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SOME ASPECTS IN N-DIMENSIONAL ALMOST PERIODIC FUNCTIONS III

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Abstract

The properties of almost periodical functions and some new results have been published in [CA1], [CA2] and [CA3] In this paper we show some new definitions in order to analize some singularities. For this functions we find some uniqueness sets in $\mathbb{R}$ and $\mathbb{R}^n$. The paper finishes analizing the relation of this functions and the function *sinc*.

Keywords: Almost periodic functions, structure theorem, Radon transform.

Resumen

Las propiedades de las funciones cuasiperiódicas y algunos resultados nuevos se han presentado en [CA1], [CA2] y [CA3]. En este artículo variamos un poco la definición para incluir cierto tipo de singularidades y encontramos para estas funciones algunos conjuntos numerables de unicidad en $\mathbb{R}$ y en $\mathbb{R}^n$. El artículo termina analizando la relación entre estas funciones y la función *sinc*.

Palabras clave: Funciones cuasiperiódicas, teorema de estructura, transformada de Radon.

Mathematics Subject Classification: 42A75,43A60,35A22,46F12.

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1 Some notations and reminders

Elementary properties of some sets of almost periodic functions have been published in [Ca], [CO], [A-P], [BO], [COR] This paper is a natural continuation of [CA1], [CA2] and [CA3]. We keep the basic notations and results.

Let us summarize some important results:
$f:\mathbb{R}^N \to \mathbb{R}$ is an almost periodic function if $\forall\varepsilon > 0$ there is a $N-$dimensional vector L whose entries are positive and satisfies that $\forall y$ in $\mathbb{R}^N$ there is an T in the $N-$dimensional box $[y, y+L]$ (component wise) such that $|f[x+T]-f[x]| < \varepsilon$ for all x in $\mathbb{R}^N$.

Let $x \in \mathbb{R}^N$, $x[[i]]$ denotes the i-th component of x. We write $x > 0$ if $x[[i]] > 0$, $i = 1, \ldots, N$.

If x, y are in $\mathbb{R}^N$ we write:

$$|x-y| := \begin{pmatrix} |x[[1]]-y[[1]]| \\ \vdots \\ |x[[N]]-y[[N]]| \end{pmatrix}.$$

In the case of the usual functions sin, cos, exp, sinc, we write: $\sin : \mathbb{R}^N \to \mathbb{R}$ as

$$\sin\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} := \sin(x_1) * \ldots * \sin(x_N)$$

and the same definition holds for the other functions. In general we extend in the multiplicative way any finite family of functions.

A set $E \subset \mathbb{R}^N$ is called relatively dense (r.d) if there is an $L \in \mathbb{R}^N$, $L > 0$ such that for all $a \in \mathbb{R}^N$, $[a, a+L] \cap E \neq \emptyset$.

There are many examples of r.d sets, for instance:

- $\mathbb{Z}$ and $p\mathbb{Z}$, wsich that $p \in \mathbb{R}$ and $p \notin \mathbb{Z}$, are r.d in $\mathbb{R}$.
- $\mathbb{Z}^N$, $p_1\mathbb{Z} \times \ldots \times p_N\mathbb{Z}$, $p_i \notin \mathbb{Z}$, $i = 1, \ldots, N$ are r.d in $\mathbb{R}^N$.
- If A is an r.d set in $\mathbb{R}^N$ and B is an r.d set in $\mathbb{R}^M$ then $A \times B$ is an r.d set in $\mathbb{R}^{N+M}$.
- If A is an r.d set in $\mathbb{R}^N$ and $\pi_i : \mathbb{R}^N \to \mathbb{R}$ is the i-th projection then $\pi_i[A]$ is an r.d set in $\mathbb{R}$.
- If $f : \mathbb{R}^N \to \mathbb{R}^N$ is an isometry then $f[A]$ is an r.d set for any A r.d set in $\mathbb{R}^N$.
- Let G in $\mathbb{R}^N$ a discrete non trivial additive subgroup then G is r.d. also $a+G$ is r.d. for all a in $\mathbb{R}^N$.

$C_b(\mathbb{R}^N, \mathbb{R})$ denotes the set of all bounded functions from $\mathbb{R}^N \to \mathbb{R}$ endowed with the norm $\|\cdot\|_\infty$

$f[x_- + m]$ denotes the function $x \to f[x+m]$, m fixed.

We use the following definition:
Let $f : \mathbb{R}^N \to \mathbb{R}$ be an almost periodic function; f is said to have Bochner compact range (BCR) if for any $N-$dimensional sequence $(x_n)_{n\in\mathbb{N}}$ there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and $x_0 \in \mathbb{R}^N$ such that $f[x_- + x_{n_k}] \to f[x_- + x_0]$ uniformly when $k \to \infty$.

We proved in those papers results like:

- Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function, f is almost periodic iff $A = \{f[x_- \pm y], y \in \mathbb{R}^N\}$ is relatively compact in $C(\mathbb{R}^N, \|\cdot\|_\infty)$.
- f is almost periodic iff for any sequence $(y_n)_{n\in\mathbb{N}}$ there is a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ and a function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f[x_- + y_{n_k}] \to g$ in $C(\mathbb{R}^N, \|\cdot\|_\infty)$.
- Let $f : \mathbb{R}^N \to \mathbb{R}$ be a uniformly continuous bounded function, $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ be a sequence such that $f[x_- + y_n] \to g[x_-]$ uniformly, and let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ be a sequence such that $x_n \to x_0$. Then $f[x_- + y_n + x_n] \to g[x_- + x_0]$.
- Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous bounded function, and let $E \subset \mathbb{R}^N$, E r.d and $\bigcup_{y\in E}\{f[x_- + y]\}$ relatively compact in $C_b(\mathbb{R}^N, \|\cdot\|_\infty)$. Then f is uniformly continuous .
- (Haraux condition) Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous bounded function, $E \subset \mathbb{R}^N$, E r.d and $\cup_{y\in E}\{f[x_- + y]\}$ relatively compact in $C_b(\mathbb{R}^N, \|\cdot\|_\infty)$, then f is almost periodic.
- Let $f : \mathbb{R}^N \to \mathbb{R}$ be an almost periodic function that it attains its maximum and minimum. Then for any sequence $(x_n)_{n\in\mathbb{N}}$ there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and $x_0 \in \mathbb{R}^N$ such that $f'[x_- + x_{n_k}] \to f[x_- + x_0]$ uniformly.
- Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic function, f is periodic if and only if f has Bochner compact range.

2 Periodic and almost periodic functions and its relations to some sets

It is well known that any non trivial additive subgroup G of $\mathbb{R}^N$ such that for all $x > 0$, there exists $g \in G$ with $0 < g < x$ (lexicographic) is dense in $\mathbb{R}^N$. From that result it follows immediately that $\{n + m * r\}$ is dense in $\mathbb{R}$ with n, m integers and r irrational. Without difficulties it is easy to prove the same result in $\mathbb{R}^N$ with n, m in $\mathbb{Z}^N$ and r in $\mathbb{R}^N, r[[i]]$ irrational for $i = 1, \dots, N$, $m * r$ denotes the componentwise multiplication. Interesting though is that from the above results it follows that:

- $\{\sin(n), n \in \mathbb{Z}\}$ and $\{\cos(n), n \in \mathbb{Z}\}$ are dense in $[-1, 1]$.
- $\{|\sin(n)|, n \in \mathbb{Z}\}$ and $\{|\cos(n)|, n \in \mathbb{Z}\}$ are dense in $[0, 1]$.
- $\{\sin(n), n \in G\}$ and $\{\cos(n), n \in G\}$ are dense in $[-1, 1]$, where G is any non trivial additive subgroup of $\mathbb{R}$ such that for all $x > 0$, there is $g \in G$ with $0 < g < x$.

The above statements can be formulated in $\mathbb{R}^N$, for example: $\{\sin(n), n \in \mathbb{Z}^N\}$ is dense in $[-1,1]$.

Definition 1 *Let G be any discreet non trivial additive group of $\mathbb{R}^N$. $L \subset \mathbb{R}^N$ is called a lattice —determined by G— if $L = G$ or there exists $a \in \mathbb{R}^N$ with $L = a + G$.*

It is easy to prove that any n-dimensional lattice is r.d.

In $\mathbb{R}$ a lattice G has the form: $G = a + p\mathbb{Z}$, for a, p in $\mathbb{R}$.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two periodic, non trivial, continuous functions, then f/g is a continuous function except for a lattice L, $L = \{x \in \mathbb{R}/g(x) = 0\}$.

If f, g have measurable periods T_1, T_2, then f/g is periodic$-$measurable means $T_1/T_2 \in \mathbb{Q}-$.

If f, g have no measurable periods then f/g is almost almost periodic (a.a.p). Here, non measurable means $T_1/T_2 \notin \mathbb{Q}-$.

Let $A_p := \{g : \mathbb{R} \to \mathbb{R},\ g$ continuous of period $p\}$.

Theorem 1 *If p in $\mathbb{R}$ is an irrational number then $\mathbb{Z}$ is a uniqueness set for A_p.*

Proof: $B = \{n + m * p/n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Then $f(x = n + m * p) = f(n)$ for all $n, m \in \mathbb{Z}$. ■

Theorem 2 *Let $f \in A_p$, with a uniqueness set E, then $f(x_{\_} + z) \in A_p$ for all $z \in \mathbb{R}$ with the same uniqueness set E.*

As a matter of fact sometimes if $f \in A_p$, f an odd function, there is $z \in \mathbb{R}$ with $f(x_{\_} + z)$ an even function.

Some examples are:

- $\sin(x_{\_})$ and $z = \pi/2$;
- $\sum_{k=0}^{p} a_k \sin((2k+1)x)$ and $z = \pi/2$, $a_k \in \mathbb{R}$, $k = 0, \ldots, p$.
- For the odd function: $\sin(x_{\_}) + \sin(2x_{\_}) + \sin(3x_{\_}) + \sin(4x_{\_})$ there is not such a z.

Some graphics illustrate this situation in Figures 1, 2 and 3.

Theorem 3 *If we take in consideration in A_p only the even functions we obtain that $\mathbb{N}_0$ is a uniqueness set for this class of functions.*

As examples we have:

- $\{\sin(n), n \in \mathbb{N}_0\}$ is dense in $[-1,1]$.
- $\{\cos(n), n \in \mathbb{N}_0\}$ is dense in $[-1,1]$.
- $\{|\sin(n)|, n \in \mathbb{N}_0\}$ is dense in $[0,1]$.
- $\{|\cos(n)|, n \in \mathbb{N}_0\}$ is dense in $[0,1]$.

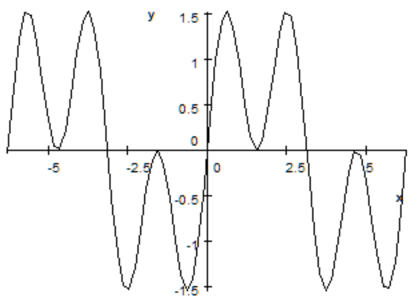

Figure 1: $\sin(x) + \sin(3 * x)$.

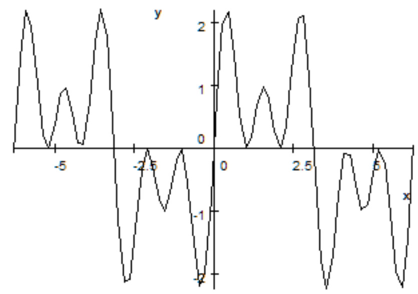

Figure 2. $\sin(x) + \sin(3 * x) + \sin(5 * x)$.

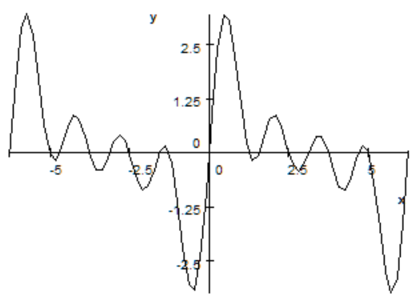

Figure 3: $\sin(x) + \sin(2 * x) + \sin(3 * x) + \sin(4 * x)$.

In the case $p \in Q$ we get:

Theorem 4 *If p in $\mathbb{R}$ is a rational number then $\mathbb{Z}r$, r irrational, is a uniqueness set for A_p.*

$\mathbb{Z}$ and $\mathbb{Z}r$ are lattices. We may summarizes the result as: let f be a continuous function of period p then there is a lattice L which is a uniqueness set for A_p.

This statement can be extended to the set of functions: $B_p := \{f/g | f, g \in A_p\}$. There are discontinuous functions on this set.

We introduce now the sets:

$$AP_p := \{f : \mathbb{R} \to \mathbb{R} | f \text{ almost periodic } \}$$

and the set of a.a. functions BB_p,

$$BB_p := \{f/g | f, g \in AP_p\}.$$

Actually, those sets are vector spaces over $\mathbb{R}$

For instance we get: $\{tan(n), n \in \mathbb{N}_0\}$ is dense in $\mathbb{R}$.

In the n-dimensional case there are several definitions of the concept of periodic function, but we work with the R-periodic concept: $f : \mathbb{R}^N \to \mathbb{R}$ is an R-periodic function if there are N linearly independent vectors e_k, $k = 1, \ldots, N$ such that: $f(x + e_k) = f(x)$, $\forall x \in \mathbb{R}^N$. The vectors e_k $k = 1, \ldots, N$ are called periods of f.

We get that if f is R-periodic and all the e_k in the definition are irrational then $\sum_{k=1}^{N} \mathbb{Z}e_k$ is an uniqueness set for the set of functions: $A_{e_i,\ldots,e_N} := \{f : \mathbb{R}^N \to \mathbb{R}$ is a continuous R-periodic function, with periods e_k, $k = 1, \ldots, N\}$ and for $B_{e_i,\ldots,e_N} := \{f/g|, f, g \in A_{e_i,\ldots,e_N}\}$; of course there are discontinuous functions on this set.

We have an inmediate generalization of Theorem 2.

Theorem 5 *Let $f \in A_{e_i,\ldots,e_N}$ with a uniqueness set E, then $f(x_- + z) \in A_{e_i,\ldots,e_N}$ for all $z \in \mathbb{R}^N$ with the same uniqueness set E.*

Theorem 6 *Let $f \in A_{e_i,\ldots,e_N}$ then there exists a lattice L such that L is a uniqueness set of $A_{e_i,\ldots,e_N}$.*

3 The relation between $sinc$ and A_p, B_p, AP_p, and BB_p

Theorem 7 *Let L be a numerable uniqueness lattice of a function f in A_p or AP_p, $L = \mathbb{Z}h$. Then $\sum_{k\in L} f(kh)\mathrm{sinc}(\frac{\pi}{h}(x - k))$ is convergent toward f. When $f \in A_p$ this convergence is uniform. When $f \in AP_p$ this convergence is uniform when restricted to compact sets. Over $\mathbb{R}^N$ it holds the same result.*

Proof: A detailed proof will appear elsewhere.
In an schematic way we proceed as follows: We associate to f a function $f_c \in C_c(\mathbb{R})$ and apply the Fourier band limited theory and Wiener-Paley like theorem.

A point wise proof in one variable is: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period π, let us consider the case f even.

Let $a_n(x_-) := f(n)\mathrm{sinc}(\pi(x - n)) + f(-n)\mathrm{sinc}(\pi(x + n))$, $n \in \mathbb{N}$, then $a_n(x_-) = (-1)^n 2\frac{f(n)}{\pi}\sin(\pi x)\frac{x}{x^2-n^2}$ from this follows the convergence over compact sets of $\sum_{n=0}^{\infty} a_n(x_-)$ toward a function g. It follows immediately that $g(n) = f(n)$ for all $n \in \mathbb{Z}$ then $f = g$.

In the odd case we have: $a_n(x_-) := f(n)\mathrm{sinc}(\pi(x - n)) + f(-n)\mathrm{sinc}(\pi(x + n))$, $n \in \mathbb{N}$, then: $a_n(x_-) = (-1)^n 2\frac{f(n)}{\pi}\sin(\pi x)\frac{n}{x^2-n^2}$ from this follows the point wise convergence.

In the general case of a continuous periodic function f of period π we get that: $f(x_-) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$, $\frac{f(x)+f(-x)}{2}$ is an even periodic function and $\frac{f(x)-f(-x)}{2}$ is an odd periodic function, by using the preceding method we get the result. The choice of the period π is irrelevant, the same with respect to the choice of the lattice $\mathbb{Z}$. ■

At this moment we do not know what happens to $\sum_{k\in L=\mathbb{Z}*p} f(kp)\mathrm{sinc}(\frac{\pi}{p}(x-k))$ when f belongs to B_p or BB_p.

However, it is that a function f in BB_p has not necessarily the property that for any sequence $(x_n) \in \mathbb{R}$ there is a subsequence (x_{n_k}) such that $f(x_- + x_{n_k}) \to g$.

An easy counterexample is: $f(x\_) := \frac{\sin(\sqrt{2}x)}{\sin(x)}$.

We define: $x_1 = \lfloor 2\pi \rfloor$, $x_2 = \lfloor 2 * 2\pi \rfloor + 0.d_1$,...,$x_n = \lfloor n * 2\pi \rfloor + 0.d_1 \dots d_{n-1}$, where $0.d_1 \dots d_{n-1}$ denotes the $n-1$ decimal expansion of the number $n * 2\pi$.

4 Some graphical examples

Let us see the graphics in the interval $[-2\pi, 2\pi]$.

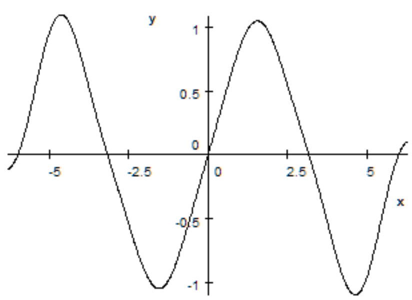

Figure 4: $\displaystyle\sum_{k=-5}^{5} \frac{\sin(k) * \sin(\pi * (x-k))}{\pi * (x-k)}$.

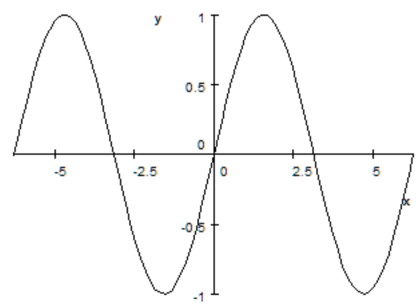

Figure 5. $\sin(x)$.

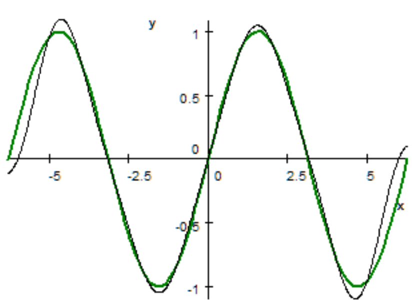

Figure 6: $\displaystyle\sum_{k=-5}^{5} \frac{\sin(k) * \sin(\pi * (x-k))}{\pi * (x-k)}$.

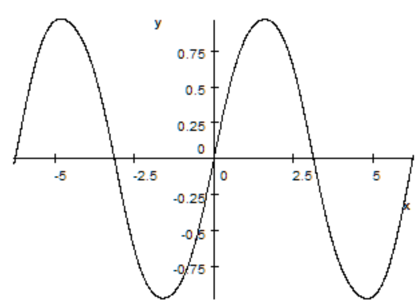

Figure 7. $\displaystyle\sum_{k=-10}^{10} \frac{\sin(k) * \sin(\pi * (x-k))}{\pi * (x-k)}$.

See the case of the tangent in $(-\pi/2, \pi/2)$ in Figure 10.

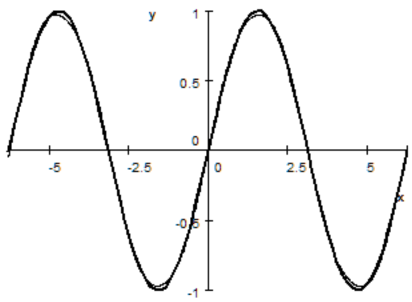

Figure 8: $\displaystyle\sum_{k=-10}^{10} \frac{\sin(k) * \sin(\pi * (x-k))}{(\pi * (x-k)}$.

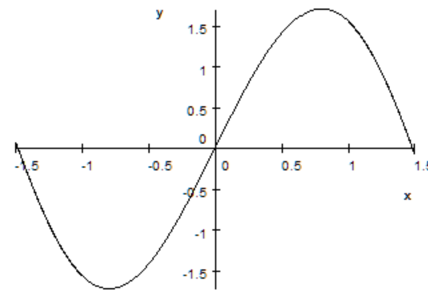

Figure 9. $\displaystyle\sum_{k=-5}^{5} \frac{\tan(k) * \sin(\pi * (x-k))}{(\pi * (x-k)}$.

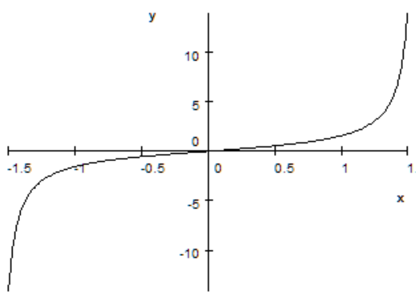

Figure 10: $\tan(x)$.

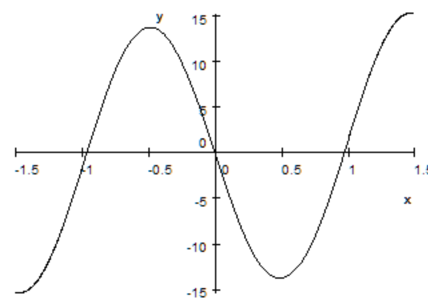

Figure 11. $\displaystyle\sum_{k=-100}^{100} \frac{\tan(k) * \sin(\pi * (x-k))}{\pi * (x-k)}$.

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