega(Z) := (\phi \mid c(\omega)\psi) = -(c(\omega)\phi \mid \psi),$$

since the right hand side is $\mathcal{A}$-linear in $\omega \in \mathcal{A}^1(M)$. Now

$$-E_\alpha(c(\theta^\alpha)\phi \mid \psi) + (c(\nabla^{LC}_{E_\alpha}\theta^\alpha)\phi \mid \psi) = E_\alpha(\theta^\alpha(Z)) - (\nabla^{LC}_{E_\alpha}\theta^\alpha)(Z) = \theta^\alpha(\nabla^{LC}_{E_\alpha}Z)$$

from the definition of a dual connection. The divergence formula (8.5) says that the right hand side equals $\operatorname{div} Z$, as claimed. □

8.5 Laplacians

Definition 8.5. Let $\nabla^E \colon \Gamma(E) \to \mathcal{A}^1(M,E) = \Gamma(T^*M \otimes E)$ be a connection on a vector bundle $E \longrightarrow M$ over a Riemannian manifold (M,g), and let $\widetilde{\nabla}^E := \nabla^{LC} \otimes \nabla^E$ be its tensor product with the Levi-Civita connection on the cotangent bundle of M. Then $\widetilde{\nabla}^E$ maps $\Gamma(T^*M \otimes E)$ to $\mathcal{A}^1(M, T^*M \otimes E) = \Gamma(T^*M \otimes T^*M \otimes E)$, so it can be composed with ∇^E. Contraction with the metric g^{-1} on T^*M gives an $\mathcal{A}$-linear map $\operatorname{Tr}_g \colon \Gamma(T^*M \otimes T^*M \otimes E) \to \Gamma(E)$. The composition of these three maps yields the following operator on $\Gamma(E)$:

$$\Delta^E := -\operatorname{Tr}_g \circ \widetilde{\nabla}^E \circ \nabla^E,$$

called the **Laplacian** associated to the connection ∇^E. The minus sign is a convention[7] which assures that Δ^E is a positive operator whenever ∇^E is a Hermitian connection.

To express Δ^E in a more tractable form, we shall compute the result of contracting the operator $\widetilde{\nabla}^E \circ \nabla^E$ with two vector fields $X, Y \in \mathfrak{X}(M)$. If $s \in \Gamma(E)$, we can write $\nabla^E s = \beta^k \otimes s_k$, and so

$$\begin{aligned}
\iota_Y \iota_X(\widetilde{\nabla}^E \nabla^E s) &= \iota_Y \widetilde{\nabla}^E_X(\beta^k \otimes s_k) = \iota_Y(\beta^k \otimes \nabla^E_X s_k + \nabla_X \beta^k \otimes s_k) \\
&= \beta^k(Y)\, \nabla^E_X s_k + (\nabla_X \beta^k)(Y)\, s_k \\
&= \nabla^E_X(\beta^k(Y) s_k) - X(\beta^k(Y)) + (\nabla_X \beta^k)(Y)\, s_k \\
&= \nabla^E_X(\beta^k(Y) s_k) - \beta^k(\nabla_X Y)\, s_k = \nabla^E_X(\nabla^E_Y s) - \iota(\nabla_X Y)\nabla^E s
\end{aligned}$$

where we have written the Levi-Civita connection simply as ∇. In other words,

$$\iota_Y \iota_X(\widetilde{\nabla}^E \circ \nabla^E) = \nabla^E_X \nabla^E_Y - \nabla^E_{\nabla_X Y}.$$

[7] Actually, the usual convention is to use the opposite sign; however, this results in operators, such as $\sum_j \partial_j^2$ on $\mathbb{R}^n$, which are negative definite.

Since $g^{-1} = g^{ij}\partial_i \cdot \partial^j$ on a chart domain, we get immediately from (7.12) the local expression for the Laplacian:

$$\Delta^E = -g^{ij}\big(\nabla^E_{\partial_i}\nabla^E_{\partial_j} - \Gamma^k_{ij}\nabla^E_{\partial_k}\big). \tag{8.10}$$

With a local orthonormal basis of vector fields $\{e_1, \dots, e_n\}$, this takes a slightly simpler form:[8]

$$\Delta^E = -\sum_{\alpha=1}^{n}\big(\nabla^E_{e_\alpha}\nabla^E_{e_\alpha} - \nabla^E_{\nabla_{e_\alpha}e_\alpha}\big). \tag{8.11}$$

It is immediate from (8.10) or (8.11) that the Laplacian Δ^E is a *second-order differential operator.*

Exercise 8.4. Use the formulas of Exercise 7.11 to give an alternative derivation of (8.10).

Exercise 8.5. Show that the Laplacian Δ_0 associated to the standard connection d on the trivial line bundle $\mathbb{S}^2 \times \mathbb{C} \longrightarrow \mathbb{S}^2$ is

$$\Delta_0 = -\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi^2}\right),$$

i.e., the Laplace–Beltrami operator on $\mathcal{A}^0(\mathbb{S}^2)$.

Proposition 8.8. *Let ∇^E be a Hermitian connection on a vector bundle E over a Riemannian manifold (M, g). Then its Laplacian is a formally selfadjoint and positive operator on $\Gamma(E)$.*

Proof. The strategy of the proof should by now be familiar: we choose sections $s, t \in \Gamma(E)$ and with them construct a vector field $Z \in \mathfrak{X}(M)$ such that $(s\,|\,\Delta^E t) = \operatorname{div} Z + \operatorname{Tr}(\nabla^E s\,|\,\nabla^E t)$. (The last term is locally expressed as $\sum_{\alpha=1}^n(\nabla^E_{e_\alpha}s\,|\,\nabla^E_{e_\alpha}t)$, where $\{e_1, \dots, e_n\}$ are orthonormal vector fields.) Multiplying these functions by Ω and integrating over M, we get the relation

$$\langle\!\langle s \mid \Delta^E t\rangle\!\rangle = \langle\!\langle \nabla^E s \mid \nabla^E t\rangle\!\rangle, \tag{8.12}$$

where these brackets denote integrated inner products on $\Gamma(E)$ and $\mathcal{A}^1(M, E)$ respectively.[9] Positivity follows from setting $s = t$, and formal selfadjointness follows by repeating the argument with the rôles of s and t interchanged.

The vector field Z is defined by

$$g(Z, X) := -(s \mid \nabla^E_X t) \qquad \text{for} \quad X \in \mathfrak{X}(M).$$

Its divergence may be computed with the formula (8.5):

$$\begin{aligned}\operatorname{div} Z = \theta^\alpha(\nabla_{e_\alpha}Z) &= g(\nabla_{e_\alpha}Z, e_\alpha) = -g(Z, \nabla_{e_\alpha}e_\alpha) + e_\alpha\, g(Z, e_\alpha)\\ &= (s \mid \nabla^E_{\nabla_{e_\alpha}e_\alpha}t) - e_\alpha(s \mid \nabla^E_{e_\alpha}t) = (s \mid \Delta^E t) - (\nabla^E_{e_\alpha}s \mid \nabla^E_{e_\alpha}t),\end{aligned}$$

(with summation over α), so $\operatorname{div} Z = (s \mid \Delta^E t) - \operatorname{Tr}(\nabla^E s \mid \nabla^E t)$ as claimed. □

[8] We rename the vector fields E_α to e_α temporarily to avoid a notational clash with the exponent E denoting a vector bundle.

[9] On account of (8.12), the Laplacian is often denoted $\nabla^*\nabla$, as in [39], for instance.

We already know a second-order differential operator on the bundle $\Lambda^\bullet T^*M \longrightarrow M$, namely the "Hodge Laplacian", which we now write Δ_{Hodge}. It is natural to ask whether this operator equals the Laplacian Δ^{LC} associated to the Levi-Civita connection on the exterior bundle. It turns out that they are *not the same*: indeed, they differ by a differential operator of order zero, that is essentially the *Ricci tensor* of Riemannian geometry [8, 36, 39].

Definition 8.6. The **Ricci tensor** on a Riemannian manifold is the symmetric tensor Ric of bidegree $(2,0)$ obtained from the Riemann curvature tensor R by defining $\mathrm{Ric}(X,Y)$ as the trace of the $\mathcal{A}$-bilinear form $(W,Z) \mapsto (W \mid R(Z,Y)X)$ on $\mathfrak{X}(M)$. Locally, $\mathrm{Ric}(X,Y) = dx^k(R(\partial_k,Y)X)$. In terms of the components of the Riemann curvature tensor, we get $\mathrm{Ric}(\partial_i,\partial_j) = R^k_{ikj}$.

By contracting with the metric, we obtain the **curvature scalar**[10] $K := \mathrm{Tr}_g \circ \mathrm{Ric}$ of the Riemannian manifold (M,g); notice that $K \in C^\infty(M)$. Locally, $K = g^{ij} R^k_{ikj}$.

Exercise 8.6. Show that the Riemannian curvature tensor on the sphere $\mathbb{S}^2$ satisfies $R^\phi_{\theta\phi\theta} = 1$ and $R^\theta_{\phi\theta\phi} = \sin^2\theta$, and deduce that $\mathrm{Ric} = g$: the Ricci tensor coincides with the metric on $\mathbb{S}^2$. Conclude that $K = 2$ on $\mathbb{S}^2$: the sphere is a surface of constant (Gaussian) curvature.

We can identify Ric with a tensor of bidegree $(0,2)$ on M, also called Ric, via the metric g; or better, with the element of $\Gamma(\mathrm{End}\, T^*M)$ defined by $\mathrm{Ric}(\omega)(X) := \mathrm{Ric}(\omega^\sharp, X)$. The relation between the Hodge Laplacian and the connection Laplacian, as operators on $\mathcal{A}^\bullet(M)$, is then given by the so-called *Weitzenböck formula*:

$$\Delta_{\text{Hodge}} = \Delta^{LC} + \mathrm{Ric}\,. \tag{8.13}$$

We shall not prove this here, though we have the means to do so; consult [9] or [39] for the details.

Two things are notable about the formula (8.13). First of all, $\Delta_{\text{Hodge}} = (d+\delta)^2$ is the square of the Dirac operator $d+\delta$ on the Clifford module $\mathcal{A}^\bullet(M)$, so the formula says that, for the de Rham complex at least, the square of the Dirac operator is almost, but not quite, a Laplacian; more precisely, it differs from the Laplacian by a differential operator of lower order that, by the way, depends only on the curvature of the connection which defines both the Dirac operator and the Laplacian. This is one of a family of formulae due variously to Bochner, Weitzenböck and Lichnerowicz, which say that $\not{D}^2 - \Delta$ is a curvature-dependent multiplication operator.

The second noteworthy feature is that Δ_{Hodge} and Δ^{LC} are positive (formally) selfadjoint operators. What happens if the Ricci operator is positive too? For one thing, both Δ^{LC} and Ric must then vanish on the kernel of the Hodge Laplacian, namely, on the harmonic forms. Thus, for example, on $\mathbb{S}^2$ the Ricci operator kills 0-forms and 2-forms but acts as the identity on 1-forms since the Ricci tensor coincides with the metric; the upshot of (8.13) is then that *any harmonic 1-form on* $\mathbb{S}^2$ *must be zero* (as we already noted in Section 4), and therefore $H^1_{\mathrm{dR}}(\mathbb{S}^2) = 0$. This result is an example of a *vanishing theorem*, whereby certain cohomology groups reduce to zero —a topological result— on account of positivity properties of certain analytic operators.

[10]The curvature scalar is also called the *Gaussian curvature* when $\dim M = 2$.

8.6 The Lichnerowicz formula

Proposition 8.9. *Let M be a compact spin manifold, let $D\!\!\!/^S = \hat{c} \circ \nabla^S$ denote the Dirac operator on the irreducible spinor module $\Gamma(S)$, and let Δ^S be the Laplacian associated to the spin connection ∇^S. Then*

$$(D\!\!\!/^S)^2 = \Delta^S + \frac{K}{4}, \tag{8.14}$$

where K is the curvature scalar of M.

Proof. It suffices to prove this on any chart domain; that is, we must show that, in a local coordinate basis,

$$\big(c(dx^j)\nabla^S_{\partial_j}\big)^2 = -g^{ij}\big(\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \Gamma^k_{ij}\nabla^S_{\partial_k}\big) + \tfrac14 K.$$

The left hand side is

$$\begin{aligned} c(dx^i)\nabla^S_{\partial_i} c(dx^j)\nabla^S_{\partial_j} &= c(dx^i)\, c(dx^j)\nabla^S_{\partial_i}\nabla^S_{\partial_j} + c(dx^i)\, c(\nabla_{\partial_i} dx^k)\nabla^S_{\partial_k} \\ &= c(dx^i)\, c(dx^j)\big(\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \Gamma^k_{ij}\nabla^S_{\partial_k}\big) \\ &= \tfrac12 [\![c(dx^i), c(dx^j)]\!]\big(\nabla^S_{\partial_i}\nabla^S_{\partial_j} - \Gamma^k_{ij}\nabla^S_{\partial_k}\big) + \tfrac12 c(dx^i)\, c(dx^j)[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}], \end{aligned} \tag{8.15}$$

where we have used the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ due to the zero torsion $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ of the Levi-Civita connection. Now $[\![c(dx^i), c(dx^j)]\!] = -2g^{ij}$ —recall (7.17)— and $[\nabla^S_{\partial_k}, \nabla^S_{\partial_l}] = \omega^S(\partial_k, \partial_l) = -\frac14 R_{ijkl}\, c(dx^i)\, c(dx^j)$ by (7.15), so by taking (8.10) into account we arrive at

$$(D\!\!\!/^S)^2 = \Delta^S - \tfrac18 R_{ijkl}\, c(dx^k)\, c(dx^l)\, c(dx^i)\, c(dx^j).$$

It remains only to check that the second term on the right reduces to $\frac14 K$. We may rewrite it as $\frac18 R_{jikl}\, c(dx^k)\, c(dx^l)\, c(dx^i)\, c(dx^j)$ since R_{ijkl} is antisymmetric in the indices i, j by (5.29). Since the antisymmetrization of R_{jikl} in the indices i, k, l vanishes by (5.28), it follows from Exercise 8.7 below that this term reduces to

$$\tfrac18 R_{jikl}\big(-g^{kl}\, c(dx^i)\, c(dx^j) + g^{il}\, c(dx^k)\, c(dx^j) - g^{ik}\, c(dx^l)\, c(dx^j)\big),$$

and since $R_{jikl}g^{kl} = 0$ by antisymmetry of R_{jikl} in k, l, again by (5.29), this in turn reduces to

$$\tfrac14 R_{ijkl} g^{ik}\, c(dx^l)\, c(dx^j) = \tfrac14 R^k_{jki}\, c(dx^i)\, c(dx^j) = \tfrac14 g^{ij} R^k_{ikj} = \tfrac14 K,$$

due to the symmetry $\mathrm{Ric}_{ij} := R^k_{ikj}$ of the Ricci tensor. □

Exercise 8.7. If u, v, w are three vectors in a Euclidean vector space (V, q), denote their antisymmetrized product by $a := \frac16(uvw + vwu + wuv - uwv - wvu - vuw) \in C\ell(V, q)$. Show that $uvw = a - q(v, w)u + q(u, w)v - q(u, v)w$.

The identity (8.14) is due to Lichnerowicz [40]. The proof generalizes in a straightforward manner [9, 27, 39] to the case of a twisted Clifford module. The only differences consist in replacing the Laplacian Δ^S by Δ^F, where $F = W \otimes S$, and the curvature term $\frac12 c(dx^i)\, c(dx^j)\omega^S(\partial_i, \partial_j)$ in (8.15) by $\frac12 c(dx^i)\, c(dx^j)\omega^F(\partial_i, \partial_j)$. Now by Proposition 7.9, the curvature ω^F can be decomposed as $\omega^F = \omega^W + \omega^S$, where ω^W is the curvature of some compatible connection on W. This yields an extra term of the form $\frac{1}{2}c(dx^i)\, c(dx^j)\omega^W(\partial_i, \partial_j)$, which, in view of (6.10), we may write as $Q(\omega^W)$, where $Q\colon \mathcal{A}^\bullet(M,W) \to \mathbb{C}\ell(M)$ is the extension of the quantization map of Definition 6.12 to the (trivial) Clifford module $\mathcal{A}^\bullet(M,W)$. The result is a generalization of (8.14) known as the *Bochner–Weitzenböck formula*:

$$(D\!\!\!/^F)^2 = \Delta^F + \tfrac{1}{4}K + Q(\omega^W).$$

Exercise 8.8. When $F = \Lambda^\bullet T^*M \simeq S \otimes S$, verify (8.13) by showing that $\operatorname{Ric} - \frac{1}{4}K = Q(\omega_0)$ for a suitable $\omega_0 \in \mathcal{A}^2(M,S)$.

9 The Dirac operator on the Riemann sphere

This section is devoted to a detailed exploration of a single but fundamental example: the Dirac operator on the irreducible spinor module over the sphere $\mathbb{S}^2$. While the sphere is undoubtedly the simplest possible even-dimensional compact spin manifold, its Dirac operator exemplifies the full complexity of the general case while remaining directly accessible by elementary computations. We give here an account of its action on spinors, show its equivariance under the Lie group $SU(2)$ of symmetries of the spinor module, compute its spectrum and exhibit a full set of eigenspinors.

Surprisingly, such an account is not to be found in the current literature on spinors, so this exposition breaks some new ground. The ingredients have been available for a long time, and there is no reason why this story could not have been told thirty years ago. Indeed, before the geometrical theory of Dirac operators was developed at all, the eigenspinors for the Dirac operator on the sphere were considered (in 1938) by Schrödinger [47], who put his finger on the basic module property (7.10) of the spin connection (albeit not in so many words). A generation later, Newman and Penrose [43] introduced a family of functions on the sphere that they called "spinor harmonics", which generalize the ordinary spherical harmonics and constitute the eigenspinors, as we shall see.

In order to keep the development of the Dirac operator on $\mathbb{S}^2$ fairly self-contained, we begin by reviewing some elementary notions. Everything can be done with elementary calculus, provided one takes great care to get the signs and the constants right from the beginning.

9.1 Coordinates on the Riemann sphere

The most direct way to reveal the Dirac operator on the sphere is to regard it as the *Riemann* sphere $\mathbb{CP}^1$, and use complex coordinates. Recall that $\mathbb{CP}^1$ may be described by homogeneous coordinates $[z^0 : z^1]$ with z^0, z^1 not both zero, and is covered by two chart domains $U_0 := \{\, [z^0 : z^1] : z^0 \neq 0 \,\}$ and $U_1 := \{\, [z^0 : z^1] : z^1 \neq 0 \,\}$. We shall use the local complex coordinates

$$z := z^1/z^0 \quad \text{on } U_0, \qquad \zeta := z^0/z^1 \quad \text{on } U_1.$$

The coordinate change is just $z = \zeta^{-1}$, $\zeta = z^{-1}$ on $U_0 \cap U_1$.

We identify $\mathbb{CP}^1$ with the sphere $\mathbb{S}^2$ of unit vectors $(u^1, u^2, u^3) \in \mathbb{R}^3$ by the *stereographic projections*:

$$z = \frac{u^1 + iu^2}{1 - u^3} \quad \text{on } \mathbb{S}^2 \setminus \{N\}, \qquad \zeta = \frac{u^1 - iu^2}{1 + u^3} \quad \text{on } \mathbb{S}^2 \setminus \{S\}$$

where $N = (0, 0, 1)$ and $S = (0, 0, -1)$ are the north and south poles. This identifies U_0 with $\mathbb{S}^2 \setminus \{N\}$ and U_1 with $\mathbb{S}^2 \setminus \{S\}$. It is important to notice that both stereographic projections are *orientation reversing*.

We may also use the standard spherical coordinates (θ, ϕ), satisfying $u^1 = \sin\theta\cos\phi$, $u^2 = \sin\theta\sin\phi$, $u^3 = \cos\theta$. The stereographic projections are then expressed as

$$\begin{aligned} z &= e^{i\phi}\cot\tfrac{\theta}{2} = e^{i\phi}\frac{\sin\theta}{1-\cos\theta} = e^{i\phi}\frac{1+\cos\theta}{\sin\theta}, \\ \zeta &= e^{-i\phi}\tan\tfrac{\theta}{2} = e^{-i\phi}\frac{\sin\theta}{1+\cos\theta} = e^{-i\phi}\frac{1-\cos\theta}{\sin\theta}. \end{aligned} \tag{9.1}$$

The positive functions (2.6) on U_0 and U_1 are given by

$$Q_0(z) := 1 + z\bar{z} = \frac{2}{1-\cos\theta}, \qquad Q_1(\zeta) := 1 + \zeta\bar{\zeta} = \frac{2}{1+\cos\theta}. \tag{9.2}$$

Observe that $Q_0(z)/Q_1(z^{-1}) = z\bar{z}$.

The local 1-forms dz, $d\bar{z}$ on U_0 and $d\zeta$, $d\bar{\zeta}$ on U_1 are given —recall (2.14)— by

$$dz = \frac{e^{i\phi}}{1-\cos\theta}(-d\theta + i\sin\theta\,d\phi), \qquad d\zeta = \frac{e^{-i\phi}}{1+\cos\theta}(d\theta - i\sin\theta\,d\phi),$$

and their complex conjugates; notice that $d\zeta = -e^{-2i\phi}Q_1/Q_0\,dz$ on $U_0 \cap U_1$. The basic local vector fields are given by

$$\frac{\partial}{\partial z} = e^{-i\phi}\frac{1-\cos\theta}{2}\left(-\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right), \quad \frac{\partial}{\partial\zeta} = e^{i\phi}\frac{1+\cos\theta}{2}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right), \tag{9.3}$$

and their complex conjugates; now $\partial/\partial\zeta = -e^{+2i\phi}Q_0/Q_1\,\partial/\partial z$ on $U_0 \cap U_1$.

The Riemannian metric g on $\mathbb{S}^2$ is the usual one:

$$g = d\theta^2 + \sin^2\theta\,d\phi^2 = 4dz \cdot d\bar{z}(1 + z\bar{z})^2 = 4d\zeta \cdot d\bar{\zeta}(1 + \zeta\bar{\zeta})^2 \tag{9.4}$$

(although (2.13) differs from this by a factor of two). The Riemannian volume form is $\Omega = \sin\theta\,d\theta \wedge d\phi = -2i\,Q_0^{-2}\,dz \wedge d\bar{z} = -2i\,Q_1^{-2}\,d\zeta \wedge d\bar{\zeta}$, where the minus sign [compare with (2.12)] indicates the reversal of orientation in passing from (θ, ϕ) coordinates to $(z, \bar{z})$ or $(\zeta, \bar{\zeta})$ coordinates.

9.2 Sections and gauge transformations

A section of a complex line bundle $L \longrightarrow \mathbb{S}^2$ is given by *a pair of local sections* over U_0, U_1 respectively. Once we have chosen basic sections $s_0 \in \Gamma(U_0, L)$ and $s_1 \in \Gamma(U_1, L)$ which are nonvanishing, any global section $s \in \Gamma(L)$ is determined by a pair of smooth functions f_0, f_1 on $\mathbb{C}$ such that

$$s(x) = f_0(z)\, s_0(x) \quad \text{for } x \in U_0, \qquad s(x) = f_1(\zeta)\, s_1(x) \quad \text{for } x \in U_1,$$

where z, ζ are the respective coordinates of the point $x \in \mathbb{S}^2$. The basic local sections are related by the transition function of the line bundle $s_0 = g_{01} s_1$ (since the sphere is covered by only two charts, one transition function suffices), which implies a corresponding relation between $f_0(z)$ and $f_1(\zeta)$, called a *gauge transformation*. In fine: a global section is determined by a related pair of functions, and the line bundle is identified by the particular gauge transformation relating them.

We start with a brief mention of the *holomorphic* line bundles. The *tautological line bundle* $L \longrightarrow \mathbb{CP}^1$ is defined, as in (5.12), by the fibres $L_x := \{\, (\lambda z^0, \lambda z^1) \in \mathbb{C}^2 : \lambda \in \mathbb{C} \,\}$ for $x = [z^0 \colon z^1]$; its basic sections are defined as $\tilde{s}_0(x) := (1, z)$ for $x \in U_0$, $\tilde{s}_1(x) := (\zeta, 1)$ for $x \in U_1$. This yields $\tilde{s}_1(x) = z^{-1} \tilde{s}_0(x)$ for $x \in U_0 \cap U_1$; the transition functions are therefore $\tilde{g}_{10}(z) = z^{-1}$, $\tilde{g}_{01}(\zeta) = \zeta^{-1}$, which of course are holomorphic on $U_0 \cap U_1$.

The hermitian metric on L is obtained from the natural inclusion of the fibres in $\mathbb{C}^2$; this means that $(\tilde{s}_0 \mid \tilde{s}_0) = \|(1, z)\|^2 = 1 + z\bar{z} = Q_0(z)$ and $(\tilde{s}_1 \mid \tilde{s}_1) = \|(\zeta, 1)\|^2 = 1 + \zeta\bar{\zeta} = Q_1(\zeta)$. Its dual, the *hyperplane bundle* $H \longrightarrow \mathbb{CP}^1$ has local sections $\tilde{\sigma}_0$, $\tilde{\sigma}_1$ with $(\tilde{\sigma}_0 \mid \tilde{\sigma}_0) = Q_0^{-1}$ on U_0, $(\tilde{\sigma}_1 \mid \tilde{\sigma}_1) = Q_1^{-1}$ on U_1 —recall (5.13)— and so they extend smoothly to global sections on $\mathbb{S}^2$ just by setting $\tilde{\sigma}_0(N) := 0$, $\tilde{\sigma}_1(S) := 0$. The (holomorphic) transition functions are $\tilde{g}_{01}(\zeta) = \zeta$ and $\tilde{g}_{10}(z) = z$. A global holomorphic section $\tilde{\sigma} \in \mathcal{O}(\mathbb{CP}^1, H)$ is given by a pair of holomorphic functions f_0, f_1 with $\tilde{\sigma}(x) = f_0(z)\, \tilde{\sigma}_0(x)$ and $\tilde{\sigma}(x) = f_1(\zeta)\, \tilde{\sigma}_1(x)$ for *all* x. That means that f_0, f_1 are entire functions whose possible singularities at infinity can only be poles, and $f_0(z) = z f_1(z^{-1})$ for $z \in \mathbb{C}^\times$; this relation identifies a Taylor series and a Laurent series, and can only hold if f_0, f_1 are of the form $f_0(z) = a + bz$, $f_1(\zeta) = b + a\zeta$ for some $a, b \in \mathbb{C}$: we have once again established that $\mathcal{O}(\mathbb{CP}^1, H) \simeq \mathbb{C}^2$.

From now on, we shall normalize all basic sections and thus use only $U(1)$-valued transition functions. Thus we replace the basic sections of L by

$$s_0 := Q_0(z)^{-1/2} \tilde{s}_0, \quad s_1 := Q_1(\zeta)^{-1/2} \tilde{s}_1,$$

and likewise $\sigma_0 := Q_0(z)^{1/2} \tilde{\sigma}_0$, $\sigma_1 := Q_0(\zeta)^{1/2} \tilde{\sigma}_1$. This gives

$$s_1(x) = z^{-1} \sqrt{\frac{Q_0(z)}{Q_1(z^{-1})}}\, s_0(x) = \frac{\sqrt{z\bar{z}}}{z} s_0(x) = (\bar{z}/z)^{1/2} s_0(x),$$

so the transition function is $g_{10}(z) = (\bar{z}/z)^{1/2} = e^{-i\phi}$. A section $s \colon \mathbb{CP}^1 \to L$ is then given by a pair of functions (f_0, f_1) such that $f_0(z) s_0(x) \equiv f_1(\zeta) s_1(x)$ on $U_0 \cap U_1$, i.e., such that

$$f_0(z) \equiv (\bar{z}/z)^{1/2} f_1(z^{-1}), \qquad f_1(\zeta) \equiv (\bar{\zeta}/\zeta)^{1/2} f_0(\zeta^{-1}), \tag{9.5}$$

where the second equation is of course redundant. The formula (9.5) exhibits the $U(1)$ gauge transformation of the tautological line bundle.

It is clear from (9.5) that f_0 and f_1 cannot both be holomorphic, and in general neither is holomorphic. Therefore, it would be more correct to write $f_0(z,\bar{z})$ and $f_1(\zeta,\bar{\zeta})$ to signal the dependence of these smooth functions on both real coordinates. We shall do so whenever the need arises.

For the hyperplane bundle $H \longrightarrow \mathbb{CP}^1$, the very same argument shows that a global section is given by a pair of functions (h_0, h_1) such that $h_0(z)\sigma_0(x) \equiv h_1(\zeta)\sigma_1(x)$ on $U_0 \cap U_1$, and which are therefore related by the gauge transformation

$$h_0(z) \equiv (z/\bar{z})^{1/2} h_1(z^{-1}), \qquad h_1(\zeta) \equiv (\zeta/\bar{\zeta})^{1/2} h_0(\zeta^{-1}). \tag{9.6}$$

Exercise 9.1. Show that for any Hermitian line bundle $E \longrightarrow \mathbb{CP}^1$, the sections in $\Gamma(E)$ are described by pairs of functions (f_0, f_1) satisfying the relation $f_0(z) \equiv (z/\bar{z})^{k/2} f_1(z^{-1})$, where $k \in \mathbb{Z}$ and $k[H]$ is the Chern class of E.

We may decompose the complexified tangent bundle as $T_{\mathbb{C}}\mathbb{S}^2 = T^{1,0}\mathbb{S}^2 \oplus T^{0,1}\mathbb{S}^2$, where the sections of the *holomorphic tangent bundle* $T^{1,0}\mathbb{S}^2$ are locally of the form $f_0(z,\bar{z})\,\partial/\partial z$ or $f_1(\zeta,\bar{\zeta})\,\partial/\partial\zeta$. [The sections of the "antiholomorphic tangent bundle" $T^{0,1}\mathbb{S}^2$ are of the form $h_0(z,\bar{z})\,\partial/\partial\bar{z}$ or $h_1(\zeta,\bar{\zeta})\,\partial/\partial\bar{\zeta}$.] Then $T^{1,0}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is a Hermitian line bundle, under the metric determined by

$$\langle \partial/\partial z \mid \partial/\partial z \rangle := g(\partial/\partial\bar{z}, \partial/\partial z) = \frac{4}{(1+z\bar{z})^2}.$$

As normalized local sections we take

$$E_z := \tfrac{1}{2} Q_0(z) \frac{\partial}{\partial z} \quad \text{over } U_0, \qquad -E_\zeta := -\tfrac{1}{2} Q_1(\zeta) \frac{\partial}{\partial \zeta} \quad \text{over } U_1.$$

Since $z = \zeta^{-1}$ gives $\partial/\partial z = -\zeta^2\,\partial/\partial\zeta$, we find that $E_z = -\zeta^2 Q_0(z)/Q_1(\zeta)\,E_\zeta = (\bar{z}/z)(-E_\zeta)$. Thus a section of the holomorphic tangent bundle is given by a pair of functions f_0, f_1 satisfying $f_0(z)\,E_z \equiv f_1(\zeta)(-E_\zeta)$ on $U_0 \cap U_1$, or equivalently

$$f_0(z) \equiv (z/\bar{z})\, f_1(z^{-1}).$$

This establishes that $[T^{1,0}\mathbb{S}^2] = 2[H]$ in $\check{H}(\mathbb{S}^2, \mathbb{Z})$, i.e., the line bundles $T^{1,0}\mathbb{S}^2$ and $H \otimes H$ are equivalent.

Exercise 9.2. Conclude from $\overline{E}_z = (z/\bar{z})(-\overline{E}_\zeta)$ that the complex line bundle $T^{0,1}\mathbb{S}^2$ is equivalent to $L \otimes L$.

Exercise 9.3. Write $T^*_{\mathbb{C}}\mathbb{S}^2 = \Lambda^{1,0}T^*\mathbb{S}^2 \oplus \Lambda^{0,1}T^*\mathbb{S}^2$, where $\mathcal{A}^{1,0}(\mathbb{S}^2) = \Gamma(\Lambda^{1,0}T^*\mathbb{S}^2)$ has elements $f_0(z,\bar{z})\,dz = f_1(\zeta,\bar{\zeta})\,d\zeta$ and $\mathcal{A}^{0,1}(\mathbb{S}^2) = \Gamma(\Lambda^{0,1}T^*\mathbb{S}^2)$ consists of all $h_0(z,\bar{z})\,d\bar{z} = h_1(\zeta,\bar{\zeta})\,d\bar{\zeta}$. Use g^{-1} to define Hermitian metrics on both these line bundles, write down suitable normalized local sections, and verify that $\Lambda^{1,0}T^*\mathbb{S}^2 \sim L \otimes L$ and $\Lambda^{0,1}T^*\mathbb{S}^2 \sim H \otimes H$.

Definition 9.1. Let $S \longrightarrow \mathbb{S}^2$ denote the irreducible spinor bundle. Since $T^*_{\mathbb{C}}\mathbb{S}^2 \sim S \otimes S^*$, we expect that $S \sim L \oplus H$, and that $S^+ \sim L$, $S^- \sim H$. Due to the noncanonical nature of the spinor bundle, we bypass the construction of basic local sections and define a **spinor** ψ over $\mathbb{S}^2$ directly as *a pair of functions on each chart*, denoted $\psi^\pm_N(z,\bar z)$ and $\psi^\pm_S(\zeta,\bar\zeta)$ respectively, satisfying the gauge transformation rules:

$$\begin{aligned} \psi^+_N(z,\bar z) &\equiv (\bar z/z)^{1/2}\psi^+_S(z^{-1},\bar z^{-1}), & \psi^-_N(z,\bar z) &\equiv (z/\bar z)^{1/2}\psi^-_S(z^{-1},\bar z^{-1}), \\ \psi^+_S(\zeta,\bar\zeta) &\equiv (\bar\zeta/\zeta)^{1/2}\psi^+_N(\zeta^{-1},\bar\zeta^{-1}), & \psi^-_S(\zeta,\bar\zeta) &\equiv (\zeta/\bar\zeta)^{1/2}\psi^-_N(\zeta^{-1},\bar\zeta^{-1}). \end{aligned} \tag{9.7}$$

It is immediate from (9.5) and (9.6) that (ψ^+_N,ψ^+_S) determines a section of L and (ψ^-_N,ψ^-_S) determines a section of H.[1]

9.3 The spin connection over the sphere

Lemma 9.1. *The Levi-Civita connection on the sphere is determined by the local formulae*

$$\nabla_{\partial_i}\partial_j = -\frac{2}{1+x_1^2+x_2^2}(x_i\partial_j + x_j\partial_i - \delta_{ij}x^k\,\partial_k), \tag{9.8}$$

with $x_1 \equiv x^1$, $x_2 \equiv x^2$, where $x^1+ix^2 = z$ on U_0 and $x^1+ix^2=\zeta$ on U_1.

Proof. The metric (9.4) is given by $g = 4(1+x_1^2+x_2^2)^{-2}(dx_1^2+dx_2^2)$ on both charts, so $g_{ij} = 4(1+x_1^2+x_2^2)^{-2}\delta_{ij}$ and $g^{ij} = \frac14(1+x_1^2+x_2^2)^2\delta^{ij}$. Also, $\partial_k g_{ij} = -16x_k(1+x_1^2+x_2^2)^{-3}\delta_{ij}$. From (7.13) we get at once

$$\Gamma^k_{ij} = -\frac{2}{1+x_1^2+x_2^2}(x_i\delta^k_j + x_j\delta^k_i - x^k\delta_{ij}),$$

from which (9.8) follows. ☐

Local orthonormal bases of vector fields (on both charts) are given by $E_1 := \frac12(1+x_1^2+x_2^2)\partial/\partial x^1$, $E_2 := \frac12(1+x_1^2+x_2^2)\partial/\partial x^2$. With these bases, we get

$$\nabla_{\partial_i}E_\alpha = x_i\partial_\alpha + \tfrac12(1+x_1^2+x_2^2)\nabla_{\partial_i}\partial_\alpha = \frac{2}{1+x_1^2+x_2^2}(\delta_{i\alpha}x^\beta\,E_\beta - x_\alpha E_i),$$

or equivalently,

$$\widetilde\Gamma^\beta_{i\alpha} = \frac{2}{1+x_1^2+x_2^2}(\delta_{i\alpha}x^\beta - \delta^\beta_i x_\alpha).$$

The spin connection components are given by (7.18) as $\omega_i = \frac14\widetilde\Gamma^\beta_{i\alpha}\,\gamma^\alpha\gamma_\beta$, which yields

$$\begin{aligned} \omega_1 &= \frac{1}{2(1+x_1^2+x_2^2)}(x^\beta\gamma^1\gamma_\beta - x_\alpha\gamma^\alpha\gamma_1) = \frac{x_2}{1+x_1^2+x_2^2}\gamma^1\gamma_2, \\ \omega_2 &= \frac{1}{2(1+x_1^2+x_2^2)}(x^\beta\gamma^2\gamma_\beta - x_\alpha\gamma^\alpha\gamma_2) = -\frac{x_1}{1+x_1^2+x_2^2}\gamma^1\gamma_2, \end{aligned} \tag{9.9}$$

[1] We label the local spinor coefficients N and S (north and south) in order to reduce the clutter of numerical indices.

To return to complex notation, we notice that $(\omega_1 + i\omega_2)(z) = -\frac{iz}{1+z\bar z}\gamma^1\gamma_2$ over U_0 while $(\omega_1 + i\omega_2)(\zeta) = -\frac{i\zeta}{1+\zeta\bar\zeta}\gamma^1\gamma_2$ over U_1. These are related by a gauge transformation: $(\omega_1 + i\omega_2)(z) = (z/\bar z)\,(\omega_1 + i\omega_2)(\zeta)$.

9.4 The Dirac operator over the sphere

Definition 9.2. We fix a local Clifford action on the spinor module by choosing a particular matrix function $\widetilde{H} := [\tilde h^r_\beta(z)]$ such that $\widetilde{H}^t\widetilde{H} = G^{-1} \equiv [g^{ij}(z)] = \frac14(1+z\bar z)^2\, I$ over U_0. Let us (arbitrarily) choose the positive square root $\widetilde{H} := G^{-1/2} = \frac12(1+z\bar z)\, I$. The **Dirac operator** on the chart U_0 is then defined as

$$\begin{aligned} D\!\!\!/_N &:= c(dx^j)\nabla^S_{\partial_j} = \tilde h^j_\beta(z)\gamma^\beta(\partial_j + \omega_j(z)) \\ &= \tfrac12(1+z\bar z)(\gamma^\beta\partial_\beta + \gamma^\beta\omega_\beta(z)) \\ &= \tfrac12(1+z\bar z)(\gamma^1\,\partial/\partial x^1 + \gamma^2\,\partial/\partial x^2) - \tfrac12(x_1\gamma^1 + x_2\gamma^2). \end{aligned}$$

Here we have used the identities $\gamma^1\gamma^1\gamma^2 = -\gamma^2$ and $\gamma^2\gamma^1\gamma^2 = +\gamma^1$.

It turns out that the Dirac operator over the chart U_1, which we shall write as $D\!\!\!/_S$, is now completely determined: the Dirac operator on the spinor module is determined by its restriction to a single chart, no matter how small![2] Indeed, this is a characteristic feature of the spinor module [19]. However, in order to preserve the illusion of a symmetrical treatment of both charts, we shall anticipate the corresponding formulas for $D\!\!\!/_S$. We therefore choose the matrix $[\tilde h^r_\beta(\zeta)]$ to be the *negative* square root of $G^{-1} \equiv [g^{ij}(\zeta)] = \frac14(1+\zeta\bar\zeta)^2\, I$, that is, $\tilde h^r_\beta(\zeta) := -\frac12(1+\zeta\bar\zeta)\delta^r_\beta$. This leads to

$$D\!\!\!/_S = -\tfrac12(1+\zeta\bar\zeta)(\gamma^1\,\partial/\partial x^1 + \gamma^2\,\partial/\partial x^2) + \tfrac12(x_1\gamma^1 + x_2\gamma^2). \tag{9.10}$$

Explicitly, $\partial/\partial x^1 = \partial/\partial z + \partial/\partial\bar z$ and $\partial/\partial x^2 = i(\partial/\partial z - \partial/\partial\bar z)$ on U_0, and similar formulas hold on U_1 with z replaced by ζ. This allows us to express the Dirac operators properly in complex coordinates:

$$\begin{aligned} D\!\!\!/_N &= \tfrac12(1+z\bar z)\biggl((\gamma^1 + i\gamma^2)\frac{\partial}{\partial z} + (\gamma^1 - i\gamma^2)\frac{\partial}{\partial\bar z}\biggr) - \tfrac14\bigl(z(\gamma^1 - i\gamma^2) + \bar z(\gamma^1 + i\gamma^2)\bigr), \\ D\!\!\!/_S &= -\tfrac12(1+\zeta\bar\zeta)\biggl((\gamma^1 + i\gamma^2)\frac{\partial}{\partial\zeta} + (\gamma^1 - i\gamma^2)\frac{\partial}{\partial\bar\zeta}\biggr) + \tfrac14\bigl(\zeta(\gamma^1 - i\gamma^2) + \bar\zeta(\gamma^1 + i\gamma^2)\bigr). \end{aligned} \tag{9.11}$$

These formulae are still a bit cumbersome; to obtain a simpler picture, we need to select a particular representation for the operators γ^1 and γ^2.

[2] On more general compact spin manifolds, the restriction of the Dirac operator to a single chart determines its restriction to neighbouring charts through its interaction with the gauge transformations; this in turn determine their neighbours, and so on.

Definition 9.3. The Fock space of $\mathbb{R}^2$ is the two-dimensional complex superspace $\Lambda^0\mathbb{C} \oplus \Lambda^1\mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$. We may represent γ^1 and γ^2 as anticommuting *odd* operators of square -1; a suitable choice is

$$\gamma^1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^2 := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{9.12}$$

The grading operator is $i\gamma^1\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since $\gamma^1 + i\gamma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $\gamma^1 - i\gamma^2 = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$, the expressions (9.11) simplify to

$$\not{D}_N = \begin{pmatrix} 0 & \eth_z \\ -\bar{\eth}_z & 0 \end{pmatrix}, \qquad \not{D}_S = \begin{pmatrix} 0 & -\eth_\zeta \\ \bar{\eth}_\zeta & 0 \end{pmatrix}, \tag{9.13}$$

where $\eth_z$ is the first-order differential operator[3]

$$\eth_z := (1 + z\bar{z})\frac{\partial}{\partial z} - \tfrac{1}{2}\bar{z}. \tag{9.14}$$

The operator $\eth_z$ was introduced by Newman and Penrose [43] and further studied by Goldberg *et al.* [30], with particular attention to its eigenfunctions. Since

$$\eth_z\psi = (1 + z\bar{z})\frac{\partial\psi}{\partial z} - \frac{1}{2}\frac{\partial}{\partial z}(1 + z\bar{z})\psi = (1 + z\bar{z})^{3/2}\frac{\partial}{\partial z}\big((1 + z\bar{z})^{-1/2}\psi\big), \tag{9.15}$$

we get the identity $\eth_z = Q_0^{3/2}(\partial/\partial z)Q_0^{-1/2}$ in the algebra of differential operators on U_0.

The formal selfadjointness of $\not{D}$ is not perhaps apparent from (9.13), since the term $-\frac{1}{2}\bar{z}$ in (9.14) seems to have the adjoint $-\frac{1}{2}z$ rather than $+\frac{1}{2}z$. However, the inner product of spinors $\langle\!\langle \phi \,|\, \psi \rangle\!\rangle$ involves integration over the sphere, i.e., integration over $\mathbb{C}$ with respect to to the area form $2i(1 + z\bar{z})^{-2}\, dz \wedge d\bar{z}$. Thus, if ϕ, ψ are two spinors, then (9.15) gives

$$(1 + z\bar{z})^{-2}\overline{\phi_N^+}\eth_z\psi_N^- = (1 + z\bar{z})^{-1/2}\overline{\phi_N^+}\frac{\partial}{\partial z}\big((1 + z\bar{z})^{-1/2}\psi_N^-\big),$$

and it follows that $\langle\!\langle \phi^+ \mid \eth_z\psi^- \rangle\!\rangle = -\langle\!\langle \bar{\eth}_z\phi^+ \mid \psi^- \rangle\!\rangle$ on integrating by parts.

Proposition 9.2. *The restriction of the Dirac operator to U_1 is determined by its restriction to U_0.*

Proof. We must show that the form (9.13) of the operator $\not{D}_S$ is completely determined by that of $\not{D}_N$. This is possible because these operators act respectively on functions $\psi_S^\pm$ and $\psi_N^\pm$ which are linked by the gauge transformations (9.7), and because the Dirac operator is odd, so the gauge transformations for both spinor parities must be invoked.

[3] The letter ð, from the Icelandic alphabet, is pronounced "edth".

Given a spinor ψ with components ψ^+ and ψ^-, let $\phi = \not{D}\psi$. Then $(\eth_z \psi_N^-)(z,\bar z) = \phi_N^+(z,\bar z) = (\bar z/z)^{1/2}\phi_S^+(z^{-1},\bar z^{-1})$. On the other hand, with $\zeta = z^{-1}$ we get

$$\begin{aligned}\frac{\partial \psi_N^-}{\partial z}(z,\bar z) &= \frac{\partial}{\partial z}\big((z/\bar z)^{1/2}\psi_S^-(z^{-1},\bar z^{-1})\big)\\ &= \tfrac12 (z\bar z)^{-1/2}\psi_S^-(z^{-1},\bar z^{-1}) - (z/\bar z)^{1/2} z^{-2}\frac{\partial \psi_S^-}{\partial \zeta}(z^{-1},\bar z^{-1})\\ &= (\zeta\bar\zeta)^{1/2}\Big(-\zeta\frac{\partial \psi_S^-}{\partial\zeta}(\zeta,\bar\zeta) + \tfrac12\psi_S^-(\zeta,\bar\zeta)\Big),\end{aligned}$$

and so the operator $\eth_z$ transforms as follows:

$$\begin{aligned}(\eth_z\psi_N^-)(z,\bar z) &= (1+z\bar z)\frac{\partial\psi_N^-}{\partial z}(z,\bar z) - \tfrac12\bar z\psi_N^-(z,\bar z)\\ &= (\zeta\bar\zeta)^{-1/2}(1+\zeta\bar\zeta)\Big(-\zeta\frac{\partial\psi_S^-}{\partial\zeta}(\zeta,\bar\zeta)+\tfrac12\psi_S^-(\zeta,\bar\zeta)\Big) - \frac{(\bar\zeta/\zeta)^{1/2}}{2\bar\zeta}\psi_S^-(\zeta,\bar\zeta)\\ &= (\zeta/\bar\zeta)^{1/2}\Big(-(1+\zeta\bar\zeta)\frac{\partial\psi_S^-}{\partial\zeta}(\zeta,\bar\zeta) + \tfrac12(\zeta^{-1}+\bar\zeta)\psi_S^-(\zeta,\bar\zeta) - \tfrac12\zeta^{-1}\psi_S^-(\zeta,\bar\zeta)\Big)\\ &= -(\zeta/\bar\zeta)^{1/2}\eth_\zeta\psi_S^-(\zeta,\bar\zeta) = -(\bar z/z)^{1/2}\eth_\zeta\psi_S^-(z^{-1},\bar z^{-1}).\end{aligned}\tag{9.16}$$

We conclude that $\phi_S^+ = -\eth_\zeta\psi_S^-$. If we apply complex conjugation to (9.16) and replace ψ_N^- by ψ_N^+ and ψ_S^- by ψ_S^+, we find that

$$\phi_N^-(z,\bar z) = -(\bar\eth_z\psi_N^+)(z,\bar z) = (z/\bar z)^{1/2}\bar\eth_\zeta\psi_S^+(z^{-1},\bar z^{-1}),$$

and it follows that $\phi_S^- = \bar\eth_\zeta\psi_S^+$.

We have recovered the expression of (9.13) for $\not{D}_S$ from the corresponding for $\not{D}_N$ and from (9.7), without using the recipe (9.10) for $\not{D}_S$. This means that the "choice" $\tilde h_\beta^r(\zeta) := -\frac12(1+\zeta\bar\zeta)\delta_\beta^r$ which led to (9.10) is actually forced by the action of $\not{D}$ on spinors. $\square$

Exercise 9.4. Use (9.1) and (9.3) to obtain expressions for $\eth_z$ and $\eth_\zeta$ in spherical coordinates (θ,ϕ). Check that $e^{i\phi}\eth_z = -e^{-i\phi}\eth_\zeta - \csc\theta$ on $U_0\cap U_1$.

Proposition 9.2 shows that the only freedom in the choice of the Dirac operator, once the metric and the spin connection are given, lies in selecting the matrix function $\widetilde H = [\tilde h_\beta^r(z)]$ which gives the local Clifford action on U_0. The condition $\widetilde H^t\widetilde H = G^{-1}$ fixes the matrix $\widetilde H$ up to premultiplication by an arbitrary $SO(2)$ matrix function on U_0. We may alternatively think of this as the *freedom to select any local orthonormal bases of tangent vectors* compatible with the orientation of $\mathbb{S}^2$, since this amounts to picking a section of the $SO(2)$-frame bundle. To express this in our complex-coordinate notation, we must bear in mind that the stereographic projections $(\theta,\phi)\mapsto(z,\bar z)$ and $(\theta,\phi)\mapsto(\zeta,\bar\zeta)$ are *orientation reversing*; such a local orthonormal basis is therefore given by

$$\begin{pmatrix}E_1\\E_2\end{pmatrix} := \begin{pmatrix}-\cos\alpha & \sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\partial/\partial\theta\\ \csc\theta\,\partial/\partial\phi\end{pmatrix},\tag{9.17}$$

where $\alpha(\theta,\phi)$ is a smooth real-valued function which we call the **spin gauge** [25].

Exercise 9.5. Check that $E_z = \frac{1}{2}(E_1 - iE_2)$ if and only if $\alpha \equiv -\phi$, and $-E_\zeta = \frac{1}{2}(E_1 - iE_2)$ if and only if $\alpha \equiv \phi$.

Exercise 9.6. Compute the mixed Christoffel symbols $\widetilde{\Gamma}^\beta_{i\alpha}$ with the spin gauge (9.17) (use Exercise 7.16) and verify that $\widetilde{\Gamma}^2_{\theta 1} = \partial\alpha/\partial\theta$, $\widetilde{\Gamma}^2_{\phi 1} = \partial\alpha/\partial\phi - \cos\theta$.

Exercise 9.7. Use the results of the Exercises 9.5 and 9.6 to obtain the following local expressions in spherical coordinates for the spin connection ∇^S on the chart domains U_0 and U_1:

$$\nabla^S_\theta = \frac{\partial}{\partial\theta}, \qquad \nabla^S_\phi = \frac{\partial}{\partial\phi} \mp \tfrac{1}{2}(1 \pm \cos\theta)\gamma^1\gamma^2 \tag{9.18}$$

where the upper signs are for U_0 and the lower signs are for U_1.

Exercise 9.8. Compute the Dirac operator in spherical coordinates with the spin gauge (9.17), using the results of Exercise 7.16: check that $\not{D}$ is given by

$$\begin{aligned}
\not{D}_- &= e^{i\alpha}\biggl(-\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{2\sin\theta}\Bigl(\frac{\partial\alpha}{\partial\phi} - \cos\theta\Bigr) - \frac{i}{2}\frac{\partial\alpha}{\partial\theta}\biggr), \\
\not{D}_+ &= e^{-i\alpha}\biggl(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{1}{2\sin\theta}\Bigl(\frac{\partial\alpha}{\partial\phi} - \cos\theta\Bigr) - \frac{i}{2}\frac{\partial\alpha}{\partial\theta}\biggr),
\end{aligned}$$

with the conventions of (8.7) and (9.12).[4]

9.5 The spinor Laplacian

Lemma 9.3. *The spinor Laplacian Δ^S on the sphere $\mathbb{S}^2$ is given locally by*

$$\Delta^S = -\Bigl(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi^2}\Bigr) + \Bigl(\frac{1 \pm \cos\theta}{2\sin\theta}\Bigr)^2 \pm \frac{1 \pm \cos\theta}{\sin^2\theta}\gamma^1\gamma^2\frac{\partial}{\partial\phi}, \tag{9.19}$$

where the upper signs are for U_0 and the lower signs are for U_1.

Proof. It suffices to check that the general local formula (8.10) for the Laplacian specializes, in view of (7.14) and (9.18), to

$$\Delta^S = -\frac{\partial^2}{\partial\theta^2} - \frac{1}{\sin^2\theta}\Bigl(\frac{\partial}{\partial\phi} \mp \tfrac{1}{2}(1 \pm \cos\theta)\gamma^1\gamma^2\Bigr)^2 - \frac{1}{\sin^2\theta}\Bigl(\sin\theta\cos\theta\frac{\partial}{\partial\theta}\Bigr),$$

from which (9.19) is immediate. □

It is convenient to rewrite the spinor Laplacian in complex coordinates. From (9.3) we derive

$$\frac{\partial}{\partial\phi} = i\Bigl(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Bigr) = -i\Bigl(\zeta\frac{\partial}{\partial\zeta} - \bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}\Bigr) \qquad \text{on} \quad U_0 \cap U_1.$$

[4] The formulae (9.18) have been obtained by Dray [25] as the expressions of the Newman–Penrose operators $-\eth$ and $\bar{\eth}$ on quantities of "spin-weights" $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively.

Using (9.2) and (9.3) it is readily checked that

$$(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} = (1+\zeta\bar{\zeta})^2 \frac{\partial^2}{\partial\zeta\,\partial\bar{\zeta}} = \frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi^2}.$$

Since $(1+\cos\theta/\sin\theta)^2 = z\bar{z}$ and $(1-\cos\theta/\sin\theta)^2 = \zeta\bar{\zeta}$ by (9.1), we arrive at

$$\begin{aligned} \Delta^S &= -(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} + \tfrac14 z\bar{z} + \tfrac12(1+z\bar{z})\, i\gamma^1\gamma^2\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big) \quad \text{over } U_0, \\ &= -(1+\zeta\bar{\zeta})^2 \frac{\partial^2}{\partial\zeta\,\partial\bar{\zeta}} + \tfrac14 \zeta\bar{\zeta} + \tfrac12(1+\zeta\bar{\zeta})\, i\gamma^1\gamma^2\Big(\zeta\frac{\partial}{\partial\zeta} - \bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}\Big) \quad \text{over } U_1. \end{aligned} \tag{9.20}$$

Lemma 9.4. *The Dirac operator and the spinor Laplacian on $\mathbb{S}^2$ are related by*

$$D\!\!\!/^2 = \Delta^S + \tfrac12.$$

Proof. This follows, of course, from the Lichnerowicz formula (8.14), since the sphere has constant scalar curvature $K \equiv 2$; but it is instructive to make a direct verification. From (9.14) we get

$$\begin{aligned} -\eth_z\bar{\eth}_z &= \Big((1+z\bar{z})\frac{\partial}{\partial z} - \tfrac12\bar{z}\Big)\Big(-(1+z\bar{z})\frac{\partial}{\partial\bar{z}} + \tfrac12 z\Big) \\ &= -(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} + \tfrac12(1+z\bar{z})\Big(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial\bar{z}}\Big) - \tfrac14 z\bar{z} + (1+z\bar{z})\Big(\tfrac12 - \bar{z}\frac{\partial}{\partial\bar{z}}\Big) \\ &= -(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} + \tfrac12 + \tfrac14 z\bar{z} + \tfrac12(1+z\bar{z})\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big), \end{aligned} \tag{9.21}$$

whereas

$$\begin{aligned} -\bar{\eth}_z\eth_z &= \Big(-(1+z\bar{z})\frac{\partial}{\partial\bar{z}} + \tfrac12 z\Big)\Big((1+z\bar{z})\frac{\partial}{\partial z} - \tfrac12\bar{z}\Big) \\ &= -(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} + \tfrac12 + \tfrac14 z\bar{z} - \tfrac12(1+z\bar{z})\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big) \end{aligned} \tag{9.22}$$

by a similar calculation. The different signs of the last terms on the right in (9.21) and (9.22) signify the presence of the grading operator $i\gamma^1\gamma^2$ in the representation (9.12). Thus

$$\begin{aligned} D\!\!\!/_N^2 &= \begin{pmatrix} -\eth_z\bar{\eth}_z & 0 \\ 0 & -\bar{\eth}_z\eth_z \end{pmatrix} \\ &= -(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} + \tfrac12 + \tfrac14 z\bar{z} + \tfrac12(1+z\bar{z})\, i\gamma^1\gamma^2\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big). \end{aligned} \tag{9.23}$$

There is an identical formula for $D\!\!\!/_S$, on replacing z by ζ. Now a glance at (9.20) gives the Lichnerowicz formula $D\!\!\!/^2 = \Delta^S + \tfrac12$. $\square$

9.6 The $SU(2)$ action on the spinor bundle

Definition 9.4. The Lie group $SU(2)$ of unitary matrices of determinant 1,

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \qquad \text{with} \quad \alpha\bar\alpha + \beta\bar\beta = 1,$$

acts transitively on the sphere $\mathbb{S}^2$ by rotations. The identification $g \leftrightarrow (\alpha, \beta) \in \mathbb{C}^2$ shows that $SU(2)$ is topologically the sphere $\mathbb{S}^3$, so it is compact. If $[z^0 \colon z^1]$ are homogeneous coordinates of a point in $\mathbb{CP}^1$, the action is given by $g \cdot [z^0 \colon z^1] = [\alpha z^0 + \beta z^1 : -\bar\beta z^0 + \bar\alpha z^1]$. On the charts U_0 and U_1, the action of $SU(2)$ is given by *Möbius transformations*:

$$g \cdot z = \frac{\alpha z + \beta}{-\bar\beta z + \bar\alpha}, \qquad g' \cdot \zeta = \frac{\bar\alpha\zeta - \bar\beta}{\beta\zeta + \alpha}, \tag{9.24}$$

which clearly satisfies $(g \cdot z)^{-1} = g' \cdot \zeta$. Notice that $g \mapsto g'$ is an (inner) automorphism of $SU(2)$, since

$$g' := \begin{pmatrix} \bar\alpha & -\bar\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Exercise 9.9. Any isometry (rotation or reflection) of the sphere $\mathbb{S}^2$ takes circles to circles and takes antipodal pairs of points to antipodal pairs of points. Conversely, any smooth bijective transformation of $\mathbb{S}^2$ with these two geometrical properties is either a rotation or a reflection. Prove this converse, using the fact that a circle-preserving transformation of $\mathbb{S}^2$ corresponds, under stereographic projection, to a circle-preserving transformation of $\mathbb{C}_\infty$, which is either a Möbius transformation $z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$ or a conjugate Möbius transformation $z \mapsto (\alpha\bar z + \beta)/(\gamma\bar z + \delta)$; and that a Möbius transformation is determined by the images of three points in $\mathbb{C}_\infty$.

Exercise 9.10. The antipode of $z = e^{i\phi}\cot\frac{1}{2}\theta$ is $e^{i(\pi+\phi)}\cot\frac{1}{2}(\pi-\theta) = -1/\bar z$. Use this fact and the preceding exercise to prove that any rotation of $\mathbb{S}^2$ is given by a Möbius transformation of the form $z \mapsto (\alpha z + \beta)/(-\bar\beta z + \bar\alpha)$.

Definition 9.5. The elements of $SU(2)$ are conveniently described in terms of the *Pauli matrices*:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\vec n = (n^1, n^2, n^3)$ denote Cartesian coordinates in $\mathbb{R}^3$ of a point $\vec n \in \mathbb{S}^2$, we write $\vec n \cdot \vec\sigma := n^1\sigma^1 + n^2\sigma^2 + n^3\sigma^3$; then any $g \in SU(2)$ may be written in the form

$$g = \exp\bigl(\tfrac{1}{2}i\psi\, \vec n \cdot \vec\sigma\bigr) = \cos\tfrac{1}{2}\psi + i\sin\tfrac{1}{2}\psi\, \vec n \cdot \vec\sigma \tag{9.25}$$

with $\vec n \in \mathbb{S}^2$ and $-\pi < \psi \leq \pi$. This is called the *angle-axis parametrization* of $SU(2)$. We may identify the sphere $\mathbb{S}^2$ with the submanifold $\{\, g : \operatorname{Tr} g = 0 \,\}$ of $SU(2)$, where $\vec n \in \mathbb{S}^2$ corresponds to $i\vec n \cdot \vec\sigma \in SU(2)$; the rotation action ρ of $SU(2)$ on the sphere is given by conjugation: $i\vec n \cdot \vec\sigma \mapsto g(i\vec n \cdot \vec\sigma)g^{-1} =: i\rho(g)\vec n \cdot \vec\sigma$. The isotropy subgroup of the point $\vec n$ consists of elements of the form (9.25) with ψ arbitrary, which form a subgroup isomorphic to $U(1)$. Thus $\mathbb{S}^2 \approx SU(2)/U(1)$ as a homogeneous space.

Exercise 9.11. Show that $\rho(g)\vec{n} = \vec{n}$ if and only if g is of the form (9.25) for some $\psi \in \mathbb{R}$.

Exercise 9.12. Show that there is a group isomorphism between Spin(3) and $SU(2)$, such that if $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{R}^3$, then the elements e_2e_3, e_3e_1, $e_1e_2 \in \mathrm{Spin}(3)$ correspond respectively to $i\sigma_1, i\sigma_2, i\sigma_3 \in SU(2)$.

The quotient mapping $\eta\colon SU(2) \to \mathbb{S}^2$ may be described more economically by regarding $\mathbb{S}^2$ as $\mathbb{CP}^1$ and proceeding as follows.

Definition 9.6. The **Hopf fibration** $\eta\colon SU(2) \to \mathbb{CP}^1$ is the map given by

$$\eta \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} := \frac{\beta}{\bar{\alpha}}. \tag{9.26}$$

It is immediate that

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \overset{\eta}{\longmapsto} \frac{\beta}{\bar{\alpha}},$$

so $SU(2) \xrightarrow{\eta} \mathbb{CP}^1$ is a principal $U(1)$-bundle, where the free right action of $U(1)$ on the fibres $\eta^{-1}(\beta/\bar{\alpha})$ is given simply by multiplication on the right by the diagonal elements of $SU(2)$.

Exercise 9.13. The Hopf fibration decomposes the sphere $\mathbb{S}^3 \approx SU(2)$ into a disjoint union of circles (the fibres), any two of which are *linked*. Regard $\mathbb{S}^3$ as $\mathbb{R}^3 \uplus \{\infty\}$ via the *stereographic projection* $(\alpha, \beta) \mapsto (w, t)$ where

$$w := \frac{\alpha}{1 - \Im\beta} \in \mathbb{C}, \qquad t := \frac{\Re\beta}{1 - \Im\beta} \in \mathbb{R}.$$

Check that $\chi(w, t) := (2t + i(w\bar{w} + t^2 - 1))/2\bar{w} \in \mathbb{C}_\infty$ is the expression in (w, t)-coordinates of the Hopf fibration. Deduce that the equation $\chi(w, t) = z$ represents the circle obtained by cutting the sphere $w\bar{w} + t^2 - i(\bar{z}w - z\bar{w}) = 1$ with the equatorial plane $\bar{z}w + z\bar{w} - 2t = 0$; in particular, $\chi(w, t) = \infty$ is the t-axis (including ∞) and $\chi(w, t) = 0$ is the unit circle in the $w\bar{w}$-plane. Show that any other circle $\chi(w, t) = z$ cuts the $w\bar{w}$-plane obliquely at two points, one inside and one outside the unit circle; conclude that the circles $\chi(w, t) = 0$ and $\chi(w, t) = z$ are linked.

The group $SU(2)$ acts on itself by left translations $\lambda(g)\colon h \mapsto gh$; since these commute with right translations by $U(1)$, they define a left action on the quotient manifold $\mathbb{CP}^1$, which is just the aforementioned rotation action. Indeed, if $h = (1 + z\bar{z})^{-1} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}$, then $\eta(h) = z$, and from

$$\frac{1}{1 + z\bar{z}} \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix} = \frac{1}{1 + z\bar{z}} \begin{pmatrix} \alpha - \beta\bar{z} & \alpha z + \beta \\ -\bar{\beta} - \bar{\alpha}\bar{z} & -\bar{\beta}z + \bar{\alpha} \end{pmatrix}$$

it follows that $\eta(gh) = g \cdot \eta(h) \equiv \rho(g)\eta(h)$ for $g, h \in SU(2)$. Schematically, we get a commutative diagram

$$\begin{array}{ccc} SU(2) & \xrightarrow{\lambda(g)} & SU(2) \\ {\scriptstyle\eta}\big\downarrow & & \big\uparrow{\scriptstyle\eta} \\ \mathbb{S}^2 & \xrightarrow{\rho(g)} & \mathbb{S}^2 \end{array}$$

which says that each pair of maps $(\lambda(g), \rho(g))$ is a morphism of the principal $U(1)$-bundle $SU(2) \xrightarrow{\eta} \mathbb{CP}^1$.

We may say that the group $SU(2)$ acts "equivariantly" on the principal bundle. The corresponding type of group action on vector bundles is defined as follows.

Definition 9.7. A **homogeneous vector bundle** with symmetry group G (a Lie group)[5] is a vector bundle $E \longrightarrow M$ together with a pair of (left) actions $\tau\colon G \times E \to E$ and $\rho\colon G \times M \to M$ such that each $(\tau(g), \rho(g))$ is a vector bundle morphism on $E \longrightarrow M$. We shall call such a pair a *bundle action* of G.

A *Hermitian homogeneous vector bundle* is a Hermitian vector bundle with a bundle action for which each $\tau(g)_x \in \mathrm{End}(E_x)$ is unitary. If $E = E^+ \oplus E^-$ is a superbundle, we say the bundle action of G is *even* if $\tau(g)_x \in \mathrm{End}^+(E_x)$ for each $x \in M$; in other words, if both subbundles $E^\pm \longrightarrow M$ are G-homogeneous under the bundle action (τ, ρ).

We seek to define an even action of $SU(2)$ on the spinor bundle $S \longrightarrow \mathbb{S}^2$. This can be pictured as a commutative diagram

$$\begin{array}{ccc} S^\pm & \xrightarrow{\tau(g)} & S^\pm \\ {\scriptstyle\pi^\pm}\big\downarrow & & \big\uparrow{\scriptstyle\pi^\pm} \\ \mathbb{S}^2 & \xrightarrow{\rho(g)} & \mathbb{S}^2 \end{array}$$

which, in terms of the spinor components $\psi^\pm \in \Gamma(S^\pm)$, means that

$$\begin{aligned} \tau(g)\psi_N^\pm(z, \bar z) &= A_N^\pm(g, z)\, \psi_N^\pm(g^{-1} \cdot z, (g^{-1} \cdot z)^-), \\ \tau(g)\psi_S^\pm(\zeta, \bar\zeta) &= A_S^\pm(g', \zeta)\, \psi_S^\pm(g'^{-1} \cdot \zeta, (g'^{-1} \cdot \zeta)^-), \end{aligned} \tag{9.27}$$

for $g \in SU(2)$. By unitarity, the multipliers $A^\pm$ must be $U(1)$-valued functions; and they must satisfy the following consistency conditions in order that (9.27) define a group action:

$$A_N^\pm(gh, z) = A_N^\pm(g, z) A_N^\pm(h, g^{-1} \cdot z), \tag{9.28}$$

$$A_S^\pm(g'h', \zeta) = A_S^\pm(g', \zeta) A_S^\pm(h', g'^{-1} \cdot \zeta). \tag{9.29}$$

Exercise 9.14. Show that $A(g, z) := (\beta \bar z + \bar\alpha)^k / (\bar\beta z + \alpha)^k$ is a formal solution to the equation $A(gh, z) = A(g, z) A(h, g^{-1} \cdot z)$, for $g, h \in SU(2)$, $z \in \mathbb{C}_\infty$; and that this solution is well-defined provided $2k$ is an integer.

Bearing in mind that

$$g^{-1} \cdot z = \frac{\bar\alpha z - \beta}{\bar\beta z + \alpha}, \qquad g'^{-1} \cdot \zeta = \frac{\alpha\zeta + \bar\beta}{-\beta\zeta + \bar\alpha},$$

the gauge transformation rule (9.7) for the spinors $\tau(g)\psi$ yields

$$A_S^+(g', \zeta)\psi_S^+\left(\frac{\alpha\zeta + \bar\beta}{-\beta\zeta + \bar\alpha}, \frac{\bar\alpha\bar\zeta + \beta}{-\bar\beta\bar\zeta + \alpha}\right) = (\bar\zeta/\zeta)^{1/2} A_N^+(g, \zeta^{-1})\psi_N^+\left(\frac{\bar\alpha - \beta\zeta}{\bar\beta + \alpha\zeta}, \frac{\alpha - \bar\beta\bar\zeta}{\beta + \bar\alpha\bar\zeta}\right). \tag{9.30}$$

[5]This is often called a *G-vector bundle*, for short.

The gauge transformation

$$\psi_S^+\left(\frac{\alpha\zeta+\bar\beta}{-\beta\zeta+\bar\alpha}, \frac{\bar\alpha\bar\zeta+\beta}{-\bar\beta\bar\zeta+\alpha}\right) = \left(\frac{\bar\alpha\bar\zeta+\beta}{-\bar\beta\bar\zeta+\alpha} \Big/ \frac{\alpha\zeta+\bar\beta}{-\beta\zeta+\bar\alpha}\right)^{1/2} \psi_N^+\left(\frac{\bar\alpha-\beta\zeta}{\bar\beta+\alpha\zeta}, \frac{\alpha-\bar\beta\bar\zeta}{\beta+\bar\alpha\bar\zeta}\right)$$

shows that the coefficients in (9.30) must satisfy the relation

$$\frac{A_S^+(g',\zeta)}{A_N^+(g,\zeta^{-1})} = \left(\frac{\bar\zeta(\alpha\zeta+\bar\beta)(-\bar\beta\bar\zeta+\alpha)}{\zeta(-\beta\zeta+\bar\alpha)(\bar\alpha\bar\zeta+\beta)}\right)^{1/2} = \left(\frac{(\alpha+\bar\beta\zeta^{-1})(-\bar\beta\bar\zeta+\alpha)}{(-\beta\zeta+\bar\alpha)(\bar\alpha+\beta\bar\zeta^{-1})}\right)^{1/2},$$

or equivalently

$$\frac{A_S^+(g',\zeta)}{A_N^+(g,z)} = \left(\frac{-\bar\beta\bar\zeta+\alpha}{-\beta\zeta+\bar\alpha}\right)^{1/2} \Big/ \left(\frac{\beta\bar z+\bar\alpha}{\bar\beta z+\alpha}\right)^{1/2}. \tag{9.31}$$

The following solution of (9.29) is therefore consistent with the spinor gauge transformations:

$$A_N^+(g,z) := \left(\frac{\beta\bar z+\bar\alpha}{\bar\beta z+\alpha}\right)^{1/2}, \qquad A_S^+(g',\zeta) := \left(\frac{-\bar\beta\bar\zeta+\alpha}{-\beta\zeta+\bar\alpha}\right)^{1/2}. \tag{9.32}$$

Substituting these in (9.27) yields a bundle action of $SU(2)$ on $S^+ \longrightarrow \mathbb{S}^2$.

The same procedure leads to a bundle action of $SU(2)$ on $S^- \longrightarrow \mathbb{S}^2$. One need only replace the term $(\bar\zeta/\zeta)^{1/2}$ in (9.30) by $(\zeta/\bar\zeta)^{1/2}$ when invoking the gauge transformation rule (9.7); this leads to the choice of $A_N^-(g,z)$ and $A_S^-(g',\zeta)$ as the complex conjugates of (9.32):

$$A_N^-(g,z) := \left(\frac{\beta\bar z+\bar\alpha}{\bar\beta z+\alpha}\right)^{-1/2}, \qquad A_S^-(g',\zeta) := \left(\frac{-\bar\beta\bar\zeta+\alpha}{-\beta\zeta+\bar\alpha}\right)^{-1/2}. \tag{9.33}$$

We summarize the foregoing in a definition.

Definition 9.8. The Lie group $SU(2)$ acts on the spinor bundle via (τ,ρ), where ρ is the rotation action (9.24) on the Riemann sphere, and τ is given by (9.27), where the multipliers $A_N^\pm$ and $A_S^\pm$ are defined by (9.32) and (9.33).

9.7 Equivariance of the Dirac operator

Lemma 9.5. *Let $T \in \operatorname{End}^+(\Gamma(U_0,S))$ be a transformation of the form*

$$\begin{aligned}(T\psi_N^+)(z,\bar z) &:= a(z,\bar z)^{1/2}\psi_N^+\big(b(z),\overline{b(z)}\big),\\ (T\psi_N^-)(z,\bar z) &:= a(z,\bar z)^{-1/2}\psi_N^-\big(b(z),\overline{b(z)}\big).\end{aligned} \tag{9.34}$$

where a is a smooth function on $\mathbb{C}^\times$ and b is a rational function on $\mathbb{C}_\infty$. Then $T\eth_z = \eth_z T$ as operators from $\Gamma(U_0,S^-)$ to $\Gamma(U_0,S^+)$ if and only if a, b satisfy the pair of differential equations

$$\begin{aligned}(1+z\bar z)\frac{db}{dz} &= a(z,\bar z)(1+b(z)\overline{b(z)}),\\ (1+z\bar z)\frac{\partial a}{\partial z} &= a(z,\bar z)^2\overline{b(z)} - \bar z a(z,\bar z).\end{aligned} \tag{9.35}$$

Proof. It is enough to notice that

$$T(\eth_z\psi_N^-)(z,\bar z) = a^{1/2}(1+b\bar b)\frac{\partial\psi_N^-}{\partial z}(b,\bar b) - \tfrac12 a^{1/2}\bar b\psi_N^-(b,\bar b),$$

whereas

$$\eth_z(T\psi_N^-)(z,\bar z) = (1+z\bar z)\Big\{-\tfrac12 a^{-3/2}\frac{\partial a}{\partial z}\psi_N^-(b,\bar b) + a^{-1/2}\frac{db}{dz}\frac{\partial\psi_N^-}{\partial z}(b,\bar b)\Big\} - \tfrac12\bar z a^{-1/2}\psi_N^-(b,\bar b),$$

using both halves of (9.34); and then to equate coefficients of ψ_N^- and $\partial\psi_N^-/\partial z$. □

Proposition 9.6. *The Dirac operator on the sphere is equivariant under the action of $SU(2)$ on the spinor bundle, i.e., $\tau(g)D\!\!\!/ = D\!\!\!/\tau(g)$ on $\Gamma(S)$, for all $g \in SU(2)$.*

Proof. To show that $\tau(g)D\!\!\!/ = D\!\!\!/\tau(g)$ for all g, it is enough to check this for g belonging to a collection of one-parameter subgroups which generate $SU(2)$. Since $\alpha\bar\alpha + \beta\bar\beta = 1$, we can write $\alpha = \exp(\frac{i}{2}\phi + \frac{i}{2}\psi)\cos\frac12\theta$, $\beta = \exp(\frac{i}{2}\phi - \frac{i}{2}\psi)\sin\frac12\theta$; we thereby see that any $g \in SU(2)$ is of the form $k(\phi)h(\theta)k(\psi)$, where[6]

$$k(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2}\end{pmatrix}, \qquad h(\theta) = \begin{pmatrix} \cos\frac12\theta & \sin\frac12\theta \\ -\sin\frac12\theta & \cos\frac12\theta\end{pmatrix}. \tag{9.36}$$

Now $\tau(k(t))$ is of the form (9.34) with $a(z,\bar z) = e^{-it}$, $b(z) = e^{-it}z$, so the equations (9.35) reduce to the identities

$$(1+z\bar z)e^{-it} = e^{-it}(1+z\bar z), \qquad 0 = e^{-2it}e^{it}\bar z - \bar z e^{-it}.$$

On the other hand, $\tau(h(t))$ is of the form (9.34) with

$$a(z,\bar z) = \frac{\bar z\sin\frac12 t + \cos\frac12 t}{z\sin\frac12 t + \cos\frac12 t}, \qquad b(z) = \frac{z\cos\frac12 t - \sin\frac12 t}{z\sin\frac12 t + \cos\frac12 t}. \tag{9.37}$$

for which

$$\frac{\partial a}{\partial z} = -\sin\tfrac12 t\,\frac{\bar z\sin\frac12 t + \cos\frac12 t}{(z\sin\frac12 t + \cos\frac12 t)^2}, \qquad \frac{db}{dz} = \frac{1}{(z\sin\frac12 t + \cos\frac12 t)^2}. \tag{9.38}$$

From this it is easy to check that (9.37) satisfies the equations (9.35) for all $t \in \mathbb{R}$.

Thus $\tau(g)\eth_z\psi_N^- = \eth_z\tau(g)\psi_N^-$ for all $g \in SU(2)$. By applying complex conjugation, we obtain $\bar\eth_z\tau(g^{-1}) = \tau(g^{-1})\bar\eth_z$ on functions ψ_N^+, and so $\tau(g)D\!\!\!/_N = D\!\!\!/_N\tau(g)$ for all g. Replacing z by ζ and g by g' and adjusting a few signs, the same calculations show that $\tau(g)D\!\!\!/_S = D\!\!\!/_S\tau(g)$ for all g, as expected. □

Exercise 9.15. Verify directly that (9.37) satisfies the equations (9.35).

[6] The parameters (ϕ,θ,ψ) in this product are the so-called *Euler angles* for the group $SU(2)$.

9.8 Angular momentum operators

Definition 9.9. The homomorphisms $t \mapsto g(\vec{n};t) := \exp(\frac{1}{2}it\,\vec{n}\cdot\vec{\sigma}) = \cos\frac{1}{2}t + i\sin\frac{1}{2}t\,\vec{n}\cdot\vec{\sigma}$, for each $\vec{n} \in \mathbb{S}^2$, yield all one-parameter subgroups of $SU(2)$. The infinitesimal generators of these subgroups are $-\frac{i}{2}\vec{n}\cdot\vec{\sigma} \in \mathfrak{su}(2)$, where $\mathfrak{su}(2)$ is the Lie algebra of antihermitian 2×2 matrices. The corresponding generator $J_{\vec{n}}$ of the spinor action of this subgroup is

$$-i(J_{\vec{n}}\psi_N^\pm)(z,\bar{z}) := \frac{d}{dt}\bigg|_{t=0} \tau(g(\vec{n};t))\psi_N^\pm(z,\bar{z}) = \frac{d}{dt}\bigg|_{t=0} a_t(z,\bar{z})^{\pm 1/2}\psi_N^\pm\big(b_t(z), \overline{b_t(z)}\big), \tag{9.39}$$

where the coefficient $(-i)$ is inserted for convenience, so that $J_{\vec{n}}$ is formally selfadjoint (rather than skewadjoint).

For the three cardinal directions, where $\vec{n}\cdot\vec{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$, we write the generators simply as J_1, J_2, J_3 respectively; these are commonly called the *angular momentum generators*.[7] As before, we write $J_\pm := J_1 \pm iJ_2$.

With the notations $\dot{a}_0(z,\bar{z}) := \frac{d}{dt}\big|_{t=0} a_t(z,\bar{z})$ and $\dot{b}_0(z) := \frac{d}{dt}\big|_{t=0} b_t(z)$, the definition (9.39) simplifies to

$$J_{\vec{n}} = -i\Big(\dot{b}_0(z)\frac{\partial}{\partial z} + \overline{\dot{b}_0(z)}\frac{\partial}{\partial \bar{z}}\Big) + \tfrac{1}{2}\dot{a}_0(z,\bar{z})\,\gamma^1\gamma^2 \tag{9.40}$$

on recalling that $i\gamma^1\gamma^2 = \pm 1$ on $\Gamma(S^\pm)$.

Thus, for the one-parameter subgroup $\{\, k(-t) : t \in \mathbb{R} \,\}$, where $a_t(z,\bar{z}) = e^{it}$ and $b_t(z) = e^{it}z$, we get $\dot{a}_0(z,\bar{z}) = i$, $\dot{b}_0(z) = iz$, and so

$$J_3 = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + \tfrac{1}{2}\,i\gamma^1\gamma^2 = -i\frac{\partial}{\partial \phi} + \tfrac{1}{2}\,i\gamma^1\gamma^2. \tag{9.41}$$

For the one-parameter subgroup $\{\, h(-t) : t \in \mathbb{R} \,\}$, $a_t(z,\bar{z})$ and $b_t(z)$ are given by (9.37) with t replaced by $-t$. In this case $\dot{a}_0(z,\bar{z}) = \frac{1}{2}(z-\bar{z})$ and $\dot{b}_0(z) = \frac{1}{2}(z^2+1)$, leading to

$$J_2 = \tfrac{i}{2}(z^2+1)\frac{\partial}{\partial z} + \tfrac{i}{2}(\bar{z}^2+1)\frac{\partial}{\partial \bar{z}} - \tfrac{1}{4}(z-\bar{z})\,\gamma^1\gamma^2.$$

The one-parameter subgroup $\{\, \cos\frac{1}{2}t + i\sin\frac{1}{2}t\,\sigma_1 : t \in \mathbb{R} \,\}$ is generated by $\frac{i}{2}\sigma_1$, and $a_t(z,\bar{z})$, $b_t(z)$ are now given by

$$a_t(z,\bar{z}) = \frac{\cos\frac{1}{2}t + i\bar{z}\sin\frac{1}{2}t}{\cos\frac{1}{2}t - iz\sin\frac{1}{2}t}, \qquad b_t(z) = \frac{z\cos\frac{1}{2}t - i\sin\frac{1}{2}t}{\cos\frac{1}{2}t - iz\sin\frac{1}{2}t},$$

for which $\dot{a}_0(z,\bar{z}) = \frac{i}{2}(\bar{z}+z)$ and $\dot{b}_0(z) = \frac{i}{2}(z^2-1)$. Therefore

$$J_1 = -\tfrac{1}{2}(z^2-1)\frac{\partial}{\partial z} + \tfrac{1}{2}(\bar{z}^2-1)\frac{\partial}{\partial \bar{z}} - \tfrac{i}{4}(z+\bar{z})\,\gamma^1\gamma^2.$$

[7] The suitability of interpreting these generators as angular momentum operators for a magnetic monopole is discussed at length in [10].

The operators $J_\pm$ are therefore given by:

$$\begin{aligned} J_+ &= J_1 + iJ_2 = -z^2\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar z} - \tfrac12 z\, i\gamma^1\gamma^2, \\ J_- &= J_1 - iJ_2 = \frac{\partial}{\partial z} + \bar z^2\frac{\partial}{\partial \bar z} - \tfrac12 \bar z\, i\gamma^1\gamma^2. \end{aligned} \tag{9.42}$$

It follows from this and from (9.41) that $[J_+, J_-] = 2J_3$. One also obtains that $J_1^2 + J_2^2 = J_-J_+ + \frac12[J_+, J_-] = J_-J_+ + J_3$.

Exercise 9.16. Show that the operators $J_\pm$ are given in spherical coordinates by

$$J_\pm = e^{\pm i\phi}\biggl(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi} - \frac{1+\cos\theta}{2\sin\theta} i\gamma^1\gamma^2\biggr).$$

These operators arise in the theory of the magnetic monopole [10, 25] when the monopole parameter[8] $\mu = eg/\hbar c$ takes the value $\mu = \frac12$.

Exercise 9.17. Show that over U_1 the angular momentum generators satisfy the analogue of (9.40), with z replaced by ζ. Verify that

$$\begin{aligned} J_3 &= -\zeta\frac{\partial}{\partial\zeta} + \bar\zeta\frac{\partial}{\partial\bar\zeta} - \tfrac12\, i\gamma^1\gamma^2, \\ J_+ &= \frac{\partial}{\partial\zeta} + \bar\zeta^2\frac{\partial}{\partial\bar\zeta} - \tfrac12\bar\zeta\, i\gamma^1\gamma^2, \\ J_- &= -\zeta^2\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\bar\zeta} - \tfrac12\zeta\, i\gamma^1\gamma^2, \end{aligned} \tag{9.43}$$

in the coordinates $(\zeta, \bar\zeta)$ over U_1.

There is one more operator worthy of mention, namely that corresponding to the "Casimir element" $X_1^2+X_2^2+X_3^2$, where $\{X_1, X_2, X_3\}$ is an orthonormal basis for the Lie algebra $\mathfrak{su}(2)$. [The Casimir element belongs to the enveloping algebra $\mathfrak{U}(\mathfrak{su}(2))$.] The image of this element under a representation of the Lie algebra is called a "Casimir operator".

Definition 9.10. The **Casimir operator** for the spinor bundle action of $SU(2)$ is the operator defined by $C := J_1^2 + J_2^2 + J_3^2 = J_-J_+ + J_3(J_3+1)$ on $\Gamma(S)$.[9]

Proposition 9.7. *The Casimir operator satisfies the relations*

$$C = \Delta^S + \tfrac14 = D\!\!\!/^2 - \tfrac14. \tag{9.44}$$

[8]The constants are c, the speed of light; $\hbar$, Planck's constant; e, the electric charge of a particle whose total angular momentum is J; and g, the magnetic charge of the monopole. The condition that 2μ be an integer is the quantization condition of Dirac [24], and arises from the necessary description of monopoles by complex line bundles over the sphere.

[9]Strictly speaking the Casimir operator should be $-C$, on account of the factor $(-i)$ in (9.39); but we change the sign to obtain a positive operator. On $\Gamma(S)$, it is formally selfadjoint.

Proof. Using (9.41) and (9.42), we compute that

$$\begin{aligned} C &= J_- J_+ + J_3(J_3+1) \\ &= -(1+z\bar{z})^2 \frac{\partial^2}{\partial z\,\partial\bar{z}} + \tfrac{1}{4}(1+z\bar{z}) + \tfrac{1}{2}(1+z\bar{z})\, i\gamma^1\gamma^2\Big(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}\Big) \end{aligned}$$

over U_0, with an analogous formula over U_1. A glance at (9.20) and (9.23) is enough to verify (9.44). □

This result shows that even in the simplest example of a spinor bundle over a compact manifold, the three operators commonly referred to as "the Laplacian" are distinct, and must be carefully distinguished. Though they only differ by constants, this has the important consequence that their spectra are not the same. A similar shifting of the Laplacian occurs in harmonic analysis on compact Lie groups [59], where the Casimir satisfies $C = \Delta + \frac{1}{12}\dim G$. For $SU(2)$, a 3-dimensional group, this leads us to expect $C = \Delta + \frac{1}{4}$, which is nicely confirmed by Proposition 9.7.

9.9 Spinor harmonics

We come, finally, to the matter of diagonalizing the Dirac operator by finding an explicit basis of eigenspinors for $D\!\!\!\!/$. In view of the $SU(2)$ symmetry, we may suspect, by analogy with the diagonalization of the Hodge–Dirac operator in Section 4, that the members of this basis should be closely related to the spherical harmonics $Y_{lm}(\theta,\phi)$ on $\mathbb{S}^2$. We may also anticipate that at some point we shall need to use some heavy artillery from the representation theory of $SU(2)$. However, we begin in a fairly pedestrian manner, with some polynomial calculations over the chart U_0.

Lemma 9.8. *The identity*

$$\eth_z\big((1+z\bar{z})^{-l} z^r(-\bar{z})^s\big) = (1+z\bar{z})^{-l}\big((l+\tfrac{1}{2}-r) z^r(-\bar{z})^{s+1} + r z^{r-1}(-\bar{z})^s\big)$$

holds for all $r,s\in\mathbb{N}$ and $l\in\mathbb{R}$. □

Exercise 9.18. Prove Lemma 9.8, and show also that the conjugate identity holds: $-\bar{\eth}_z\big((1+z\bar{z})^{-l} z^r(-\bar{z})^s\big) = (1+z\bar{z})^{-l}\big((l+\frac{1}{2}-s) z^{r+1}(-\bar{z})^s + s z^r(-\bar{z})^{s-1}\big)$.

These calculations show one method of finding eigenspinors: take for ψ_N^- a linear combination of several terms of the form $(1+z\bar{z})^{-l} z^r(-\bar{z})^s$, with a common value for the difference of exponents $(r-s)$, and choose the coefficients cleverly enough that the result of applying $\eth_z$ closely resembles a multiple of the original function. However, since we wish these functions to be components of *spinors*, we must first consider the effect of the gauge transformation rules.

Lemma 9.9. *Let $\phi\colon \mathbb{C}\to\mathbb{C}$ be a smooth function of the form*

$$\phi(z,\bar{z}) := (1+z\bar{z})^{-l} \sum_{r,s\in\mathbb{N}} a(r,s) z^r(-\bar{z})^s.$$

Then ϕ represents a section in $\Gamma(U_0, S^\pm)$ if and only if $l + \frac{1}{2}$ is a positive integer, and $a(r,s) = 0$ for $r > l \mp \frac{1}{2}$ or $s > l \pm \frac{1}{2}$. Moreover, the coefficients must satisfy the symmetry relations $a(r,s) = (-1)^{l \pm \frac{1}{2}} a(l \mp \frac{1}{2} - r, l \pm \frac{1}{2} - s)$.

Proof. Suppose that ϕ represents a section in $\Gamma(U_0, S^+)$. Then by (9.7) we obtain

$$\begin{aligned}\phi(z,\bar{z}) = (\bar{z}/z)^{1/2}\phi(z^{-1},\bar{z}^{-1}) &= z^{-1/2}\bar{z}^{1/2}(z\bar{z})^l(1+z\bar{z})^{-l}\sum_{r,s\in\mathbb{N}} a(r,s)z^{-r}(-\bar{z})^{-s}\\ &= (1+z\bar{z})^{-l}(-1)^{l+\frac{1}{2}}\sum_{r,s\in\mathbb{N}} a(r,s)z^{l-\frac{1}{2}-r}(-\bar{z})^{l+\frac{1}{2}-s},\end{aligned}$$

where the exponents in the sum on the right hand side must also be nonnegative integers. Thus $l - \frac{1}{2} \in \mathbb{N}$, and the nonnegativity of the exponents on the right guarantees that $r \in \{0, 1, \dots, l - \frac{1}{2}\}$ while $s \in \{0, 1, \dots, l + \frac{1}{2}\}$. The argument for sections in $\Gamma(U_0, S^-)$ is similar. □

The structure of the symmetry relations among the coefficients, and the allowed ranges of the exponents, suggests the introduction of the following spinors.

Definition 9.11. For each $l \in \mathbb{N} + \frac{1}{2} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$, and for each $m \in \{-l, -l+1, \dots, l-1, l\}$,[10] let $Y'_{lm} \in \Gamma(S)$ be the spinors whose components over U_0 are $2^{-1/2}Y^\pm_{lm}(z,\bar{z})$, where

$$\begin{aligned}Y^+_{lm}(z,\bar{z}) &:= C_{lm}(1+z\bar{z})^{-l}\sum_{r-s=m-\frac{1}{2}}\binom{l-\frac{1}{2}}{r}\binom{l+\frac{1}{2}}{s}z^r(-\bar{z})^s,\\ Y^-_{lm}(z,\bar{z}) &:= C_{lm}(1+z\bar{z})^{-l}\sum_{r-s=m+\frac{1}{2}}\binom{l+\frac{1}{2}}{r}\binom{l-\frac{1}{2}}{s}z^r(-\bar{z})^s,\end{aligned} \tag{9.45}$$

where the constants C_{lm} are defined as[11]

$$C_{lm} := (-1)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!\,(l-m)!}{(l+\frac{1}{2})!\,(l-\frac{1}{2})!}}. \tag{9.46}$$

Also, let $Y''_{lm} \in \Gamma(S)$ be the spinor whose components are $2^{-1/2}Y^+_{lm}(z,\bar{z})$ and $-2^{-1/2}Y^-_{lm}(z,\bar{z})$.

The simplest examples of (9.45) are

$$Y'_{\frac{1}{2},\frac{1}{2}} := \frac{1}{\sqrt{4\pi}}\begin{pmatrix}1/\sqrt{1+z\bar{z}}\\ z/\sqrt{1+z\bar{z}}\end{pmatrix}, \qquad Y'_{\frac{1}{2},-\frac{1}{2}} := \frac{1}{\sqrt{4\pi}}\begin{pmatrix}\bar{z}/\sqrt{1+z\bar{z}}\\ -1/\sqrt{1+z\bar{z}}\end{pmatrix}.$$

[10]The range of allowed values of m is obtained by listing the possibilities for $(r-s)$ in (9.45) and adjusting by $\frac{1}{2}$. Notice that each m is a half-integer.

[11]The precise form of the constants C_{lm} is obtained by looking ahead to the normalization $\langle\langle Y'_{lm} | Y'_{lm}\rangle\rangle = 1$; for the present, we need only that the constants for Y^+_{lm} and Y^-_{lm} be the same.

The functions Y^+_{lm} and Y^-_{lm} were introduced by Newman and Penrose [43], using the notations ${}_{-\frac12}Y_{lm}$ and ${}_{\frac12}Y_{lm}$ respectively. In fact, these appear as a subfamily of functions ${}_sY_{lm}$ with $s \in \{-l, -l+1, \dots, l-1, l\}$ which, for l, m, s integers, they called "spin-s spherical harmonics"; for $s = 0$ they reduce to the everyday spherical harmonics Y_{lm} on $\mathbb{S}^2$. Newman and Penrose also noted that their formulas make sense when l, m, s are all half-integers, and christened such functions "spinor harmonics"; they were investigated further by Goldberg *et al* [30]. Later, Dray [25] showed that these same functions occur as the spinor components in the theory of the magnetic monopole [10].

Lemma 9.10. $\eth_z Y^-_{lm} = (l+\frac12)Y^+_{lm}$ *and* $-\bar\eth_z Y^+_{lm} = (l+\frac12)Y^-_{lm}$.

Proof. Using Lemma 9.8, we find that $\eth_z Y^-_{lm}(z,\bar z)$ equals

$$\begin{aligned} &C_{lm}(1+z\bar z)^{-l} \sum_{r-s=m+\frac12} \binom{l+\frac12}{r}\binom{l-\frac12}{s}\{(l+\tfrac12-r)z^r(-\bar z)^{s+1} + rz^{r-1}(-\bar z)^s\} \\ &= C_{lm}(1+z\bar z)^{-l} \sum_{j-k=m-\frac12} \biggl\{(l+\tfrac12-j)\binom{l+\frac12}{j}\binom{l-\frac12}{k-1} \\ &\qquad\qquad + (j+1)\binom{l+\frac12}{j+1}\binom{l-\frac12}{k}\biggr\} z^j(-\bar z)^k. \end{aligned} \tag{9.47}$$

The term in braces can be simplified, using the binomial identities

$$k\binom{r}{k} = r\binom{r-1}{k-1}, \qquad (r-k)\binom{r}{k} = r\binom{r-1}{k},$$

to the form

$$(l+\tfrac12)\binom{l-\frac12}{j}\binom{l-\frac12}{k-1} + (l+\tfrac12)\binom{l-\frac12}{j}\binom{l-\frac12}{k} = (l+\tfrac12)\binom{l-\frac12}{j}\binom{l+\frac12}{k},$$

so the right hand side of (9.47) equals $(l+\frac12)Y^+_{lm}$. □

Corollary 9.11. *The spinors Y'_{lm} and Y''_{lm} are eigenspinors for the Dirac operator, with nonzero integer eigenvalues $\pm(l+\frac12)$:*

$$\not{D} Y'_{lm} = (l+\tfrac12)Y'_{lm}, \qquad \not{D} Y''_{lm} = -(l+\tfrac12)Y''_{lm},$$

and each eigenvalue $\pm(l+\frac12)$ has multiplicity $(2l+1)$.

Proof. Just observe that

$$\begin{pmatrix} 0 & \eth_z \\ -\bar\eth_z & 0 \end{pmatrix}\begin{pmatrix} Y^+_{lm} \\ Y^-_{lm} \end{pmatrix} = \begin{pmatrix} (l+\frac12)Y^+_{lm} \\ (l+\frac12)Y^-_{lm} \end{pmatrix}, \qquad \begin{pmatrix} 0 & \eth_z \\ -\bar\eth_z & 0 \end{pmatrix}\begin{pmatrix} Y^+_{lm} \\ -Y^-_{lm} \end{pmatrix} = \begin{pmatrix} -(l+\frac12)Y^+_{lm} \\ (l+\frac12)Y^-_{lm} \end{pmatrix}.$$

The multiplicity is just the number of possibilities for the index m, i.e., the $(2l+1)$ elements of $\{-l, -l+1, \dots, l-1, l\}$. □

Exercise 9.19. For a fixed $l \in \mathbb{N} + \frac12$, express $J_3 Y'_{lm}$, $J_+ Y'_{lm}$ and $J_- Y'_{lm}$ as linear combinations of the spinors Y'_{ln} with $n \in \{-l, -l+1, \dots, l-1, l\}$.

9.10 The spectrum of the Dirac operator

Definition 9.12. A *representative function* for the group $SU(2)$ is a function in $L^2(SU(2))$ which may appear as a matrix element in some finite-dimensional unitary representation of the group. It is a linear combination of the functions $\mathcal{D}^j_{mn}$, indexed by $j \in \frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ and $m, n \in \{-j, -j+1, \dots, j-1, j\}$, which are defined in terms of the Euler-angle presentation $g = k(\alpha)h(\beta)k(\gamma) \in SU(2)$ by

$$\begin{aligned}\mathcal{D}^j_{mn}(\alpha,\beta,\gamma) := {} & \sqrt{\frac{(j+n)!\,(j-n)!}{(j+m)!\,(j-m)!}}\, e^{i(n\alpha+m\gamma)}(\sin\tfrac{1}{2}\beta)^{2j} \\ & \times \sum_r (-1)^{j+m-r}\binom{j+m}{r}\binom{j-m}{r-m-n}(\cot\tfrac{1}{2}\beta)^{2r-m-n}. \end{aligned} \tag{9.48}$$

The Hilbert space $L^2(SU(2))$ is described by defining the Haar measure on $SU(2)$ in terms of the Euler angles as $dg = (16\pi^2)^{-1}\sin\beta\,d\alpha\,d\beta\,d\gamma$, and the *Parseval–Plancherel formula* [37] shows that the functions $\mathcal{D}^j_{mn}$ form an orthogonal basis for this Hilbert space:

$$\int_{SU(2)} |h(g)|^2\,dg = \sum_{2j=0}^{\infty}(2j+1)\sum_{m,n=-j}^{j}\bigl|(\mathcal{D}^j_{mn} \mid h)\bigr|^2.$$

On comparing the definitions (9.45), (9.46) of the functions $Y^{\pm}_{lm}(z,\bar z)$ with (9.48), we see from $z = e^{i\phi}\cot\frac{1}{2}\theta$ that

$$Y^+_{lm}(z,\bar z) = \sqrt{\frac{2l+1}{4\pi}}\,\mathcal{D}^l_{-\frac{1}{2},m}(\phi,\theta,\phi), \quad Y^-_{lm}(z,\bar z) = \sqrt{\frac{2l+1}{4\pi}}\,\mathcal{D}^l_{\frac{1}{2},m}(\phi,\theta,\phi).$$

Exercise 9.20. Show that, with $\zeta = e^{-i\phi}\tan\frac{1}{2}\theta$, the formulae

$$Y^+_{lm}(\zeta,\bar\zeta) = \sqrt{\frac{2l+1}{4\pi}}\,\mathcal{D}^l_{-\frac{1}{2},m}(\phi,\theta,-\phi), \quad Y^-_{lm}(\zeta,\bar\zeta) = \sqrt{\frac{2l+1}{4\pi}}\,\mathcal{D}^l_{\frac{1}{2},m}(\phi,\theta,-\phi)$$

express $Y^{\pm}_{lm}$ in terms of the $SU(2)$ representative functions over U_1.

We now (at last!) fix the normalization of the inner product of spinors by

$$\|\psi\|^2 = \langle\!\langle \psi \mid \psi \rangle\!\rangle := \frac{1}{4\pi}\int_{\mathbb{S}^2} (\overline{\psi^+}\psi^+ + \overline{\psi^-}\psi^-)\,\sin\theta\,d\theta\wedge d\phi. \tag{9.49}$$

Proposition 9.12. *The spinors* $\{\, Y'_{lm}, Y''_{lm} : l \in \mathbb{N}+\frac{1}{2}, m \in \{-l,\dots,l\}\,\}$ *form an orthonormal basis for the Hilbert space* $L^2(S)$.

Proof. We associate to each spinor $\psi \in \Gamma(S)$ a pair of functions $h^{\pm}$ on $SU(2)$ by

$$h^{\pm}(\phi,\theta,\psi) := \sqrt{4\pi}e^{\pm i(\phi-\psi)/2}\,\psi^{\pm}_N(z,\bar z) = \sqrt{4\pi}e^{\mp i(\phi+\psi)/2}\,\psi^{\pm}_S(\zeta,\bar\zeta).$$

By integration over the ψ variable, we see that h^+ is orthogonal to $\mathcal{D}^j_{mn}$ unless $m = -\frac{1}{2}$, and h^- is orthogonal to $\mathcal{D}^j_{mn}$ unless $m = +\frac{1}{2}$. The Parseval–Plancherel formula then shows that

$$
\begin{aligned}
\|\psi\|^2 &= \frac{1}{4\pi} \int_{SU(2)} \bigl(|h^+(g)|^2 + |h^-(g)|^2\bigr)\, dg \\
&= \sum_{l,m} \frac{(2l+1)}{4\pi} \Bigl(\bigl|(\mathcal{D}^l_{-\frac{1}{2},m} \mid h^+)\bigr|^2 + \bigl|(\mathcal{D}^l_{\frac{1}{2},m} \mid h^-)\bigr|^2\Bigr) \\
&= \sum_{l,m} \left|\frac{1}{4\pi} \int_{\mathbb{S}^2} \overline{Y^+_{lm}} \psi^+ \,\Omega\right|^2 + \left|\frac{1}{4\pi} \int_{\mathbb{S}^2} \overline{Y^-_{lm}} \psi^- \,\Omega\right|^2 \\
&= \sum_{l,m} \bigl|\langle\!\langle Y'_{lm} \mid \psi\rangle\!\rangle\bigr|^2 + \bigl|\langle\!\langle Y''_{lm} \mid \psi\rangle\!\rangle\bigr|^2. \qquad (9.50)
\end{aligned}
$$

This is a Parseval identity for the orthonormal family $\{Y'_{lm}, Y''_{lm}\}$, and so establishes completeness of this family in $L^2(S)$. □

Corollary 9.13. *The spectra of the Dirac operator, its square, the Casimir operator and the spinor Laplacian are given by*

$$
\begin{aligned}
\operatorname{sp}(D\!\!\!/) &= \{\, \pm(l + \tfrac{1}{2}) : l \in \mathbb{N} + \tfrac{1}{2} \,\} = \mathbb{Z} \setminus \{0\}, \\
\operatorname{sp}(D\!\!\!/^2) &= \{\, (l + \tfrac{1}{2})^2 : l \in \mathbb{N} + \tfrac{1}{2} \,\}, \\
\operatorname{sp}(C) &= \{\, l(l+1) : l \in \mathbb{N} + \tfrac{1}{2} \,\}, \\
\operatorname{sp}(\Delta^S) &= \{\, l^2 + l - \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \,\}.
\end{aligned}
$$

The respective multiplicities are: $2l+1$ *for the eigenvalue* $\pm(l+\frac{1}{2})$ *of* $D\!\!\!/$, *and* $2(2l+1)$ *for each listed eigenvalue of* $D\!\!\!/^2$, C *and* Δ^S.

Proof. The eigenvalues of $D\!\!\!/$ are those given by Corollary 9.11; the completeness relation (9.50) shows that there are no others. The eigenvalues of C and Δ^S follow from (9.43). Notice that the Casimir eigenvalues have the form $l(l+1)$ (compare the spectrum of the Hodge Laplacian), but with l half-integral in the present case. □

We recall from (4.16) the definition of the **index** of (any) Dirac operator:

$$
\operatorname{ind} D\!\!\!/ := \dim(\ker D\!\!\!/_+) - \dim(\ker D\!\!\!/_-),
$$

We end with an important result.

Corollary 9.14. *For the spinor module over* $\mathbb{S}^2$, *the index of the Dirac operator is zero.* □

The Atiyah–Singer index theorem [9, 28, 39, 42] asserts the existence of a characteristic class whose integral coincides with the index of the Dirac operator. In fact, the characteristic form is given by $\hat{A}(R) := \det^{-1/2}(j(R))$, where R is the Riemannian curvature and $j(x) := (\sinh \frac{1}{2}x)/\frac{1}{2}x$. Since the power series $j(x)$ is *even*, however, only forms of degree $4k$ appear in the expansion of $\hat{A}(R)$ by degrees; in particular, the second-degree component is zero. Thus for $\mathbb{S}^2$ or indeed any compact two-dimensional manifold M, the integral $(2\pi i)^{-1} \int_M \hat{A}(R)$ (called the "$\hat{A}$-genus of M") vanishes, and on that basis the vanishing of the index of the Dirac operator is to be expected.

To get a Dirac operator with a nontrivial kernel, one only has to twist the irreducible spinor module with a complex line bundle, such as H or L. In that context, the eigenspinors (9.45) yield explicit solutions of the Seiberg–Witten equations [60].

10 Construction of representations of $SU(2)$

The theory of the Dirac operator on the Riemann sphere, developed in the previous chapter, has a direct application to the construction of the irreducible unitary representations of the group $SU(2)$. There are three main routes to that goal. In view of the Peter–Weyl theorem, an irreducible representation of a compact Lie group is finite-dimensional, unitarizable (that is, the representation space may be provided with an inner product that is invariant under the group action) and lies within the left regular representation of the group. The first route to the irreducible unitary representations is an algebraic construction called the "theorem of the highest weight" [13, 35, 37] which yields an abstract description of all such representations up to unitary equivalence, but does not *exhibit* them concretely. Secondly, the famous Borel–Weil theorem [11, 50] constructs the representation spaces as spaces of holomorphic sections of certain line bundles over coadjoint orbits of the compact Lie group in question; this construction depends on the fact that these orbits are complex manifolds, and therefore carry a spin^c structure. It turns out that the maximal-dimensional coadjoint orbits in fact carry a *spin* structure, compatible with the group action, and this opens a third path to the construction of representations, whereby the representation spaces are the kernel spaces of certain Dirac operators. (Indeed, this last recipe is related to the Borel–Weil–Bott construction by a "twisting" with a certain line bundle, so the second and third routes are thereby equivalent.) Here we shall not attempt to give the construction for all compact groups, but rather shall build up the particular example of $SU(2)$, that contains all the features of the general theory.

10.1 Characters of the maximal torus

Any compact connected Lie group G contains a maximal torus T, and any two such tori are conjugate, by a basic theorem of Weyl [13]. For $G = SU(2)$, we may take T to be the subgroup of diagonal unitary matrices $k(\phi)$ of (9.36), so that $T \simeq U(1)$. Its characters are

$$\chi_m(k(\phi)) = \chi_m \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = e^{im\phi/2} \tag{10.1}$$

where m is necessarily an integer in order that χ_m be a well-defined homomorphism from T to $U(1)$. The Hopf map η of (9.26) allows us to regard $\mathbb{S}^2$ as the homogeneous space $SU(2)/T$, whereby $SU(2) \longrightarrow \mathbb{S}^2$ becomes a principal $U(1)$-bundle.

Let $x \in \mathbb{S}^2$; if it is not the north or south pole, let $\zeta = z^{-1}$ be its local coordinates. Two particular local sections of this principal bundle are $\gamma_0 \in \Gamma(U_0, SU(2))$ and $\gamma_1 \in \Gamma(U_1, SU(2))$, given by

$$\gamma_0(x) := \frac{1}{\sqrt{1 + z\bar{z}}} \begin{pmatrix} \bar{z} & 1 \\ -1 & z \end{pmatrix}, \qquad \gamma_1(x) := \frac{1}{\sqrt{1 + \zeta\bar{\zeta}}} \begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix}.$$

It is immediate that $\eta(\gamma_0(x)) = z^{-1} = \zeta = \eta(\gamma_1(x))$ for $x \in U_0 \cap U_1$, which verifies that these are sections and implies that $\gamma_0(x) = \gamma_1(x)h(\zeta)$ with $h(\zeta) \in T$; a quick calculation yields

$$h(\zeta) = \begin{pmatrix} \sqrt{\zeta/\bar{\zeta}} & 0 \\ 0 & \sqrt{\bar{\zeta}/\zeta} \end{pmatrix},$$

so that $\chi_m(h(\zeta)) = (\zeta/\bar{\zeta})^{m/2}$.

Every complex line bundle over $\mathbb{S}^2$ may be constructed as an associated bundle to this principal bundle via a suitable character of the maximal torus T. This can be seen directly by exhibiting these associated bundles, since we have already classified all line bundles over $\mathbb{S}^2$ in Proposition 5.13. However, it should be said that one can take a more principled point of view. On each such line bundle, we shall eventually construct an equivariant bundle action of $SU(2)$; the equivalence classes of such actions generate a commutative semiring (under the Whitney sum and tensor product of equivariant vector bundles), and by formal subtraction (the Grothendieck construction) they generate an abelian group, denoted $K_{SU(2)}(\mathbb{S}^2)$ [48]. The characters of the maximal torus T form a ring $R(T)$, isomorphic to $\mathbb{Z}$, and the matching of equivariant line bundles and characters yields a canonical isomorphism $K_{SU(2)}(\mathbb{S}^2) \simeq R(T)$.

The line bundle $L^m \longrightarrow \mathbb{S}^2$ associated to $SU(2) \longrightarrow \mathbb{S}^2$ via the character χ_m of T is given by (1.2): the space L^m has elements $[g, v]$ with $g \in SU(2)$, $v \in \mathbb{C}$, and $[gh, v] = [g, \chi_m(h)v]$ whenever $h \in T$. The local sections $s_0 \in \Gamma(U_0, L^m)$ and $s_1 \in \Gamma(U_1, L^m)$ defined by $s_j(x) := [\gamma_j(x), 1]$ suffice to determine the class of L^m. Indeed,

$$s_0(x) = [\gamma_1(x)h(\zeta), 1] = [\gamma_1(x), \chi_m(h(\zeta))] = \chi_m(h(\zeta))s_1(x) = (\zeta/\bar{\zeta})^{m/2}s_1(x),$$

so the transition function of L^m is $g_{01}(\zeta) = (\zeta/\bar{\zeta})^{m/2} = e^{-im\phi/2}$. As before, we represent a general section $s\colon \mathbb{S}^2 \to L^m$ by a pair of functions (f_0, f_1) satisfying $f_0(z)s_0(x) = f_1(\zeta)s_1(x)$ on $U_0 \cap U_1$. We therefore get the gauge transformation rule

$$f_1(\zeta) \equiv (\bar{\zeta}/\zeta)^{m/2} f_0(\zeta^{-1}).$$

From Exercise 9.1 (compare also (9.5) and (9.6), which are particular cases), we obtain that the Chern class of L^m is $m[L] = -m[H]$, so that L^m is indeed equivalent to the m-th tensor power of the tautological line bundle, as the notation had anticipated.[1]

[1] The isomorphism $K_{SU(2)}(\mathbb{S}^2) \simeq R(T)$ is thus given by $m[L] \mapsto \chi_m$.

10.2 Twisting connections

In subsection 9.3 we have seen that the *spin connection* ∇^S is defined over U_1 (say) by $\nabla^S_{\partial_i} = \partial_i + \omega_i$, where $\zeta = x^1 + ix^2$ and ω_1, ω_2 are given by (9.9). Since $\partial/\partial\zeta = \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial/\partial\bar\zeta = \frac{1}{2}(\partial_1 + i\partial_2)$, we may rewrite (9.9) as[2]

$$\nabla^S_{\partial/\partial\zeta} = \frac{\partial}{\partial\zeta} + \frac{i\bar\zeta}{2(1+\zeta\bar\zeta)}\gamma^1\gamma_2, \qquad \nabla^S_{\partial/\partial\bar\zeta} = \frac{\partial}{\partial\bar\zeta} - \frac{i\zeta}{2(1+\zeta\bar\zeta)}\gamma^1\gamma_2.$$

If we use the presentation (9.12) of γ^1 and γ^2, in which the grading operator $i\gamma^1\gamma^2$ is a diagonal matrix, these become

$$\nabla^{S\pm}_{\partial/\partial\zeta} = \frac{\partial}{\partial\zeta} \pm \frac{\bar\zeta}{2(1+\zeta\bar\zeta)}, \qquad \nabla^{S\pm}_{\partial/\partial\bar\zeta} = \frac{\partial}{\partial\bar\zeta} \mp \frac{\zeta}{2(1+\zeta\bar\zeta)}. \tag{10.2}$$

Now $S^+ = L$, the tautological bundle over $\mathbb{S}^2$. If $f_1, \dots, f_m$, with $m \in \mathbb{N}$, are local sections in $\Gamma(U_1, L)$, then their product is a local section in $\Gamma(U_1, L^m)$. Denote by $\nabla^{(m)}$ the tensor product of m copies of ∇^{S^+}; this is a connection on the vector bundle $L^m \longrightarrow \mathbb{S}^2$, for which

$$\nabla^{(m)}_X(f_1 \dots f_m) = \sum_{j=1}^m f_1 \dots f_{j-1}\, \nabla^{S\pm}_X(f_j)\, f_{j+1} \dots f_m,$$

from which it follows that

$$\nabla^{(m)}_{\partial/\partial\zeta} = \frac{\partial}{\partial\zeta} + \frac{m\bar\zeta}{2(1+\zeta\bar\zeta)}, \qquad \nabla^{(m)}_{\partial/\partial\bar\zeta} = \frac{\partial}{\partial\bar\zeta} - \frac{m\zeta}{2(1+\zeta\bar\zeta)} \tag{10.3}$$

on $\Gamma(U_1, L^m)$.

Exercise 10.1. If m is a negative integer, let $\nabla^{(m)}$ be the tensor product of $(-m)$ copies of ∇^{S^-}, i.e., the connection on $L^m = H^{-m}$ dual to the connection $\nabla^{(-m)}$ on L^{-m}; show that (10.3) holds also in this case.

Exercise 10.2. The curvature ω_m of the connection $\nabla^{(m)}$ is determined by the relation $\omega_m(\partial/\partial\zeta, \partial/\partial\bar\zeta) = [\nabla^{(m)}_{\partial/\partial\zeta}, \nabla^{(m)}_{\partial/\partial\bar\zeta}]$. Check that $\omega_m = -m(1+\zeta\bar\zeta)^{-2}\, d\zeta \wedge d\bar\zeta$.

Definition 10.1. Let $S \longrightarrow \mathbb{S}^2$ denote the irreducible spinor bundle; let $L^m \longrightarrow \mathbb{S}^2$ be the complex line bundle with first Chern class $m[L]$; we call $S \otimes L^m \longrightarrow \mathbb{S}^2$ a *twisted spinor bundle.* Clearly $S \otimes L^m \sim L^{m+1} \oplus L^{m-1}$. The **twisted spin connection** $\widetilde\nabla := \nabla^S \otimes 1 + 1 \otimes \nabla^{(m)}$ is determined, in view of (10.2) and (10.3), by

$$\widetilde\nabla^{\pm}_{\partial/\partial\zeta} = \frac{\partial}{\partial\zeta} + \frac{(m\pm 1)\bar\zeta}{2(1+\zeta\bar\zeta)}, \qquad \widetilde\nabla^{\pm}_{\partial/\partial\bar\zeta} = \frac{\partial}{\partial\bar\zeta} - \frac{(m\pm 1)\zeta}{2(1+\zeta\bar\zeta)}. \tag{10.4}$$

[2] In this Section, we shall write explicit formulas over the chart domain U_1 with local coordinates $(\zeta, \bar\zeta)$; we leave the corresponding formulas for U_0 to the reader.

10.3 Twisted Dirac operators

Definition 10.2. The *Dirac operator* $D\!\!\!/_m$ on the twisted spinor bundle $S \otimes L^m \longrightarrow \mathbb{S}^2$ is given, in accordance with Definition 8.3, by $D\!\!\!/_m := c(dx^1)\widetilde{\nabla}_{\partial_1} + c(dx^2)\widetilde{\nabla}_{\partial_2} = c(d\zeta)\widetilde{\nabla}_{\partial/\partial\zeta} + c(d\bar{\zeta})\widetilde{\nabla}_{\partial/\partial\bar{\zeta}}$. From (9.10), we get

$$c(d\zeta) = -\tfrac{1}{2}(1+\zeta\bar{\zeta})(\gamma^1 + i\gamma^2) = \begin{pmatrix} 0 & -(1+\zeta\bar{\zeta}) \\ 0 & 0 \end{pmatrix},$$
$$c(d\bar{\zeta}) = -\tfrac{1}{2}(1+\zeta\bar{\zeta})(\gamma^1 - i\gamma^2) = \begin{pmatrix} 0 & 0 \\ 1+\zeta\bar{\zeta} & 0 \end{pmatrix},$$

so that $D\!\!\!/_m = c(d\zeta)\widetilde{\nabla}^-_{\partial/\partial\zeta} + c(d\bar{\zeta})\widetilde{\nabla}^+_{\partial/\partial\bar{\zeta}}$; explicitly,

$$\begin{aligned} D\!\!\!/_m &= \begin{pmatrix} 0 & -(1+\zeta\bar{\zeta})\partial/\partial\zeta - \tfrac{1}{2}(m-1)\bar{\zeta} \\ (1+\zeta\bar{\zeta})\partial/\partial\bar{\zeta} - \tfrac{1}{2}(m+1)\zeta & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\eth_\zeta - \tfrac{1}{2}m\bar{\zeta} \\ \bar{\eth}_\zeta - \tfrac{1}{2}m\zeta & 0 \end{pmatrix} =: \begin{pmatrix} 0 & D\!\!\!/^-_m \\ D\!\!\!/^+_m & 0 \end{pmatrix}. \end{aligned} \tag{10.5}$$

From (9.15) we get $\eth_\zeta\psi + \frac{1}{2}m\bar{\zeta}\psi = Q_1(\zeta)\partial\psi/\partial\zeta + \frac{1}{2}(m-1)(\partial Q_1/\partial\zeta)\psi$, where we have written $Q_1(\zeta) = 1+\zeta\bar{\zeta}$. Also, $\bar{\eth}_\zeta\psi - \frac{1}{2}m\zeta\psi = Q_1(\zeta)\partial\psi/\partial\bar{\zeta} - \frac{1}{2}(m+1)(\partial Q_1/\partial\bar{\zeta})\psi$. Thus,

$$\begin{aligned} D\!\!\!/^-_m &= -Q_1(\zeta)^{-(m-3)/2}\,\frac{\partial}{\partial\zeta}\,Q_1(\zeta)^{(m-1)/2}, \\ D\!\!\!/^+_m &= Q_1(\zeta)^{(m+3)/2}\,\frac{\partial}{\partial\bar{\zeta}}\,Q_1(\zeta)^{-(m+1)/2}. \end{aligned} \tag{10.6}$$

The kernel of $D\!\!\!/^-_m$ is a subspace of $\Gamma(L^{m-1})$ and the kernel of $D\!\!\!/^+_m$ is a subspace of $\Gamma(L^{m+1})$. Suppose that $\psi \in \Gamma(L^{m+1})$, and let $\psi_S(\zeta,\bar{\zeta})$ and $\psi_N(z,\bar{z})$ be its component functions over U_1 and U_0 respectively. These are related by the gauge transformation rule $\psi_N(\zeta^{-1},\bar{\zeta}^{-1}) \equiv (\zeta/\bar{\zeta})^{(m+1)/2}\psi_S(\zeta,\bar{\zeta})$. From (10.6), $\psi \in \ker D\!\!\!/^+_m$ iff

$$a(\zeta) := (1+\zeta\bar{\zeta})^{-(m+1)/2}\psi_S(\zeta,\bar{\zeta}) \tag{10.7}$$

is an entire *holomorphic* function on $\mathbb{C}$. Moreover,

$$\psi_N(\zeta^{-1},\bar{\zeta}^{-1}) = ((1+\zeta\bar{\zeta})\zeta/\bar{\zeta})^{(m+1)/2}a(\zeta)$$

must be regular at $\zeta = \infty$. Since $Q_1(\zeta)^{(m+1)/2} = O(|\zeta|)^{m+1}$, for nonnegative m this is only possible if $a(\zeta) \equiv 0$. If m is negative, the entire function $a(\zeta)$ is $O(|\zeta|)^{|m|-1}$ as $|\zeta| \to \infty$, so that $a(\zeta)$ is a polynomial of degree at most $|m|-1$. To sum up:

$$\dim\ker D\!\!\!/^+_m = \begin{cases} 0, & \text{if } m \geq 0, \\ |m|, & \text{if } m < 0. \end{cases}$$

The kernel of $\not{D}_m^-$ is found similarly. Suppose that $\phi \in \Gamma(L^{m-1})$, with component functions $\phi_S(\zeta, \bar\zeta)$ and $\phi_N(z, \bar z)$ related by $\phi_N(\zeta^{-1}, \bar\zeta^{-1}) \equiv (\zeta/\bar\zeta)^{(m-1)/2}\phi_S(\zeta, \bar\zeta)$. Then $\phi \in \ker \not{D}_m^-$ iff

$$b(\bar\zeta) := (1 + \zeta\bar\zeta)^{(m-1)/2}\phi_S(\zeta, \bar\zeta) \tag{10.8}$$

is *antiholomorphic*. The regularity at ∞ of $\phi_N(\zeta^{-1}, \bar\zeta^{-1}) = ((1+\zeta\bar\zeta)\bar\zeta/\zeta)^{-(m-1)/2} b(\bar\zeta)$, together with $Q_1(\zeta)^{-(m-1)/2} = O(|\zeta|)^{1-m}$, shows that $b(\bar\zeta) \equiv 0$ for m negative or zero, while for m positive $b(\bar\zeta)$ is a polynomial in $\bar\zeta$ of degree at most $m - 1$. In fine:

$$\dim\ker \not{D}_m^- = \begin{cases} m, & \text{if } m > 0, \\ 0, & \text{if } m \le 0. \end{cases}$$

We have in particular shown the following result.

Proposition 10.1. *The index of the Dirac operator $\not{D}_m$ on the twisted spinor bundle $S \otimes L^m \longrightarrow \mathbb{S}^2$ is given by*

$$\operatorname{ind} \not{D}_m = \dim\ker \not{D}_m^+ - \dim\ker \not{D}_m^- = -m,$$

that is, by the integer that labels the first Chern class of the twisting line bundle L^m. □

10.4 The group action on the twisted spinor bundles

Proposition 10.2. *A bundle action of $SU(2)$ on $S \otimes L^m \longrightarrow \mathbb{S}^2$ is given by the formula (9.27), where $\psi^\pm \in \Gamma(L^{m\pm1})$, and where the multipliers $A_N^\pm$, $A_S^\pm$ are determined by*

$$A_N^\pm(g, z) := \left(\frac{\beta\bar z + \bar\alpha}{\bar\beta z + \alpha}\right)^{(m\pm1)/2}, \qquad A_S^\pm(g', \zeta) := \left(\frac{-\bar\beta\bar\zeta + \alpha}{-\beta\zeta + \bar\alpha}\right)^{(m\pm1)/2}.$$

Proof. The analysis of the $SU(2)$-action on the untwisted spinor bundle, given in Section 9.6, may be repeated verbatim, except that the gauge transformation factor $(\bar\zeta/\zeta)^{1/2}$ in (9.30) must be replaced by $(\bar\zeta/\zeta)^{(m\pm1)/2}$. This leads to the necessary and sufficient condition (9.31), with the exponent $1/2$ replaced by $(m \pm 1)/2$, and the cocycles (10.2) satisfy that condition. □

Proposition 10.3. *The Dirac operator $\not{D}_m$ is equivariant under the action of $SU(2)$ on the twisted spinor bundle $S \otimes L^m \longrightarrow \mathbb{S}^2$ determined by (10.2).*

Proof. Let $T \in \operatorname{End}^+(\Gamma(S \otimes L^m))$ be an operator that commutes with $\not{D}_m$ and is of the form $(T\psi_S^\pm)(\zeta, \bar\zeta) = c_\pm(\zeta, \bar\zeta)\psi_S^\pm\big(b(\zeta), \overline{b(\zeta)}\big)$. Then, as in Lemma 9.5, the identity $T\not{D}_m^- = \not{D}_m^- T$ —in $\operatorname{Hom}(\Gamma(L^{m-1}), \Gamma(L^{m+1}))$— yields the relations

$$\begin{aligned} Q_1(\zeta)c_-(\zeta, \bar\zeta)b'(\zeta) &= Q_1(b(\zeta))c_+(\zeta, \bar\zeta), \\ Q_1(\zeta)\frac{\partial c_-}{\partial\zeta} &= \tfrac12(m-1)\big(\overline{b(\zeta)}c_+(\zeta, \bar\zeta) - \bar\zeta c_-(\zeta, \bar\zeta)\big). \end{aligned} \tag{10.9}$$

Similarly, from $T\not{D}_m^+ = \not{D}_m^+ T$ in $\operatorname{Hom}(\Gamma(L^{m+1}), \Gamma(L^{m-1}))$ we get

$$\begin{aligned} Q_1(\zeta)c_+(\zeta,\bar\zeta)\overline{b'(\zeta)} &= Q_1(b(\zeta))c_-(\zeta,\bar\zeta), \\ Q_1(\zeta)\frac{\partial c_+}{\partial\bar\zeta} &= -\tfrac12(m+1)\big(b(\zeta)c_-(\zeta,\bar\zeta) - \zeta c_+(\zeta,\bar\zeta)\big). \end{aligned} \tag{10.10}$$

We check that these conditions are satisfied when $T = T(g)$ is the operator corresponding to the diagonal element $g = k(\phi)$ of $SU(2)$, determined by (10.2). Here $b(\zeta) = g'^{-1} \cdot \zeta = e^{i\phi}\zeta$. Clearly $Q_1(b(\zeta)) \equiv Q_1(\zeta)$, so that (10.9) and (10.10) reduce to the conditions $c_+(\zeta,\bar\zeta) = e^{i\phi}c_-(\zeta,\bar\zeta)$, $\partial c_-/\partial\zeta = 0$, and $\partial c_+/\partial\bar\zeta = 0$, which together imply that c_- and c_+ are independent of ζ and $\bar\zeta$. Since the prescription $c_\pm(\zeta,\bar\zeta) := e^{i(m\pm1)\phi/2}$, dictated by (10.2), obeys $c_+ = e^{i\phi}c_-$, we see that $T(g)$ commutes with $\not{D}_m$.

In the same way, it is easy to show that the operator $T(h(\theta))$, associated to $h(\theta)$ of (9.36) by $b(\zeta) = h(\theta)'^{-1} \cdot \zeta$ and by (10.2), commutes with $\not{D}_m$. Since elements of the form $k(\phi)$ and $h(\theta)$ generate $SU(2)$, and since $g \mapsto T(g)$ is a homomorphism on account of the cocycle relations (9.29), we conclude that $T(g)\not{D}_m = \not{D}_m T(g)$ for all $g \in SU(2)$. □

Exercise 10.3. Complete the proof of the previous Proposition by verifying that the coefficient functions of $T(h(\theta))$ satisfy (10.9) and (10.10).

Corollary 10.4. *The bundle action of $SU(2)$ on $S \otimes L^m \longrightarrow \mathbb{S}^2$ determined by (9.27) and (10.2) restricts to a finite-dimensional unitary representation ρ_m of $SU(2)$ on the kernel of the Dirac operator $\not{D}_m$.* □

To exhibit the representation ρ_m, it is useful to notice that with $g'^{-1}\cdot\zeta = (\alpha\zeta+\bar\beta)/(-\beta\zeta+\bar\alpha)$, one has $Q_1(g'^{-1} \cdot \zeta) = Q_1(\zeta)|-\beta\zeta+\bar\alpha|^{-2}$ on account of $\alpha\bar\alpha + \beta\bar\beta = 1$. If $m > 0$, then for $\psi \in \ker \not{D}_m^- \subset \Gamma(L^{m-1})$, the bundle action (9.27) specializes to

$$\rho_m\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \psi_S(\zeta,\bar\zeta) = \left(\frac{-\bar\beta\bar\zeta+\alpha}{-\beta\zeta+\bar\alpha}\right)^{(m-1)/2} \psi_S\biggl(\frac{\alpha\zeta+\bar\beta}{-\beta\zeta+\bar\alpha}, \frac{\bar\alpha\bar\zeta+\beta}{-\bar\beta\bar\zeta+\alpha}\biggr). \tag{10.11}$$

Using (10.8), we can write $\psi_S(\zeta,\bar\zeta) = Q_1(\zeta)^{-(m-1)/2}b(\bar\zeta)$ with b a polynomial in $\bar\zeta$ of degree less than m, and the previous formula simplifies to

$$\rho_m\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \big[Q_1(\zeta)^{-(m-1)/2}b(\bar\zeta)\big] = Q_1(\zeta)^{-(m-1)/2}(-\bar\beta\bar\zeta+\alpha)^{m-1}\, b\biggl(\frac{\bar\alpha\bar\zeta+\beta}{-\bar\beta\bar\zeta+\alpha}\biggr).$$

The right hand side is clearly also $Q_1^{-(m-1)/2}$ times a polynomial in $\bar\zeta$ of degree less than m. If we define $\psi_k(\zeta,\bar\zeta) := Q_1(\zeta)^{-(m-1)/2}\bar\zeta^k$, for $k = 0, 1, \dots, m-1$, these form an orthogonal basis for $\ker \not{D}_m^-$. For these basis vectors,

$$\rho_m\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \psi_k(\zeta,\bar\zeta) = Q_1(\zeta)^{-(m-1)/2}(\bar\alpha\bar\zeta+\beta)^k(-\bar\beta\bar\zeta+\alpha)^{m-k-1}, \tag{10.12}$$

from which the matrix elements of the representation may be computed at once.

When $m < 0$, we obtain similar formulas for ρ_m. Indeed, for $\psi \in \ker D\!\!\!/_m^+ \subset \Gamma(L^{m+1})$, we obtain (10.11) with the exponent $(m-1)/2$ replaced by $(m+1)/2$. Since, by (10.7), $\psi_S(\zeta, \bar{\zeta}) = Q_1(\zeta)^{(m+1)/2} a(\zeta)$ with a a polynomial in ζ of degree less than m, we get

$$\rho_m \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \big[Q_1(\zeta)^{(m+1)/2} a(\zeta)\big] = Q_1(\zeta)^{(m+1)/2} (-\beta\zeta + \bar{\alpha})^{|m|-1} \, a\bigg(\frac{\alpha\zeta + \bar{\beta}}{-\beta\zeta + \bar{\alpha}}\bigg), \tag{10.13}$$

and on the orthogonal basis $\psi_k(\zeta, \bar{\zeta}) := Q_1(\zeta)^{-(|m|-1)/2} \zeta^k$ $(k = 0, 1, \dots, m-1)$ for $\ker D\!\!\!/_m^+$, we find that

$$\rho_m \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \psi_k(\zeta, \bar{\zeta}) = Q_1(\zeta)^{-(|m|-1)/2} (\alpha\zeta + \bar{\beta})^k (-\beta\zeta + \bar{\alpha})^{|m|-k-1}. \tag{10.14}$$

From the explicit formulae (10.12) and (10.14), it is evident that ρ_{-m} is the conjugate representation to ρ_m.

Proposition 10.5. *The representation ρ_m is irreducible.*

Proof. If $m = 0$, there is nothing to prove. The case $m < 0$ mirrors the case $m > 0$, on interchanging $D\!\!\!/_m^+$ with $D\!\!\!/_m^-$; thus we may take $m > 0$. Then $\ker D\!\!\!/_m = \ker D\!\!\!/_m^-$ is an m-dimensional Hilbert space with the orthogonal basis $\{\, \psi_k : k = 0, 1, \dots, m-1 \,\}$. It is immediate from (10.12) that this basis consists of joint eigenvectors for $\{\, \rho_m(g) : g \in T \,\}$; indeed, with $g = k(\phi)$ we get

$$\rho_m(k(\phi))\psi_k = e^{i(m-1)\phi/2} e^{-ik\phi} \psi_k, \tag{10.15}$$

so the eigenvalues are generally distinct, and therefore any operator S on $\ker D\!\!\!/_m^-$ that commutes with each $\rho_m(g)$ must have a diagonal matrix in this basis. On the other hand, the one-parameter subgroup of operators $t \mapsto \rho_m(h(t))$ mingles the basis elements ψ_k: for $h(t)$, one has $\alpha = \cos \frac{1}{2}t$, $\beta = \sin \frac{1}{2}t$, and from (10.12) we get $(d/dt)\big|_{t=0} \rho_m(h(t))\psi_k = \frac{1}{2}k\psi_{k-1} - \frac{1}{2}(m-k-1)\psi_{k+1}$ if $1 \le k \le m-2$. A similar calculation with the subgroup $h'(t)$ for which $\alpha = \cos \frac{1}{2}t$ and $\beta = i \sin \frac{1}{2}t$ gives $(d/dt)\big|_{t=0} \rho_m(h'(t))\psi_k = \frac{1}{2}ik\psi_{k-1} + \frac{1}{2}i(m-k-1)\psi_{k+1}$. Thus if S commutes with every $\rho_m(g)$, it commutes with the "ladder operator" $\psi_k \mapsto k\psi_{k-1}$, and hence S is scalar. Irreducibility of ρ_m now follows from Schur's lemma. □

In order to identify the unitary irreducible representation ρ_m of $SU(2)$, it suffices to compute its character. Since the character is a class function, it is determined by its restriction to a maximal torus, so for $m > 0$ we obtain it immediately from (10.15):

$$\chi_m(k(\phi)) = \sum_{k=0}^{m-1} e^{i(m-1)\phi/2} e^{-ik\phi} = \frac{e^{im\phi/2}}{e^{i\phi/2}} \frac{1 - e^{-im\phi}}{1 - e^{-i\phi}} = \frac{e^{im\phi/2} - e^{-im\phi/2}}{e^{i\phi/2} - e^{-i\phi/2}}.$$

When $m < 0$, we similarly get from (10.14)

$$\rho_m(k(\phi))\psi_k = e^{i(m+1)\phi/2} e^{ik\phi} \psi_k,$$

and so its character is:

$$\chi_m(k(\phi)) = \sum_{k=0}^{|m|-1} e^{i(m+1)\phi/2} e^{ik\phi} = \frac{e^{im\phi/2}}{e^{-i\phi/2}} \frac{1 - e^{im\phi}}{1 - e^{i\phi}} = \frac{e^{im\phi/2} - e^{-im\phi/2}}{e^{i\phi/2} - e^{-i\phi/2}}.$$

Therefore the representations ρ_{-m} and ρ_m are equivalent.

This equivalence is to be expected on general grounds. The *Weyl group* of $SU(2)$, namely the normalizer subgroup of the maximal torus T modulo T itself, is just the two-element group $W \simeq \mathbb{Z}_2$, since the only nontrivial isomorphism σ of the diagonal subgroup T is the interchange of diagonal elements, that can be implemented by conjugating with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By (10.1), $\chi_m \circ \sigma = \chi_{-m}$; so two characters of T lead to equivalent representations of $SU(2)$ iff they lie in the same orbit under the action of the Weyl group. This exemplifies the general relation [13, 49]:

$$R(G) \simeq R(T)^W.$$

This construction of the unitary irreducible representations of $SU(2)$ directly from the equivariant Dirac operator exemplifies the "universal quantization map" of Vergne [55]:

$$Q\colon K_{SU(2)}(\mathbb{S}^2) \to R(SU(2)),$$

which is already an instance of the index theorem of Atiyah, Segal and Singer [5, 6].

10.5 The Borel–Weil theorem

The construction of the irreducible unitary representations of $SU(2)$ on the kernel spaces of Dirac operators, developed in the preceding sections, produces two families of equivalent representations, according as one twists the standard bundle with a tensor power of the tautological bundle (L^m, with $m > 0$), or with a tensor power of the hyperplane bundle ($H^n \sim L^{-n}$, with $n > 0$). Now only the latter line bundles admit nonzero holomorphic sections, while only the former admit nonzero antiholomorphic sections. The celebrated construction of Borel and Weil does not deal with spinor bundles, but rather uses the fact that the coadjoint orbits of compact Lie groups are *complex manifolds* and constructs the desired representations on the *spaces of holomorphic sections* of holomorphic line bundles over those orbits. By a simple application of Liouville's theorem —see Section 5.6 for the details in the case $M = \mathbb{CP}^(m)$— the compactness of the orbit implies that the space of holomorphic sections is finite-dimensional. Indeed, in many cases, that of the trivial line bundle for instance, the space of holomorphic sections reduces to zero. A variant of the Borel–Weil construction produces representations on space of antiholomorphic sections of the respective dual line bundles.

There is a simple relationship between the spinor-based and the Borel–Weil constructions: one passes from one to the other by twisting or untwisting with a *fixed* line bundle that can be obtained directly from the spin structure of the maximal coadjoint orbit G/T. Indeed, this relationship is parallel to the replacement of the spin structure by a spin^c structure on a spin manifold: for that, one needs to identify a particular principal $U(1)$-bundle that combines with the spin structure to yield a spin^c structure. (See the discussion in Section 7.3, and also Appendix D of [39].)

If M is a complex manifold of complex dimension m, let $K := \Lambda^{m,0}T^*M$; then $K \longrightarrow M$ is the so-called *canonical line bundle*, and $\Gamma(K) = \mathcal{A}^{m,0}(M)$. The first Chern class of the dual bundle K^*, turns out [41] to be the element of $\check{H}^2(M,\mathbb{Z})$ whose modulo-2 reduction is the Stiefel–Whitney class $w_2(M)$; thus $w_2(M) + j_*(c_1(K)) = 0$ in $\check{H}^2(M,\mathbb{Z}_2)$, in the notation of Section 7.3. If M is spin, so that $w_2(M) = 0$, then $j_*(c_1(K)) = 0$ also, so that $c_1(K)$ is even; in other words, there is a complex line bundle $K^{1/2} \longrightarrow M$ such that $K^{1/2} \otimes K^{1/2} \sim K$. In principle, the "square root of the canonical bundle" $K^{1/2}$ may depend on the spin structure chosen for M.

For the case $M = \mathbb{S}^2$, we know that $K \sim L \otimes L = L^2$ by Exercise 9.3. Thus we can take $K^{1/2} := L$. Its dual line bundle is $K^{-1/2} := H$. Therefore we find that

$$S \otimes K^{-1/2} = (L \oplus H) \otimes H \sim E \oplus H^2 \sim \Lambda^{0,\bullet}T^*\mathbb{S}^2$$

where $E = \mathbb{S}^2 \times \mathbb{C}$ denotes the trivial line bundle, and $\Lambda^{0,\bullet}T^*\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is the complex vector bundle whose smooth sections form the module $\mathcal{A}^{0,\bullet}(\mathbb{S}^2) = \mathcal{A}^{0,0}(\mathbb{S}^2) \oplus \mathcal{A}^{0,1}(\mathbb{S}^2)$. Over U_1, such a section may be written as $f(\zeta,\bar\zeta) + h(\zeta,\bar\zeta)\,d\bar\zeta$.

Thus $\mathcal{A}^{0,\bullet}(\mathbb{S}^2) = \Gamma(S \otimes L^{-1})$ is the domain of the twisted Dirac operator $D\!\!\!/_{-1}$. Of course, (10.5) for $m = -1$ simplifies to

$$D\!\!\!/_{-1} = \begin{pmatrix} 0 & -Q_1(\zeta)\partial/\partial\zeta + \bar\zeta \\ Q_1(\zeta)\partial/\partial\bar\zeta & 0 \end{pmatrix}.$$

The operator $D\!\!\!/^+_{-1} \colon \mathcal{A}^{0,0}(\mathbb{S}^2) \to \mathcal{A}^{0,1}(\mathbb{S}^2)$ has as kernel the *holomorphic* functions in $\mathcal{A}^{0,0}(\mathbb{S}^2) = C^\infty(\mathbb{S}^2)$, which are just the constant functions, by Liouville's theorem. The factor $Q_1(\zeta) = 1 + \zeta\bar\zeta$ is a normalization factor, that takes account of the metric on $\mathcal{A}^{0,1}(\mathbb{S}^2)$.

To see that, consider the twisted spinor bundle $S \otimes L^m$ for any negative m, say $m = -(2j+1)$ with j a nonnegative half-integer.[3] Clearly $S \otimes L^m = \Lambda^{0,\bullet}T^*\mathbb{S}^2 \otimes H^{2j} = H^{2j} \oplus H^{2j+2}$, so that $S^+ \otimes L^m = H^{2j}$ admits holomorphic sections. If s_1 denotes a section of H normalized by $(s_1 \,|\, s_1) = 1$, then s_1^{2j} is a normalized section of H^{2j}. We may select s_1 as $s_1 := Q_1(\zeta)^{1/2}\sigma_1$, where σ_1 is the *holomorphic section* of $H \longrightarrow \mathbb{CP}^(1)$ of Section 5.6: the normalization follows from (5.13), whereby we have $(\sigma_1 \mid \sigma_1) = Q_1(\zeta)^{-1}$.

It is convenient[4] to choose the metric $4g^{-1} = Q_1(\zeta)^2(\partial/\partial\zeta)\cdot(\partial/\partial\bar\zeta)$ on $T^*_{\mathbb{C}}M$, so that the normalized section of $\mathcal{A}^{0,1}(\mathbb{S}^2)$ is just $s_1^2 = Q_1(\zeta)^{-1}\,d\bar\zeta$. It follows that $\sigma_1^2 = Q_1(\zeta)^{-2}\,d\bar\zeta$.

Any section of $\mathcal{A}^{0,1}(\mathbb{S}^2)$ is of the form $h(\zeta,\bar\zeta)\,d\bar\zeta = Q_1(\zeta)h(\zeta,\bar\zeta)\,s_1^2$, and a section of $S \otimes L^{-1} = L^0 \oplus L^{-2}$ can be written as $\phi = f(\zeta,\bar\zeta) + h(\zeta,\bar\zeta)\,d\bar\zeta$. Then the Dirac operator $D\!\!\!/_{-1}$ can be identified as follows. Firstly,

$$D\!\!\!/^+_{-1} f = \Bigl(Q_1(\zeta)\frac{\partial f}{\partial\bar\zeta}\Bigr)s_1^2 = \frac{\partial f}{\partial\bar\zeta}\,d\bar\zeta = \bar\partial f$$

[3] By now it should be clear that it is preferable to label representations of $SU(2)$ by nonnegative half-integers j rather than the integers m.

[4] Without this normalization, the operator $\bar\partial$ in the following formulae should be replaced by $\sqrt{2}\,\bar\partial$; this convention is adopted in [28], for instance.

where $\bar\partial$ is the *Dolbeault operator* of Section 2.1.

The twisted Dirac operator $\not{D}_m$, with $m = -(2j+1)$, is

$$\not{D}_m = \begin{pmatrix} 0 & -Q_1(\zeta)\partial/\partial\zeta + (j+1)\bar\zeta \\ Q_1(\zeta)\partial/\partial\bar\zeta + j\zeta & 0 \end{pmatrix},$$

by rewriting (10.5). An element of $\Gamma(H^{2j})$ can be written (over U_1) as $f(\zeta,\bar\zeta)\,\sigma_1^{2j}$. Then

$$\begin{aligned}
\not{D}_m^+(f(\zeta,\bar\zeta)\,\sigma_1^{2j}) &= \not{D}_m^+(Q_1(\zeta)^{-j} f(\zeta,\bar\zeta)\,s_1^{2j}) \\
&= \Big(Q_1(\zeta)\frac{\partial}{\partial\bar\zeta} + j\zeta\Big)\big[Q_1(\zeta)^{-j} f(\zeta,\bar\zeta)\big]\,s_1^{2j+2} = Q_1(\zeta)^{-j+1}\frac{\partial f}{\partial\zeta}\,s_1^{2j+2} \\
&= Q_1(\zeta)^2\frac{\partial f}{\partial\zeta}\,\sigma_1^{2j+2} = \frac{\partial f}{\partial\zeta}\,d\bar\zeta\otimes\sigma_1^{2j} = \bar\partial f\otimes\sigma_1^{2j}. \qquad (10.16)
\end{aligned}$$

The Dolbeault operator extends to $\mathcal{A}^{0,\bullet}(\mathbb{S}^2, H^{2j}) := \mathcal{A}^{0,\bullet}(\mathbb{S}^2)\otimes\Gamma(H^{2j})$ by setting $\bar\partial(\omega\otimes s) := (\bar\partial\omega)\otimes s$ for $s\in\Gamma(H^{2j})$. With this understanding,[5] we have the relation $\not{D}_{-2j-1}^+ = \bar\partial$ as operators from $\mathcal{A}^{0,0}(\mathbb{S}^2, H^{2j})$ to $\mathcal{A}^{0,1}(\mathbb{S}^2, H^{2j})$.

In order to identify $\not{D}_m^-$, we must compute the adjoint of the Dolbeault operator. We consider first the case $m = -1$.

Definition 10.3. The adjoint of $\bar\partial\colon \mathcal{A}^{0,0}(\mathbb{S}^2)\to\mathcal{A}^{0,1}(\mathbb{S}^2)$ is the operator $\bar\partial^*\colon \mathcal{A}^{0,1}(\mathbb{S}^2)\to\mathcal{A}^{0,0}(\mathbb{S}^2)$ given by

$$\langle\!\langle \bar\partial^*(h\,d\bar\zeta) \mid f\rangle\!\rangle := \langle\!\langle h\,d\bar\zeta \mid \bar\partial f\rangle\!\rangle$$

where $\langle\!\langle\cdot\mid\cdot\rangle\!\rangle$ denotes the integrated inner product in either space of sections, defined as in (9.49) by integrating the appropriate pairing of sections over $\mathbb{S}^2$ with respect to the volume form $(4\pi)^{-1}\sin\theta\,d\theta\wedge d\phi = (2\pi i)^{-1}Q_1(\zeta)^{-2}\,d\zeta\wedge d\bar\zeta$. We find that

$$\begin{aligned}
\langle\!\langle \bar\partial^*(h\,d\bar\zeta)\mid f\rangle\!\rangle = \langle\!\langle h\,d\bar\zeta\mid\bar\partial f\rangle\!\rangle &= \frac{1}{2\pi i}\int_{\mathbb{C}} 4g^{-1}\Big(\bar h\,d\zeta, \frac{\partial f}{\partial\bar\zeta}\,d\bar\zeta\Big)Q_1(\zeta)^{-2}\,d\zeta\wedge d\bar\zeta \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \bar h\,\frac{\partial f}{\partial\bar\zeta}\,d\zeta\wedge d\bar\zeta = -\frac{1}{2\pi i}\int_{\mathbb{C}}\overline{\frac{\partial h}{\partial\zeta}}\,f\,d\zeta\wedge d\bar\zeta \\
&= -\langle\!\langle Q_1(\zeta)^2\,\partial h/\partial\bar\zeta\mid f\rangle\!\rangle,
\end{aligned}$$

on integrating by parts, so that $\bar\partial^*(h\,d\bar\zeta) = -(1+\zeta\bar\zeta)^2\,\partial h/\partial\bar\zeta$.

From this it follows easily that

$$\not{D}_{-1}^-(h\,d\bar\zeta) = \not{D}_{-1}^-(Q_1 h\,s_1^2) = (-Q_1\partial/\partial\zeta + \bar\zeta)(Q_1 h) = -Q_1^2\,\partial h/\partial\bar\zeta = \bar\partial^*(h\,d\bar\zeta). \qquad (10.17)$$

Thus $\not{D}_{-1}^- = \bar\partial^*$ on $\mathcal{A}^{0,1}(\mathbb{S}^2)$.

Exercise 10.4. Show $\not{D}_{-2j-1}^- = \bar\partial^*$ as operators from $\mathcal{A}^{0,1}(\mathbb{S}^2, H^{2j})$ to $\mathcal{A}^{0,0}(\mathbb{S}^2, H^{2j})$.

[5] We could have written $\bar\partial_{(2j)}$ to denote the extended operator, but we suppress the index to reduce notational clutter.

We summarize these calculations as follows.

Proposition 10.6. *The Dirac operator on the twisted spinor bundle $S \otimes L^{-2j-1}$ equals the sum of the Dolbeault operator and its adjoint on the twisted bundle $\Lambda^{0,\bullet}T^*\mathbb{S}^2 \otimes H^{2j}$.*

Proof. The formulae (10.16) and (10.17), plus the previous exercise, establish that $D\!\!\!/^{+}_{-2j-1} = \bar\partial$ and $D\!\!\!/^{-}_{-2j-1} = \bar\partial^*$, thus

$$D\!\!\!/_{-2j-1} = \bar\partial + \bar\partial^*$$

on the module $\mathcal{A}^{0,\bullet}(\mathbb{S}^2, H^{2j})$. $\square$

The kernel of the Dirac operator $D\!\!\!/_{-2j-1}$ thus coincides with $\ker\bar\partial \oplus \ker\bar\partial^*$. The second summand is zero since $\ker D\!\!\!/^{-}_{-2j-1} = 0$. Thus $\ker D\!\!\!/^{+}_{-2j-1} = \ker\bar\partial$ consists of sections $\psi_S(\zeta,\bar\zeta)\, s_1^{2j} = f(\zeta,\bar\zeta)\,\sigma_1^{2j} \in \mathcal{A}^{0,0}(\mathbb{S}^2, H^{2j})$ for which $\partial f/\partial\bar\zeta = 0$; these are precisely the *holomorphic* sections $f(\zeta)\,\sigma_1^{2j} \in \mathcal{O}(H^{2j})$. Since

$$\psi_S(\zeta,\bar\zeta)\, s_1^{2j} = f(\zeta)\,\sigma_1^{2j} = f(\zeta)Q_1(\zeta)^{-j}\, s_1^{2j}$$

it follows from (10.7) that $f(\zeta)$ is a polynomial of degree at most $2j$. That is to say, we have shown that $\dim\mathcal{O}(H^{2j}) = 2j+1$. By suppressing the factors $Q_1(\zeta)^{-j}$ in (10.13) and (10.14), we arrive at the following representation π_j of $SU(2)$ on $\mathcal{O}(H^{2j})$, that is by construction equivalent to ρ_{-2j-1}:

$$\pi_j\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}[f(\zeta)] = (-\beta\zeta+\bar\alpha)^{2j}\, f\!\left(\frac{\alpha\zeta+\bar\beta}{-\beta\zeta+\bar\alpha}\right),$$

and on the orthogonal basis $\{\, \xi_k := \zeta^k\,\sigma_1^{2j} : k = 0,1,\dots,2j \,\}$ for $\mathcal{O}(H^{2j})$, we find that

$$\pi_j\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}\xi_k = (\alpha\zeta+\bar\beta)^k(-\beta\zeta+\bar\alpha)^{2j-k}\,\sigma_1^{2j}.$$

This completes the passage from the Dirac-operator-kernel construction of the irreducible representations of $SU(2)$ to their realization on finite-dimensional spaces of holomorphic sections of line bundles over $\mathbb{S}^2$. The content of the foregoing construction is the **Borel–Weil theorem** for the compact group $SU(2)$, that we may now state as follows.

Theorem 10.7. *Every irreducible unitary representation of $SU(2)$ can be realized on a space of holomorphic sections of a holomorphic Hermitian line bundle over $\mathbb{S}^2$. This line bundle is determined, up to equivalence, by a character χ of a maximal torus of $SU(2)$ modulo the action of the Weyl group $\mathbb{Z}_2$. Moreover, if $E \longrightarrow \mathbb{S}^2$ is a holomorphic Hermitian line bundle whose Chern class is $m[H]$ with $m > 0$, then $\mathcal{O}(E)$ carries the representation corresponding to the character χ_m of the maximal torus.* $\square$

This result is of course well known; what we have done is to show how it arises from the equivariant index of the Dirac operator on the flag manifold $\mathbb{S}^2$. We remark that our construction is equivalent to the standard induced representation recipe for each π_j, though this is usually achieved by studying the complexification of the compact Lie group in question [56].

A Calculus on manifolds

In this Appendix, we briefly review the concepts and notations of calculus on manifolds, with emphasis on the algebraic formulae which arise in differential geometry. Proofs are left to the reader. General references are the books of Abraham, Marsden and Ratiu [1], Crampin and Pirani [20], Singer and Thorpe [51], and Spivak [52].

A.1 Differential manifolds

Definition A.1. A **differential manifold** of finite dimension n is a paracompact Hausdorff topological space M together with a family (or "atlas") of *local charts* $\{ (U_j, \phi_j) : j \in J \}$ such that $\mathcal{U} := \{ U_j : j \in J \}$ is a locally finite open covering of M, $\phi_j \colon U_j \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, and the *transition functions*

$$\phi_i \circ \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

are smooth. A second atlas $\{ (V_k, \psi_k) : k \in K \}$ is declared equivalent to the first if every $\phi_i \circ \psi_k^{-1}$ is smooth (the "differentiable structure" of M is actually an equivalence class of atlases). If M is compact, a finite atlas may be chosen.

If $n = 2m$ is even, we can regard the chart maps ϕ_j as having images in $\mathbb{C}^m$. We say that M is a *complex manifold* if the transition functions are *holomorphic* maps between open subsets of $\mathbb{C}^m$.

Definition A.2. If M, N are two differential manifolds, a continuous map $f \colon M \to N$ is *smooth* if for any pair of local charts (U, ϕ) for M and (V, ψ) for N, the composite map $\psi \circ f \circ \phi^{-1} \colon \phi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth. When $N = \mathbb{R}$, the set of all smooth functions on M is a commutative algebra over $\mathbb{R}$, which we denote by $C^\infty(M, \mathbb{R})$; when $N = \mathbb{C}$, the smooth complex-valued functions on M forms a commutative $\mathbb{C}$-algebra, $C^\infty(M, \mathbb{C})$. We often write simply $C^\infty(M)$, if it is clear from the context whether real-valued or complex-valued functions are to be used.

A *diffeomorphism* between M and N is a bijective smooth function $f \colon M \to N$ whose inverse $f^{-1} \colon N \to M$ is also smooth. If such an f exists, we say that M and N are diffeomorphic.

If (U_j, ϕ_j) is a local chart for M, we define $x^1, \dots, x^n \in C^\infty(U_j, \mathbb{R})$ by $x^k := \mathrm{pr}_k \circ \phi_j$, where $\mathrm{pr}_k \colon \mathbb{R}^n \to \mathbb{R}$ is the k-th coordinate projector. We say $(x^1, \dots, x^n)$ is a system of *local coordinates* for M on the chart domain U_j.

The following two lemmas show that smooth functions are abundant.

Lemma A.1. *If M is a differential manifold, and if V, W are two open subsets of M with $\overline{V} \subset W$, there exists $f \in C^\infty(M, \mathbb{R})$ such that $\operatorname{supp} f \subset W$, $f \equiv 1$ on V, and $0 \leq f \leq 1$ on $W \setminus \overline{V}$.* □

Lemma A.2. *If M is a differential manifold, with atlas $\{ (U_j, \phi_j) : j \in J \}$, there exists a smooth **partition of unity** subordinate to the locally finite covering $\mathcal{U}$, that is, a family $\{ f_j : j \in J \} \subset C^\infty(M, \mathbb{R})$ with $0 \leq f_j \leq 1$ and $\operatorname{supp} f_j \subset U_j$ for each j, such that $\sum_{j \in J} f_j(x) = 1$ for all $x \in M$ (the sum is finite for each x).* □

If M, N are two manifolds with respective atlases $\{(U_j, \phi_j)\}$, $\{(V_k, \psi_k)\}$, the *product manifold* is the cartesian product $M \times N$ with atlas $\{(U_j \times V_k, \phi_j \times \psi_k)\}$; its dimension is $\dim(M \times N) = \dim M + \dim N$.

A.2 Tangent spaces

Definition A.3. Let M be a differential manifold, $x \in M$. Let $C^\infty(M, x)$ be the set of all smooth functions $f \colon V_f \to \mathbb{R}$ whose domain is an open neighbourhood of x in M; this is a commutative algebra that includes $C^\infty(M)$. Indeed, $C^\infty(M) = \bigcap_{x \in M} C^\infty(M, x)$.

A *tangent vector* at x is an $\mathbb{R}$-linear map $v \colon C^\infty(M, x) \to \mathbb{R}$ which satisfies the "local Leibniz rule":

$$v(fg) = v(f)\, g(x) + f(x)\, v(g), \qquad \text{for all} \quad f, g \in C^\infty(M, x).$$

These form a real vector space $T_x M$.

If (U_j, ϕ_j) is a local chart for M, with local coordinates $(x^1, \dots, x^n)$, then the directional derivatives $\frac{\partial}{\partial x^j}\big|_x \colon f \mapsto D_j(f \circ \phi^{-1})(\phi(x))$ form a basis for $T_x M$; in particular, $\dim T_x M = n$.

If $\gamma \colon I \to M$ is a smooth curve, whose domain is an interval $I \subseteq \mathbb{R}$, with $\gamma(t_0) = x$, its *velocity vector* at x is $\dot\gamma(t_0) \in T_x M$ defined by $\dot\gamma(t_0)(f) := (f \circ \gamma)'(t_0)$.

Definition A.4. If $f \colon M \to N$ is smooth, and $x \in M$, the *tangent mapping* $T_x f \colon T_x M \to T_{f(x)} N$ is the $\mathbb{R}$-linear map

$$T_x f(v) \colon h \mapsto v(h \circ f),$$

for $v \in T_x M$, $h \in C^\infty(N, f(x))$. If $(y^1, \dots, y^r)$ is a system of local coordinates near $f(x) \in N$, the matrix of $T_x f$ is has entries $\partial f^k / \partial x^j|_x := \frac{\partial}{\partial x^j}\big|_x (y^k \circ f)$.

Definition A.5. A smooth mapping $f \colon M \to N$ is an *immersion* if for all $x \in M$, the tangent map $T_x f \colon T_x M \to T_{f(x)} N$ is injective. If each $T_x f$ is surjective, f is a *submersion*.

If f is both an immersion and a submersion, then $\dim M = \dim N$, and the Jacobian matrix of $T_x f$ (in local coordinates) is invertible, for each x; by the inverse function theorem, f is a diffeomorphism between a neighbourhood of x and a neighbourhood of $f(x)$, for each x; we say f is a local diffeomorphism. However, f need not be injective or surjective on all of M, so it need not be a global diffeomorphism.

A.3 Vector fields

Definition A.6. Let M be a differential manifold. A **vector field** on M is a $\mathbb{R}$-linear operator $X \colon C^\infty(M) \to C^\infty(M)$, which is a *derivation* of this algebra, that is, it satisfies the Leibniz rule:

$$X(fg) = (Xf)\, g + f\, (Xg), \qquad \text{for all} \quad f, g \in C^\infty(M).$$

These derivations form a vector space denoted by $\mathfrak{X}(M)$. This is in fact a *module* for the algebra $C^\infty(M)$, if we define[1] $fX \in \mathfrak{X}(M)$ by $fX(h) := f(Xh)$.

[1] There is a notational difficulty here, since expressions like fXh, while not ambiguous, are confusing. For this reason, the derivation action of the vector field X on the function h is sometimes written $X \cdot h$ rather than Xh; then the module structure can be defined by $(fX) \cdot h := f(X \cdot h)$, and so forth.

If $Y \in \mathfrak{X}(M)$ and $x \in M$, the recipe $Y_x f := (Yf)(x)$ defines a tangent vector $Y_x \in T_x M$.

If (U, ϕ) is a local chart of M, any vector field $X \in \mathfrak{X}(M)$ determines a vector field $X|_U \in \mathfrak{X}(U)$ by restriction. If $(x^1, \dots, x^n)$ is the local coordinate system for this chart, we obtain n linearly independent local vector fields $\frac{\partial}{\partial x^j} \in \mathfrak{X}(U)$ by writing

$$\frac{\partial}{\partial x^j}(f) := D_j(f \circ \phi^{-1}) \circ \phi.$$

These form a basis for the $C^\infty(U)$-module $\mathfrak{X}(U)$, that is, every $X \in \mathfrak{X}(U)$ is of the form $X = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$, with $a^1, \dots, a^n \in C^\infty(U)$.

Definition A.7. The **Lie bracket** of two vector fields $X, Y \in \mathfrak{X}(M)$ is defined as

$$[X, Y] : f \mapsto X(Yf) - Y(Xf).$$

It is easy to check that this is a derivation of $C^\infty(M)$; it is clearly skewsymmetric, and it satisfies the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{A.1}$$

Thus $\mathfrak{X}(M)$ is an (infinite-dimensional) Lie algebra.

Definition A.8. If $\tau\colon M \to N$ is a diffeomorphism, and $X \in \mathfrak{X}(M)$, we define the *pushout* $\tau_* X \in \mathfrak{X}(N)$ by:

$$\tau_* X(h) := X(h \circ \tau) \circ \tau^{-1} \qquad \text{for all} \quad h \in C^\infty(N).$$

For each $x \in M$, we find that $(\tau_* X)_{\tau(x)} = T_x\tau(X_x)$. Note that this last formula makes sense if $\tau\colon M \to N$ is smooth and surjective, but not necessary invertible.

Lemma A.3. *The pushout $\tau_*\colon \mathfrak{X}(M) \to \mathfrak{X}(N)$ is a Lie algebra homomorphism, i.e., τ_* is linear and $\tau_*[X, Y] = [\tau_* X, \tau_* Y]$ for $X, Y \in \mathfrak{X}(M)$. If $\sigma\colon N \to R$ is a diffeomorphism, then $(\sigma \circ \tau)_* = \sigma_* \circ \tau_*$.* $\square$

Definition A.9. An **integral curve** of a vector field $X \in \mathfrak{X}(M)$ is a smooth curve $\gamma\colon I \to M$ such that $\dot\gamma(t) = X_{\gamma(t)}$ for all $t \in I$.

One can always find a unique integral curve γ_x for X satisfying $\gamma(0) = x$ in some maximal interval $I_x \ni 0$, by the existence and uniqueness theorem for first-order ordinary differential equations. We say the vector field X is *complete* if $I_x = \mathbb{R}$ for all $x \in M$; if M is compact, every vector field is complete. Write $\phi_t(x) := \gamma_x(t)$; then $\phi_t\colon M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$, and $\phi_t \circ \phi_s = \phi_{t+s}$ for all t, s.

The one-parameter group of diffeomorphisms $\{\phi_t\}$ is called the *flow* generated by the vector field X. The vector field may be recovered from the flow by noticing that

$$Xf = \lim_{t \to 0} \frac{f \circ \phi_t - f}{t} = \left.\frac{d}{dt}\right|_{t=0} (f \circ \phi_t).$$

A.4 Lie groups

Definition A.10. A **Lie group** is a differential manifold G which is also a group, for which the multiplication $(g,h) \mapsto gh : G \times G \to G$ and the inversion $g \mapsto g^{-1} : G \to G$ are smooth maps.

The *left translations* $\lambda_g \colon h \mapsto gh$ and the *right translations* $\rho_g \colon h \mapsto hg$ are diffeomorphisms from G onto G. We usual write e to denote the identity element of G.

A finite-dimensional (real or complex) vector space V is an additive Lie group. The group $GL_{\mathbb{R}}(V)$ of invertible linear operators on V is a Lie group (since it is a dense open subset of the vector space $\mathrm{End}_{\mathbb{R}}(V)$); if V is a complex vector space, $GL_{\mathbb{C}}(V)$ is a Lie group. We write $GL(n,\mathbb{R}) := GL_{\mathbb{R}}(\mathbb{R}^n)$ and $GL(m,\mathbb{C}) := GL_{\mathbb{C}}(\mathbb{C}^m)$.

Definition A.11. A vector field $X \in \mathfrak{X}(G)$ on a Lie group G is *left-invariant* if $(\lambda_g)_* X = X$ for all $g \in G$. In that case, X is determined by its value at the identity, $X_e \in T_e G$.

A **Lie algebra** is a real vector space with a skewsymmetric bilinear operation $[\cdot,\cdot]$ satisfying the Jacobi identity (A.1). Since the Lie bracket $[X,Y]$ of two left-invariant vector fields X, Y is also left-invariant, $T_e G$ becomes a (finite-dimensional) Lie algebra by defining $[X_e, Y_e] := [X,Y]_e$. We usually write $\mathfrak{g} := T_e G$ to denote this Lie algebra.

Definition A.12. If G is a Lie group with Lie algebra $\mathfrak{g}$, and if $X \in \mathfrak{g}$, let γ_X be the integral curve of the corresponding left-invariant vector field such that $\gamma_X(0) = e$. Then $\gamma_X(s+t) = \gamma_X(s)\,\gamma_X(t)$ for all $s,t \in \mathbb{R}$, so that $t \mapsto \gamma_X(t)$ is a *one-parameter subgroup* of G; also $\gamma_{tX}(1) = \gamma_X(t)$ for $t \in \mathbb{R}$.

We write $\exp X := \gamma_X(1)$. This defines the **exponential map** $\exp\colon \mathfrak{g} \to G$, which satisfies $\exp tX = \gamma_X(t)$, and thus $t \mapsto \exp tX$ is a homomorphism.

Unless G is abelian, exp is not a homomorphism of the additive group $\mathfrak{g}$ into G. However, there is the important *Campbell–Baker–Hausdorff formula*:

$$\exp tX\ \exp tY = \exp\bigl(t(X+Y) + \tfrac{1}{2}t^2[X,Y] + O(t^3)\bigr),$$

and its corollary:

$$\exp tX\ \exp tY\ \exp(-tX) = \exp\bigl(tY + t^2[X,Y] + O(t^3)\bigr). \tag{A.2}$$

If G is a closed subgroup of $GL_{\mathbb{R}}(V)$, the Lie algebra can be identified with the subspace of operators $\{\, X \in \mathrm{End}_{\mathbb{R}}(V) : \exp tX \in G \text{ for all } t \in \mathbb{R} \,\}$, and the Lie bracket becomes $[X,Y] = XY - YX \in \mathrm{End}_{\mathbb{R}}(V)$. If W is a complex vector space, the Lie algebra of a subgroup of $GL_{\mathbb{C}}(W)$ is likewise identified to a subspace of $\mathrm{End}_{\mathbb{C}}(W)$.

Definition A.13. A smooth **left action** of a Lie group G on a differential manifold M is a smooth map $\Phi\colon G \times M \to M : (g,x) \mapsto \Phi(g,x) \equiv g \cdot x$, such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$, for $g,h \in G$, $x \in M$.

Thus $\Phi_g \colon x \mapsto g \cdot x$ is a diffeomorphism of M for each $g \in G$.

If $x \in M$, the *isotropy subgroup* for x is $G_x := \{\, h \in G : h \cdot x = x \,\}$, and the **orbit** of x is $G \cdot x := \{\, g \cdot x \in M : g \in G \,\} \subseteq M$. The natural bijection $g \cdot x \mapsto gG_x$ between $G \cdot x$ and the left-coset space G/G_x is a diffeomorphism when G/G_x is given a natural differential structure for which the quotient map $\eta\colon G \to G/G_x$ is a submersion.

We say that the action of G on M is *free* if no group element except e leaves any point fixed; in that case, all isotropy groups are trivial and all orbits are diffeomorphic to G.

We say that that the action of G on M is *transitive* if there is only one orbit. If so, and if H is the isotropy subgroup of some point of M, then M is diffeomorphic to G/H (and the action of G on M corresponds to permutation of the left-cosets $g'H \mapsto gg'H$). A manifold with a transitive G-action is called a *homogeneous space* for the group G.

Definition A.14. A smooth **right action** of a Lie group G on a differential manifold M is similarly defined, as a smooth map $M \times G \to M : (x, g) \mapsto x \cdot g$ such that $x \cdot e = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$, for $g, h \in G$, $x \in M$.

If H is a closed subgroup of G, then H acts (on the right) on G by right translations $g \mapsto gh$; the orbits of this action are the cosets gH.

Definition A.15. The **adjoint action** of a Lie group G on its Lie algebra $\mathfrak{g}$ is the map $(g, X) \mapsto \mathrm{Ad}(g)X$ given by

$$\mathrm{Ad}(g)X := \frac{d}{dt}\bigg|_{t=0} g(\exp tX)g^{-1}.$$

It can be deduced from (A.2) that $\mathrm{Ad}(g)X \in \mathfrak{g}$. The map $g \mapsto \mathrm{Ad}(g)$ is a homomorphism $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$.

Definition A.16. A **representation** of a Lie group G on a vector space V is a homomorphism $\rho\colon G \to GL(V)$ for which $(g, v) \mapsto \rho(g)v$ is a smooth map from $G \times V$ to V.

The *derived representation* of its Lie algebra $\mathfrak{g}$ is the linear map $\dot\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ given by

$$\dot\rho(X)v := \frac{d}{dt}\bigg|_{t=0} \rho(\exp tX)v.$$

In particular, the derived representation ad of the adjoint representation Ad is given, on account of (A.2), as $\mathrm{ad}(X)Y = [X, Y]$.

A.5 Fibre bundles

Definition A.17. A **fibre bundle** is a triple (E, M, π), more usually written as $E \xrightarrow{\pi} M$, where E and M are differential manifolds (called, respectively, the *total space* and the *base space*), and $\pi\colon E \to B$ is a surjective submersion,[2] such that each *fibre* $E_x := \pi^{-1}(\{x\})$ is diffeomorphic to a fixed manifold F, called the "typical fibre", and which is *locally trivial* in the following sense. If $\mathcal{U} = \{\, U_j : j \in J \,\}$ is a covering of the base space by chart domains, there are diffeomorphisms

$$\psi_j\colon \pi^{-1}(U_j) \to U_j \times F \tag{A.3}$$

(called "local trivializations") such that $\pi(\psi_j^{-1}(x, v)) = x$ for all $x \in U_j$, $v \in F$.

[2] We shall often omit explicit mention of the submersion π and simply write $E \longrightarrow M$ to denote the fibre bundle.

Definition A.18. If $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ are two fibre bundles, a **bundle morphism** is a pair of smooth maps (τ, σ), with $\tau\colon E \to E'$ and $\sigma\colon M \to M'$, such that $\pi' \circ \tau = \sigma \circ \pi$. (We also say that τ is a **lifting** of the map σ between the base spaces.)

A **bundle equivalence** is a bundle morphism (τ, σ) such that both τ and σ are diffeomorphisms.

A fibre bundle $E \longrightarrow M$ is *trivial* if it is equivalent to the product bundle $M \times F \xrightarrow{\mathrm{pr}_1} M$ via a bundle morphism (τ, id_M). Thus (A.3) says that any fibre bundle is locally a trivial bundle.

Definition A.19. A **vector bundle** is a fibre bundle $E \xrightarrow{\pi} M$ whose typical fibre is a (real or complex) vector space V, and whose fibres E_x are vector spaces of the same dimension, such that the maps $V \to E_x : v \mapsto \psi_j^{-1}(x, v)$ are linear isomorphisms.

The dimension $\dim V$ is called the *rank* of the vector bundle.

The **tangent bundle** $TM \longrightarrow M$ of a manifold M has total space $TM := \{\, (x, v) : v \in T_x M \,\}$, with $\pi(x, v) := x$. The local trivialization $\psi_j\colon \pi^{-1}(U_j) \to U_j \times \mathbb{R}^n$ is given by $\psi_j(x, v) := (x; v^1, \dots, v^n)$, with $v = \sum_k v^k \frac{\partial}{\partial x^k}\big|_x$. The atlas $\{\, (\pi^{-1}(U_j), (\phi_j \times \mathrm{id}) \circ \psi_j) : j \in J \,\}$ makes TM a $2n$-dimensional manifold. The fibre at $x \in M$ is the tangent space $T_x M$.

The **cotangent bundle** $T^*M \longrightarrow M$ is formed similarly; its fibres are the dual spaces $T_x^* M := (T_x M)^*$. We define $\{dx^1|_x, \dots, dx^n|_x\}$ as the dual basis in $T_x^* M$ to the basis $\{\frac{\partial}{\partial x^1}\big|_x, \dots, \frac{\partial}{\partial x^n}\big|_x\}$ of $T_x M$, and if $\xi = \sum_k \xi_k \, dx^k|_x$, then $\psi_j(x, \xi) := (x; \xi_1, \dots, \xi_n)$.

Definition A.20. A smooth **section** of a fibre bundle $E \xrightarrow{\pi} M$ is a smooth map $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$, i.e., $s(x) \in E_x$ for each $x \in M$. We denote the totality of smooth sections by $\Gamma(M, E)$, or simply by $\Gamma(E)$ if the base space M is understood; it is a $C^\infty(M)$-*module*, where the action of $C^\infty(M)$ is just scalar multiplication on each fibre:

$$(fs)(x) := f(x)\, s(x),$$

for $s \in \Gamma(E)$, $f \in C^\infty(M)$.

If $U \subset M$ is open, a smooth map $s\colon U \to \pi^{-1}(U)$ satisfying $\pi(s(x)) = x$ for $x \in U$ is called a *local section* of $E \xrightarrow{\pi} M$; all such maps form a vector space $\Gamma(U, E)$, which is a module over $C^\infty(U)$.

From the definition, it is easy to see that a smooth section of the tangent bundle $TM \longrightarrow M$ can be written in local coordinates on a chart domain U_j as $X = \sum_{k=1}^n a^k \frac{\partial}{\partial x^k}$, with each $a^k \in C^\infty(U)$. Thus X is nothing other than a vector field on M; we have $\Gamma(TM) = \mathfrak{X}(M)$ as $C^\infty(M)$-modules.

A section of a trivial fibre bundle $M \times F \xrightarrow{\mathrm{pr}_1} M$ is of the form $s(x) = (x, f(x))$ where $f\colon M \to F$ is a smooth map. Thus sections of more general bundles can be thought of as "functions" which take values in different sets at each point of their domains.

A.6 Tensors and differential forms

Definition A.21. A **differential 1-form** on a differential manifold M is a map $\alpha\colon \mathfrak{X}(M) \to C^\infty(M)$ that is $C^\infty(M)$-*linear*, that is:

$$\alpha(X+Y) = \alpha(X) + \alpha(Y), \qquad \alpha(fX) = f\,\alpha(X),$$

for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. These form a real vector space $\mathcal{A}^1(M)$, which becomes a $C^\infty(M)$-module on defining $f\alpha : X \mapsto f\,\alpha(X)$.

If $x \in M$, then $\alpha(Y)(x) = f\alpha(Y)(x)$ if f is any smooth function with $f(x) = 1$, whose support is an (arbitrary small) neighbourhood of x. Thus $\alpha(Y)(x)$ depends only on Y_x, and so $Y_x \mapsto \alpha(Y)(x)$ is an element α_x of the dual space T_x^*M of T_xM; by definition, $\alpha(Y)(x) = \alpha_x(Y_x)$.

Hence α can be identified with the section $x \mapsto \alpha_x$ of the cotangent bundle $T^*M \longrightarrow M$; and $\Gamma(T^*M) = \mathcal{A}^1(M)$ as $C^\infty(M)$-modules. In local coordinates over U, we can write $\alpha = \sum_{k=1}^n f_k\,dx^k$ with each $f_k \in C^\infty(U)$.

Definition A.22. A **tensor** of *bidegree* (p,q) on a manifold M is a multilinear map

$$T\colon \mathfrak{X}(M)^p \times \mathcal{A}^1(M)^q \to C^\infty(M)$$

such that $T(X_1,\dots,X_p,\alpha^1,\dots,\alpha^q)$ is $C^\infty(M)$-linear in each X_j and each α^k. The tensor is called covariant if $q = 0$, or contravariant if $p = 0$. Any such tensor T defines a smooth section of a vector bundle over M whose fibre at $x \in M$ is $(T_x^*M)^{\otimes p} \otimes (T_xM)^{\otimes q}$.

Definition A.23. A **Riemannian metric** on M is a tensor g of bidegree $(2,0)$ that is symmetric, i.e., $g(X,Y) = g(Y,X)$ for all $X, Y \in \mathfrak{X}(M)$, and positive definite: $g(X,X) > 0$ for nonzero $X \in \mathfrak{X}(M)$. Locally, we may take $g = g_{ij}\,dx^i \cdot dx^j$, where $[g_{ij}]$ is a positive-definite symmetric matrix of elements of $C^\infty(U)$ and $dx^i \cdot dx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$; a Riemannian metric may be defined globally on M by taking $g = \sum_j f_j g^{(j)}$, where $g^{(j)}$ is a metric on the chart domain U_j, and the functions f_j form a smooth partition of unity on M.

The pair (M,g), consisting of a differential manifold with a Riemannian metric g, is called a **Riemannian manifold**.

A *Hermitian metric* on M is a tensor h of bidegree $(2,0)$ with values in $C^\infty(M,\mathbb{C})$, such that $h(X,Y) = \overline{h(Y,X)}$ for $X, Y \in \mathfrak{X}(M)$ and h is positive definite. One often writes $(X \mid Y)$ instead of $h(X,Y)$. The same partition-of-unity argument shows that any manifold can be given a Hermitian metric.

Definition A.24. A **differential** k**-form** on M is a covariant tensor $\omega\colon \mathfrak{X}(M)^k \to C^\infty(M)$ that is *alternating*, which means that $\omega(X_{\sigma(1)},\dots,X_{\sigma(k)}) = (-1)^\sigma \omega(X_1,\dots,X_k)$ for $\sigma \in S_k$. The totality of k-forms on M is denoted $\mathcal{A}^k(M)$, and is a $C^\infty(M)$-module.

Let $\Lambda^k T^*M \longrightarrow M$ be the vector bundle whose fibre at x is $\Lambda^k T_x^*M$, the k-th exterior power of T_x^*M; then $\mathcal{A}^k(M) = \Gamma(\Lambda^k T^*M)$.

The direct sum $\mathcal{A}^\bullet(M) := \bigoplus_{k=0}^n \mathcal{A}^k(M) = \Gamma(\Lambda^\bullet T^*M)$ is a $\mathbb{Z}$-graded $C^\infty(M)$-module. The zero-degree term is $\mathcal{A}^0(M) := C^\infty(M)$. Under the exterior product, $\mathcal{A}^\bullet(M)$ is an algebra;

in fact, it is $\mathbb{Z}_2$-graded by the parity of the degree k, and is a supercommutative superalgebra, since this property holds in each fibre of $\Lambda^\bullet T^*M \longrightarrow M$. Thus $\omega \wedge \eta = (-1)^{\sharp\omega\,\sharp\eta}\eta \wedge \omega$ where $\sharp\omega = k$ for $\omega \in \mathcal{A}^k(M)$.

Thus if $\omega \in \mathcal{A}^k(M)$, $\eta \in \mathcal{A}^l(M)$, and $X_1, \dots, X_{k+l} \in \mathfrak{X}(M)$, then

$$(\omega \wedge \eta)(X_1, \dots, X_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}),$$

and in particular, $(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$ for $\alpha, \beta \in \mathcal{A}^1(M)$. Moreover, if $\alpha^1, \dots, \alpha^k \in \mathcal{A}^1(M)$ then $\alpha^1 \wedge \dots \wedge \alpha^k \in \mathcal{A}^k(M)$, with

$$(\alpha^1 \wedge \dots \wedge \alpha^k)(X_1, \dots, X_k) = \det[\alpha^i(X_j)].$$

In local coordinates, an element of $\mathcal{A}^k(U)$ is of the form $\omega = \sum_{|J|=k} f_J\, dx^{j_1} \wedge \dots \wedge dx^{j_k}$, where $J = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$.

A.7 Calculus of differential forms

Definition A.25. If $\tau\colon M \to N$ is a diffeomorphism, and $\omega \in \mathcal{A}^k(N)$, we define the **pullback** $\tau^*\omega \in \mathcal{A}^k(M)$ by:

$$\tau^*\omega(X_1, \dots, X_k) := \omega(\tau_* X_1, \dots, \tau_* X_k) \circ \tau.$$

In particular, $\tau^*\beta(X) := \beta(\tau_* X) \circ \tau$ for $\beta \in \mathcal{A}^1(N)$. Thus $(\tau^*\beta)_x(X_x) := \beta_{\tau(x)}(T_x\tau(X_x))$, so that the linear map $\beta_{\tau(x)} \mapsto (\tau^*\beta)_x : T^*_{\tau(x)}N \to T^*_x M$ is the transpose of the tangent map $T_x\tau\colon T_xM \to T_{\tau(x)}N$.

If $\tau\colon M \to N$ is any smooth map, not necessarily a diffeomorphism, the pullback $\tau^*\omega$ of $\omega \in \mathcal{A}^k(N)$ is likewise defined by transposition:

$$(\tau^*\omega)_x((X_1)_x, \dots, (X_k)_x) := \omega_{\tau(x)}(T_x\tau((X_1)_x), \dots, T_x\tau((X_k)_x)).$$

For $k = 0$, we get simply: $\tau^* f := f \circ \tau$.

Lemma A.4. *The pullback $\tau^*\colon \mathcal{A}^\bullet(N) \to \mathcal{A}^\bullet(M)$ is a degree-preserving homomorphism of exterior algebras, that is, τ^* is linear and $\tau^*(\omega \wedge \eta) = \tau^*\omega \wedge \tau^*\eta$ for $\omega, \eta \in \mathcal{A}^\bullet(N)$. If $\sigma\colon N \to R$ is a smooth map, then $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$.* $\square$

Definition A.26. The **contraction** of a k-form $\omega \in \mathcal{A}^k(M)$ with a vector field $X \in \mathfrak{X}(M)$ is the $(k-1)$-form $\iota(X)\omega \equiv \iota_X\omega$ defined as

$$\iota_X\omega(X_1, \dots, X_{k-1}) := \omega(X, X_1, \dots, X_{k-1}).$$

For $f \in \mathcal{A}^0(M)$, we set $\iota_X f := 0$. If $\alpha \in \mathcal{A}^1(M)$, then $\iota_X\alpha = \alpha(X) \in C^\infty(M)$.

Lemma A.5. *Contraction with $X \in \mathfrak{X}(M)$ is an odd derivation of the graded algebra $\mathcal{A}^\bullet(M)$, that is,*

$$\iota_X(\omega \wedge \eta) = \iota_X\omega \wedge \eta + (-1)^{\sharp\omega}\omega \wedge \iota_X\eta.$$

In particular,

$$\iota_X(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{j=1}^{k}(-1)^{j-1}\alpha^j(X)\,(\alpha^1 \wedge \cdot \overset{j}{\vee} \cdot \wedge \alpha^k)$$

for $\alpha_1, \dots, \alpha_k \in \mathcal{A}^1(M)$. □

Proposition A.6. *There is a unique operator $d\colon \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$, called* **exterior derivation**, *such that:*

1. *d is an odd derivation of degree $+1$, that is, $d(\mathcal{A}^k(M)) \subset \mathcal{A}^{k+1}(M)$, and*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\sharp\omega}\omega \wedge d\eta;$$

2. *$df(X) = Xf$ for $f \in \mathcal{A}^0(M) = C^\infty(M)$;*
3. *$d^2 \equiv d \circ d = 0$;*
4. *d is natural with respect to restrictions, that is, if $U \subset M$ is open and $\omega \in \mathcal{A}^\bullet(U)$, then $d(\omega|_U) = (d\omega)\big|_U$.* □

In local coordinates, $d\big(\sum_J f_J\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\big) = \sum_J df_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$, where, for $f \in C^\infty(U)$, $df = \sum_{j=1}^n (\partial f/\partial x^j)\, dx^j$.

Lemma A.7. *If $\omega \in \mathcal{A}^k(M)$, its exterior derivative $d\omega \in \mathcal{A}^{k+1}(M)$ is given by*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{j=1}^{k+1}(-1)^{j-1}X_j(\omega(X_1, . \overset{j}{\vee} ., X_{k+1})) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, . \overset{i}{\vee} . \overset{j}{\vee} ., X_{k+1})). \end{aligned}$$

In particular, if $\alpha \in \mathcal{A}^1(M)$, then $d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$. □

Lemma A.8. *The exterior derivative commutes with pullbacks: if $\tau\colon M \to N$ is a smooth map, and $\omega \in \mathcal{A}^k(N)$, then $d(\tau^*\omega) = \tau^*(d\omega)$ in $\mathcal{A}^{k+1}(M)$.* □

Definition A.27. If $X \in \mathfrak{X}(M)$ is a complete vector field, whose flow is $\{\phi_t\}$, the **Lie derivative** of a differential form $\omega \in \mathcal{A}^\bullet(M)$ is the form $\mathcal{L}_X\omega$ defined by

$$\mathcal{L}_X\omega := \frac{d}{dt}\bigg|_{t=0} \phi_t^*\omega = \lim_{t\to 0} \frac{\phi_t^*\omega - \omega}{t}. \tag{A.4}$$

Notice that $\mathcal{L}_X f = Xf$ for $f \in \mathcal{A}^0(M)$.

If R is a contravariant tensor, we interpret $\phi_t^* R$ as the pushout $\phi_{-t*}R$ by the inverse diffeomorphism $\phi_{-t} = \phi_t^{-1}$; with this convention, (A.4) makes sense when ω is any tensor, and $\mathfrak{L}_X\omega$ is a tensor of the same bidegree. In particular, if Y is a vector field,

$$\mathfrak{L}_X Y := \frac{d}{dt}\bigg|_{t=0} \phi_{-t*}Y = \frac{d}{dt}\bigg|_{t=0} Y(\bullet \circ \phi_{-t}) \circ \phi_t = [X, Y].$$

Lemma A.9. *The Lie derivative $\mathfrak{L}_X \colon \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ is an $\mathbb{R}$-linear map satisfying:*

1. *$\mathfrak{L}_X$ is an even derivation of degree 0, that is, $\mathfrak{L}_X(\mathcal{A}^k(M)) \subset \mathcal{A}^k(M)$, and*
$$\mathfrak{L}_X(\omega \wedge \eta) = \mathfrak{L}_X\omega \wedge \eta + \omega \wedge \mathfrak{L}_X\eta; \tag{A.5}$$
2. *in particular, $\mathfrak{L}_X(f\omega) = (Xf)\,\omega + f\,\mathfrak{L}_X\omega$;*
3. *$\mathfrak{L}_X(d\omega) = d(\mathfrak{L}_X\omega)$; for all $\omega, \eta \in \mathcal{A}^\bullet(M)$, $f \in C^\infty(M)$.* $\square$

Lemma A.10. *The Lie derivative $\mathfrak{L}_X \colon \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ is given by* **Cartan's formula***:*

$$\mathfrak{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega),$$

i.e., $\mathfrak{L}_X = \iota_X \circ d + d \circ \iota_X$, for $X \in \mathfrak{X}(M)$. $\square$

Corollary A.11. *For any $\omega \in \mathcal{A}^k(M)$, $\mathfrak{L}_X\omega$ is given by the formula:*

$$\mathfrak{L}_X\omega(X_1, \dots, X_k) = X(\omega(X_1, \dots, X_k)) - \sum_{j=1}^{k} \omega(X_1, \dots, [X, X_j], \dots, X_k)).$$

In particular, $\mathfrak{L}_X\alpha(Y) = X(\alpha(Y)) - \alpha([X, Y])$ for $\alpha \in \mathcal{A}^1(M)$, $X, Y \in \mathfrak{X}(M)$. $\square$

Notice that this last formula can be rearranged as: $\mathfrak{L}_X(\alpha(Y)) = \mathfrak{L}_X\alpha(Y) + \alpha(\mathfrak{L}_X Y)$. In other words, the Leibniz rule (A.5) for $\mathfrak{L}_X$ is valid not only for exterior products of forms, but also for pairings of forms and vector fields. In passing, we note also that the Jacobi identity (A.1) for vector fields can be written as $\mathfrak{L}_X([Y, Z]) = [\mathfrak{L}_X Y, Z] + [Y, \mathfrak{L}_X Z]$.

Lemma A.12. *The map $X \mapsto \mathfrak{L}_X$ from $\mathfrak{X}(M)$ to* $\operatorname{End}(\mathcal{A}^\bullet(M))$ *is a Lie algebra homomorphism; that is,*

$$\mathfrak{L}_{[X,Y]}\omega = \mathfrak{L}_X(\mathfrak{L}_Y\omega) - \mathfrak{L}_Y(\mathfrak{L}_X\omega) \tag{A.6}$$

for all $X, Y \in \mathfrak{X}(M)$, $\omega \in \mathcal{A}^\bullet(M)$. $\square$

Note that if we substitute a vector field Z for ω in (A.6), we again get the Jacobi identity.

A.8 The de Rham complex

Definition A.28. A **cochain complex** $(C^\bullet, d)$ is a sequence of abelian groups and homomorphisms

$$C^0 \xrightarrow{d_0} C^1 \to \cdots \to C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} C^{n+2} \to \cdots$$

such that $d_{n+1} \circ d_n = 0$ for all n. (The complex may terminate: it may happen that for some N, $C^n = 0$ for $n > N$.) We usually suppress the index of d and write simply that $d^2 = 0$ at all stages. The elements of C^n are called "n-cochains": $c \in C^n$ is an *n-cocycle* if $dc = 0$; it is an *n-coboundary* if $c = db$ for some $b \in C^{n-1}$. The condition $d^2 = 0$ says that the totality of n-coboundaries $B^n(C^\bullet)$ is a subgroup of the group $Z^n(C^\bullet)$ of n-cocycles. The quotient group

$$H^n(C^\bullet) := Z^n(C^\bullet)/B^n(C^\bullet)$$

is called the n-th **cohomology group** of the complex.

A *morphism* between two complexes $(C^\bullet, d)$ and $(K^\bullet, d')$ is a set of homomorphisms $f_n \colon C^n \to K^n$ which intertwines the d-maps, i.e., $f_{n+1} \circ d_n = d'_n \circ f_n$ for all n. Thus $f_n(Z^n(C^\bullet)) \subseteq Z^n(K^\bullet)$ and $f_n(B^n(C^\bullet)) \subseteq B^n(K^\bullet)$, so that f induces a homomorphism $H^n f \colon H^n(C^\bullet) \to H^n(K^\bullet)$.

The components C^n of a complex may have more structure than that of an abelian group: they could be vector spaces, modules over a commutative ring, etc.[3] The cohomology groups $H^n(C^\bullet, d)$ inherit a similar structure.

Definition A.29. The **de Rham complex** of an n-dimensional differential manifold M is the terminating complex

$$\mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \to \cdots \to \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \to \cdots \xrightarrow{d} \mathcal{A}^n(M)$$

of $C^\infty(M)$-modules; here d is the exterior derivation. We say a k-form ω is *closed* if $d\omega = 0$, and that ω is *exact* if $\omega = d\eta$ for some $(k-1)$-form η; thus $Z^k_{\mathrm{dR}}(M) := Z^k(\mathcal{A}^\bullet(M), d)$ comprises the closed k-forms and $B^k_{\mathrm{dR}}(M) := B^k(\mathcal{A}^\bullet(M), d)$ comprises the exact k-forms. The k-th *de Rham cohomology group*

$$H^k_{\mathrm{dR}}(M) := H^k(\mathcal{A}^\bullet(M), d)$$

is a real vector space. We shall denote by $[\omega] \in H^k_{\mathrm{dR}}(M)$ the class of $\omega \in Z^k_{\mathrm{dR}}(M)$.

By the de Rham theorems —see, for instance, [23]— if M is compact then $H^k_{\mathrm{dR}}(M) \simeq H^k(M, \mathbb{R})$, where the latter is the k-th *singular cohomology group*, which is a finite-dimensional real vector space depending only on the topology of M.

The most important single fact about de Rham cohomology is the following proposition, often called the **Poincaré lemma**.

Proposition A.13. *Suppose that U is a* contractible *manifold, i.e., for some $x_0 \in U$, there is a smooth map $f \colon [0,1] \times U \to U$ such that $f(0,x) = x_0$ and $f(1,x) = x$, for $x \in U$. Then $H^0_{\mathrm{dR}}(U) = \mathbb{R}$ and $H^k_{\mathrm{dR}}(U) = 0$ for $k > 0$.*

[3] In fact, a complex can be formed from objects C^n and morphisms d_n in any abelian category.

Proof. A contractible manifold has an atlas with a single chart (U, ϕ), so we can suppose that $U \subseteq \mathbb{R}^n$, that U is star-shaped about x_0, and that $f(t, x) = (1 - t)x_0 + tx$. If $\omega \in \mathcal{A}^k(U)$, let $\eta := \iota_{\partial/\partial t}(f^*\omega) \in \mathcal{A}^{k-1}([0, 1] \times U)$. Now define $h_k \colon \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ by $h_k(\omega) := \int_0^1 \eta \, dt$. Then one can check that the maps h_k form a "cochain homotopy", i.e., that $h_{k+1} \circ d + d \circ h_k = \mathrm{id}$ for each $k > 0$, and $h_1 \circ d = \mathrm{id} - x_0$. The triviality of the cohomology groups follows at once, since $d\omega = 0$ implies $\omega = d(h_k\omega)$ for $k > 0$, and $df = 0$ implies $f \equiv f(x_0)$ for $f \in \mathcal{A}^0(M)$. □

A.9 Volume forms and integrals

Definition A.30. A **volume form** on an n-dimensional manifold M is a real n-form $\nu \in \mathcal{A}^n(M)$ which is *nonvanishing*, i.e., $\nu_x \neq 0$ in $\Lambda^n T_x^* M$ for all $x \in M$. A volume form need not exist; we say that M is **orientable** if one exists.

We say that two volume forms μ, ν on M are equivalent if $\mu = f\nu$ for some $f \in C^\infty(M)$ with $f(x) > 0$ for all $x \in M$. (This is clearly an equivalence relation.) An equivalence class for this relation is called an *orientation* on M; a pair (M, ν) consisting of a manifold and a volume form ν in a given equivalence class is called an *oriented manifold*.

In a local coordinate system on a chart domain U, we have $\nu = h \, dx^1 \wedge \cdots \wedge dx^n$ with $h \in C^\infty(U, \mathbb{R})$ nonvanishing. Under a change of local coordinates in the overlap of two chart domains, h is multiplied by a Jacobian factor, $\det[\partial y^i/\partial x^j]$.

Lemma A.14. *A differential manifold M is orientable if and only if M has an atlas $\{(U_j, \phi_j)\}$ all of whose transition functions $\phi_i \circ \phi_j^{-1}$ have positive Jacobians.* □

On an oriented manifold (M, ν), we therefore may and shall always choose an atlas such that in every local coordinate system we have $\nu = h \, dx^1 \wedge \cdots \wedge dx^n$ with $h > 0$. (We say that the corresponding charts are "positively oriented".)

Proposition A.15. *Let (M, ν) be an oriented manifold. Then there is a unique linear form $\int_M \colon \mathcal{A}^n(M) \to \mathbb{R}$, called the **integral** over M, such that if $\eta \in \mathcal{A}^n(M)$ vanishes outside the domain of a positively oriented chart (U, ϕ) with local coordinate system $(x^1, \dots, x^n)$, and if $\eta = f \, dx^1 \wedge \cdots \wedge dx^n$, then*

$$\int_M \eta = \int_{\phi(U)} f(x^1, \dots, x^n) \, dx^1 \dots dx^n,$$

where the right hand side is a Lebesgue integral on $\mathbb{R}^n$.

Proof. Uniqueness of $\int_M$ follows from the change-of-variables formula for multiple Lebesgue integrals; existence follows by writing $\eta = \sum_j f_j \eta_j$ where each η_j is supported in a chart domain and $\{f_j\}$ is a partition of unity. □

Lemma A.16. *If (M, ν) and (N, ρ) are two oriented n-dimensional manifolds and if $\tau \colon M \to N$ is an orientation-preserving diffeomorphism (i.e., $\tau^*\rho$ is equivalent to ν), then $\int_M \tau^*\eta = \int_N \eta$ for all $\eta \in \mathcal{A}^n(N)$.* □

An n-form on M is closed, since $\mathcal{A}^{n+1}(M) = 0$. If $\eta = d\zeta$ is an exact n-form, with compact support in the domain of an oriented chart (U, ϕ), then

$$\int_M \eta = \int_U d\zeta = \int_{\phi(U)} \psi^*(d\zeta) = \int_{\phi(U)} d(\psi^*\zeta),$$

where $\psi = \phi^{-1} \colon \phi(U) \to U$. Since $\psi^*\zeta = \sum_j g_j \, dx^1 \wedge \cdots \overset{j}{\vee} \cdots \wedge dx^n$ for some $g_j \in C^\infty(U)$ having compact supports in U, we conclude that $d(\psi^*\zeta) = \bigl(\sum_j \partial g_j/\partial x^j\bigr) \, dx^1 \wedge \cdots \wedge dx^n$; therefore, $\int_U d\zeta = 0$ by the fundamental theorem of calculus. By a partition-of-unity argument, we obtain $\int_M d\zeta = 0$ for any $\zeta \in \mathcal{A}^{n-1}(M)$, so *the integral vanishes on exact n-forms*. (This is the "boundaryless" case of Stokes' theorem.) In consequence, $[\eta] \mapsto \int_M \eta$ is a well-defined linear form on $H^n_{\mathrm{dR}}(M)$.

References

[1] R. Abraham, J. E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Springer, Berlin, 1988.

[2] H. Araki, "Bogoliubov automorphisms and Fock representations of canonical anticommutation relations", Contemp. Math. **62** (1987), 23–141.

[3] M. F. Atiyah, *K-Theory*, Benjamin, New York, 1967.

[4] M. F. Atiyah, R. Bott and A. Shapiro, "Clifford Modules", Topology **3** (1964), 3–38.

[5] M. F. Atiyah and I. M. Singer, "The index of elliptic operators. I", Ann. Math. **87** (1968), 484–530.

[6] M. F. Atiyah and G. B. Segal, "The index of elliptic operators. II", Ann. Math. **87** (1968), 531–545.

[7] F. A. Berezin, *The Method of Second Quantization*, Academic Press, New York, 1966.

[8] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math. **194**, Springer, Berlin, 1971.

[9] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer, Berlin, 1992.

[10] L. C. Biedenharn and J. D. Louck, *The Racah–Wigner Algebra in Quantum Theory*, Addison-Wesley, Reading, MA, 1981.

[11] R. Bott, "Homogeneous vector bundles", Ann. Math. **66** (1957), 203–248.

[12] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer, Berlin, 1982.

[13] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer, Berlin, 1985.

[14] J.-L. Brylinski, *Loop Spaces, Characteristic Classes, and Geometric Quantization*, Birkhäuser, Boston, 1993.

[15] P. Candelas, "Lectures on Complex Manifolds", in *Superstrings and Grand Unification*, T. Pradhan, ed., World Scientific, Singapore, 1988.

[16] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia Univ. Press, New York, 1954.

[17] L. Conlon, *Differentiable Manifolds: A First Course*, Birkhäuser, Boston, 1993.

[18] A. Connes, *Noncommutative Geometry*, Academic Press, London, 1994.

[19] A. Connes, private communication.

[20] M. Crampin and F. A. E. Pirani, *Applicable Differential Geometry*, Cambridge Univ. Press, Cambridge, 1986.

[21] L. Dabrowski and A. Trautman, "Spinor structures on spheres and projective spaces", J. Math. Phys. **27** (1986), 2022–2028.

[22] J. A. Dieudonné, *Treatise on Analysis*, Vol. **2**, Academic Press, New York, 1970.

[23] J. A. Dieudonné, *Eléments d'Analyse*, Vol. **9**, Gauthier–Villars, Paris, 1982.

[24] P. A. M. Dirac, "Quantized singularities in the electromagnetic field", Proc. Roy. Soc. A **133** (1931), 60–72.

[25] T. Dray, "The relationship between monopole harmonics and spin-weighted spherical harmonics", J. Math. Phys. **26** (1985), 1030–1033.

[26] G. B. Folland, "Harmonic analysis of the de Rham complex on the sphere", J. reine angew. Math. **398** (1989), 130–143.

[27] J. E. Gilbert and M. A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge Univ. Press, Cambridge, 1991.

[28] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah–Singer Index Theorem*, Publish or Perish, Wilmington, DL, 1984.

[29] P. B. Gilkey, "The index theorem and Clifford modules", in *Forty More Years of Ramifications: Spectral Asymptotics and its Applications*, S. A. Fulling and F. J. Narcowich, eds., Dept. of Math., Texas A&M University, College Station, TX, 1991; pp. 265–317.

[30] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich and E. C. G. Sudarshan, "Spin-s spherical harmonics and $\eth$", J. Math. Phys. **8** (1967), 2155–2161.

[31] J. M. Gracia-Bondía and J. C. Várilly, "QED in external fields from the spin representation", J. Math. Phys. **35** (1994), 3340–3367.

[32] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[33] W. V. D. Hodge, *The Theory and Applications of Harmonic Integrals*, Cambridge Univ. Press, Cambridge, 1952.

[34] R. Howe and E. C. Tan, *Nonabelian Harmonic Analysis: Applications of $SL(2,\mathbb{R})$*, Springer, Berlin, 1992.

[35] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer, Berlin, 1970.

[36] W. Klingenberg, *Riemannian Geometry*, de Gruyter, Berlin, 1982.

[37] A. W. Knapp, *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton, NJ, 1986.

[38] B. Kostant, "Quantization and unitary representations", in *Lectures in Modern Analysis and Applications III*, Lecture Notes in Mathematics **170**, Springer, Berlin, 1970; pp. 87–208.

[39] H. B. Lawson and M. L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, NJ, 1989.

[40] A. Lichnerowicz, "Spineurs harmoniques", C. R. Acad. Sci. Paris **257A** (1963), 7–9.

[41] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton Univ. Press, Princeton, NJ, 1974.

[42] C. Nash, *Differential Topology and Quantum Field Theory*, Academic Press, London, 1991.

[43] E. T. Newman and R. Penrose, "Note on the Bondi–Metzner–Sachs group", J. Math. Phys. **7** (1966), 863–870.

[44] A. Pressley and G. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.

[45] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1972.

[46] M. A. Rieffel, "Morita equivalence for C^*-algebras and W^*-algebras", J. Pure Appl. Algebra **5** (1974), 51–96.

[47] E. Schrödinger, "Eigenschwingungen des sphärischen Raumes", Commentationes Pontificia Academia Scientiarum **2** (1938), 321–364.

[48] G. B. Segal, "Equivariant K-theory", Publ. Math. IHES **34** (1968), 129–151.

[49] G. B. Segal, "The representation ring of a compact Lie group", Publ. Math. IHES **34** (1968), 113–128.

[50] J.-P. Serre, "Représentations linéaires et espaces homogénes kählériens des groupes de Lie compacts", in *Séminaire Bourbaki*, Exposé **100**, 1954.

[51] I. M. Singer and J. Thorpe, *Lecture Notes on Elementary Topology and Geometry*, Scott–Foresman, Glenview, IL, 1967.

[52] M. D. Spivak, *A Comprehensive Introduction to Differential Geometry*, 5 vols., Publish or Perish, Berkeley, 1979.

[53] M. E. Taylor, *Noncommutative Harmonic Analysis*, Amer. Math. Soc., Providence, RI, 1986.

[54] J. C. Várilly and J. M. Gracia-Bondía, "Connes' noncommutative differential geometry and the Standard Model", J. Geom. Phys. **12** (1993), 223–301.

[55] M. Vergne, "Geometric quantization and equivariant cohomology", in *ECM: Proceedings of the First European Congress of Mathematics*, A. Joseph, F. Mignot, F. Murat, B. Prum and R. Rentschler, eds., Progress in Mathematics **119**, Birkhäuser, Boston, 1994; pp. 249–295.

[56] N. R. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.

[57] A. Weil, *Introduction à l'étude des variétés kähleriennes*, 2nd edition, Hermann, Paris, 1971.

[58] R. O. Wells, *Differential Analysis on Complex Manifolds*, Springer, Berlin, 1980.

[59] N. J. Wildberger, "Hypergroups, symmetric spaces and wrapping maps", preprint, Sydney, 1995.

[60] E. Witten, "Monopoles and four-manifolds", Math. Res. Lett. **1** (1994), 769–796.

[61] J. Wolf, "Essential selfadjointness for the Dirac operator and its square", Indiana Univ. Math. J. **22** (1972), 611–640.