RELATIONS OF K-TH DERIVATIVE OF DIRAC DELTA IN HYPERCONE WITH ULTRAHYPERBOLIC OPERATOR

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Abstract

In this paper we prove that the generalized functions $\delta^{(k)}(P_+) - \delta^{(k)}(P)$, $\delta^{(k)}(P_-) - \delta^{(k)}(-P)$ and $\delta^{(k)}(P) - \delta^{(k)}(P)$ are concentrated in the vertex of the cone $P = 0$ and we find their relationship with the ultrahyperbolic operator iterated $(k+1 - \frac{n}{2})$ times under condition $k \geq \frac{n}{2} - 1$.

Keywords: distributions, generalized functions, distributions spaces, properties of distributions.

1 Introduction

Let $x = (x_1, x_2, \cdots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$.

Consider a quadratic form in n variables defined by

$$P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$ (1)
where \( p + q = n \) is the dimension of the space.

We call \( \varphi(x) \) the \( C^\infty \) functions with compact support defined from \( \mathbb{R}^n \) to \( \mathbb{R} \) ([2], page 4).

From [1], page 253, formula (2), the distribution \( P_+^\lambda \) is defined by

\[
\left( P_+^\lambda, \varphi \right) = \int_{\mathbb{R}^n_+} (P(x))^\lambda \varphi(x) dx
\]  

(2)

where \( \lambda \) is a complex number and \( dx = dx_1 dx_2 \ldots dx_n \). For \( \text{Real}(\lambda) \geq 0 \), this integral converges and is analytic function of \( \lambda \). Analytic continuation to \( \text{Real}(\lambda) < 0 \) can be used to extend the definition of \( (P_+^\lambda, \varphi) \). Further from [1], page 254, we have,

\[
\left( P_+^\lambda, \varphi \right) = \int_0^\infty v^{\lambda + \frac{p+q}{2} - 1} \Phi_\lambda(v) dv
\]  

(3)

where

\[
\Phi_\lambda(u) = \frac{1}{4} \int_0^\infty \frac{t^{p-1}}{(1-t)^q} \phi_1(u, tu) dt
\]  

(4)

\[
\phi(r, s) = \phi_1(u, v)
\]  

(5)

\[
\phi(r, s) = \int \varphi d\Omega_p d\Omega_q.
\]  

(6)

\[
r = \sqrt{x_1^2 + \cdots + x_p^2},
\]  

(7)

\[
s = \sqrt{x_{p+1}^2 + \cdots + x_{p+q}^2},
\]  

(8)

\( d\Omega_p \) and \( d\Omega_q \) are elements of surface are on the unit sphere in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively.

Similarly we can also defined the generalized \( P_-^\lambda \) by

\[
\left( P_-^\lambda, \varphi \right) = \int_{\mathbb{R}^n_-} (-P(x))^\lambda \varphi(x) dx.
\]  

(9)

Further we obtain

\[
\left( P_-^\lambda, \varphi \right) = \int_0^\infty v^{\lambda + \frac{p+q}{2} - 1} \Phi_\lambda(v) dv
\]  

(10)

where

\[
p\Phi_\lambda(u) = \frac{1}{4} \int_0^\infty \frac{t^{p-1}}{(1-t)^q} \phi_1(vt, v) dt.
\]  

(11)

From (1) the \( P = 0 \) hypersurface is a hypercone with a singular point (the vertex) at the origin.

On the other hand, from [1], page 249, we have,

\[
\left( \delta^{(k)}(P), \varphi \right) = \int_0^\infty \left[ \left( \frac{\partial}{2s \partial s} \right)^k \left\{ s^{q-2} \phi(r, s) \right\} \right]_{s=r} r^{p-1} dr
\]  

(12)
and
\[
\left( \delta^{(k)}(P), \varphi \right) = (-1)^k \int_0^\infty \left[ \left( \frac{\partial}{2r \partial r} \right)^k \left\{ r^{p-2} \frac{\phi(r,s)}{2} \right\} \right]_{r=s} s^{q-1} ds \tag{13}
\]
where \( \phi(r,s) \) is defined by the equation (6).

Also from [1], page 250, the generalized functions \( \delta_1^{(k)}(P) \) and \( \delta_2^{(k)}(P) \) are defined by
\[
\left( \delta_1^{(k)}(P), \varphi \right) = \int_0^\infty \left[ \left( \frac{\partial}{2s \partial s} \right)^k \left\{ s^{q-2} \frac{\phi(r,s)}{2} \right\} \right]_{s=r} r^{p-1} dr \tag{14}
\]
and
\[
\left( \delta_2^{(k)}(P), \varphi \right) = (-1)^k \int_0^\infty \left[ \left( \frac{\partial}{2r \partial r} \right)^k \left\{ r^{p-2} \frac{\phi(r,s)}{2} \right\} \right]_{r=s} s^{q-1} ds \tag{15}
\]
where \( \phi(r,s) \) is \( r^{1-p} s^{1-q} \) multiplied by the integral of \( \varphi \) over the surface \( x_1^2 + x_2^2 + \cdots + x_p^2 = r^2 \) and \( x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2 = s^2 \).

The integrals converges and coincide for
\[
k < \frac{p + q - 2}{2}. \tag{16}
\]

If, on the other hand,
\[
k \geq \frac{p + q - 2}{2}, \tag{17}
\]
these integrals must be understood in the sense of their regularization (see [1], page 250).

Now in general \( \delta_1^{(k)}(P) \) and \( \delta_2^{(k)}(P) \) may not be the same generalized function.

Note that the definition of these generalized functions implies that in any case
\[
\delta_2^{(k)}(P) = (-1)^k \delta_1^{(k)}(-P). \tag{18}
\]

From [1], page 278, the following formulae are valid,
\[
\delta^{(k)}(P_+) = (-1)^k k! \mathcal{R} |s_{\lambda=-k-1} P_+^\lambda \tag{19}
\]
and
\[
\delta^{(k)}(P_-) = (-1)^k k! \mathcal{R} |s_{\lambda=-k-1} P_-^\lambda. \tag{20}
\]

On the other hand, from [1], page 278, for odd \( n \), as well as for even \( n \) and \( k < \frac{n}{2} - 1 \) we have,
\[
\delta^{(k)}(P_+) = \delta_1^{(k)}(P) = \delta^{(k)}(P) \tag{21}
\]
and
\[
\delta^{(k)}(P_-) = \delta_1^{(k)}(-P). \tag{22}
\]

While in the case of even dimension and \( k \geq \frac{n}{2} - 1 \),
\[
\delta^{(k)}(P_+) - \delta_1^{(k)}(P) \tag{23}
\]
and
\[ \delta^{(k)}(P_-) = \delta^{(k)}_1(-P) \] (24)
are generalized functions concentrated at the vertex of the \( P = 0 \) cone ([1], page 279).

From [1], page 279 we have:

If \( p \) and \( q \) are both even and if \( k \geq \frac{n}{2} - 1 \), then
\[ (-1)^k \delta^{(k)}(P_+) - \delta^{(k)}(P_-) = a_{q,n,k} L^{k+1} \{ \delta(x) \} \] (25)
while in all other cases
\[ \delta^{(k)}(P_-) = (-1)^k \delta^{(k)}(P_+) . \] (26)

In (25)
\[ a_{q,n,k} = \frac{(-1)^{\frac{n}{2}} \pi^{\frac{n}{2}}}{4^{k+1} (k - \frac{n}{2} + 1)!} \] (27)
and \( L^j \) is a linear homogeneous differential operation iterated \( j \) times defined by the following formula
\[ L^j = \left\{ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right\}^j. \] (28)

The operator \( L = \left\{ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right\} \) is often called ultrahyperbolic.

From [1], page 255, \( (P^\lambda_+, \varphi) \) has two sets of singularities namely
\[ \lambda = -1, -2, -3, \ldots \] (29)
and
\[ \lambda = -\frac{n}{2}, -\frac{n}{2} - 1, \ldots \] (30)
and from [1], pages 256-269 and page 352 we have ([4], page 139, formula (2.27)):
\[ \mathcal{R}|_{s=\lambda=\frac{n}{2}+k} P^\lambda_+ = \frac{(-1)^k}{k!} \delta^{(k)}_1(P) \text{ if } p \text{ is even and } q \text{ odd}, \] (31)
\[ \mathcal{R}|_{s=\lambda=\frac{n}{2}+k} P^\lambda_+ = \frac{(-1)^k}{k!} \delta^{(k)}_1(P) \text{ if } p \text{ is odd and } q \text{ even}, \] (32)
\[ \mathcal{R}|_{s=\lambda=-\frac{n}{2}+k} P^\lambda_+ = 0 \text{ if } p \text{ is even and } q \text{ odd} \] (33)
and
\[ \mathcal{R}|_{s=\lambda=-\frac{n}{2}+k} P^\lambda_+ = \frac{(-1)^{\frac{n}{2}} \pi^{\frac{n}{2}} L^k}{4^n k! (\frac{n}{2} + k)} \{ \delta(x) \} \text{ if } p \text{ is odd and } q \text{ even}. \] (34)
where \( L^k \) is defined by the formula (28).

Similarly \( (P^\lambda_+, \varphi) \) has singularities in the same points that \( (P^\lambda_+, \varphi) \) and taking into account all that we have above about \( P^\lambda_+ \) remains true also for \( P^\lambda_- \) except that \( p \) and \( q \) must interchanged, and in all the formulae \( \delta^{(k)}_1(P) \) must be replaced by
\[ \delta^{(k)}_1(-P) = (-1)^k \delta^{(k)}_2(P) \] (35)
and \((L)\) by \((-L)\) (see ([1]), pages 279 and 352) we have,

\[
\mathcal{R}[s_{\lambda=-k-1}P_{-}^\lambda] = \frac{(-1)^k}{k!}\delta_1^{(k)}(-P) \quad \text{if } p \text{ is odd and } q \text{ even},
\]

\[
\mathcal{R}[s_{\lambda=-k-1}P_{+}^\lambda] = \frac{(-1)^k}{k!}\delta_1^{(k)}(-P) \quad \text{if } p \text{ is even and } q \text{ odd},
\]

\[
\mathcal{R}[s_{\lambda=-\frac{k}{2}-k}P_{-}^\lambda] = 0 \quad \text{if } p \text{ is odd and } q \text{ even}
\]

and

\[
\mathcal{R}[s_{\lambda=-\frac{\lambda}{2}-k}P_{+}^\lambda] = \frac{(-1)^{\frac{\lambda}{2}+k-1}}{4^{k\frac{n}{2}}\Gamma(\frac{n}{2}+k)}(-L)^k \{\delta(x)\} \quad \text{if } p \text{ is even and } q \text{ odd}.
\]

If the dimension \(n\) of the space is even and \(p\) and \(q\) are even, \(P_{+}^\lambda\) has simple poles at \(\lambda = -\frac{n}{2} - k\), where \(k\) is a non-negative integer, and the residues are given by ([1], p.268 and [4], p.141)

\[
\mathcal{R}[s_{\lambda=-\frac{n}{2}-k, k=0,2,\ldots}P_{+}^\lambda] = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(P) + \\
\frac{(-1)^{\frac{n}{2}+k-1}}{4^{k\frac{n}{2}}\Gamma(\frac{n}{2}+k)}L^k \{\delta(x)\},
\]

where \(L^k\) is defined by (28).

If, on the other hand, \(p\) and \(q\) are odd, \(P_{+}^\lambda\) has pole of order 2 at \(\lambda = -\frac{n}{2} - k\) and from [1], p.269 and [4], p.143, we have

\[
\mathcal{R}[s_{\lambda=-\frac{n}{2}-k}P_{+}^\lambda] = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(P) + \\
\frac{(-1)^{\frac{n}{2}+k-1}}{2^{2k}\Gamma(\frac{n}{2}+k)}\left[\psi\left(\frac{n}{2}\right) - \psi\left(\frac{n}{2}\right)^{k\frac{n}{2}}\right] \cdot L^k \{\delta(x)\},
\]

where

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.
\]

and \(\Gamma(x)\) is the function gamma defined by

\[
\Gamma(x) = \int_0^\infty e^{-z}z^{x-1}dz.
\]

([3], Vol.I, p.344).

For integral and half-integral values of the argument, \(\psi(x)\) is given by

\[
\psi(k) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k-1},
\]

\[
\psi\left(k + \frac{1}{2}\right) = -\gamma - 2\ln(2) + 2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right),
\]

where \(\gamma\) is Euler’s constant.
Similarly
\[ \mathcal{R}[s_{\lambda = -\frac{\pi}{2} - k}] = \frac{(-1)^{\frac{n}{2} + k - 1}}{\Gamma(\frac{n}{2} + k)} \delta_1^{\frac{n}{2} + k - 1}(-P) + \frac{(-1)^{\frac{n}{2} + \frac{p}{2} \pi + \frac{q}{2}}}{4^k k! \Gamma(\frac{n}{2} + k)} (-L)^k \{ \delta(x) \} \] (47)
if \( p \) and \( q \) are even, and
\[ \mathcal{R}[s_{\lambda = -\frac{\pi}{2} - k}] = \frac{(-1)^{\frac{n}{2} + k - 1}}{\Gamma(\frac{n}{2} + k)} \delta_1^{\frac{n}{2} + k - 1}(-P) + \frac{(-1)^{\frac{n}{2} + 1} \pi^{\frac{n}{2} - 1}}{2^{2k} k! \Gamma(\frac{n}{2} + k)} \left[ \psi\left(\frac{q}{2}\right) - \psi\left(\frac{n}{2}\right) \right] (-L)^k \{ \delta(x) \} \] (48)
if \( p \) and \( q \) are odd.

2 Relations of \( k \)-th derivative of Dirac delta in hypercone with ultrahyperbolic operator

In this paragraph we prove that generalized functions \( \delta^{(k)}(P_+) - \delta^{(k)}_1(P) \) and \( \delta^{(k)}(P_-) - \delta^{(k)}_1(-P) \) are concentrated in the vertex of the cone \( P = 0 \).

**Theorem 1** Let \( k \) be non-negative integer and \( n \) even dimension of the space then the following formulae are valid,
\[ \delta^{(k)}(P_+) - \delta^{(k)}_1(P) = B_{k,p,q} L^{k - \frac{n}{2} + 1} \text{ if } k \geq \frac{n}{2} - 1 \] (49)
where
\[ B_{k,p,q} = \frac{(-1)^k (-1)^{\frac{n}{2} \pi + \frac{p}{2}}}{4^{k - \frac{n}{2} + 1} (k - \frac{n}{2} - 1)!} \text{ for } p \text{ and } q \text{ are both even}, \] (50)
and
\[ B_{k,p,q} = \frac{(-1)^k (-1)^{\frac{n}{2} \pi + \frac{p}{2} - 1}}{4^{k - \frac{n}{2} + 1} (k - \frac{n}{2} + 1)!}. \] (51)

PROOF: From (41), (47) and considering the formulae (19) and (20) under conditions \( k \geq \frac{n}{2} - 1 \), and when \( p \) and \( q \) are even, we have
\[ \delta^{(k)}(P_+) - \delta^{(k)}_1(P) = (-1)^k a_{q,n,k} L^{k - \frac{n}{2} + 1} \{ \delta(x) \}. \] (52)
where \( a_{q,n,k} \) is defined by (27).

Similarly from (42), (48) and considering the formulae (19) and (20) under conditions \( k \geq \frac{n}{2} - 1 \), and when \( p \) and \( q \) are odd, we have
\[ \delta^{(k)}(P_+) - \delta^{(k)}_1(P) = \frac{(-1)^k (-1)^{\frac{n}{2} \pi + \frac{p}{2} - 1}}{4^{k - \frac{n}{2} + 1} (k - \frac{n}{2} + 1)!}. \] (53)

\[ \left[ \psi\left(\frac{q}{2}\right) - \psi\left(\frac{n}{2}\right) \right] L^{k - \frac{n}{2} + 1} \{ \delta(x) \} \text{ for } p \text{ and } q \text{ are both odd.} \]
From (52) and (53) we obtain the formula (49),(50) and (51) which proves the theorem.

The formula (49) represent a relation between \( \delta^{(k)}(P_+) - \delta_1^{(k)}(-P) \) and the ultrahyperbolic operator iterated \( k - \frac{n}{2} + 1 \) times under condition \( k \geq \frac{n}{2} - 1 \).

**Theorem 2** Let \( k \) be non-negative integer and \( n \) even dimension of the space, then the following formulae are valid:

\[
\delta^{(k)}(P_+) - \delta_1^{(k)}(-P) = D_{k,p,q} L^{k-\frac{n}{2}+1} \{ \delta(x) \}
\]

where

\[
D_{k,p,q} = \frac{(-1)^{\frac{n}{2}+1} \pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2} + 1)!}
\]

and

\[
\left[ \psi(\frac{a}{2}) - \psi(\frac{\pi}{2}) \right] . L^{k-\frac{n}{2}+1} \{ \delta(x) \} \text{ for } p \text{ and } q \text{ are both odd}
\]

**Proof:** From (41),(47) and considering the formulae (19) and (20) under conditions \( k \geq \frac{n}{2} - 1 \), and when \( p \) and \( q \) are even, we have:

\[
\delta^{(k)}(P_+) - \delta_1^{(k)}(-P) = (-1)^{a_{q,n,k}} L^{k-\frac{n}{2}+1} \{ \delta(x) \}
\]

where \( a_{q,n,k} \) is defined by (27)

Similarly from (42), (48) and considering the formulae (19) and (20) under conditions \( k \geq \frac{n}{2} - 1 \), and when \( p \) and \( q \) are odd, we have:

\[
\delta^{(k)}(P_+) - \delta_1^{(k)}(-P) = \frac{(-1)^{\frac{n}{2}+1} \pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2} + 1)!}
\]

\[
\left[ \psi(\frac{a}{2}) - \psi(\frac{\pi}{2}) \right] . L^{k-\frac{n}{2}+1} \{ \delta(x) \} \text{ for } p \text{ and } q \text{ are both odd}
\]

From the formulae (57) and (58) we obtain the formulae (54),(55) and (56) which proves the theorem.

The formula (54) represents a relation between \( \delta^{(k)}(P_+) - \delta_1^{(k)}(-P) \) with the ultrahyperbolic operator iterated \( k - \frac{n}{2} + 1 \) times under condition \( k \geq \frac{n}{2} - 1 \).

**Theorem 3** Let \( k \) be non-negative integer and \( n \) even dimension of the space then the following formulae are valid:

\[
\delta_1^{(k)}(P) - \delta_2^{(k)}(P) = A_{k,p,q} L^{k-\frac{n}{2}+1} \{ \delta(x) \}
\]

where

\[
A_{k,p,q} = \frac{(-1)^{k}(\frac{n}{2}+1) \pi^{\frac{n}{2}+1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2} + 1)!} \text{ for } p \text{ and } q \text{ are both even}
\]
and

\[ D_{k,p,q} = \frac{(-1)^{\frac{k+1}{2}} \pi^{\frac{k}{2} - 1}}{4^{k - \frac{k}{2} + 1} (k - \frac{k}{2} + 1)!}. \]  

(61)

\[ \left[ \psi\left(\frac{\pi}{2}\right) - \psi\left(\frac{\pi}{2}\right) \right] . L^{k - \frac{k}{2} + 1} \{ \delta(x) \} \] for \( p \) and \( q \) are both odd

**Proof:** From (49) and (54) using (25), (50) and (60) under conditions \( k \geq \frac{n}{2} - 1 \), and when \( p \) and \( q \) are even, we have,

\[
\frac{(-1)^{k}(-1)^{\frac{k}{2}} \pi^{\frac{k}{2} - 1}}{4^{k - \frac{k}{2} + 1} (k - \frac{k}{2} + 1)!} L^{k - \frac{k}{2} + 1} \{ \delta(x) \} = \delta^{(k)}(P_{+}) - (-1)^{k} \delta^{(k)}(P_{-}) = \\
\delta_{1}^{(k)}(P) - \delta_{2}^{(k)}(P) + \frac{(-1)^{k}(-1)^{\frac{k}{2}} \pi^{\frac{k}{2}}}{4^{k - \frac{k}{2} + 1} (k - \frac{k}{2} + 1)!} L^{k - \frac{k}{2} + 1} \{ \delta(x) \} +
\]

(62)

therefore

\[
\delta_{1}^{(k)}(P) - \delta_{2}^{(k)}(P) = \frac{(-1)^{k}(-1)^{\frac{k}{2}} \pi^{\frac{k}{2}}}{4^{k - \frac{k}{2} + 1} (k - \frac{k}{2} + 1)!} L^{k - \frac{k}{2} + 1} \{ \delta(x) \} .
\]

(63)

Similarly from (49) and (54) using (26), (51) and (56) under conditions \( k \geq \frac{n}{2} - 1 \), and when \( p \) and \( q \) are odd, we have,

\[
\delta_{1}^{(k)}(P) - \delta_{2}^{(k)}(P) = \delta^{(k)}(P_{+}) - (-1)^{k} \delta^{(k)}(P_{-}) + \\
+ \frac{(-1)^{k}(-1)^{\frac{k+1}{2}} \pi^{\frac{k}{2} - 1}}{4^{k - \frac{k}{2} + 1} (k - \frac{k}{2} + 1)!} \left[ \psi\left(\frac{\pi}{2}\right) - \psi\left(\frac{\pi}{2}\right) + \psi\left(\frac{\pi}{2}\right) - \psi\left(\frac{\pi}{2}\right) \right] . L^{k - \frac{k}{2} + 1} \{ \delta(x) \} =
\]

(64)

\[
= \frac{(-1)^{k}(-1)^{\frac{k+1}{2}} \pi^{\frac{k}{2} - 1}}{4^{k - \frac{k}{2} + 1} (k - \frac{k}{2} + 1)!} \left[ \psi\left(\frac{\pi}{2}\right) - \psi\left(\frac{\pi}{2}\right) \right] . L^{k - \frac{k}{2} + 1} \{ \delta(x) \}
\]

From the formulae (63) and (64) we obtain the formulae (59), (60) and (61) which proves the theorem. 

The formula (59) represent a relation between \( \delta_{1}^{(k)}(P) - \delta_{2}^{(k)}(P) \) with the ultrahyperbolic operator iterated \( k - \frac{k}{2} + 1 \) times under condition \( k \geq \frac{n}{2} - 1 \).

**References**

