Groups with f-generics in NTP\(_2\) and PRC fields

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Abstract

We study groups with f-generic types definable in bounded PRC fields. Along the way, we generalize part of the basic theory of definably amenable NIP groups to NTP\(_2\) theories and prove variations on Hrushovski’s stabilizer theorem.

1 Introduction

A field is PRC if every absolutely irreducible variety which has zeros in every real closed extension has a zero in the field. Hence PRC fields generalize both the notions of real closed fields and of pseudo algebraically closed fields (PAC). It was shown in [Mon17] that bounded PRC fields are NTP\(_2\), a notion which generalizes the more studied concepts of dependent theories and simple theories. Since bounded PAC fields have been a very inspirational example of a simple unstable field, and real closed fields are one of the most well studied dependent fields, bounded PRC fields appear to be examples of NTP\(_2\) fields, the study of which can be very telling about which properties one can and cannot expect of an NTP\(_2\) theory.

In this paper we try to understand definable groups in a bounded PRC field, assuming in addition existence of f-generic types (a slightly weaker assumption than definable amenability). We prove that such a group is isogeneous with a finite index subgroup of a quantifier-free definable groups (Theorem 6.3). In fact, that latter group admits a definable covering by multicells on which the group operation is algebraic. This generalizes similar

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results proved in [HP94] by Hrushovski and Pillay for (not necessarily f-
generic) groups definable in both pseudofinite fields and real closed fields. Our theorem applies in particular to all solvable groups.

In order to prove this result we need to develop various tools. In Section 2 we prove two new versions of Hrushovski’s Stabilizer Theorem from [Hru12]. In particular, we manage to give a slightly simpler proof at the cost of losing some optimality in the hypothesis. In Section 3 we prove some results about groups definable in an \( \text{NTP}_2 \) theory admitting f-generic types. We generalize some basic statements proved in [CS16] for definably amenable NIP groups. In Section 4 we recall some results on PRC fields and prove that the expansion of a bounded PRC field obtained by adding all quantifier-free externally definable sets has elimination of quantifiers.

The sketch of the proof of the main theorem is as follows: After an initial reduction to groups of finite index, we use the same ideas of the first sections of [HP94] to show that given a group \( G \) with f-generics definable in a bounded PRC field, there is an algebraic group \( H \) and a (relatively) definable isomorphism between type definable subgroups \( G_{M}^{00} \) of \( G \) and \( K \) of \( H \). The isomorphism is achieved from the maximum type definable over \( M \) subgroup \( G_{M}^{00} \) of \( G \), a fact uses very strongly that \( G \) has f-generics and the Stabilizer Theorem. In Section 5 we show that for any such \( K \) (a type definable subgroup of an algebraic group), if \( \overline{K} \) denotes the topological closure of \( K \), then \( \overline{K}/K \) is profinite. The proof then continues adapting the proofs in [HP94] for the pseudofinite case and for the real closed case to complete the proof of the bounded PRC case.

## 2 Stabilizer theorems

Let \( M \) be a model and let \( G \) be an \( M \)-definable group. Let \( \mu \) be an \( M \)-invariant ideal of subsets of \( G \) which is invariant by left translations by elements of \( G \). We say that a type \( p(x) \) in \( G \) is \( \mu \)-wide if it is not contained in a set \( D \in \mu \). A key concept we will need is Hrushovski’s definition of an S1 ideal.

**Definition 2.1.** An \( A \)-invariant ideal \( \mu \) has the S1 property if whenever \( (a_j)_{j \in \omega} \) is an \( A \)-indiscernible sequence and \( \phi(x, y) \) is a formula, then if \( \phi(x, a_i) \land \phi(x, a_j) \) is in \( \mu \) for some/all \( i \neq j \), then \( \phi(x, a_i) \) is in \( \mu \) for some/any \( i \).

We will say that the ideal \( \mu \) is S1 on the \( A \)-definable set \( X \) if \( X \) is not in \( \mu \) and the property above holds for formulas \( \phi(x, a_i) \) included in \( X \). Finally,
we say that \( \mu \) is \( S_1 \) on a partial type \( \pi(x) \) if \( \pi(x) \) is \( \mu \)-wide and included in a definable set in which \( \mu \) is \( S_1 \).

The following results all appear in [Hru12].

**Fact 2.2.** Let \( p \) be a type and assume that \( \mu \) has the \( S_1 \) property. Then for any type \( q \) the relation

\[
R(a, b) \iff p(x)a^{-1} \cap q(x)b^{-1} \text{ is } \mu\text{-wide,}
\]

where we identify a type with its realizations in the monster model, is a stable relation.

**Fact 2.3.** Let \( \mu \) be an \( M \)-invariant ideal which is \( S_1 \) on some set \( X \). Then for any type \( p(x) \) whose realizations are contained in \( X \), if \( p(x) \) is \( \mu \)-wide then \( p(x) \) does not fork over \( M \).

Finally, the following is Lemma 2.3 in [Hru12].

**Fact 2.4.** Let \( p, q \) be complete types over a model \( M \) and let \( R(x, y) \) be a stable \( M \)-invariant relation in the realizations of \( p(x) \times q(y) \). Then the truth value of \( R(a, b) \) is constant for all \( a \models p(x) \) and \( b \models q(y) \) as long as either \( tp(a/Mb) \) or \( tp(b/Ma) \) does not fork over \( M \).

We consider a second ideal \( \lambda \) of subsets of \( G \) with the property that \( \mu \) is \( S_1 \) on any set in \( \lambda \). This ideal will only be truly used in Theorem 2.13; everywhere else, one may take \( \lambda \) to be the ideal of all definable sets on which \( \mu \) is \( S_1 \). We assume that \( \lambda \) is also invariant under left translations by elements of \( G \). A type which is not \( \lambda \)-wide will be called medium, and we will refer to \( \mu \)-wide types by “wide”. Note that if \( p \) is medium, then \( \mu \) is \( S_1 \) on \( p \), and if \( a \models p \) and \( tp(a/Mb) \) is wide, then \( tp(a/Mb) \) does not fork over \( M \).

If \( q \) and \( r \) are wide types, then we define \( St(q, r) := \{ g : gp \cap r \text{ is wide} \} \). If \( p \) is wide, we will refer to \( St(p, p) \) by \( St(p) \) and \( St_r(p) = \{ g : pg \cap p \text{ is wide} \} \). Hence \( g \in St(p) \) if and only if there is some \( a \models p \), \( tp(a/Mp) \) wide and \( ga \models p \) (then also \( tp(ga/Mg) \) is wide by \( G \)-invariance of \( \mu \)). Observe that \( St(p) \) is stable under inversion. Finally, \( Stab(p) \) is the subgroup generated by \( St(p) \).

If \( p \) and \( q \) are two types, we let \( p \times_{nf} q = \{ (a, b) : a \models p, b \models q, tp(b/Ma) \) does not fork over \( M \} \).

We recall one version of Hrushovski’s stabilizer theorem from [Hru12].
Fact 2.5 ([Hru12]). Let $\mu$ be an $M$-invariant ideal on $G$ stable under left and right multiplication. Let $X \subseteq G$ be a symmetric $M$-definable set such that $\mu$ is $S1$ on $X^3$. Let $q$ be a wide type over $M$ concentrating on $X$. Assume

(F) There are $a,b \models q$ such that $tp(a/Mb)$ and $tp(b/Ma)$ are both non-forking over $M$.

Then there is a wide type-definable subgroup $S$ of $G$. We have $S = (q^{-1}q)^2$ and $qq^{-1}q$ is a coset of $S$. Moreover $S$ is normal in the group generated by $X$ and $S \setminus (q^{-1}q)$ is included in a union of non-wide $M$-definable sets.

We will not actually use this theorem, but some modified versions of it, which we prove in this section. Theorem 2.10 below is very close to Fact 2.5. The proof is of course very much inspired, at times literally copied, from that of Hrushovski. The main difference is that we assume the ideal to be $S1$ on up to four products of the type and its inverse (instead of three), and this allows us to simplify slightly the arguments. Furthermore, we drop assumption (F) and under assumption (B1), we forgo right-invariance.

The proof in [Hru12] operates by acting on the right on $q$, we decide to act on the left, which explains some differences in the statements.

We will need a stronger version of Fact 2.2, where we restrict the requirement that $\mu$ has the $S1$ property in all sets.

Lemma 2.6. Let $p, q$ be medium, then the relation $R(g, h)$ defined as “$gp \caphq$ is wide” is a stable relation.

Proof. Note that by invariance of $\mu$, every translate of $p$ and $q$ is medium. Let $(g_i, h_i : i \in \mathbb{Z})$ be an indiscernible sequence and assume that $R(g_i, h_j)$ holds if and only if $i \leq j$.

Case 1: $g_0p \cap g_1p \cap h_2q$ is wide.

We then have that for all $i > 0$, $g_0p \cap g_i p \cap h_{i+1}q$ is wide by indiscernibility. Also for $i < j$, we have $(g_i p \cap h_{i+1}q) \cap (g_{i+2} p \cap h_{i+3}q)$ is not wide as already $h_{i+1}q \cap g_{i+2}p$ is not wide. Therefore the sequence $(g_0p \cap (g_{2i} p \cap h_{2i+1}q) : i > 1)$ contradicts the $S1$ property inside $g_0p$.

Case 2: $g_0p \cap g_1p \cap h_2q$ is not wide.

We know that for all $i < 2$, $g_i p \cap h_{2i}q$ is wide. Hence the sequence $(h_{2i}q \cap g_i p : i < 2)$ contradicts the $S1$ property inside $h_{2i}q$. \qed
Lemma 2.7. Let $q, r$ be medium and wide, and let $p \in St(q, r)$. Take $(a, b) \in p \times_{nf} p$, then $a^{-1}b, b^{-1}a \in St(q)$.

Proof. Take $(a, b) \in p \times_{nf} p$. Since $St(q)$ is stable under inverses, it suffices to show that $a^{-1}bq \cap q$ is wide, which is equivalent to $bq \cap aq$ is wide. As $q$ is medium, by stability it is enough to prove this for one pair $(a, b) \in p \times_{nf} p$. Take $(a_i : i < \omega)$ an indiscernible sequence in $p$ such that $tp(a_i/Ma_0)$ is non-forking over $M$. Then $a_iq \cap r$ is wide for all $i$, as $p \in St(q, r)$. As $r$ is medium, it follows that $a_0q \cap a_1q \cap r$ is wide. In particular $a_0q \cap a_1q$ is wide, as required.

Lemma 2.8. Let $p$ be wide and medium, and let $q \in St(p)$, take $(a, b) \models q \times_{nf} q$, then $a^{-1}b, b^{-1}a \in St(p)$. If $\mu$ is right invariant and if $q \in St_r(p)$, then $ab^{-1}, ba^{-1} \in St_r(p)$.

Proof. The first part follows from the previous lemma by taking $q, r$ there to be $p$ here. The second part of the statement is proved in the same way by multiplying on the right.

We will also show the following.

Lemma 2.9. Let $p$ and $r$ be medium types, let $(a, b) \models p \times_{nf} p$, and assume that $p^{-1}r$ is medium. Then $ba^{-1} \in St(r)$.

Proof. We need to show that $a^{-1}r \cap b^{-1}r$ is wide. Let $(a_i)_{i<\omega}$ be an indiscernible sequence of realizations of $p$ such that $tp(a_n/Ma_{<n})$ is wide for all $n$ (and hence non-forking over $M$ as $p$ is medium). By stability, it is enough to show that $a_0^{-1}r \cap a_1^{-1}r$ is wide. The type-definable sets $(a_i^{-1}r)_{i<\omega}$ are wide and included in $p^{-1}r$ which is medium by hypothesis, so by the S1 property $a_0^{-1}r \cap a_1^{-1}r$ is wide as required.

Theorem 2.10. Let $\mu$ be an $M$-invariant ideal on $G$ stable under left multiplication. Let $p \in S_G(M)$ be wide. Assume either (B1) or (B2), where:

(B1) For some symmetric definable set $X \in p$, $\mu$ is S1 on $X^4$;

(B2) $\mu$ is S1 on $(pp^{-1})^2$ and invariant under (left and) right multiplication;

Then $Stab(p) = St(p)^2 = (pp^{-1})^2$ is a connected, wide type-definable group on which $\mu$ is S1. Furthermore $Stab(p) \setminus St(p)$ is included in a union of non-wide $M$-definable sets.
Proof. Here we take for $\lambda$ the ideal of all definable sets on which $\mu$ is S1, so a type is medium if $\mu$ is S1 on it. In particular, under either of (B1) or (B2), we have that both $p$ and $p^{-1}p$ are medium.

The proof will proceed by a series of steps. Only in the beginning will there be differences depending on whether (B1) or (B2) is assumed.

Claim 1: Let $(a, b) \in p \times_{nf} p$, then $ba^{-1} \in St(p)$.

Proof: This follows immediately from Lemma 2.9.

Claim 1': If (B1) holds, then for any $(a, b) \in p \times_{nf} p$ we have $a^{-1}b \in St(p)$.

Proof: By symmetry of $X$, we have that $p^2$ is medium, so the result follows from Lemma 2.9 with $p = p^{-1}$ and $r = p$.

Take now $(a, b) \in p \times_{nf} p$, $tp(b/Ma)$ wide. We define $q = tp(a^{-1}b/M)$ under assumption (B1) and $q = tp(ba^{-1}/M)$ under assumption (B2). Then in both cases $q \in St(p)$, $q$ is wide (using right-invariance in the (B2) case) and medium. Notice that under either assumption $p^{-1}q$ is medium: under (B1) $p^{-1}q \subseteq X^3$ and under (B2) $p^{-1}q \subseteq p^{-1}pp^{-1}$.

So Lemma 2.9 implies

Claim 2: Let $(a, b) \in p \times_{nf} p$, then $ba^{-1} \in St(q)$.

Claim 3: Let $(b, c) \in Stab(q) \times_{nf} q$, then $bc \in St(p)$.

Proof: As $St(q)$ is stable under inverse, we can write $b = b_1 \cdots b_n$, with each $b_i \in St(q)$. We show the result by induction on $n$. For $n = 0$, it follows from the fact that $q \in St(p)$.

Assume we know it for $n - 1$ and take $b = b_1 \cdots b_n$. We have to show that $b_n^{-1} \cdots b_1^{-1}p \cap cp$ is wide. As $b_n \in St(q)$, there is $c' \models q$, $tp(c'/Mb_n)$ wide such that $b_n c' \models q$. We may also assume that $tp(c'/Mb_0 \cdots b_n)$ is wide. Then by translation invariance, $tp(b_n c'/Mb_0 \cdots b_n)$ is wide. By induction, $b_n^{-1} \cdots b_1^{-1}p \cap b_n c'p$ is wide, then so is $b_n^{-1} \cdots b_1^{-1}p \cap c'p$ and we conclude by stability.

Claim 4: Let $a, b \models p$, then $ab^{-1} \in St(q)^2$.

Proof: Take $c \models p$ such that $tp(c/Mab)$ is non-forking over $M$. Write $ab^{-1} = (ac^{-1})(cb^{-1})$. By Claim 2 and the fact that $St(q)$ is closed under inverses, both $ac^{-1}$ and $cb^{-1}$ are in $St(q)$ and the claim follows.

Claim 5: $Stab(p) = Stab(q) = (pp^{-1})^2$ is wide and medium.

Proof: By Claim 3, we have $Stab(q) \subseteq St(p)^2 \subseteq (pp^{-1})^2$. By Claim 4, $pp^{-1} \subseteq Stab(q)$ so also $(pp^{-1})^2 \subseteq Stab(q)$, hence $(pp^{-1})^2 = St(p)^2 = Stab(q)$. Finally, since $Stab(q)$ is a subgroup, we have $Stab(p) = St(p)^2 = Stab(q)$. By hypothesis $(pp^{-1})^2$ is medium, and it is wide since it contains $q$. 

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All that is left to prove is that \( \text{Stab}(p) \) has no type-definable over \( M \) proper subgroup of bounded index, and that any wide type in \( \text{Stab}(p) \) lies in \( \text{St}(p) \).

Let \( T \leq \text{Stab}(p) \) be a type-definable over \( M \) subgroup of bounded index. We have \( pp^{-1} \subseteq \text{Stab}(p) \), hence for \( a \models p \), \( p \subseteq \text{Stab}(p)a \). So \( p \) lies in a right coset \( S_p \) of \( \text{Stab}(p) \). This coset is \( M \)-invariant and hence type-definable over \( M \). All right cosets of \( T \) in \( S_p \) are type-definable over \( M \) and as \( p \) is a complete type over \( M \), it must lie entirely within one of them. Therefore \( pp^{-1} \subseteq T \) and \( T = \text{Stab}(p) \).

Now, let \( s \) be a wide type in \( \text{Stab}(p) = \text{Stab}(q) \). By Claim 3, for any \( b \in \text{Stab}(q) \) and \( c \models q \) with \( \text{tp}(c/Mb) \) wide, \( bp \cap cp \) is wide. By stability, the same holds assuming instead that \( \text{tp}(b/Mc) \) is wide. Let \( c \models q \) and \( b \models s \) such that \( \text{tp}(b/cM) \) is wide. Then by left invariance, \( \text{tp}(cb/cM) \) is wide. But we also have \( cb \in \text{Stab}(q) \), hence \( cbp \cap cp \) is wide. From which it follows that \( bp \cap p \) is wide, so \( s \) lies in \( \text{St}(p) \), as required.

The proofs of the following two propositions are taken essentially without change from [Hru12].

**Proposition 2.11.** Under assumption (B1), \( \text{Stab}(p) \) is normal and of bounded index in the group generated by \( X \) and \( X^n \) is medium for all \( n \).

**Proof.** Write \( S = \text{Stab}(p) \). Let \( r \) be a type over \( M \) of elements of \( X \). Then the image of \( r \) in \( G/S \) is bounded. Indeed, assume not, then we can find an indiscernible sequence \( (a_i : i < \omega) \) of realizations of \( r \) such that the cosets \( a_iS \) are pairwise disjoint. Hence so are the types \( a_i pp^{-1} \) (as \( pp^{-1} \subseteq S \)), but this contradicts S1 inside \( X^3 \). As \( r \) is a complete type over \( M \) it must be included in one left coset of \( S \). Applying the same reasoning to \( r^{-1} \), we see that \( r \) is also included in a unique right coset of \( S \). Thus \( X/S \) is bounded and if \( c, c', \models r \), then \( cSc^{-1} = c'Sc'^{-1} =: S' \) is type-definable over \( M \).

We now claim that \( p^{-1} \) has bounded image in \( G/S' \): for if not, we would have an \( M \)-indiscernible sequence \( (a_i : i < \omega) \) of realizations of \( p \) with \( a_i^{-1}cSc^{-1} \) pairwise disjoint and again \( a_i^{-1}cpp^{-1} \) would be pairwise disjoint contradicting S1 in \( X^4 \). Hence \( p^{-1} \) lies entirely within one left coset of \( S' \), and \( pp^{-1} \subseteq S' \). Therefore \( S \leq S' \). We also have \( S \leq S'^{-1} \) and then \( S = S' \).

We have shown that \( S \) is normalized by \( X \) and has bounded index in it. It follows that \( S \) has bounded index in any \( X^n \), thus \( X^n \) is medium. \( \square \)

**Proposition 2.12.** If we assume that both conditions (B1) and (B2) (equivalently (B1) and right-invariance) hold, then \( pp^{-1}p \) is a coset of \( \text{Stab}(p) \).
Proof. Let \( c \models p \). By the previous proposition \( \text{Stab}(p) \) is normal in the group generated by \( X \). Since \( pp^{-1} \subseteq \text{Stab}(p) \), \( p \) lies entirely within one coset of \( \text{Stab}(p) \) and hence \( pp^{-1}p \subseteq \text{Stab}(p)c \). Conversely, take any \( a \in \text{Stab}(p)c \) and let \( b \models p \) such that \( \text{tp}(b/Ma) \) is wide. Then \( ba^{-1} \in \text{Stab}(p) \) and \( \text{tp}(ba^{-1}/M) \) is wide by right-invariance. By Theorem 2.10 any wide type in \( \text{Stab}(p) \) is in \( \text{St}(p) \), so \( ba^{-1} \in \text{St}(p) \subseteq pp^{-1} \). So \( a = ab^{-1}b \in pp^{-1}p \).

We now wish to relax the hypothesis that \( \mu \) is \( S1 \) on \( (pp^{-1})^2 \) and assume only that \( \mu \) is \( S1 \) on generic products in \( p^{-1}p \) (see condition (B) below). We will need however to make extra technical hypothesis (A) and (F).

Theorem 2.13. Let \( \mu \) and \( \lambda \) be \( M \)-invariant ideals on \( G \) as above, stable under left and right multiplication, and such that \( \mu \) is \( S1 \) in any \( X \in \lambda \).

Assume we are given a wide and medium type \( p \) in \( G \) and the following conditions are satisfied:

(A) for any types \( q, r \), if for some \((c,d) \models q \times_{n_f} r \), \( \text{tp}(cd/M) \) or \( \text{tp}(dc/M) \) is medium, then \( q \) is medium;

(B) for any \((a,b) \in p \times_{n_f} p \), \( \text{tp}(a^{-1}b/M) \) is medium;

(F) there are \((a,b) \models p \times_{n_f} p \) such that \( \text{tp}(a/Mb) \) does not fork over \( M \).

Then \( \text{Stab}(p) = \text{St}(p)^2 = (pp^{-1})^2 \) is a connected type-definable, wide and medium group. Also \( \text{Stab}(p) \setminus \text{St}(p) \) is contained in a union of non-wide \( M \)-definable sets.

Proof. Condition (A) implies that if \( q \) is a medium type, then both \( \text{St}(q) \) and \( \text{St}_{r}(q) \) are medium.

Claim 1: If \((a,b) \in p \times_{n_f} p \), then \( ba^{-1} \in \text{St}(p) \).
Proof: By Lemma 2.9.

Claim 1': If \((a,b) \in p \times_{n_f} p \), then \( a^{-1}b \in \text{St}_{r}(p) \).
Proof: By Claim 1, we have that if \((a,b) \in p \times_{n_f} p \), then \( ba^{-1} \in \text{St}(p) \), in particular \( \text{tp}(ba^{-1}/M) \) is medium. We can then repeat the argument of Lemma 2.9 by multiplying on the right to show Claim 1'.

Let \( \mu' \) be the ideal defined by \( \phi(x) \in \mu' \iff \phi(x^{-1}) \in \mu \). Then \( \mu' \) is \( M \)-invariant, invariant under left and right multiplication and is \( S1 \) on any inverse of a medium type. We will write \( \text{St}' \), \( \text{Stab}' \) for the stabilizers with respect to \( \mu' \).
Let \((a, b) \models p \times p, \tp(b/Ma)\) wide (hence non-forking over \(M\)) and \(q = \tp(ab^{-1}/M)\). Then \(q\) is \(\mu'\)-wide and is in \(St(p)\), as \(St(p)\) is closed under inverses, and thus \(q\) and \(q^{-1}\) are medium. Also if \((c, d) \models q \times_{n_f} q\), then \(\tp(c^{-1}d/M) \in St(p)\) by Lemma 2.8. In particular \(\tp(c^{-1}d/M)\) is medium.

Claim 2: If \((b, c) \in \text{Stab}'(q) \times_{n_f} q\), then \(bc \in St(p)\).

Proof: As \(St'(q)\) is stable under inverse, we can write \(b = b_1 \cdots b_n\), with each \(b_i \in St'(q)\). We show the result by induction on \(n\). For \(n = 0\), it is clear.

Assume we know it for \(n - 1\) and take \(b = b_1 \cdots b_n\). We have to show that \(b_{n}^{-1} \cdots b_{1}^{-1}p \cap cp\) is wide. As \(b_n \in St'(q)\), there is \(c' \models q\), \(\tp(c'/Mb_n)\) \(\mu'\)-wide such that \(b_n c' \models q\). We may also assume that \(\tp(c'/Mb_1 \ldots b_n)\) is \(\mu'\)-wide. Then by translation invariance, \(\tp(b_n c'/Mb_1 \ldots b_n)\) is \(\mu'\)-wide. By induction, \(b_{n}^{-1} \cdots b_{1}^{-1}p \cap b_n c'p\) is wide. We conclude by stability.

Claim 3: There is \((a, b) \models p \times_{n_f} q, \tp(b/Ma)\) \(\mu'\)-wide, such that \(\tp(a^{-1}b/M)\) and its inverse are medium.

Proof: By (F) there is \((c, d) \in p \times_{n_f} p\) such that also \(\tp(c/Md)\) does not fork over \(M\). Let \(r = \tp(d^{-1}c/M)\). Let \(a \models p\) and choose \(b_0\) such that \(\tp(a, b_0/M) = \tp(d, c/M)\). Then \(a^{-1}b_0 \models r\) and \(\tp(b_0/Ma)\) does not fork over \(M\). Now choose \(b_1 \models p\) such that \(\tp(b_1/Mb_0)\) is wide and \(\tp(b_0b_1^{-1}/M) = q\). We can furthermore assume that \(\tp(b_1/Mab_0)\) is wide. By translation invariance, \(\tp(b_0b_1^{-1}/Ma)\) is \(\mu'\)-wide. Now pick \(b_2\) such that \(\tp(b_2/Mab_0b_1)\) is non-forking over \(M\) and \(\tp(b_1, b_2/M) = \tp(c, d/M)\) so that \(\tp(b_1^{-1}b_2/M) = r^{-1}\). By transitivity of non-forking, we have \(\tp(b_1^{-1}b_2/Mab_0)\) is non-forking over \(M\). Hence \((a^{-1}b_0, b_1^{-1}b_2) \models r \times_{n_f} r^{-1}\).

By Claim 1' and the since \(St_r(p)\) is stable under inversion, \(r \in St_r(p)\) and by Lemma 2.8, \(a^{-1}b_0b_1^{-1}b_2\) is also in \(St_r(p)\). It follows that \(\tp(a^{-1}b_0b_1^{-1}b_2/M)\) and its inverse are medium. By hypothesis (A), \(\tp(a^{-1}b_0b_1^{-1}/M)\) and its inverse are medium.

Claim 4: If \((a, b) \models p \times_{n_f} p\), then \(ab^{-1} \in St'(q)\).

The proof is similar to that of Lemma 2.9. As there, we can take \((a_i : i < \omega)\) an indiscernible sequence in \(p\) with \(\tp(a_n/Ma_{<n})\) wide and it is enough to show that \(a_0^{-1}q \cap a_1^{-1}q\) is \(\mu'\)-wide. By Claim 3, there is \(b \models q\) with \(\tp(b/Ma_{<\omega})\) \(\mu'\)-wide, \((a_i)_{i<\omega}\) indiscernible over \(M\) and \(r = \tp(a_0^{-1}b/M)\) and its inverse are medium. Also \(a_0^{-1}b \in a_0^{-1}q \cap r\). By translation invariance, \(\tp(a_0^{-1}b/Ma_0)\) is \(\mu'\)-wide, hence \(a_0^{-1}q \cap r\) is wide. By indiscernibility, \(a_i^{-1}q \cap r\) is \(\mu'\)-wide for all \(i\). As \(r^{-1}\) is medium, it follows that \(a_0^{-1}q \cap a_1^{-1}q\) is \(\mu'\)-wide.

Now we can conclude: we have by Claim 2, \(\text{Stab}'(q) \subseteq St(p)^2 \subseteq (pp^{-1})^2\). Let \(a, b \models p\) and choose \(c \models p\) such that \(\tp(b/Mc)\) and \(\tp(c/Mb)\) do not fork
over $M$ (using (F)). We can furthermore assume that $\text{tp}(c/Mab)$ does not fork over $M$. Then $(a, c) \models p \times_{nf} p$ and $(c, b) \models p \times_{nf} p$ and $ab^{-1} = (ac^{-1})(cb^{-1})$. By Claim 4, both $ac^{-1}$ and $cb^{-1}$ are in $St'(q)$, therefore $ab^{-1} \in St'(q)$. We thus have $pp^{-1} \subseteq St'(q)$. Therefore $St'(q) = St(p)^2 = (pp^{-1})^2$ and as $St'(q)$ is a subgroup, $St'(q) = Stab(p)$. Type-definability of $Stab(p)$ is clear, so is wideness. The fact that $Stab(p) = Stab(q)$ is medium follows from Claim 2 and property (A).

Connectedness is proved as in Claim 6 of Theorem 2.10. Finally, the fact that any wide type in $Stab(p)$ lies in $St(p)$ is proved as Claim 7 in Theorem 2.10 replacing $Stab(q)$ there by $Stab'(q)$.

The following lemma will be useful later to check that the hypothesis of the theorem are satisfied.

**Lemma 2.14.** Assume that $\mu$ is left invariant and condition (A) holds. Let $q, r$ be medium and wide types. Let $p \in St(q, r)$ be a wide type and take $(a, b) \in p \times_{nf} p$. Then $\text{tp}(a^{-1}b/M)$ is medium.

**Proof.** We show that $a^{-1}b \in St(q)$, i.e., that $aq \cap bq$ is wide. As $q$ is medium, by stability, it is enough to show this for some pair $(a, b) \in p \times_{nf} p$. Take $(a_i : i < \omega)$ an indiscernible sequence in $p$ with $\text{tp}(a_n/Ma_{<n})$ wide and it is enough to show that $a_0q \cap a_1q$ is wide. By assumption $a_0q \cap r$ is wide. As $r$ is medium, by the S1 property, $a_0q \cap a_1q \cap r$ is wide, hence $a_0q \cap a_1q$ is wide as required.

3 Groups with f-generics in NTP$_2$

In this section we will use Theorem 2.10 to prove Theorem 3.18, which is a stabilizer theorem for strong f-generic types in a group $G$ definable in an NTP$_2$ theory (see Definition 3.3).

We work here with a complete theory $T$ and let $\mathcal{U}$ denote a monster model of $T$.

We recall the definition of NTP$_2$.

**Definition 3.1.** We say that $\phi(\bar{x}, \bar{y})$ has TP$_2$ if there are $(a_{ij})_{i,j<\omega}$ in $\mathcal{U}$ and $k \in \omega$ such that:

1. $\{\phi(\bar{x}, a_{ij})_{j \in \omega}\}$ is $k$-inconsistent for all $l < \omega$.

2. For all $f : \omega \rightarrow \omega$, $\{\phi(\bar{x}, a_{l,f(l)}) : l \in \omega\}$ is consistent.
A formula $\phi(\bar{x}, \bar{y})$ is $NTP_2$ if it does not have $TP_2$. The theory $T$ is $NTP_2$ if no formula has $TP_2$.

We will assume throughout this section that $T$ is $NTP_2$. Let $G$ be a $\emptyset$-definable group. Recall that an extension base is a set $A$ such that no $p \in S(A)$ forks over $A$. We will use the following results (the first three are from [CK12] and the fourth one from [BYC14]).

**Fact 3.2.** Let $T$ be an $NTP_2$ theory and $A$ an extension base.

1. For any $b$, there is an $A$-indiscernible sequence $(b_i : i < \omega)$ such that for any formula $\phi(x; b)$ which divides over $A$, the partial type $\{\phi(x; b_i) : i < \omega\}$ is inconsistent.

2. A formula forks over $A$ if and only if it divides over $A$.

3. Condition (F) is satisfied: given any type $p$ over $A$, there are $a, b \models p$ such that $tp(a/Ab)$ and $tp(b/Aa)$ are non-forking over $A$.

4. The ideal of formulas which do not fork over $A$ is has the $S1$ property.

**Definition 3.3.** A global type $p \in S_G(\mathcal{U})$ is strongly (left) $f$-generic over $A$ if for all $g \in G(\mathcal{U})$, $g \cdot p$ does not fork over $A$.

It is strongly bi-$f$-generic if for all $g, h \in G(\mathcal{U})$, $g \cdot p \cdot h$ does not fork over $A$.

It is proved in [HP11] that a definable group in an NIP theory is definably amenable (that is, admits a definable $G$-invariant measure on definable sets) if and only if it admits a strong $f$-generic type over some model. The theory of definably amenable NIP groups was studied in [HPP08], [HP11] and [CS16] (amongst other papers). In particular, the paper [CS16] characterizes in various ways formulas which extend to strong $f$-generic types. We generalize here those results to the $NTP_2$ context, assuming that $G$ admits a strong $f$-generic type. The proofs are very similar to those in [CS16].

First, we generalize Proposition 5.11 (i) of [HP11], with essentially the same proof.

**Lemma 3.4.** If for some model $M$, $G$ admits a strongly $f$-generic type over $M$, then the same is true over any extension base $A$.  


Proof. We expand the structure by adding a new sort \( S \) which, as a set, is a copy of the group \( G \) and we put all \( G \)-invariant relations on it. So \( S \) becomes a homogeneous space for \( G \) and any point of \( S \) gives rise to a definable bijection between \( S \) and \( G \). This expanded structure is \( \text{NTP}_2 \), and is conservative: it does not add any definable sets to the main sort. For \( A \subseteq U \), there is a strongly f-generic type over \( A \) if and only if the formula \( x_S = x_S \) in the expanded structure does not fork over \( A \). (See [HP11, Proposition 5.11] or [Sim15, Lemma 8.19].)

Now assume that \( x_S = x_S \) does not fork over some \( M \subseteq U \) and let \( A \subseteq U \) be an extension base. Let \( \tilde{N} \) be an \( |M|^+ \)-saturated model of the expanded theory containing \( A \).

Claim. In this expansion, the type \( \text{tp}(M/A) \) does not fork over \( A \).

Proof. Assume it did. Then by definition it implies a disjunction of formulas, each dividing over \( A \). As the expansion is conservative, we may assume that those formulas have parameters in the main sort. But then we can forget about the additional sort and use the fact that \( \text{tp}(M/A) \) does not fork over \( A \) in the original structure as \( A \) is an extension base. 

There is therefore \( M' \equiv_A M \) such that \( \text{tp}(M'/\tilde{N}) \) does not fork over \( A \). By assumption, there is some \( d \in S \) such that \( \text{tp}(d/M'/\tilde{N}) \) does not fork over \( M' \). Then by transitivity of non-forking, \( \text{tp}(d/\tilde{N}) \) does not fork over \( A \) as required.

Lemma 3.5. Let \( A \subseteq N \), \( N \) is \( |A|^+ \)-saturated. Assume that \( p \in S(U) \) is strongly f-generic over \( A \). Let \( a \models p|_N \) and \( b \models p|_{Na} \). Then \( \text{tp}(ba^{-1}/N) \) extends to a global type, strongly bi-f-generic over \( A \).

Proof. Let \( g, h \in G(N) \). Then \( \text{tp}(gb/Na) \) does not fork over \( A \) and neither does \( \text{tp}(ha/N) \). By transitivity of non-forking, \( \text{tp}(gb, ha/N) \) does not fork over \( A \). Hence \( \text{tp}(gba^{-1}h^{-1}/N) \) does not fork over \( A \). Since \( g, h \) were arbitrary in \( G(N) \), this shows that \( \text{tp}(ba^{-1}/N) \) is strongly bi-f-generic over \( A \).

Since \( N \) is \( |A|^+ \)-saturated, \( \text{tp}(ba^{-1}/N) \) extends to a global type strongly bi-f-generic over \( A \). (This is a closed condition and any finite part of it can be dragged down into \( N \)).

We will say that the group \( G \) has strong f-generics if it has a strongly f-generic type over some/any extension base. By Lemma 3.5 it would then also have a strong bi-f-generic type over any extension base.
Definition 3.6. Let $\phi(x) \in L(A)$ be a formula. We say that $\phi(x)$ is $f$-generic over $A$ if no (left) translate of $\phi(x)$ forks over $A$. We say that $\phi(x)$ $G$-divides over $A$ if for some $A$-indiscernible sequence $(g_i : i < \omega)$ of elements of $G$, the partial type $\{g_i \cdot \phi(x) : i < \omega\}$ is inconsistent.

Lemma 3.7. Let $A$ be an extension base and $\phi(x) \in L(A)$. Then $\phi(x)$ is $f$-generic over $A$ if and only if it does not $G$-divide over $A$.

Proof. If for some $g \in G$, $\phi(g^{-1}x)$ forks over $A$, then it divides over $A$ and there is an $A$-indiscernible sequence $(g_i : i < \omega)$ such that $\{\phi(g_i^{-1}x) : i < \omega\}$ is inconsistent. This shows that $\phi(x)$ $G$-divides over $A$. Conversely, if $\phi(x)$ $G$-divides over $A$ as witnessed by $(g_i : i < \omega)$, then $\phi(g_0^{-1}x)$ divides over $A$.

Let $A \subseteq B$ be two extension bases over which $\phi(x;a)$ is defined, then $\phi(x;a)$ $G$-divides over $A$ if and only if it $G$-divides over $B$ so the same is true for $f$-generic. From now on, we drop the “over $A$” when talking about $f$-generic formulas.

Lemma 3.8. Assume that the formula $\phi(x;b)$ forks over $A$ and that $\text{tp}(g/Ab)$ does not fork over $A$. Then $\phi(gx;b)$ forks over $A$.

Proof. Assume that $\phi(gx;b)$ does not fork over $A$ and let $c \models \phi(gx;b)$ with $c \downarrow_A Abg$. Then $c \downarrow_A Abg$. We also have $g \downarrow_A Ab$ by hypothesis. By transitivity, $gc \downarrow_A Ab$. Since $gc \models \phi(x;b)$, we get that $\phi(x;b)$ does not fork over $A$.

Proposition 3.9. Let $A$ be an extension base, $A \subseteq B$ and $\phi(x) \in L(B)$. Let $q$ be a global type strongly $f$-generic over $A$ and $g \models q|_B$. Then $\phi(x)$ extends to a global type strongly $f$-generic over $A$ if and only if $g^{-1} : \phi(x)$ does not fork over $A$.

Proof. Assume that $\phi(x)$ does not extend to a global type strongly $f$-generic over $A$. Then there are elements $g_i$, $i < n$ in $G(\mathcal{U})$ and formulas $\phi_i(x;b) \in L(\mathcal{U})$ each forking over $A$ such that $\phi(x) \vdash \bigvee_{i<n} \phi_i(g_ix;b)$. We can assume that $g$ realizes $q$ over $Bb\{g_i\}_{i<n}$. We have then that $\phi(gx) \vdash \bigvee_{i<n} \phi_i(g,gx;b)$. Now, $\text{tp}(g,g/Ab)$ does not fork over $A$ for each $i < n$. By Lemma 3.8, this implies that $\phi_i(g_i,gx;b)$ forks over $A$. Hence $\phi(gx) = g^{-1} : \phi(x)$ forks over $A$.

Conversely, if $\phi(x)$ extends to some global type strongly $f$-generic over $A$, then no translate of $\phi(x)$ forks over $A$ and in particular $g^{-1} : \phi(x)$ does not fork over $A$. 

\[ \square \]
The previous results combine into the following equivalences.

**Proposition 3.10.** Let $A$ be an extension base and assume that there is a global type $q$ strongly $f$-generic over $A$. Let $\phi(x) \in L(A)$ and let $g$ realize $q$ over $A$. The following are equivalent:

1. $\phi(x)$ is $f$-generic;
2. $\phi(x)$ does not $G$-divide over $A$;
3. $g^{-1} \cdot \phi(x)$ does not fork over $A$;
4. $\phi(x)$ extends to a global type strongly $f$-generic over $A$.

As usual, we extend definitions from definable sets to types: we define a type to be $f$-generic if it contains only $f$-generic formulas.

**Proposition 3.11.** Let $A$ be an extension base and assume that there is a global $f$-generic type $q$. Let $\phi(x) \in L(A)$ and let $g$ realize $q$ over $A$. Then $\phi(x)$ is $f$-generic if and only if $g^{-1} \cdot \phi(x)$ does not fork over $A$.

**Proof.** If $\phi(x)$ is $f$-generic, then $g^{-1} \cdot \phi(x)$ does not fork over $A$ by definition.

Conversely, assume that $\phi(x)$ does $G$-divide and let $(g_i : i < \omega)$ be an $A$-indiscernible sequence witnessing it. Let $\hat{q} = q|_A$.

**Claim.** The partial type $\bigcup g_i^{-1} \cdot \hat{q}$ is consistent.

**Proof.** If not, then there is a formula $\psi(x) \in \hat{q}(x)$ such that $\{g_i^{-1} \cdot \psi(x) : i < \omega\}$ is inconsistent. Then $g_0^{-1} \cdot \psi(x)$ divides over $A$, contradicting the assumption on $q$. □

Let $h$ realize $\bigcup g_i^{-1} \cdot \hat{q}$, so $g_i \cdot h \models \hat{q}$ for each $i$. Notice that $\{h^{-1}g_i^{-1} \cdot \phi(x) : i < \omega\}$ is still $k$-inconsistent for some $k$, and $g^{-1} \cdot \phi(x)$ divides over $A$ as required. □

**Corollary 3.12.** Assume that there is a global $f$-generic type, then the family $\mu$ of non-$f$-generic formulas is an ideal.

**Proof.** Let $q$ be a global $f$-generic type. Let $\phi(x)$ and $\psi(x)$ be non-$f$-generic and take $M$ a model over which both are defined. Let $g \models q|_M$ as in the previous proposition. Then $g^{-1} \cdot \phi(x)$ and $g^{-1} \cdot \psi(x)$ both fork over $M$, hence so does $g^{-1} \cdot (\phi(x) \lor \psi(x))$—as forking equals dividing over $M$—which implies that $\phi(x) \lor \psi(x)$ is not $f$-generic. □

**Question 3.13.** Assume that there is a global $f$-generic type, then is there a strongly $f$-generic type?
Notice that the ideal $\mu$ of non-f-generic formulas is $\emptyset$-invariant and invariant by translations on the left and on the right. It is however not S1 in general. For this we have to work with $\mu_A$.

Assume that $G$ has a strong f-generic type over $A$. Let $\mu_A$ be the ideal of formulas $\phi(x) \in L(U)$ which do not extend to a global type strongly f-generic over $A$. Then $\mu_A$ is $A$-invariant, left-$G$-invariant over $A$. By Proposition 3.10, $\mu$ and $\mu_A$ agree on $L(A)$.

**Lemma 3.14.** The ideal $\mu_A$ is S1.

*Proof.* Assume that $(a_i : i < \omega)$ is an $A$-indiscernible sequence such that $\phi(x; a_i)$ extends to a type strongly f-generic over $A$. Let $q$ be strongly f-generic over $A$ and let $g$ realize $q$ over $Aa_{<\omega}$ such that $(a_i)_{i<\omega}$ is indiscernible over $Ag$. Then $g^{-1} \cdot \phi(x; a_i)$ is non-forking over $A$ for all $i$. As the non-forking ideal is S1 in NTP$_2$ theories, also $g^{-1} \cdot \left( \phi(x; a_0) \land \phi(x; a_1) \right)$ is non-forking over $A$. By Proposition 3.10, $\phi(x; a_0) \land \phi(x; a_1)$ is $\mu_A$-wide. \(\square\)

### 3.1 Stabilizers of strong f-generic types

We will need the following definitions.

**Definition 3.15.** Let $G$ be a definable group, and $M$ be a model over which $G$ is definable.

We will say that a subset $X \subset G$ is *generic* if finitely many translates cover $G$.

If $H$ is a type definable (with parameters in $M_0$) subgroup of $G$ (or more generally an automorphism invariant subgroup), we will say that $H$ has *bounded index in $G$* if we have that the cardinality of $G(M^*)/H(M^*)$ is smaller than the cardinality of $M^*$ for some saturated model $M^*$ extending $M$.

Finally, we define $G^0_M$ to be the smallest type definable over $M$ subgroup of bounded index and we define $G^\infty_M$ to be the smallest $M$-invariant subgroup of $G$ of bounded index.

**Lemma 3.16.** Let $X$ be an f-generic definable set. Then $XX^{-1}$ is generic.

*Proof.* Let $(a_i : i < n)$ be a maximal sequence such that the sets $(a_i X : i < n)$ are disjoint, which must exist by f-genericity of $X$. Take any $b \in G$. Then for some $i < n$, $bX \cap a_iX \neq \emptyset$. Hence $b \in a_iXX^{-1}$ and $\bigcup_{i<n} a_iXX^{-1} = G$. \(\square\)
Lemma 3.17. Let $H < G$ be a type-definable group. Assume that $H$ is $\mu$-wide (i.e., every definable set containing it is $\mu$-wide), then $H$ has bounded index.

Proof. Let $X$ be a definable set containing $H$. Then there is a definable set $Y$ containing $H$ such that $YY^{-1} \subseteq X$. By hypothesis, $Y$ is f-generic and the previous lemma implies that $YY^{-1}$ is generic and therefore $X$ is generic. \(\square\)

In the following statement, $\mu_M$ is the ideal of formulas which do not extend to a global type, strongly f-generic over $M$.

Theorem 3.18. Assume that $G$ has strong f-generics. Let $p \in S_G(M)$ be f-generic.

Then $G_M^{00} = G_M^{\infty} = St_{\mu_M}(p)^2 = (pp^{-1})^2$ and $G_M^{00} \setminus St_{\mu_M}(p)$ is contained in a union of non-wide $M$-definable sets.

Proof. The ideal $\mu_M$ is $G$-invariant (by left multiplication), $M$-invariant and $S_1$ on $G$ by Lemma 3.14. We can apply Theorem 2.10 with hypothesis (B1) to deduce that $S = (pp^{-1})^2$ is a wide subgroup. As $p$ knows in which $G_M^{\infty}$ coset it lies, we must have $S \leq G_M^{\infty}$. On the other hand, by Lemma 3.17, $S$ has bounded index, hence $G_M^{00} \leq S$. It follows that those three subgroups are equal. The last statement also follows from Theorem 2.10. \(\square\)

3.2 Definably amenable groups

A definable group $G$ is definably amenable if for some (equiv. any) model $M$, there is a left-invariant Keisler measure on $M$-definable subsets of $G$. (See e.g. [Sim15, Chapter 8].)

Fact 3.19 ([Sim15], Lemma 7.5). Let $\mu$ be a measure over $M$ and $(b_i : i < \omega)$ an indiscernible sequence in $M$. Let $\phi(x;y)$ be a formula and $r > 0$ such that $\mu(\phi(x;b_i)) \geq r$ for all $i < \omega$. Then the partial type $\{\phi(x;b_i) : i < \omega\}$ is consistent.

Proposition 3.20. Let $G$ be a definably amenable $NTP_2$ group, then $G$ has strong f-generics.

Proof. Fix a model $M$ and $\mu$ a $G$-invariant measure on $M$-definable sets. Let $M \prec^+ N$ and assume that $\mu$ does not extend to a measure over $N$ which is both $G$-invariant and non-forking over $M$. By compactness, there are $\epsilon > 0$
and finitely many formulas $\phi_i(x; d)$, $i < n$, each forking over $M$ such that any $G$-invariant extension $\tilde{\mu}$ of $\mu$ satisfies $\bigvee_{i<n} \tilde{\mu}(\phi_i(x; d)) > \epsilon$. Take $(d_j : j < \omega)$ an indiscernible sequence in $tp(d/M)$ which witnesses dividing as given by Fact 3.2, (1). The condition that $\tilde{\mu}$ extends $\mu$ and is $G$-invariant is invariant under $Aut(N/M)$, therefore for every $j$, we also have $\bigvee_{i<n} \tilde{\mu}(\phi_i(x; d_j)) > \epsilon$. So up to taking a subsequence, for some $i < n$, we have $\bigwedge_{j<\omega} \tilde{\mu}(\phi_i(x; d_j)) > \epsilon$. But this contradicts Fact 3.19 and the property of $(d_j)_{j<\omega}$.

\[ \square \]

**Corollary 3.21.** Any solvable or pseudofinite $NTP_2$ group has strong $f$-generics.

## 4 PRC fields

A field $M$ of characteristic zero is pseudo real closed (PRC) if $M$ is existentially closed (relative to the language of rings) in every totally real regular extension $N$ of $M$. Equivalently, if given any absolutely irreducible variety $V$ defined over $M$, if $V$ has a simple $M'_{\text{r}}$—rational point for every real closure $M'_{\text{r}}$ of $M$, then $V$ has an $M$-rational point.

Prestel showed in Theorem 4.1 of [Pre82] that the class of PRC fields is axiomatizable in the language of fields. We have the following properties of PRC fields.

**Fact 4.1.** Let $M$ be a PRC field.

1. [Pre82, Proposition 1.4] If $<$ is an order on $M$, then $M$ is dense in $(M', \bar{<})$, the real closure of $M$ respect to the order $<$.

2. [Pre82, Proposition 1.6] If $<_i$ and $<_j$ are different orders on $M$, then $<_i$ and $<_j$ induce different topologies.

In this section we are interested in the class of bounded PRC fields. A field $M$ is bounded if for any integer $n$, $M$ has finitely many extensions of degree $n$. This implies in particular that all the orders which make $M$ into an ordered field are definable ([Mon17, Lemma 3.5]), and that there are finitely many of those.

### 4.1 Preliminaries on bounded PRC fields

We fix a bounded PRC field $K$ which is not algebraically closed and a countable elementary substructure $K_0$ of $K$. So there is $n \in \mathbb{N}$ such that $K$ has
exactly \( n \) distinct orders which are moreover definable (see Remark 3.2 of [Mon17]). Let \( \{<_1, \ldots, <_n\} \) be the orders on \( K \). If \( n = 0 \), then \( K \) is a PAC field, so we suppose from now on that \( n \geq 1 \).

We will work over \( K_0 \), thus we denote by \( \mathcal{L}_{\text{ring}} \) the language of rings with constant symbols for the elements of \( K_0 \), \( \mathcal{L}^{(i)}_{\text{ring}} := \mathcal{L}_{\text{ring}} \cup \{<_i\} \) and \( \mathcal{L} := \mathcal{L}_{\text{ring}} \cup \{<_1, \ldots, <_n\} \). We let \( T_{\text{pre}} := Th_{\mathcal{L}_{\text{ring}}}(K) = Th_{\mathcal{L}}(K) \). By Corollary 3.6 of [Mon17], \( T_{\text{pre}} \) is model complete. If \( M \) is a model of \( T_{\text{pre}} \), we denote by \( M^{(i)} \) the real closure of \( M \) with respect to \( <_i \).

The following is a direct consequence of the “Approximation Theorem for \( V \)-topologies” ([PZ78, Theorem 4.1]), and of the Fact 4.1.

**Fact 4.2.** Let \( (M, <_1, \ldots, <_n) \) be a model of \( T_{\text{pre}} \). Let \( A \) be a subset of \( M \) and for every order \( <_i \), let \( p^{(i)} \) be a quantifier-free \( \mathcal{L}^{(i)}_{\text{ring}} \)-type in \( M^{(i)} \) (so a consistent set of polynomial \( <_i \)-inequalities). Then \( \bigcup_{i=1}^n p^{(i)} \) is a consistent type in \( \mathcal{L} \).

Notice that the quantifier free \( \mathcal{L} \)-types all have the same form as the conclusion of Fact 4.2. We have the following amalgamation theorems for types:

**Fact 4.3** ([Mon17], Theorem 3.21). Let \( (M, <_1, \ldots, <_n) \) be a model of \( T_{\text{pre}} \). Let \( E = acl(E) \subseteq M \). Let \( a_1, a_2, c_1, c_2 \) be tuples of \( M \) such that \( E(a_1)^{\text{alg}} \cap E(a_2)^{\text{alg}} = E^{\text{alg}} \) and \( tp_{\mathcal{L}}(c_1/E) = tp_{\mathcal{L}}(c_2/E) \). Assume that there is \( c \) \( ACF \)-independent of \( \{a_1, a_2\} \) over \( E \) realizing \( qftp_{\mathcal{L}}(c_1/E(a_1)) \cup qftp_{\mathcal{L}}(c_2/E(a_2)) \). Then \( tp_{\mathcal{L}}(c/Ea_1) \cup tp_{\mathcal{L}}(c_2/Ea_2) \cup qftp_{\mathcal{L}}(c/E(a_1, a_2)) \) is consistent.

We now recall some other model theoretic properties of \( T_{\text{pre}} \).

**Fact 4.4** ([Mon17], Theorem 4.21). The theory \( T_{\text{pre}} \) is \( NTP_2 \).

**Fact 4.5** ([Mon17], Theorem 4.35). In \( T_{\text{pre}} \), all sets are extensions bases and forking equals dividing.

### 4.2 The multi-topology

**Definition 4.6.** Let \( (M, <_1, \ldots, <_n) \) be a model of \( T_{\text{pre}} \), \( A \subseteq M \) and let \( X \subseteq M^m \) be \( \mathcal{L}_{\text{ring}}(A) \)-definable. Then \( \dim(X) = \max\{\text{trdeg}(\bar{x}/A) : \bar{x} \in X\} \). This is a good notion of dimension, since \( acl(A) = dcl(A) = A^{\text{alg}} \cap M \) ([Mon17, Lemma 2.6]). We will say that \( \bar{a} \in X \) is a generic point of \( X \) over \( A \) if \( \dim(X) = \text{trdeg}(\bar{a}/A) \).
Definition 4.7. (Multi-topology) Let \((M, <_1, \ldots, <_n)\) be a model of \(T_{\text{prc}}\). Denote by \(\tau_i\) the topology induced in \(M\) by the order \(<_i\). By Fact 4.1 (2), if \(i \neq j\), then \(\tau_i \neq \tau_j\).

A definable subset of \(M\) of the form \(I = \bigcap_{i=1}^n (I^i \cap M)\) with \(I^i\) a non-empty \(<_i\)-open interval in \(M^{(i)}\) is called a multi-interval.

Notice that by Fact 4.2 every multi-interval is non-empty and if \(I\) is a multi-interval, then each \(I^i\) is \(<_i\)-dense in \(I\).

We define the multi-topology \(\tau\) as the topology in \(M\) generated by the multi-intervals and \(\tau^m\) its product topology in \(M^m\). Observe that if \(V\) is \(\tau_i\)-open, then it is \(\tau\)-open. We call a multi-box in \(M^m\) a set of the form \(C = \bigcap_{i=1}^n (C^i \cap M^m)\), with \(C^i\) an \(<_i\)-box in \(M^{(i)}\).

We extend the definition of \((j_1, \ldots, j_r)\)-cells for real closed fields (see Definition 2.3 of [vdD98]) to find a definition of multi-cells in the bounded PRC-field context.

Definition 4.8. (Multi-cells) Let \(r \in \mathbb{N}\) and let \((j_1, \ldots, j_r)\) be a sequence of zeros and ones of length \(r\).

A \((j_1, \ldots, j_r)\)-multi-cell is definable subset \(C\) of \(M^r\) such that for every \(i\) there is a \((j_1, \ldots, j_r)\)-cell \(C^i\) in \(M^{(i)}\) and

\[ C = \bigcap_{i=1}^n (C^i \cap M^r). \]

A multi-cell in \(M^r\) is a \((j_1, \ldots, j_r)\)-multi-cell, for some \((j_1, \ldots, j_r)\).

Observe that the \((1)\)-multi-cells are multi-intervals and any multi-box is a \((1, \ldots, 1)\)-multi-cell.

Notice also that the open multi-cells in \(M^r\) (or cells which are open subsets of \(M^r\)) are precisely the \(\left( \begin{array}{c} 1, \ldots, 1 \\ r \end{array} \right)\)-multi-cells.

Lemma 4.9. Let \(m \in \mathbb{N}\) and let \((i_1, \ldots, i_m)\) and \((j_1, \ldots, j_m)\) be two different sequences of zeros and ones of length \(m\). Let \(C^i \in M^{(i)}\) be a \((i_1, \ldots, i_m)\)-cell and let \(C^j \in M^{(j)}\) be a \((j_1, \ldots, j_m)\)-cell. Then \(\dim(C^i \cap C^j \cap M^m) < \min\{\dim(C^i), \dim(C^j)\}\).
Proof. Let \( r_i = \dim(C^i) \) and \( r_j = \dim(C^j) \). Suppose that there is \( \bar{a} = (a_1, \ldots, a_m) \in C \) such that \( \bar{a} \) is a generic point of \( C^i \) and \( C^j \). Let \( X_i = \{ a_k : i_k = 0 \} \), \( X_j = \{ a_k : j_k = 0 \} \). Then \( r_i = m - |X_i| \) and \( r_j = m - |X_j| \). Observe that if \( a_k \in X_i \cup X_j \), then \( a_k \in \text{acl}(a_1, \ldots, a_{k-1}) \). It follows that \( \dim(C) \leq m - |X_i \cup X_j| \). Since \( (i_1, \ldots, i_m) \neq (j_1, \ldots, j_m) \), \( |X_i \cup X_j| > \max\{|X_i|, |X_j|\} \). Thus \( \dim(C) \leq m - |X_i \cup X_j| < \min\{m - |X_i|, m - |X_j|\} = \min\{r_i, r_j\} \). \( \square \)

It follows that for an intersection of two \( r \)-dimensional cells to have dimension \( r \), one needs that both cells have the same sequences of 0’s and 1’s.

**Theorem 4.10.** Let \((M, <_1, \ldots, <_n)\) be a model of \( T_{prc} \); let \( A \subseteq M \), and \( r \in \mathbb{N} \). Let \( D \subseteq M^r \) be an \( \mathcal{L}(A) \)-definable set in \( M \). Then there are \( m \in \mathbb{N} \), and \( C_1, \ldots, C_m \) with \( C_j = \bigcap_{i=1}^{m}(C_j^i \cap M^r) \) a multi-cell in \( M^r \) such that:

1. \( D \subseteq \bigcup_{j=1}^{m} C_j \);
2. \( D \cap C_j \) is \( \tau^r \)-dense in \( C_j \), for all \( 1 \leq j \leq m \);
3. for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), \( C_j^i \) is quantifier-free \( \mathcal{L}_{ring}^{(i)}(A) \)-definable in \( M^{(i)} \);
4. for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), the set \( C_j^i \cap M^r \) is \( \mathcal{L}_{ring}^{(i)}(A) \)-definable in \( M \).

**Proof.** The proof is by induction on the dimension of \( D \). The case \( \dim(D) = 1 \) follows from [Mon17, Theorem 3.13]. Suppose that \( \dim(D) = d \). As in Theorem 3.13 [Mon17] using model completeness of \( T_{prc} \) we can suppose that there is an absolutely irreducible variety \( W \) defined over \( \text{acl}(A) \) such that:

\[
M \models \forall x_1, \ldots, x_r ((x_1, \ldots, x_r) \in D) \iff (\exists \bar{y} (x_1, \ldots, x_r, \bar{y}) \in W^{\text{sim}}(M)),
\]

where \( W^{\text{sim}}(M) = \{ \bar{x} \in W(M) : \bar{x} \text{ is a simple point of } W \} \).

Let \( d = |\bar{y}| \), for each \( i \in \{1, \ldots, n\} \) we define:

\[
A_i := \{ (x_1, \ldots, x_r) \in (M^{(i)})^r : \exists \bar{y} \in (M^{(i)})^d \text{ s.t. } (x_1, \ldots, x_r, \bar{y}) \in W^{\text{sim}}(M^{(i)}) \}.
\]

So \( A_i \) is \( \mathcal{L}_{ring}^{(i)}(A) \)-definable and \( D \subseteq A_i \). By cell decomposition in \( M^{(i)} \), there are \( k_i \in \mathbb{N}, \langle r \rangle \)-cells \( C_1^i, \ldots, C_{k_i}^i \) and \( X \) such that:
(1) the sets $C_{i_1}, \ldots, C_{i_{k_i}}, X^i$ are quantifier free $L^{(i)}_{\text{ring}}(A)$-definable in $M^{(i)}$;
(2) $\dim(C_j) = d$, for all $j \in \{i_1, \ldots, i_{k_i}\}$;
(3) $\dim(X^i) < d$,
(4) $A_i = \bigcup_{j=1}^{k_i} C_j \cup X^i$.

Let $X = \bigcup_{i=1}^{n} (X^i \cap M^r)$ and let

$$J := \{\sigma : \{1, \ldots, n\} \to \mathbb{N} | 1 \leq \sigma(i) \leq k_i\}.$$ 

For all $\sigma \in J$, let $C_\sigma := \bigcap_{i=1}^{n} (C_{\sigma(i)}^i \cap M^r)$, so $D \subseteq \bigcup_{\sigma \in J} C_\sigma \cup X$. We are interested in $C_\sigma$ of maximal dimension $d$, so let

$$J' := \{\sigma \in J : \dim(C_\sigma) = d\}.$$ 

Let $\sigma \in J'$. By Lemma 4.9 all the cells $C_{\sigma(i)}^i$ must have the same sequences of 0’s and 1’s and therefore $C_\sigma$ is a multi-cell in $M^r$.

**Claim.** For all $\sigma \in J'$, $D \cap C_\sigma$ is $\tau^r$-dense in $C_\sigma$.

**Proof.** Fix $\sigma \in J'$. Let $U_\sigma$ be a multi-box in $M^r$ such that $V := U_\sigma \cap C_\sigma \neq \emptyset$, we need to show that $V \cap D \neq \emptyset$. Let $z \in V$. Then $z \in A_i$ for all $i \in \{1, \ldots, n\}$. So there is $y^{(i)} \in (M^{(i)})^d$, such that $(z, y^{(i)})$ is a simple point of $W$. By Fact 4.2 we can find $(z_0, y_0) \in W(M)$ such that $(z_0, y_0)$ is arbitrary $<_i$-close to $(z, y^{(i)})$ for all $i \in \{1, \ldots, n\}$, in particular we can find $z_0 \in V \cap D$. 

Let $Y = X \cup \bigcup_{\sigma \in J \setminus J'} C_\sigma$, so $Y$ is an $L(A)$-definable set and $\dim(Y) < d$.

Then $D \subseteq \bigcup_{\sigma \in J'} C_\sigma \cup Y$ and each $C_\sigma$ satisfy (2), (3) and (4) of the theorem. Since $\dim(Y) < d$, by induction hypothesis we can apply the statement of the theorem to $Y$ instead of $D$, which completes the proof.

**Definition 4.11.** Let $(M, <_1, \ldots, <_n)$ be a model of $T_{\text{prc}}$ and $D \subseteq M^r$ a definable set. Denote by $\overline{D}$ the closure of $D$ for the $\tau^r$-topology. Observe that $\overline{D} = \bigcap_{i=1}^{n} \overline{D}_i$, where $\overline{D}_i$ is the closure of $D$ for the $\tau_i$-topology.
If \( X \subseteq M^r \) is a definable set and \( C_1, \ldots, C_m \) are the multi-cells obtained by Theorem 4.10, then \( \bigcup_{j=1}^{m} C_j \subseteq D \). This implies the following corollary.

**Corollary 4.12.** Let \((M, <_1, \ldots, <_n)\) be a model of \( T_{prc} \), let \( A \subseteq M \), and \( r \in \mathbb{N} \). Let \( D \subseteq M^r \) be an \( \mathcal{L}(A) \)-definable set in \( M \). Then there are \( m \in \mathbb{N} \), and \( C_1, \ldots, C_m \) with \( C_j = \bigcap_{i=1}^{n} \left( C^i_j \cap M^r \right) \) a multi-cell in \( M^r \) such that: \( D = m \bigcup_{j=1}^{m} C_j \) and such that for all \( 1 \leq i \leq n \), \( C^i_j \) is quantifier-free \( \mathcal{L}_{ring}(A) \)-definable in \( M \), for all \( 1 \leq j \leq m \).

**Proof.** The set \( D \) is \( \mathcal{L}(A) \)-definable (so was \( D \)) and by Theorem 4.10 there are \( C_1, \ldots, C_m \) multi-cells in \( M^r \) such that \( D = m \bigcup_{j=1}^{m} C_j \subseteq \overline{(D)} = \overline{D} \). So \( \overline{D} = m \bigcup_{j=1}^{m} C_j \). \( \square \)

**Definition 4.13.** A theory \( T \) in a language containing the language of rings and which contains the theory of fields, is **algebraically bounded** if, given any formula \( \phi(\bar{x}, y) \), there are polynomials \( f_1(\bar{x}, y), \ldots, f_n(\bar{x}, y) \in \mathbb{Z}[\bar{x}, y] \) such that, whenever \( K \) is a model of \( T \) and \( \bar{a} \) is a tuple of elements of \( K \) such that \( \phi(\bar{a}, K) := \{ y \in K : \phi(\bar{a}, y) \} \) is finite, then there is an index \( i \) such that the polynomial \( f_i(\bar{a}, y) \) is not identically 0 on \( K \) and \( \phi(\bar{a}, K) \) is contained in the set of roots of \( f_i(\bar{a}, y) = 0 \).

**Corollary 4.14.** The theory \( T_{prc} \) is algebraically bounded.

**Proof.** Directly from Theorem 4.10. \( \square \)

**Notation.** Let \( M \) be a structure and let \( D \subseteq M^r \) be a definable set. Let \( k < r \). We define \( \pi^M_k(D) := \{(x_1, \ldots, x_k) \in M^k : M \models \exists x_{k+1}, \ldots, x_r (x_1, \ldots, x_r) \in \bar{D}\} \).

For \( \bar{a} \in \pi^M_k(D) \), define \( D^M_{\bar{a}} := \{ \bar{y} \in M^{r-k} : (\bar{a}, \bar{y}) \in \bar{D} \} \). Define \( D^M(a_1, \ldots, a_k) := \{(a_1, \ldots, a_k, x_{k+1}, \ldots, x_r) : M \models (a_1, \ldots, a_k, x_{k+1}, \ldots, x_r) \in \bar{D}\} \). We omit \( M \) when the structure is clear.
4.3 Expansion by externally definable multi-cells

Here we will show that expanding a bounded PRC field with certain externally definable sets has elimination of quantifiers, analogous to results in [BP98] and [She09].

Definition 4.15. Let $T$ be a theory and let $M$ be a model of a theory $T$. An externally definable subset of $M^k$ is an $X \subseteq M^k$ that is equal to $\varphi(N^k,d) \cap M^k$ for some formula $\varphi$ and $d$ in some $N \succeq M$.

Definition 4.16. Let $(M,<_1,\ldots,<_n)$ be a model of $T_{prc}$. We say that $C = \bigcap_{i=1}^n C^i \cap M^r$ is an externally definable multi-cell in $M^r$ if for $i \in \{1,\ldots,n\}$, $C^i$ is the trace on $(M^{(i)})^r$ of a cell defined with exterior parameters. We say that $C$ is a multi-cell externally $\mathcal{L}(N)$-definable if $N \succeq M$ and for each $i \in \{1,\ldots,n\}$, there is $\phi_i(\bar{x}) \in \mathcal{L}(N^{(i)})$ such that $C^i = \phi_i(M)$.

Baisalov and Poizat prove in [BP98] that the theory resulting in expanding the language of any o-minimal structure with externally definable sets has elimination of quantifiers. This was generalized by Shelah to all NIP theories in [She09].

Proposition 4.17. Let $R$ be a model of $RCF$ in the ring language $\mathcal{L}_{ring}$ and consider the expansion $R^{Sh}$ of $R$ obtained by naming all externally definable sets. Then any definable subset of $R^{Sh}$ can be written as a finite union of sets of the form $U \cap D$, where $U$ is an open externally definable subset and $D$ is $\mathcal{L}_{ring}$-definable.

Proof. By [BP98] the structure $R^{Sh}$ is weakly o-minimal, so it makes sense to consider dimensions of definable sets. Let $X \subseteq R^n$ be definable in $R^{Sh}$. We prove the result by induction on the dimension of $X$. If $X$ has dimension 0, then it is finite, and the result follows.

For the inductive case, we can write $X$ as the union of an open set and a set of lower dimension, so we can assume that $X$ has dimension $d < n$. Let $\pi : R^n \to R^d$ be a coordinate projection such that $\pi(X)$ has non-empty interior (see Theorem 4.11 of [MMS00]). Then again writing $\pi(X)$ as the union of an open set and a set of smaller dimension, we may assume that $\pi(X)$ is open. For each $\bar{a} \in \pi(X)$, the fiber $X_{\bar{a}}$ is finite. By decomposing $X$ further, we may assume that it has always exactly one element. So $X$ is the graph of a function from $U := \pi(X)$ to $R^{n-d}$.

Let $R^{Sh} \prec R'$ be a sufficiently saturated elementary extension. Then by honest definitions ([Sim15, Chapter 3]), there is an $\mathcal{L}_{ring}(R')$-definable set
$X' \subseteq R'$ such that $X'(R') \subseteq X(R')$ and $X'(R) = X(R)$. Hence $X'$ is also the graph of a function from some $R'$-definable set $V$ to $R^{n-d}$, with $V(R) = U(R)$. As we are working in RCF, up to decomposing $V$ in finitely many $R'$-definable sets, we may assume that $f'$ is the function sending a point $\bar{a} \in V$ to the $k$-th solution of $P(\bar{b}, \bar{a}, \bar{Y})$, where $P(\bar{b}, \bar{T}, \bar{Y})$ is a polynomial with coordinates $\bar{b} \in R'$. Since by hypothesis, $P(\bar{b}, \bar{T}, \bar{Y})$ has a solution in $R$ for each $\bar{a}$ in the open set $U$, $P$ is definable over $R$. This implies that $X$ coincides on $U$ with the graph $\Gamma$ of an $R$-definable function. Then $X = U \times R^{n-d} \cap \Gamma$ has the required form. 

We now aim to show that the expansion of a bounded PRC field in $\mathcal{L}$ by externally definable multi-cells has elimination of quantifiers and is NTP$_2$.

**Proposition 4.18.** Let $(M, <_1, \ldots, <_n)$ be a model of $T_{prc}$. Let $A \subseteq M$ and let $D \subseteq M'$ be $\mathcal{L}(A)$-definable. Then there are $m \in \mathbb{N}$ and $C_1, \ldots, C_m$ multi-cells in $M'$, $\mathcal{L}(A)$-definable such that $D \subseteq \bigcup_{j=1}^m C_j$ and such that for every $\bar{x} \in \pi_{r-1}(D \cap C_j)$ the fiber $D_{\bar{x}}$ is $\tau$-dense in $(C_j)_{\bar{x}}$.

**Proof.** Notice that if $D = D_1 \cup D_2$ and the theorem is known for $D_1$ and $D_2$, then it follows for $D$ by taking a common refinement of the two cell decompositions obtained for $D_1$ and $D_2$.

Let $D$ be a definable set. By Theorem 4.10 for any $\bar{x} \in \pi_{r-1}(D)$ there are $k_{\bar{x}}, U_1, \ldots, U_{k_{\bar{x}}}$ multi-intervals in $M$ and a finite set $B_{\bar{x}}$ such that $D_{\bar{x}} \subseteq \bigcup_{j=1}^{k_{\bar{x}}} U_{\bar{x},j} \cup B_{\bar{x}}$, and such that $D_{\bar{x}}$ is $\tau$-dense in $U_{\bar{x},j}$, for all $j \in \{1, \ldots, k_{\bar{x}}\}$. By definition of multi-intervals $U_{\bar{x},j} = \bigcap_{i=1}^n U_{\bar{x},j,i}^i \cap M$, where $U_{\bar{x},j,i}^i$ is a $<_i$-interval in $M^{(i)}$.

For all $m_1, m_2 \in \mathbb{N}$, let $A_{m_1,m_2} := \{ \bar{x} \in \pi_{r-1}(D) : k_{\bar{x}} = m_1$ and $|B_{\bar{x}}| = m_2 \}$. By compactness there are only finitely many $(m_1, m_2)$ for which $A_{m_1,m_2}$ is non empty.

Then $A_{m_1,m_2}$ is definable with the same parameters as $D$, and $\pi_{r-1}(D) = \bigcup (m_1, m_2) A_{m_1,m_2}$ (a finite union). Since $D = \bigcup (m_1, m_2) \pi_{r-1}^{-1}(A_{m_1,m_2})$, it is enough to show that each $\pi_{r-1}^{-1}(A_{m_1,m_2})$ can be decomposed according to the conclusion of the theorem, so assume that $D = \pi_{r-1}^{-1}(A_{m_1,m_2})$ for some $(m_1, m_2)$.

Let $i \in \{1, \ldots, n\}$. For $s \in \{1, 2, 3\}$, let $f_{s,j}^i(x) : A_{m_1,m_2} \mapsto M^{(i)}$ such that:
(1) \( f_{i,j}^1(\bar{x}) = y \) if and only if \( y \) is the “\(<_i\)-smallest extremity in \( M^{(i)} \)” of the \(<_i\)-interval \( U_{\bar{x},j}^i \).

(2) \( f_{i,j}^2(\bar{x}) = y \) if and only if \( y \) is the “\(<_i\)-largest extremity in \( M^{(i)} \)” of the \(<_i\)-interval \( U_{\bar{x},j}^i \).

(3) \( f_{i,j}^3(\bar{x}) = y \) if and only if \( y \) is the \( j \)-th point in \( B_{\bar{x}} \) in the order \(<_i\).

As the structure is algebraically bounded (see Corollary 4.14), there is a definable partition of the base \( A_{m_1,m_2} = \bigcup_{t<p} X_t \) such that on each \( X_t \), each of the functions \( f_{i,j}^k \) coincides with a \(<_i\)-semi-algebraic function. Decreasing \( D \) further, we may assume that \( p = 1 \) and that all the functions \( f_{i,j}^k \) are semi-algebraic.

Now, let

\[
C_j := \{ (\bar{x}, y) \in M^r : f_{1,j}^1(\bar{x}) < y < f_{2,j}^2(\bar{x}) , \text{ for all } i \}
\]

and

\[
C_j^0 := \{ (\bar{x}, y) \in M^r : f_{3,j}^3(\bar{x}) = y \}.
\]

Then \( D \subseteq \bigcup_j C_j \cup \bigcup_j C_j^0 \) and this decomposition has the required properties.

**Definition 4.19.** Let \( \mathcal{U} \) be a monster model of \( T_{prc} \). Let \( M \) be a model of \( T_{prc} \). Let \( N \succcurlyeq M \) such that \( N \) is \( |M|^+ \)-saturated. Then \( N^{(i)} \succcurlyeq N \), and \( N^{(i)} \) is \( |M^{(i)}|^+ \)-saturated, for all \( i \in \{1, \ldots, n\} \).

Let \( \mathcal{L}^* = \mathcal{L} \cup \{ R_C(\bar{x}) : C \text{ is a multi-cell externally } \mathcal{L}(N) \text{-definable} \} \cup \{ P_D(\bar{x}) : D \text{ is } \mathcal{L}(M) \text{-definable} \} \). We define \( M_N \) to be the structure in the language \( \mathcal{L}^* \) whose universe is \( M \) and where each \( R_C \) and each \( P_D \) are interpreted as:

(1) for every \( \bar{a} \in M, M_N \models R_C(\bar{a}) \) if and only if \( \mathcal{U} \models \bar{a} \in C \),

(2) for every \( \bar{a} \in M, M_N \models P_D(\bar{a}) \) if and only if \( M \models \bar{a} \in D \)

**Theorem 4.20.** The structure \( M_N \) admits elimination of quantifiers.

**Proof.** Let \( C \) be an externally \( \mathcal{L}(N) \text{-definable multi-cell} \) and \( D \) an \( \mathcal{L}(M) \text{-definable set} \), both inside some \( M^r \). Let \( \pi \) be the projection to the first \( r-1 \) coordinates. It is enough to show that \( \pi(C \cap D) \) is quantifier-free definable in \( M_N \).
First, write \( C = \cap_{i=1}^{n} C^i \cap M^r \), where each \( C^i \) is an externally definable multi-cell in \( (M^{(i)})^r \). By Proposition 4.17, we can write each \( C^i \) as a finite union of sets of the form \( U^i \cap D^i \), where \( U^i \) is an externally definable open subset of \( (M^{(i)})^r \) and \( D^i \) is definable in \( M^{(i)} \). Then the trace of \( U^i \) on \( M^r \) is also open by density of \( M^r \) in \( (M^{(i)})^r \) and the trace of \( D^i \) on \( M^r \) is definable in \( M \). The result we want to prove is stable under taking finite unions, so we may assume that \( C^i = U^i \cap D^i \) and then by integrating \( D^i \) into \( D^i \), we may assume that \( C = C^i \cap M^r \) is open in \( M^r \).

By Proposition 4.18, we may assume that \( D \) is \( \tau \)-dense in some multi-cell \( C^{\ast} \) which contains it and such that if \( \bar{x} \in \pi(D) \), then \( D_{\bar{x}} \) is \( \tau \)-dense in the fiber \( (C^{\ast})_{\bar{x}} \). As \( C \) is \( \tau \)-open, for any \( \bar{x} \in \pi(D) \), if the fiber \( (C \cap C^{\ast})_{\bar{x}} \) is non-empty, then it is open in \( C_{\bar{x}} \) and thus also \( (D \cap C)_{\bar{x}} \) is non-empty. Therefore \( \pi(D \cap C) = \pi(D) \cap \pi(C \cap C^{\ast}) \) is quantifier-free definable in \( M_N \).

**Corollary 4.21.** The structure \( M_N \) is NTP$_2$.

**Proof.** This follows from Theorem 6.4 proved in appendix.

## 5 Type-definable subgroups of algebraic groups

In the following proposition, by a **definable ideal**, we mean an ideal \( \mu \) such that for any \( \phi(x; y) \), the set \( \{ b : \phi(x; b) \in \mu \} \) is definable.

**Proposition 5.1.** Let \( G \) be a definable group equipped with a definable (left-) \( G \)-invariant \( S1 \) ideal \( \mu \). Let \( H \leq G \) be a type-definable subgroup of \( G \) which is \( \mu \)-wide, then \( H \) is the intersection of definable subgroups of \( G \).

**Proof.** The proof follows that of Lemma 6.1 in [HP94], but since the contexts are not exactly the same we include it for completeness.

Write \( H = \bigcap_{n<\omega} H_n \), where each \( H_n \) is definable, stable under inverse and \( H_{n+1} \cdot H_{n+1} \subseteq H_n \). Let \( \delta_n(x; y) = x \in G \land y \in G \land xH_n \cap yH_n \notin \mu \). Then \( \delta_n \) is a definable, stable (as \( \mu \) is \( S1 \)), \( G \)-invariant relation. Let \( S_{\delta_n, H} \) be the set of global \( \delta_n \)-types (in variable \( x \)) which are consistent with \( H \). By stability, all \( \delta_n \)-types are definable. Recall that an element of \( S_{\delta_n, H} \) is generic if every set in it covers \( G \) in finitely many translates. By Lemma 5.16 in [HP94], there are finitely many generic types in \( S_{\delta_n, H} \).

Let \( Q_n \) be the definable set \( \{ b \in H_0 : \delta_n(x; b) \text{ is in all generic types of } S_{\delta_n, H} \} \).

**Claim.** \( H = \bigcap Q_n \).
Proof. Let \( a \in H \), then for \( b \in H \) realizing a generic type of \( S_{b_n,H} \) over \( a \), \( H \subseteq aH_n \cap bH_n \), hence \( aH_n \cap bH_n \notin \mu \) and \( a \in Q_n \).

Conversely, let \( a \in \bigcap Q_n \) and take \( b \in H \) generic over \( a \) as above. By definition of \( Q_n \), we have \( aH_n \cap bH_n \notin \mu \) for all \( n \) so that in particular it is non-empty. Hence by compactness, \( aH \cap bH \) is non-empty, so \( a \in H \).

Claim. \( HQ_n \subseteq Q_n \).

Proof. Let \( a \in H \) and \( b \in Q_n \). Let \( c \) generic over \( a, b \). We need to show that \( abH_n \cap cH_n \notin \mu \). By invariance, this is equivalent to \( bH_n \cap a^{-1}cH_n \notin \mu \). But \( a^{-1}c \) realizes a generic over \( b \), hence this follows from the fact that \( b \in Q_n \).

Finally, let \( G_n = \{ x \in H_n : xQ_n \subseteq Q_n \land x^{-1}Q_n \subseteq Q_n \} \). Then \( G_n \) is a subgroup and \( H \subseteq G_n \subseteq H_n \), so \( H = \bigcap G_n \).

Lemma 5.2. Let \((G, \star)\) be an algebraic group in an \( \aleph_1 \)-saturated real closed field \((R, <)\). Then there is an externally definable \(<\)-open subgroup \( H \leq G \) which has an invariant definable type in the expansion \( R^{ext} \), where we expand the language to include all the \( R^* \)-definable subsets of \( R \) for some saturated \( R^* \succeq R \).

Proof. As in [HPP08, Proposition 7.8], we identify a small neighborhood of \( e \) in \( G \) with a neighborhood of zero in \( R^n \). If we let \( \epsilon \) be infinitesimal with respect to \( R \), then we have

\[
|x \cdot y - (x + y)| \leq C|(x, y)|^2
\]

for some \( C \in R \) and all \( |x|, |y| \leq \epsilon \). Let \( U \) be the convex set of infinitesimals with respect to \( R \). Then \( H = \{ \bar{x}, x_i \in U \text{ for all } i \} \) is a subgroup of \( G \).

The set \( U \) is definable using parameters in \( R^* \), so \( U \) is defined by a predicate \( \bar{U} \) in \( R^{ext} \) and therefore \( H \) is also defined by a predicate \( \bar{H} \).

Let \( \bar{V} \) denote the set of elements \( x \) in \( R \) such that \( x \geq 1/n \) for some \( 0 < n < \omega \), which by compactness and saturation is also the trace in \( R \) of an \( R^* \)-definable set, so it is definable in \( R^{ext} \).

Let \( p(x_1, \ldots, x_n) \) be the type in \( \bar{H} \) saying that \( x_1 \) is as large as possible in \( \bar{U} \), and for all \( k > 1 \), \( x_k/x_{k-1} \) is infinitely small in \( \bar{V} \). Using weak o-minimality of \( R^{ext} \) we know that \( p \) determines a (definable) complete type.

We will show that \( p \) is \( \bar{H} \)-invariant, so that \( \bar{H}(R) \) and \( p \) satisfy the statement of the lemma. Let \( \bar{a} = (a_1, \ldots, a_n) \in \bar{H}(R^*) \) and let \( \bar{b} \) realize \( p \) over \( R^* \). We have to show that \( \bar{y} := \bar{a} \star \bar{b} \) realizes \( p \) over \( R^* \).
All coordinates of $\bar{a}$ and all $b_k^2$, are infinitesimal with respect to each $b_k$, so $\bar{a} \star \bar{b} = \bar{a} + \bar{b} + \bar{\epsilon}$, where $|\bar{\epsilon}| \leq C \cdot b_1^2$. Now $y_1 = b_1 + a_1 + \epsilon_1$, $a_1 \in R^*$, $b_1$ as large as possible in $U$ and $|\epsilon_1| \leq b_1^2$ which is much less than $b_1$, so $\text{tp}(y_1/R^*) \in U$ satisfies $\text{tp}(b_1/R^*)$.

In the same way, we have

$$\frac{y_k}{y_{k-1}} = \frac{b_k + a_k + \epsilon_k}{b_{k-1} + a_{k-1} + \epsilon_{k-1}}$$

hence

$$\frac{1/2b_k}{2b_{k-1}} \leq \frac{y_k}{y_{k-1}} \leq \frac{2b_k}{1/2b_{k-1}}$$

from which it follows that $y_k/y_{k-1}$ realizes over $R$ the type of an infinitesimally small element in $\hat{V}$. So $\bar{y}$ realizes $p$, as required.

**Proposition 5.3.** Let $M$ be a model of $T_{pre}$. Let $H$ be an algebraic group definable in $M$, let $K \leq H$ be a type definable subgroup and $L = K$. Then $K$ has bounded index in $L$, and $L/K$ with the logic topology is profinite.

**Proof.** Let $\overline{K^\circ}$ be the Zariski closure of $K$. Then $\overline{K^\circ}$ is an algebraic subgroup of $H$, $\overline{K^\circ}$ is type-definable and $\dim(\overline{K^\circ}) = \dim(K)$.

So replacing $H$ by $\overline{K^\circ}$ we can suppose that $\dim(H) = \dim(K) := m$. Observe that $K$ has bounded index in $L$.

Let $N \succ M$ be $|M|^+-$saturated. We now work in the structure $M_N$ defined in Definition 4.19. It is NTP$_2$ by Corollary 4.21. Suppose we have $n$-definable orders. For each $i$, we will define, in the ordered $<_i$-ring language $\mathcal{L}^{(i)}$ (using externally definable sets) a definable set $V^i$, and a type $p^i$ in $N^{(i)}$ as follows.

For each $i \in \{1, \ldots, n\}$, let $V^i$ be the $\mathcal{L}^{ext}$-definable subgroup of $H$, and let $p^i$ be the invariant $\mathcal{L}^{ext}$-definable type given by Lemma 5.2. So $V^i$ is the trace of an $N^{(i)}$-definable set in $M^{(i)}$.

Let $V := \bigcap_{i=1}^n V^i \cap M$, and let $p := \bigcup_{i=1}^n p^i$. By Fact 4.2, $V \neq \emptyset$ and $p$ is finitely consistent in $M$.

We have that $V$ is an externally definable set in $M$, and each $p_i$ is definable in $(N^{(i)})^{ext}$, so $p$ is a definable partial type in $M_N$. In a similar way we also obtain that $p$ is $V$-invariant.

So $V$ is a $\tau$-open definable subgroup of $L$, and since $K$ is $\tau$-dense in $L$, all the cosets intersect $K$ and we obtain that $V/V \cap K \cong L/K$. 28
We define an ideal $\mu$ over $V$ by $X \in \mu$ if $\overline{X} \notin p$. This ideal is definable and $V$-invariant.

**Claim.** $\mu$ is $S1$ over $V$.

**Proof.** If $X$ is a definable set, by Theorem 4.10 and Corollary 4.12 it follows that $X \in p$ if and only if $X \cup p$ is consistent. Let $\phi(x, y)$ be a formula and let $(a_j)_{j \in \omega}$ be indiscernible over $V$ such that $\phi(x, a_j) \notin \mu$, for all $j \in \omega$. Then all of the formulas $\overline{\phi(x, a_j)}$ are in $p$, and for each $j$ we have that $\phi(x, a_j) \cup p$ is consistent.

Let $c_1$ and $c_2$ be such that $(c_1 \mid p \cup \phi(x, a_1)) = (c_2 \mid p \cup \phi(x, a_2))$ and such that $c_1$ and $c_2$ are algebraically independent over $\{a_1, a_2\}$. By Fact 4.3 $\tau$-completeness of $p$ we have

$$\overline{\phi(x, a_1) \cap \phi(x, a_2)} \in p.$$

By indiscernibility

$$\overline{\phi(x, a_i) \cap \phi(x, a_j)} \in p$$

for all $i \neq j$, so $\phi(x, a_i) \cap \phi(x, a_j) \notin \mu$, for all $i \neq j$. \hfill \Box

Now, $V \cap K \in p$ so that $V \cap K$ is $\mu$-wide. It follows by Theorem 5.1 that $V \cap K$ is an intersection of definable groups. Hence $V/V \cap K$ with the $L^*$-logic topology (see Definition 4.19) is profinite, and then so is $L/K$ which is isomorphic to it.

The $L$-logic topology on $L/K$ is compact and Hausdorff and is weaker than the $L^*$-logic topology which is also compact and Hausdorff. It follows that both topologies coincide. In particular $L/K$ with the $L$-logic topology is profinite so that $K = \bigcap H_i$ where $H_i$ is $L$-definable and $H_i \cap L$ is a subgroup of $L$. \hfill \Box

## 6 Definable groups with $f$-generics in PRC

**Proposition 6.1.** Let $M$ be a model of $T_{prc}$ and let $M_0 < M$, $M$ is $|M_0|^+$-saturated. Let $G$ be a definable group in $M$ and let $p$ be a global type in $G$ strongly $f$-generic over $M_0$. Let $a \models p|M$, $b \models p|Ma$, and $c = ab$. Then $tp(c/Ma)$ if strongly $f$-generic over $M_0$ and there is an $M$-definable algebraic group $H$ and dimension-generic elements $a', b', c' \in H(U)$ such that $a' \cdot b' = c'$ and $acl(Ma) = acl(Ma')$, $acl(Mb) = acl(Mb')$ and $acl(Mc) = acl(Mc')$. 29
Proof. This is Proposition 3.1 of [HP94]. We modified the statement to additionally require that we can choose \(a, b, c\) and that we can take the set \(A\) to be inside \(M\). The reader can verify that those conditions can be met by the construction done in the proof.

Definition 6.2. Let \((M, <_1, \ldots, <_n)\) be a model of \(T_{prc}\). We say that a definable set \(X \subseteq M^m\) is multi-semialgebraic if \(X\) is a union of multi-cells in \(M^m\). Let \((G, \cdot_G)\) be an \(M\)-definable group. We say that \(G\) is multi-semialgebraic if \(G\), the graph of \(\cdot_G\) and of the inversion of \(G\) are multi-semialgebraic.

Theorem 6.3. Let \(M \models T_{prc}\) be \(\omega\)-saturated. Let \(G\) be an \(M\)-definable group with strong f-generics. Then there is a finite index \(M\)-definable subgroup \(G_1 \leq G\), a finite \(K \leq G_1\) central in \(G_1\), a multi-semialgebraic group \(H\) defined over \(M\) such that \(G_1/K\) is definably isomorphic to a finite index subgroup of \(H(M)\).

Proof. Let \(\mu_M\) be the ideal of formulas which do not extend to a strongly bi-f-generic type over \(M\). So \(\mu_M\) is \(M\)-invariant, \(S1\) and invariant under both left and right translations by elements of \(G\). Let \(q \in S(M)\) be \(\mu_M\)-wide.

By Theorem 3.18, \(Stab(q) = G^{00}_M\) and \(\mu_M\)-almost all elements of \(G^{00}_M\) are in \(St(q)\).

Let \(a \in G^{00}_M\) be such that \(tp(a/M)\) is \(\mu_M\)-wide. Let \(b \models q\) such that \(tp(b/Ma)\) is \(\mu_M\)-wide and \(tp(ab/M) = q\).

By Proposition 6.1 there is an \(M\)-definable algebraic group \((H, \cdot_H)\) and \(a', b', c' \in H\) such that \(c' = a' \cdot_H b'\), \(acl(Ma) = acl(Ma')\), \(acl(Mb) = acl(Mb')\) and \(acl(Mc) = acl(Mc')\).

We define an ideal \(\mu\) on \(G \times H\), by saying that \(D \in \mu\) if and only if \(\pi_1(D) \in \mu_M\). Then \(\mu\) is \(M\)-invariant and invariant under left and right translations. We will refer to \(\mu\)-wide as “wide”.

We define the ideal \(\lambda\) (that will define “medium” in Section 2) as the set of subsets \(X\) of \(G \times H\) for which the projections to \(G\) and \(H\) each have finite fibers. Note that \(\mu\) is \(S1\) on medium types. Define \(\tilde{p} = tp(a, a'/M)\). Then \(\tilde{p}\) is wide and medium. We will show that Theorem 2.13 can be applied to the type \(\tilde{p}\).

Claim. Condition (A) holds: If \(p, q\) are two types in \(G \times H\) and we have \((g, h) \models p \times_{nf} q\) such that either \(tp(gh/M)\) or \(tp(hg/M)\) is medium, then \(p\) is medium.
Proof. Denote $g = (g_0, g_1)$ and same for $h$. We will prove the case where we assume that $\text{tp}(gh/M)$ is medium, the other case is proved in an analogous way. Since $g_0h_0 \in \text{acl}(Mg_1h_1)$ we have $g_0 \in \text{acl}(Mg_1h_0h_1)$. As $\text{tp}(h_0h_1/Mg_0g_1)$ does not fork over $M$, this implies that $g_0 \in \text{acl}(Mg_1)$. In the same way we get $g_1 \in \text{acl}(Mg_0)$.

By Lemma 2.14, condition (B) holds. As $T_{prc}$ is NTP$^2$, condition (F) holds. We can then apply Theorem 2.13, which gives us a connected, medium, wide type-definable group $K \leq G \times H$. As $K$ is medium, its projections to $G$ and $H$ have finite fibers. As $K$ is wide, $\pi_1(K)$ is $\mu_M$-wide and hence by connectedness $\pi_1(K) = G^0_M$.

Claim. We may assume that $\pi_1$ and $\pi_2$ are injective on $K$.

Proof. For $i = 1, 2$, let $K_i = \pi_i^{-1}(e) \cap K$. Then $K_i$ is finite and normal in $K$. As $K$ is connected, $K_i$ is central in $K$ (the centralizer of $K_1$ is a relatively definable subgroup of $K$ of finite index). Let $C \leq H$ be the centralizer of $\pi_2(K_1)$ inside $H$. It is an algebraic subgroup of $H$. Then we can replace $H$ by $C/\pi_2(K_1)$ which is again an algebraic group (defined over the same parameters as $H$ and $K_1$). Thus we may assume that $K_1$ is trivial. In the same way, replacing $G$ by a subgroup of finite index, we may assume that $\pi_1(K_2)$ is central in $G$ and then replace $G$ by its quotient by $\pi_1(K_2)$.

Now choose a symmetric definable $X_0$ such that $K \subseteq X_0 \subseteq G \times H$, and such that $\pi_1$ and $\pi_2$ are injective on $X_0^4$. Replacing $G$ by a subgroup of finite index, we can assume that $\pi_1(X_0)$ generates $G$. By Proposition 2.11, $K$ is normal in the group generated by $X_0$ and $X_0^n$ is medium for any $n$. Observe that $\pi_1(X_0) \subseteq G$ is definably isomorphic to $\pi_2(X_0) \subseteq H$. In $\pi_2(X_0)$, the multi-topology $\tau$ is definable and the operations in $H$ are continuous.

Since working with projections becomes quite messy, we will abuse notation in the following way:

- Any element of $X_0$ will be written as $(x^*, x)$ where $x^* \in G$ and $x \in H$. So $x^* = \pi_1(\pi_2^{-1}(x))$ for any element $x$ in $\pi_2(X_0) \subseteq H$. We will also do this for sets, so that $A^* = \pi_1(\pi_2^{-1}(A))$ for any $A \subset \pi_2(X_0)$.

- All non $^*$-elements will be assumed to belong to $H$. We will use Greek letters for elements in $G$ which may not be in $\pi_1(X_0)$.

- We will identify $K$ with $\pi_2(K)$. 

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• We will mostly be working inside $H$, so we will drop the index in $\cdot_H$.

In $H$ we have that $\pi_2(X_0) \cap \bar{K}$ is generic in $\bar{K}$ and by Proposition 5.3, $\bar{K}/K$ is profinite, so there is a $\mathcal{L}$-definable set $X \subseteq \pi_2(X_0)$ such that $K_1 := X \cap \bar{K}$ is a subgroup of finite index of $\bar{K}$, and $K_1^*$ is a type-definable subgroup of bounded index of $G$ ($G_{M}^{00} \subseteq K_1^*$). By passing to a finite index subgroup of $G$, we may assume that $X^*$ generates $G$.

**Claim.** We may assume that $K_1$ is a normal subgroup of $\bar{K}$, in fact normalized by $X$, and that $K_1^*$ is a normal subgroup of $G$.

**Proof.** Let $r$ be the smallest integer such that every $\gamma \in G$ is the $\cdot_G$-product of $r$ elements in $X$. Define $Y_1, \ldots, Y_r$ such that:

$$Y_1 := \bigcap_{\gamma \in X} \gamma X \gamma^{-1},$$

$$Y_l := \bigcap_{\gamma \in X} \gamma Y_{l-1} \gamma^{-1},$$

for $2 \leq l \leq r$.

Then $(Y_r)^*$ is normalized by $G$, hence $Y_r$ is normalized by $X$ and so is $Y_r \cap \bar{K}$. We can now replace $X$ by $Y_r$ and $K_1$ by $Y_r \cap \bar{K}$.

Now, $\bar{K}$ is the intersection of multi-semialgebraic sets in $H$. We can define a decreasing sequence $(U_k : k < \omega)$ of quantifier-free definable symmetric sets, such that:

- $U_0 = \bar{X}$;
- $(U_{m+1} \cap X)^3 \subseteq U_m \cap X$, for each $m < \omega$;
- $g(U_{m+1} \cap X)g^{-1} \subseteq U_m \cap X$, for each $m < \omega$ and $g \in X$;
- $\bar{K} = \bigcap_{k \in \omega} U_k$.

Note that by density of $X$ in $U_0$ and by continuity of the operations, we also have $(U_{m+1})^3 \subseteq U_m$ and $gU_{m+1}g^{-1} \subseteq U_m$ for all $g \in X$.

**Claim.** We may assume that $U_m$ are multi-open, for $m \geq 1$.

**Proof.** The type definable group $\bar{K}$ has non empty interior in $H$ (since it has bounded index). The operations are continuous in $H$ and by definition $X$ is dense in $U_0$. It follows that, since $U_{m+1} \cdot \bar{K} \subseteq U_m$, every point in $U_{m+1}$ has a neighborhood contained in $U_m$, so $U_{m+1}$ is entirely contained in the interior of $U_m$. Replacing each $U_m$ by its interior, we preserve the properties and the claim holds. 

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Select points \( \{ \alpha_k : k < p \} \) in \( G \) such that

\[
G = \bigcup_{k < p} \alpha_k \cdot_G (U_4 \cap X)^*.
\]

Note that for any \( x \in G \), there is \( k < p \) such that \( x \in \alpha_k \cdot_G (U_4 \cap X)^* \) and then \( x \cdot_G (U_4 \cap X)^* \subseteq \alpha_k \cdot_G (U_3 \cap X)^* \).

Let \( m \) be the smallest integer such that every \( \alpha_i \) is the \( \cdot_G \)-product of \( m \) elements in \( (U_3 \cap X)^* \).

**Claim.** For each \( i \), the conjugation map \( f_i : x \mapsto \pi_2(\pi_1^{-1}(\alpha_i \cdot_G x \cdot_G (\alpha_i)^{-1})) \) is an algebraic map from \( U_{k+m} \cap X \) to \( U_k \cap X \) for \( k \geq 3 \).

**Proof.** Let \( \alpha_i = d_1^* \cdot_G \cdots \cdot_G d_m^* \), with each \( d_l \) in \( X \). Then for any \( l \), for any \( j \geq 3 \) and \( x \in U_j \cap X \) the map \( x \mapsto d_l^{-1}xd_l \) is algebraic (as \( H \) is algebraic) and by hypothesis \( d_l^{-1}xd_l \in U_{j-1} \cap X \). Since

\[
\pi_2(\pi_1^{-1}(\alpha_i x^*(\alpha_i)^{-1})) = d_l^{-1} \cdots d_1^{-1} x \cdot d_1^{-1} \cdots d_l^{-1},
\]

the function \( f_i \) is algebraic as a composition of algebraic functions. \( \square \)

Select points \( \{ b_i : i < l \} \) in \( U_3 \cap X \) such that

\[
U_3 \cap X = \bigcup_{i \leq l} b_i \cdot (U_{m+3} \cap X).
\]

For \( j < p \) and \( r < l \), define \( \alpha_{(j,r)} \in \{ \alpha_k : k < p \} \) and \( t_{(j,r)} \in (U_3 \cap X) \) such that (where all the products are in \( G \)):

\[
\alpha_j^{-1}b_i^* \alpha_j = \alpha_{(j,r)} t_{(j,r)}^*.
\]

Let \( W = (U_3 \cap X) \times \{ 0, \ldots, p-1 \} \) and for \( k < p \), define \( W_k = (U_3 \cap X) \times \{ k \} \).

Define an equivalence relation \( E \) on \( W^2 \) by \( (x, i)E(y, j) \) if \( \alpha_i \cdot_G x^* = \alpha_j \cdot_G y^* \). We then have

\[(x, i)E(y, j) \iff (y^*) \cdot_G (x^*)^{-1} = \alpha_j^{-1} \cdot_G \alpha_i.\]

If this happens, then \( \alpha_j^{-1} \cdot_G \alpha_i \) lies in \( (U_2 \cap X)^* \) and can be written as \( w_{ij}^* \) for some \( w_{ij} \in U_2 \cap X \). When this is not the case, say that \( w_{ij} \) is undefined.
Note that we have a definably bijection \( \phi: W/E \to G \) sending \((x,i)\) to \(\alpha_i \cdot_G x^*\).

We will now define a multi-semialgebraic group, which in a way will be the \(\tau\)-topological closure of \(W/E\).

Let \(W^c = U_3 \times \{0, \ldots, p - 1\}\) and \(W^c_k = U_3 \times \{k\}\). We equip each \(W^c_k\) with the \(\tau\)-topology. Then \(W_k\) is dense in \(W^c_k\).

We now define a relation \(E^c\) on \(W^c\) as follows: given \((x,i),(y,j)\) \(\in W\), we have \((x,i)E^c(y,j)\) if and only if \(w_{ij}\) is defined and \(yx^{-1} = w_{ij}\).

**Claim.** \(E^c\) is an equivalence relation.

**Proof.** Reflexivity holds as \(w_{ii} = e\) for all \(i\). Whenever \(w_{ij}\) is defined, then so is \(w_{ji}\) and \(w_{ji} = w_{ij}^{-1}\). This implies symmetry. Finally, assume that \((x,i)E^c(y,j)\) and \((y,j)E^c(z,k)\), then \(zx^{-1} = w_{jk}w_{ij} \in U_2 \cap X\) (as \(zx^{-1} \in U_2\) and \(w_{jk}w_{ij} \in X\)). Then \(w_{ik}\) is defined and equal to \(w_{jk}w_{ij}\) and thus \((x,i)E^c(z,k)\). \(\square\)

By construction \(W/E\) embeds in \(W^c/E^c\). We now define a group structure on \(W^c/E^c\). First consider \((x,i),(y,j),(z,k)\) \(\in W\) and write \(x = b_rw\) with \(w \in U_{m+3} \cap X\). Then we have, where all the products are understood in \(G\):

\[
\alpha_i x^* \alpha_j y^* = \alpha_k z^* \\
\iff \alpha_i b_r^* w^* \alpha_j y^* = \alpha_k z^* \\
\iff \alpha_i \alpha_j \alpha_{(j,r)} t_{(j,r)}^* f_j(w)^* y^* = \alpha_k z^* \\
\iff t_{(j,r)}^* f_j(w)^* y^* (z^*)^{-1} = \alpha_{(j,r)}^{-1} \alpha_j^{-1} \alpha_i^{-1} \alpha_k.
\]

When such an equation holds, we define \(\epsilon(i,j,k,r)\) as \(\alpha_{(j,r)}^{-1} \alpha_j^{-1} \alpha_i^{-1} \alpha_k \in U_1 \cap X\). Let \(\Gamma \in W^3\) be the pullback of the graph of multiplication on \(W/E \cong G\) via the canonical projection. Then \(((x,i),(y,j),(z,k)) \in \Gamma\) if and only if \(\epsilon(i,j,k,r)\) is defined and writing \(x = b_rw\), we have:

\[
t_{(j,r)} f_j(w) y z^{-1} = \epsilon(i,j,k,r). \quad (E \Gamma)
\]

We define \(\Gamma^c\) on \(W^c\) by \(((x = b_rw,i),(y,j),(z,k)) \in \Gamma^c\) if \((E \Gamma)\) holds. We need to check that this is well defined, i.e., does not depend on the decomposition of \(x\) as \(b_rw\). So assume that \(x = b_rw = b_{s'}w'\). Then \(w' = b_{s'}^{-1}b_rw\). Assume that \(t_{(j,r)} f_j(w) y z^{-1} = \epsilon(i,j,k,r)\). On a small neighborhood of \((w,y,z)\)
we can find \((w_0, y_0, z_0)\), all points lying in \(X\) such that \(t_{(j,r)}f_j(w_0)y_0z_0^{-1} = \epsilon(i,j,k,r)\) (as all operations are continuous). Set \(w'_0 = b_s^{-1}b_tw_0\), then \(w'_0\) is close to \(w\), hence in \(U_{m+3} \cap X\) and we have \(t_{(j,s)}f_j(w'_0)y_0z_0^{-1} = \epsilon(i,j,k,s)\) (in particular \(\epsilon(i,j,k,s)\) is defined). Letting \((w_0, y_0, z_0)\) converge to \((w, y, z)\), we obtain \(t_{(j,s)}f_j(w')yz^{-1} = \epsilon(i,j,k,s)\) as required.

A similar argument shows that \(\Gamma^d\) is \(E^d\)-equivariant: if say \((z, k)E^d(z', k')\), then we have \(z' = w_{i'}z\) and we conclude as above that \((x, i), (y, j), (z, k)\) is in \(\Gamma^d\) if and only if \((x, i), (y, j), (z', k')\) is in \(\Gamma^d\). Therefore \(\Gamma^d\) induces a ternary relation on the quotient \(W^d/E^d\). Note that on each \(W^d_i \times W^d_j \times W^d_k\), \(\Gamma^d\) is the closure of \(\Gamma\).

**Claim.** \(\Gamma^d\) induces the graph of a function \(W^d/E^d \times W^d/E^d \rightarrow W^d/E^d\).

**Proof.** First, assume that \((x, i), (y, j) \in W^d\), \(x = b_ww\). Then for a given \(j\), the equation \(t_{(j,r)}f_j(w)yz^{-1} = \epsilon(i,j,k,r)\) can have at most one solution in \(z\). If we can find \((z', k')\) such that \(t_{(j,r)}f_j(w)yz^{-1} = \epsilon(i,j,k',r)\) also holds, then \(\epsilon(i,j,k',r)w_{kk'} = \epsilon(i,j,k,r)\) and so \(z^{-1}w_{kk'} = z^{-1}\) which implies \((z, k)E^d(z', k')\). This shows that the image is unique.

It remains to show existence. Take \((x, i), (y, j) \in W^d\). Take a small neighborhood \(U_*\) of the identity included in \(\bar{K}\). Then there are some \(k\) and \(r\) such that for any \(x_0 \in xU_*\) and \(y_0 \in yU_*\), there is \((z_0, k)\) with \(((x_0,i), (y_0,j), (z_0,k)) \in \Gamma\) and \(x_0\) can be written as \(b_rw_0\) with \(w_0 \in U_{m+4} \cap X\). We may also assume that for any such \(z_0\), \(z_0\bar{K} \subseteq U_3\). Then we have \(t_{(j,r)}f_j(w_0)y_0z_0^{-1} = \epsilon(i,j,k,r)\). We can then write \(x = b_rw\) for some \(w \in U_{m+3} \cap X\) and define \(z = \epsilon(i,j,k,r)^{-1}t_{(j,r)}f_j(w)y\). Then \(z \in z_0\bar{K} \subseteq U_3\) and \(((x,i), (y,j), (z,k)) \in \Gamma\). □

Let \(\odot\) the boolean function induced by \(\Gamma\) on \(W^d/E^d\). As associativity is a closed condition \(\Gamma^d\) is the closure on \(\Gamma\) on each \(W^d_i \times W^d_j \times W^d_k\), \(\odot\) is associative. Existence of inverses is proved as the existence part of the previous claim, fixing \(z = e\) and looking for \(y\). Therefore we have equipped \(W^d/E^d\) with a group structure. Write this group as \(G_0\).

The sets \(W^d_k\) are multi semialgebraic and \(E^d\) and \(\Gamma^d\) are algebraic. Strictly speaking, \(G_0\) thus constructed is not multi semialgebraic as its construction involves quotients. However one can easily remove the quotients: letting \(\pi : W \rightarrow G_0\) be the quotient map, \(G_0\) is in definable bijection with a multi semialgebraic group with underlying set

\[
W_0 \cup (W_1 \setminus \pi^{-1}(\pi(W_0))) \cup (W_2 \setminus \pi^{-1}(\pi(W_1) \cap \pi(W_0))) \cup \cdots
\]
As $G$ embeds definably into $G_0$ as a subgroup of finite index. This finishes the proof of the theorem.

6.1 Additional comments

- As already pointed out in the introduction, we in fact obtain a stronger statement than stated in Theorem 6.3: A finite index subgroup of $G$ is isogeneous with a finite index subgroup of a ‘multi-Lie group’ that is a group which admits a definable manifold structure, in the sense of the multi-topology, for which the group operations are continuous (even $C^\infty$ with respect to each order).

- Will Johnson has studied in [Joh13] the model companion of fields with $n$ distinct orderings. This is a particular case of bounded PRC fields. Johnson proves that a Lascar-invariant quantifier-free type extends to a Lascar-invariant measure. It seems likely that an adaptation of those results should show that in this case, any group with f-generics has a translation-invariant measure.

- We expect those results to generalize to groups definable in the main sort of a pseudo p-adically closed field. This will be dealt with in future work.

Appendix: Shelah expansion and NTP$_2$

**Theorem 6.4.** Let $T$ be NTP$_2$ in a language $L$ and assume that we have an expansion $T'$ of $T$ to a language $L'$ by externally definable sets. Assume furthermore that $T'$ has elimination of quantifiers in $L'$ and the only additional predicates in $L'$ are traces of externally definable NIP formulas. Then $T'$ is NTP$_2$.

**Proof.** Let $M \models T'$ be $\aleph_1$-saturated and let $M \prec N$ be $|M|^+$-saturated. The property of NTP$_2$ for formulas is preserved by finite disjunctions, but not by finite conjunctions in general. It is enough to show that a formula of the form $\phi(x; y) \land \psi(x; y)$ is NTP$_2$, where $\phi(x; y) \in L$ and $\psi(x; y) \in L'$ is such that there is an NIP $L$-formula $\tilde{\psi}(x; y; d) \in L(N)$ such that $\psi(M) = \tilde{\psi}(N; d) \cap M$.

Let $(N, M)$ denote the expansion of $N$ with a new unary predicate naming $M$. Let $(N_1, M_1) \succ (N, M)$ be a sufficiently saturated elementary extension.
By honest definitions, there is $\theta(x, y; e) \in L(M_1)$ such that $\theta(M; e) = \psi(M)$ and $\theta(M_1; e) \subseteq \tilde{\psi}(M_1; d)$. Note that the formula $\theta(x, y; e)$ could have IP.

Now assume that we are given a witness of TP$_2$ for $\phi(x; y) \land \psi(x; y)$. Namely, we have an array $(b_{i,j} : i, j < \omega)$ and some $k$ such that each line \{\[\phi(x; b_{i,j}) \land \psi(x; b_{i,j}^\eta) : i < \omega\}\} is $k$-inconsistent and for every $\eta : \omega \to \omega$, the path \{\[\phi(x; b_{\eta(j),j}) \land \psi(x; b_{\eta(j),j}) : j < \omega\]\} is consistent, hence realized by some $a_\eta \in M$. Now the properties of the array are preserved if we replace the formula $\phi(x; y) \land \psi(x; y)$ by $(\phi(x, y) \land \theta(x, y; e)$: the paths are still consistent, using the same witnesses $a_\eta$, and the lines are still $k$-inconsistent (in the structure $M_1$) by the honesty property. This shows that the formula $\phi(x, y) \land \theta(x, y; e)$ has TP$_2$ in $M_1$ which contradicts the hypothesis that $T$ is NTP$_2$. 

\section*{References}


