MINIMIZATION OF THE FIRST EIGENVALUE IN PROBLEMS INVOLVING THE BI-LAPLACIAN

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Abstract

This paper concerns the minimization of the first eigenvalue in problems involving the bi-Laplacian under either homogeneous Navier boundary conditions or homogeneous Dirichlet boundary conditions. Physically, in case of $N = 2$, our equation models the vibration of a non homogeneous plate $\Omega$ which is either hinged or clamped along the boundary. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials inside $\Omega$ so to minimize the first mode in the vibration of the corresponding plate.

Keywords: bi-Laplacian, first eigenvalue, minimization.

Resumen

Este artículo trata de la minimización del primer autovalor en problemas relativos al bi-Laplaciano bajo condiciones de frontera homogéneas de tipo Navier o Dirichlet. Físicamente, en el problema bi-dimensional, nuestra ecuación modela la vibración de una placa inhomogénea $\Omega$ fija con goznes a lo largo de su borde. Dados varios materiales (de diferentes densidades) y extensión total $|\Omega|$, investigamos cuál debe ser la localización de tales materiales en la placa para minimizar el primer modo de su vibración.

Palabras clave: bi-Laplaciano, primer autovalor, minimización.

Mathematics Subject Classification: 35P15, 47A75, 49K20.

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1 Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ and let $g_0$ be a measurable function satisfying $0 \leq g_0 \leq M$ in $\Omega$, where $M$ is a positive constant. To avoid trivial situations, we always assume $g_0 \not\equiv 0$ and $g_0 \not\equiv M$. Define $\mathcal{G}$ as the family of all measurable functions defined in $\Omega$ which are rearrangements of $g_0$. Consider the following eigenvalue problems

$$\Delta^2 u = \lambda gu, \quad \text{in } \Omega, \quad u = \Delta u = 0 \quad \text{on } \partial\Omega, \quad (1)$$

and

$$\Delta^2 v = \Lambda g v, \quad \text{in } \Omega, \quad v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad (2)$$

where $g \in \mathcal{G}$, $\lambda = \lambda_g$, $\Lambda = \Lambda_g$ are the first eigenvalues and $u$, $v$ are the corresponding eigenfunctions. The operator $\Delta^2$ stands for the usual bi-Laplacian, that is $\Delta^2 u = \Delta(\Delta u)$.

The first eigenvalue $\lambda$ of problem (1) is obtained by minimizing the associate Rayleigh quotient

$$\lambda = \inf \left\{ \frac{\int_{\Omega} (\Delta z)^2 \, dx}{\int_{\Omega} gz^2 \, dx} : \ z, \ \Delta z \in H^1_0(\Omega), \ z \not\equiv 0 \right\}. \quad (3)$$

The first eigenvalue $\Lambda$ of problem (2) is obtained by minimizing the quotient

$$\Lambda = \inf \left\{ \frac{\int_{\Omega} (\Delta z)^2 \, dx}{\int_{\Omega} gz^2 \, dx} : \ z \in H^2_0(\Omega), \ z \not\equiv 0 \right\}. \quad (4)$$

It is well known that the inferior is attained in both cases [14]. The minimum of (3) satisfies problem (1) in the weak sense, that is

$$\int_{\Omega} \Delta u \Delta z \, dx = \lambda \int_{\Omega} guz \, dx, \quad \forall z : z, \Delta z \in H^1_0(\Omega).$$

The minimum of (4) satisfies problem (2) in the sense

$$\int_{\Omega} \Delta v \Delta z \, dx = \Lambda \int_{\Omega} gvz \, dx, \quad \forall z \in H^2_0(\Omega).$$

By regularity results (see [1]) the solutions to problems (1) and (2) belong to $H^{4}_{\text{loc}}(\Omega)$.

In this paper we investigate the problems

$$\min_{g \in \mathcal{G}} \lambda_g, \quad \text{and} \quad \min_{g \in \mathcal{G}} \Lambda_g. \quad (5)$$

Let us give a motivation for the study of these problems in case of $N = 2$. Physically, our equations model the vibration of a non homogeneous plate $\Omega$ which is either hinged or clamped along the boundary $\partial\Omega$. Given several materials (with different densities) of total extension $|\Omega|$, we investigate the location of these materials inside $\Omega$ so to minimize the first mode in the vibration of the plate. The corresponding problem for second order equations has been discussed in several papers, see for example [6], [7], [9].

The paper is organized as follows. In Section 2 we collect some definitions and known results. In Section 3 we investigate the problems (5) proving results of existence and results of representation of minimizers. In case $\Omega$ is a ball we prove uniqueness for both problems.
2 Preliminaries

Denote with $|E|$ the Lebesgue measure of the (measurable) set $E \subset \mathbb{R}^N$. Given a measurable function $g_0(x)$ defined in $\Omega$ we say that $g(x)$, defined in $\Omega$, belongs to the class of rearrangements $\mathcal{G} = \mathcal{G}(\{\})$ if $|\{x \in \Omega : g(x) \geq \beta\}| = |\{x \in \Omega : g_0(x) \geq \beta\}| \quad \forall \beta \in \mathbb{R}$.

We make use of the following results.

**Lemma 2.1** Let $g \in L^1(\Omega)$ and let $u \in L^1(\Omega)$. Suppose that every level set of $u$ (that is, sets of the form $u^{-1}(\{\alpha\})$), has measure zero. Then there exists an increasing function $\phi$ such that $\phi(u)$ is a rearrangement of $g$.

**Proof.** The assertion follows by Lemma 2.9 of [4]. □

**Lemma 2.2** Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^r(\Omega)$, $r > 1$, $g_0 \not\equiv 0$, and let $\overline{\mathcal{G}}$ denote the weak closure of $\mathcal{G}$ in $L^r(\Omega)$. If $u \in L^s(\Omega)$, $s = r/(r - 1)$, $u \not\equiv 0$, and if there is an increasing function $\phi$ such that $\phi(u) \in \mathcal{G}$ then

$$
\int_\Omega g u \, dx \leq \int_\Omega \phi(u) \, dx \quad \forall g \in \overline{\mathcal{G}},
$$

and the function $\phi(u)$ is the unique maximizer relative to $\overline{\mathcal{G}}$.

**Proof.** The assertion follows by Lemma 2.4 of [4]. □

**Lemma 2.3** Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^r(\Omega)$, $r > 1$, $g_0 \not\equiv 0$, and let $u \in L^s(\Omega)$, $s = r/(r - 1)$, $u \not\equiv 0$. There exists $g \in \mathcal{G}$ such that

$$
\int_\Omega g u \, dx \leq \int_\Omega g u \, dx \quad \forall g \in \mathcal{G}.
$$

**Proof.** It follows by Lemma 2.4 of [4]. See also [5]. □

Next we recall a well known rearrangement inequality. For $u$ non-negative in $\Omega$, $u^\sharp$ denotes the decreasing Schwarz rearrangement of $u$; that is, $u^\sharp$ is defined in $\Omega^\sharp$, the ball centered in the origin with measure equal to $|\Omega|$, is radially symmetric, decreases as $|x|$ increases, and satisfies

$$
|\{x \in \Omega : u(x) \geq \beta\}| = |\{x \in \Omega^\sharp : u^\sharp(x) \geq \beta\}| \quad \forall \beta \geq 0.
$$

If $u \in H^1_0(\Omega)$ is non-negative and if $u^\sharp$ is the decreasing Schwarz rearrangement of $u$ then $u^\sharp \in H^1_0(\Omega^\sharp)$ and the inequality

$$
\int_\Omega |\nabla u^\sharp|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx
$$

holds. The case of equality in (6) has been considered in [3]. We have

**Lemma 2.4** Let $u \in H^1_0(\Omega)$ be non-negative, and suppose equality holds in (6). If

$$
|\{x \in \Omega^\sharp : \nabla u^\sharp(x) = 0, \quad 0 < u^\sharp(x) < \sup_{\Omega} u(x)\}| = 0
$$

then $u$ is a translate of $u^\sharp$.

**Proof.** See Theorem 1.1 of [3] or the monograph [13]. □
3 Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and let $M > 0$ be a real number. Let $\mathcal{G}$ be the family of all functions defined in $\Omega$ which are rearrangements of a given function $g_0$ with $0 \leq g_0(x) \leq M$, $g_0(x) \neq 0$, $g_0(x) \neq M$. For $g \in \mathcal{G}$, let $\lambda_g$ be the first eigenvalue of problem (1), and let $\Lambda_g$ be the first eigenvalue of problem (2). We investigate the problems

$$\min_{g \in \mathcal{G}} \lambda_g, \quad \text{and} \quad \min_{g \in \mathcal{G}} \Lambda_g.$$

Recalling (3) and (4) we can formulate the previous problems as

$$\min_{g \in \mathcal{G}} \lambda_g = \min \left\{ \frac{\int_\Omega (\Delta z)^2 \, dx}{\int_\Omega g \, z^2 \, dx} : g \in \mathcal{G}, \, z \in H^1_0(\Omega), \, \Delta z \in H^1_0(\Omega) \right\}, \quad (7)$$

and

$$\min_{g \in \mathcal{G}} \Lambda_g = \min \left\{ \frac{\int_\Omega (\Delta z)^2 \, dx}{\int_\Omega g \, z^2 \, dx} : g \in \mathcal{G}, \, z \in H^2_0(\Omega) \right\}. \quad (8)$$

**Theorem 3.1** Let $0 \leq g_0(x) \leq M$, $g_0(x) \neq 0$, $g_0(x) \neq M$, and let $\mathcal{G}$ be the class of all rearrangements of $g_0$. Then

a) there exists $\overline{g} \in \mathcal{G}$ such that

$$\lambda_{\overline{g}} = \min_{g \in \mathcal{G}} \lambda_g;$$

b) there exists $\tilde{g} \in \mathcal{G}$ such that

$$\Lambda_{\tilde{g}} = \min_{g \in \mathcal{G}} \Lambda_g.$$

**Proof.** We prove first part a). Let

$$I = \inf_{g \in \mathcal{G}} \lambda_g = \lim_{i \to \infty} \lambda_{g_i} = \lim_{i \to \infty} \frac{\int_\Omega (\Delta u_i)^2 \, dx}{\int_\Omega g_i u_i^2 \, dx}, \quad (9)$$

where $u_i = u_{g_i}$ is the eigenfunction corresponding to $g_i$ normalized so that

$$\int_\Omega u_i^2 \, dx = 1.$$

We may assume that the sequence $\{\lambda_{g_i}\}$ is decreasing. By (9) and the latter equation we get

$$\int_\Omega (\Delta u_i)^2 \, dx \leq \lambda_{g_i} M. \quad (10)$$

On the other side, since $u_i$ vanishes on $\partial \Omega$, by Lemma 9.17 of [11] we have

$$\|u_i\|_{H^2(\Omega)}^2 \leq C \|\Delta u_i\|_{L^2(\Omega)} \|u_i\|_{L^2(\Omega)}$$

with $C$ independent of $i$. It follows that the norms $\|u_i\|_{H^2(\Omega)}$ and $\|\Delta u_i\|_{L^2(\Omega)}$ are equivalent. This fact and (10) imply that the sequence $\{u_i\}$ is bounded in the $H^2(\Omega)$ norm and some subsequence (still denoted $\{u_i\}$) converges weakly in $H^2(\Omega)$ to a function $\overline{u}$. We
can also assume that \( \{ u_i \} \) converges strongly to \( \pi \) in \( L^{2+\epsilon}(\Omega) \) for some \( \epsilon > 0 \). Furthermore, since \( \{ g_i \} \) is bounded in \( L^\infty(\Omega) \), it must contain a subsequence (still denoted \( \{ g_i \} \)) converging weakly to \( \eta \in L^r(\Omega) \) for any \( r > 1 \). We have
\[
\int_{\Omega} g_i u_i^2 dx - \int_{\Omega} \eta \pi^2 dx = \int_{\Omega} (g_i - \eta) \pi^2 dx + \int_{\Omega} g_i (u_i^2 - \pi^2) dx.
\]
We find
\[
\lim_{i \to \infty} \int_{\Omega} (g_i - \eta) \pi^2 dx = 0,
\]
because \( \pi^2 \in L^s(\Omega) \) for some \( s > 1 \) and \( g_i \to \eta \) weakly in \( L^r(\Omega) \) for \( r = s/(s-1) \).

Moreover,
\[
\lim_{i \to \infty} \int_{\Omega} g_i (u_i^2 - \pi^2) dx = 0.
\]
The latter result can be proved by using Lebesgue’s theorem as follows. Since \( u_i \to \pi \) in \( L^2(\Omega) \), we have (up to a subsequence)
\[
\lim_{i \to \infty} g_i (u_i^2 - \pi^2) = 0 \quad \text{a.e. in } \Omega,
\]
and
\[
g_i |u_i^2 - \pi^2| \leq M(\psi^2 + \pi^2),
\]
for some integrable function \( \psi^2 \). Indeed, since \( u_i \) converges in \( L^2(\Omega) \) one can find \( \psi \in L^2(\Omega) \) such that \( u_i(x) \leq \psi(x) \) a.e. for some subsequence of \( u_i \) [10]. Hence,
\[
\lim_{i \to \infty} \int_{\Omega} g_i u_i^2 dx = \int_{\Omega} \eta \pi^2 dx.
\]
By Lemma 2.3 we can find \( \pi \in G \) such that
\[
\int_{\Omega} \eta \pi^2 dx \leq \int_{\Omega} \pi \pi^2 dx.
\]
On the other side, from the inequality
\[
0 \leq \int_{\Omega} (\Delta (u_i - \pi))^2 dx = \int_{\Omega} (\Delta u_i)^2 dx - 2 \int_{\Omega} \Delta u_i \Delta \pi dx + \int_{\Omega} (\Delta \pi)^2 dx
\]
and the weak convergence of \( \{ u_i \} \) to \( \pi \) in \( H^2(\Omega) \) we find
\[
\liminf_{i \to \infty} \int_{\Omega} (\Delta u_i)^2 dx \geq \int_{\Omega} (\Delta \pi)^2 dx.
\]
By using the latter result together with (11) and (12) we have
\[
I = \lim_{i \to \infty} \frac{\int_{\Omega} (\Delta u_i)^2 dx}{\int_{\Omega} g_i u_i^2 dx} \geq \frac{\int_{\Omega} (\Delta \pi)^2 dx}{\int_{\Omega} \eta \pi^2 dx} \geq \frac{\int_{\Omega} (\Delta \pi)^2 dx}{\int_{\Omega} \pi \pi^2 dx}.
\]
Our minimizing sequence \( u_i \) satisfies (in a weak sense)
\[
\Delta(\Delta u_i) = \lambda_i g_i u_i, \quad \Delta u_i \in H_0^1(\Omega).
\]
If we multiply by \(-\Delta u_i\) and integrate over \( \Omega \), after simplification we find
\[
\|\nabla(\Delta u_i)\|_{L^2(\Omega)} \leq \lambda_i g_i \|g_i u_i\|_{L^2(\Omega)}.
\]
Since \( \lambda_i g_i \) is decreasing, \( 0 \leq g_i \leq M \) and \( \|u_i\|_{L^2(\Omega)} = 1 \) we find that \( \|\nabla(\Delta u_i)\|_{L^2(\Omega)} \leq \lambda_i M \).

As a consequence, since \( \Delta u_i \in H_0^1(\Omega) \) we also have \( \Delta \tilde{\psi} \in H_0^1(\Omega) \). Therefore, if \( \lambda_{\tilde{\psi}} \) is the (first) eigenvalue corresponding to \( \tilde{\psi} \) in problem (1), and if \( u_{\tilde{\psi}} \) is a corresponding eigenfunction then by (3) we have
\[
\frac{\int_\Omega (\Delta \tilde{\psi})^2 \, dx}{\int_\Omega g_i \, v_i^2 \, dx} \geq \frac{\int_\Omega (\Delta u_i)^2 \, dx}{\int_\Omega g_i \, u_i^2 \, dx} = \lambda_{\tilde{\psi}} \geq I.
\]

By the latter result and (13) we must have \( I = \lambda_{\tilde{\psi}} \). Part a) of the theorem is proved.

The proof of part b) is similar. Define
\[
\tilde{I} = \inf_{g \in \mathcal{G}} \Lambda_g = \lim_{i \to \infty} \frac{\int_\Omega (\Delta v_i)^2 \, dx}{\int_\Omega g_i v_i^2 \, dx},
\]
where \( v_i = v_{g_i} \) is the eigenfunction corresponding to \( g_i \) normalized so that
\[
\int_\Omega v_i^2 \, dx = 1.
\]
Of course, \( \{g_i\} \) is not, in general, the same as for part a). Arguing as in the previous case we find that \( v_i \) is bounded in the norm of \( H^2(\Omega) \). Therefore, a subsequence (still denoted \( \{v_i\} \)) converges weakly in \( H^2(\Omega) \) to a function \( \tilde{v} \in H_0^2(\Omega) \). We can also assume that \( \{v_i\} \) converges strongly to \( \tilde{v} \) in \( L^{2+\epsilon}(\Omega) \) for some \( \epsilon > 0 \). Furthermore, \( \{g_i\} \) must contain a subsequence (still denoted \( \{g_i\} \)) converging weakly to some \( \zeta \in L^r(\Omega) \) for any \( r > 1 \). Hence,
\[
\lim_{i \to \infty} \int_\Omega g_i v_i^2 \, dx = \int_\Omega \zeta \tilde{v}^2 \, dx.
\]
By Lemma 2.3 we can find \( \tilde{g} \in \mathcal{G} \) such that
\[
\int_\Omega \zeta \tilde{v}^2 \, dx \leq \int_\Omega \tilde{g} \tilde{v}^2 \, dx.
\]
Moreover we have
\[
\liminf_{i \to \infty} \int_\Omega (\Delta v_i)^2 \, dx \geq \int_\Omega (\Delta \tilde{v})^2 \, dx.
\]
Using the last three results we find
\[
\tilde{I} = \lim_{i \to \infty} \frac{\int_\Omega (\Delta v_i)^2 \, dx}{\int_\Omega g_i v_i^2 \, dx} \geq \frac{\int_\Omega (\Delta \tilde{v})^2 \, dx}{\int_\Omega \zeta \tilde{v}^2 \, dx} \geq \frac{\int_\Omega (\Delta \tilde{v})^2 \, dx}{\int_\Omega \tilde{g} \tilde{v}^2 \, dx}.
\]
Recall that $\tilde{v} \in H_0^2(\Omega)$ and $\tilde{g} \in G$. If $\Lambda_{\tilde{g}}$ is the (first) eigenvalue corresponding to $\tilde{g}$ in problem (2), and if $v_{\tilde{g}}$ is a corresponding eigenfunction then, using (4) we have

$$\frac{\int_{\Omega}(\Delta \tilde{v})^2 dx}{\int_{\Omega} \tilde{g} \tilde{v}^2 dx} \geq \frac{\int_{\Omega}(\Delta v_{\tilde{g}})^2 dx}{\int_{\Omega} \tilde{g} v_{\tilde{g}}^2 dx} = \Lambda_{\tilde{g}} \geq \tilde{I}. \tag{15}$$

By (14) and (15) we must have $\tilde{I} = \Lambda_{\tilde{g}}$. The theorem is proved. □

We prove the so called Euler-Lagrange equation for solutions of our minimization problems. Actually, there is a difference between the two cases. Concerning problem (1), we know that the first eigenfunction does not change sign, and we can assume that it is positive in $\Omega$. Concerning problem (2), there are domains $\Omega$ such that the corresponding first eigenfunction is sign changing, and there are domains such that the corresponding first eigenfunction is positive: see [12] and references therein.

In what follows we write $\{g(x) > 0\}$ instead of $\{x \in \Omega : g(x) > 0\}$.

**Theorem 3.2** a) Suppose $\overline{g}$ is a solution to problem (7). There exists an increasing function $\phi$ such that

$$\overline{g} = \phi(u_{\overline{g}}).$$

b) Suppose $\overline{g}$ is a solution to problem (8) and that $\Omega$ is such that the corresponding first eigenfunction of problem (2) is positive. There exists an increasing function $\varphi$ such that

$$\overline{g} = \varphi(u_{\overline{g}}).$$

**Proof.** If $u_{\overline{g}}$ is the positive normalized eigenfunction corresponding to the minimizer $\overline{g}$ of problem (7), for any $g \in G$ we have

$$\frac{\int_{\Omega}(\Delta u_{\overline{g}})^2 dx}{\int_{\Omega} \overline{g} u_{\overline{g}}^2 dx} \leq \frac{\int_{\Omega}(\Delta u_{\overline{g}})^2 dx}{\int_{\Omega} g u_{\overline{g}}^2 dx}.$$

Hence,

$$\int_{\Omega} g u_{\overline{g}}^2 dx \leq \int_{\Omega} \overline{g} u_{\overline{g}}^2 dx \tag{16}$$

for all $g \in G$.

On the other side, we know that the function $u_{\overline{g}}$ satisfies the eigenvalue equation

$$\Delta^2 u_{\overline{g}} = \lambda \overline{g} u_{\overline{g}}.$$

If $-\Delta u_{\overline{g}} = v$, by the above equation we have $-\Delta v \geq 0$ in $\Omega$ and $v = 0$ on $\partial\Omega$. It follows that $v(x) > 0$ in $\Omega$. Since $-\Delta u_{\overline{g}} \geq 0$, the function $u_{\overline{g}}$ cannot have level sets of positive measure. Hence, by Lemma 2.1, inequality (16) and Lemma 2.2 we infer the existence of an increasing function $\phi_1$ such that $\overline{g} = \phi_1(u_{\overline{g}}^2)$. Thus, part a) of the theorem follows with $\phi(t) = \phi_1(t^2)$.

If $u_{\overline{g}}$ is the positive normalized eigenfunction corresponding to the minimizer $\overline{g}$ of problem (8), inequality (16) holds for all $g \in G$. Moreover, $\Delta^2 u_{\overline{g}} = \Lambda \overline{g} u_{\overline{g}}$. By this equation, the function $u_{\overline{g}}$ cannot have level sets of positive measure on $\{\overline{g}(x) > 0\}$. If the
set \( \{ \mathcal{g}(x) = 0 \} \) has zero measure, by Lemma 2.1, inequality (16) and Lemma 2.2 we infer the existence of an increasing function \( \varphi_1 \) such that \( \mathcal{g} = \varphi_1(u_2^2) \). Thus, in this case part b) of the theorem follows with \( \varphi(t) = \varphi_1(t^2) \). Otherwise, setting \( E = \{ \mathcal{g}(x) = 0 \} \), we define:

\[
S = \sup_{x \in E} (u_2^2(x))^2.
\]

By using (16) one proves that \( (u_2^2(x))^2 \geq S \) on \( \{ \mathcal{g}(x) > 0 \} \) a.e. For the proof of this result we refer to [8], Theorem 3.2. Since \( u_2 \mathcal{g} \) cannot have level sets of positive measure on \( \Omega \setminus E \), by Lemma 2.1 we infer the existence of an increasing function \( \varphi_1 : (S, \infty) \to [0, M] \) such that \( \varphi_1(u_2^2) \) is a rearrangement of \( \mathcal{g} \) on \( \Omega \setminus E \). Now we define an increasing function \( \varphi_2 \) as

\[
\varphi_2(t) = \begin{cases} 
0 & t \leq S \\
\varphi_1(t) & t > S.
\end{cases}
\]

Since \( \varphi_2(u_2^2) \) is a rearrangement of \( \mathcal{g} \) on \( \Omega \), by inequality (16) and Lemma 2.2 we infer that \( \mathcal{g} = \varphi_2(u_2^2) \). Part b) of the theorem follows taking \( \varphi(t) = \varphi_2(t^2) \). The theorem is proved. \( \Box \)

**Remarks.** Theorem 3.2 gives some information on the location of the materials in order to minimize the first eigenvalue of problem (7). Indeed, since the associate eigenfunction \( u_2 \mathcal{g} \) vanishes on the boundary \( \partial \Omega \), and \( \mathcal{g} = \varphi(u_2) \) with \( \varphi \) increasing, the material with higher density must be located where \( u_2 \mathcal{g} \) is large, that is, far from \( \partial \Omega \). The same remark holds for problem (8) in appropriate domains.

**Theorem 3.3** Let \( B \) be a ball in \( \mathbb{R}^N \), and let \( g \) be a minimizer of either problem (7) or problem (8) with \( \Omega = B \). Then \( g = g^\sharp \).

**Proof.** If \( g \) is a minimizer of problem (7) and if \( u = u_g \) is a corresponding positive eigenfunction we have

\[
\lambda_g = \frac{\int_B (\Delta u)^2 \, dx}{\int_B g u^2 \, dx}.
\] (17)

Put

\[
-\Delta u = z.
\] (18)

Then

\[
-\Delta z = \lambda_g g u.
\]

Since \( u > 0 \) in \( B \) and \( z = 0 \) on \( \partial B \) we have \( z > 0 \) in \( B \). If \( z^\sharp \) is the Schwarz decreasing rearrangement of \( z \) then \( z^\sharp \in H^1_0(B) \) and

\[
\int_B (\Delta u)^2 \, dx = \int_B (z^\sharp)^2 \, dx.
\] (19)

Furthermore, if \( \mathcal{u} \) is the solution to the problem

\[
-\Delta \mathcal{u} = z^\sharp \text{ in } B, \quad \mathcal{u} = 0 \text{ on } \partial B
\] (20)
then, by a result of G. Talenti ([15], Theorem 1, iv)) we have

\[ \varphi^* \leq \varpi \text{ in } B. \] (21)

By a well known inequality on rearrangements and (21) we find

\[ \int_B g u^2 dx \leq \int_B g^\sharp(u^\sharp)^2 dx \leq \int_B g^\sharp(\varpi)^2 dx. \] (22)

Since \((z^\sharp)^2 = (\Delta \varpi)^2\), by (19), (17) and (22) we find

\[ \lambda_g \geq \frac{\int_B (\Delta \varpi)^2 dx}{\int_B g^\sharp(\varpi)^2 dx} \geq \frac{\int_B (\Delta u^\sharp)^2 dx}{\int_B g^\sharp(u^\sharp)^2 dx} = \lambda_g^\sharp. \]

In the last step we have used the fact that \(\varpi\) is admissible (because \(\varpi = \Delta \varpi = 0\) on \(\partial B\)) and the variational characterization of \(\lambda_g\). Since \(\lambda_g\) is a minimizer, we must have \(\lambda_g^\sharp = \lambda_g\) and equality must hold in (22). In particular,

\[ \int_B g^\sharp(u^\sharp)^2 dx = \int_B g^\sharp(\varpi)^2 dx. \]

Recalling that \(g(x)\) is positive in a set of positive measure we have \(g^\sharp(x) > 0\) in a ball \(B(r_0)\) of radius \(r_0 > 0\). Therefore the previous equation and (21) imply that \(u^\sharp(0) = \varpi(0)\). An inspection of the proof of Talenti’s result [15] (see also [2]) yields \(u^\sharp(x) = \varpi(x)\) in all of \(B\). Moreover by (18), (20) with \(\varpi = u^\sharp\), and (6) we find

\[ \int_B |\nabla u|^2 dx = \int_B uzdx \leq \int_B u^\sharp z^\sharp dx = \int_B |\nabla u^\sharp|^2 dx \leq \int_B |\nabla u|^2 dx. \]

It follows that

\[ \int_B |\nabla u|^2 dx = \int_B |\nabla u^\sharp|^2 dx. \]

By Lemma 2.4 we get \(u(x) = u^\sharp(x)\) in \(B\). Furthermore, by Theorem 3.2 a) we have \(g = \phi(u)\) for some increasing function \(\phi\). This implies that \(g\) is radially symmetric and decreasing, hence \(g = g^\sharp\). The theorem is proved in this case.

Let us come to problem (8). Putting \(-\Delta v = w\) and recalling that \(\frac{\partial v}{\partial \nu} = 0\) on \(\partial B\) we find

\[ \int_B w dx = -\int_B \Delta v dx = \int_{\partial B} \frac{\partial v}{\partial \nu} d\sigma = 0. \]

This means that \(w(x)\) is sign changing in \(B\). Let \(w^\sharp(x)\) be the signed Schwarz decreasing rearrangement of \(w(x)\) and let

\[ -\Delta \varpi = w^\sharp \text{ in } B, \quad \varpi = 0 \text{ on } \partial B. \]

Since \(\int_B w^\sharp dx = 0\) the result of Talenti [15] continues to hold as observed also in [2]. Hence,

\[ u^\sharp \leq \varpi \text{ in } B. \]

Moreover, since

\[ 0 = -\int_B w dx = -\int_B w^\sharp dx = \int_B \Delta \varpi dx = \int_{\partial B} \frac{\partial \varpi}{\partial \nu} d\sigma = \frac{\partial \varpi}{\partial \nu} |\partial B|, \]

we have \(\varpi \in H^1_0(B)\). The proof continues as in the previous case. \(\square\)
References


