## Contents

1 Vector bundles and their classification 4
   1.1 Generalities on manifolds 4
   1.2 Principal bundles and vector bundles 5
   1.3 Hermitian vector bundles 7
   1.4 Operations on vector bundles 7
   1.5 Equivalent bundles 8
   1.6 Sections of vector bundles 9
   1.7 Local sections and transition functions 10
   1.8 Čech cocycles 12
   1.9 The Čech cohomology of $S^2$ 13
   1.10 Line bundle classification 14
   1.11 The second cohomology group 15
   1.12 Classification of Hermitian line bundles 16
   1.13 Classification of vector bundles 17

2 Complex projective spaces 19
   2.1 Complex manifolds 19
   2.2 Local charts for complex projective spaces 20
   2.3 The Kähler form 21
   2.4 The Fubini–Study metric 22
   2.5 The Riemann sphere 24

3 The de Rham complex and Hodge duality 24
   3.1 The de Rham complex 24
   3.2 The Riemannian volume form 25
   3.3 The Hodge star operator 26
   3.4 The Hodge Laplacian 31
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>The Hodge Laplacian on the 2-sphere</td>
<td>33</td>
</tr>
<tr>
<td>4.1</td>
<td>The rotation group in three dimensions</td>
<td>33</td>
</tr>
<tr>
<td>4.2</td>
<td>The Hodge operators on the sphere</td>
<td>35</td>
</tr>
<tr>
<td>4.3</td>
<td>Eigenvectors for the Laplacian</td>
<td>36</td>
</tr>
<tr>
<td>4.4</td>
<td>Spectrum of the Hodge Laplacian</td>
<td>38</td>
</tr>
<tr>
<td>4.5</td>
<td>The Hodge–Dirac operator</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>Connections on vector bundles</td>
<td>42</td>
</tr>
<tr>
<td>5.1</td>
<td>Modules of vector-valued forms</td>
<td>42</td>
</tr>
<tr>
<td>5.2</td>
<td>Connections</td>
<td>45</td>
</tr>
<tr>
<td>5.3</td>
<td>Curvature of a connection</td>
<td>46</td>
</tr>
<tr>
<td>5.4</td>
<td>A curvature formula</td>
<td>47</td>
</tr>
<tr>
<td>5.5</td>
<td>From de Rham cohomology to Čech cohomology</td>
<td>48</td>
</tr>
<tr>
<td>5.6</td>
<td>Line bundles over $\mathbb{C}P^n$</td>
<td>51</td>
</tr>
<tr>
<td>5.7</td>
<td>Connections on the hyperplane bundle</td>
<td>53</td>
</tr>
<tr>
<td>5.8</td>
<td>Characteristic classes</td>
<td>54</td>
</tr>
<tr>
<td>5.9</td>
<td>Chern classes and the Chern character</td>
<td>57</td>
</tr>
<tr>
<td>5.10</td>
<td>Classification of line bundles over $\mathbb{C}P^n$</td>
<td>60</td>
</tr>
<tr>
<td>5.11</td>
<td>The Levi-Civita connection on the tangent bundle</td>
<td>61</td>
</tr>
<tr>
<td>6</td>
<td>Clifford algebras</td>
<td>62</td>
</tr>
<tr>
<td>6.1</td>
<td>Superspaces and superalgebras</td>
<td>62</td>
</tr>
<tr>
<td>6.2</td>
<td>Clifford algebras</td>
<td>65</td>
</tr>
<tr>
<td>6.3</td>
<td>Clifford actions</td>
<td>66</td>
</tr>
<tr>
<td>6.4</td>
<td>Complex Clifford algebras</td>
<td>68</td>
</tr>
<tr>
<td>6.5</td>
<td>The Fock space of spinors</td>
<td>70</td>
</tr>
<tr>
<td>6.6</td>
<td>The Pin and Spin groups</td>
<td>72</td>
</tr>
<tr>
<td>6.7</td>
<td>The spin representation</td>
<td>76</td>
</tr>
<tr>
<td>7</td>
<td>Global Clifford modules</td>
<td>77</td>
</tr>
<tr>
<td>7.1</td>
<td>Clifford algebra bundles</td>
<td>77</td>
</tr>
<tr>
<td>7.2</td>
<td>Existence of spin structures</td>
<td>80</td>
</tr>
<tr>
<td>7.3</td>
<td>Spin$^c$ structures</td>
<td>83</td>
</tr>
<tr>
<td>7.4</td>
<td>The spinor module</td>
<td>85</td>
</tr>
<tr>
<td>7.5</td>
<td>The spin connection</td>
<td>86</td>
</tr>
<tr>
<td>7.6</td>
<td>Local coordinate formulas</td>
<td>89</td>
</tr>
<tr>
<td>8</td>
<td>Dirac operators and Laplacians</td>
<td>91</td>
</tr>
<tr>
<td>8.1</td>
<td>Connections and differential forms</td>
<td>91</td>
</tr>
<tr>
<td>8.2</td>
<td>Divergence of a vector field</td>
<td>93</td>
</tr>
<tr>
<td>8.3</td>
<td>The Hodge–Dirac operator revisited</td>
<td>94</td>
</tr>
<tr>
<td>8.4</td>
<td>Dirac operators</td>
<td>95</td>
</tr>
<tr>
<td>8.5</td>
<td>Laplacians</td>
<td>98</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>8.6</td>
<td>The Lichnerowicz formula</td>
<td>101</td>
</tr>
<tr>
<td>9</td>
<td><strong>The Dirac operator on the Riemann sphere</strong></td>
<td>102</td>
</tr>
<tr>
<td>9.1</td>
<td>Coordinates on the Riemann sphere</td>
<td>102</td>
</tr>
<tr>
<td>9.2</td>
<td>Sections and gauge transformations</td>
<td>104</td>
</tr>
<tr>
<td>9.3</td>
<td>The spin connection over the sphere</td>
<td>106</td>
</tr>
<tr>
<td>9.4</td>
<td>The Dirac operator over the sphere</td>
<td>107</td>
</tr>
<tr>
<td>9.5</td>
<td>The spinor Laplacian</td>
<td>110</td>
</tr>
<tr>
<td>9.6</td>
<td>The SU(2) action on the spinor bundle</td>
<td>112</td>
</tr>
<tr>
<td>9.7</td>
<td>Equivariance of the Dirac operator</td>
<td>115</td>
</tr>
<tr>
<td>9.8</td>
<td>Angular momentum operators</td>
<td>117</td>
</tr>
<tr>
<td>9.9</td>
<td>Spinor harmonics</td>
<td>119</td>
</tr>
<tr>
<td>9.10</td>
<td>The spectrum of the Dirac operator</td>
<td>122</td>
</tr>
<tr>
<td>10</td>
<td><strong>Construction of representations of SU(2)</strong></td>
<td>124</td>
</tr>
<tr>
<td>10.1</td>
<td>Characters of the maximal torus</td>
<td>124</td>
</tr>
<tr>
<td>10.2</td>
<td>Twisting connections</td>
<td>126</td>
</tr>
<tr>
<td>10.3</td>
<td>Twisted Dirac operators</td>
<td>127</td>
</tr>
<tr>
<td>10.4</td>
<td>The group action on the twisted spinor bundles</td>
<td>128</td>
</tr>
<tr>
<td>10.5</td>
<td>The Borel–Weil theorem</td>
<td>131</td>
</tr>
<tr>
<td>A</td>
<td><strong>Calculus on manifolds</strong></td>
<td>135</td>
</tr>
<tr>
<td>A.1</td>
<td>Differential manifolds</td>
<td>135</td>
</tr>
<tr>
<td>A.2</td>
<td>Tangent spaces</td>
<td>136</td>
</tr>
<tr>
<td>A.3</td>
<td>Vector fields</td>
<td>136</td>
</tr>
<tr>
<td>A.4</td>
<td>Lie groups</td>
<td>138</td>
</tr>
<tr>
<td>A.5</td>
<td>Fibre bundles</td>
<td>139</td>
</tr>
<tr>
<td>A.6</td>
<td>Tensors and differential forms</td>
<td>141</td>
</tr>
<tr>
<td>A.7</td>
<td>Calculus of differential forms</td>
<td>142</td>
</tr>
<tr>
<td>A.8</td>
<td>The de Rham complex</td>
<td>145</td>
</tr>
<tr>
<td>A.9</td>
<td>Volume forms and integrals</td>
<td>146</td>
</tr>
</tbody>
</table>
1 Vector bundles and their classification

The procedure commonly known as geometric quantization associates a Hilbert space to certain symplectic manifolds, and thereby allows a bridge to be built from the algebra of functions on the manifold to an operator algebra. It has been used, with considerable success, to reconstruct unitary representations of Lie groups from their symplectic homogeneous spaces; this is known as the Kirillov orbit method. An alternative method of constructing these representations, at least for compact groups, arises from the study of Dirac operators on these same homogeneous manifolds. The purpose of these notes is to clarify the rôle of the Dirac operator. Hopefully, this will enable us to build a bridge between geometric quantization and noncommutative geometry, in which the key concept is a generalization of the Dirac operator to “noncommutative manifolds”.

The main geometric objects with which we shall be concerned are vector bundles with some extra algebraic structure, such as a metric or a symplectic structure (or both). Roughly speaking, a vector bundle consists of the disjoint union $E$ of a collection of isomorphic vector spaces $E_x$, indexed by the points of a differential manifold $M$, together with some smoothness conditions relating $E$ and $M$; an example is the tangent bundle of the manifold $M$. The algebraic structure of the “fibres” $E_x$ transfers to give an interesting algebraic structure to the manifold $E$. The set of all possible vector bundles of a given species on a manifold $M$, up to a suitable equivalence relation, holds important information about the topology of the manifold $M$ (in fact, it holds all the topological information we need to know for the purposes of noncommutative geometry) and we begin our study with this classification.

We have gathered in Appendix A a compendium of definitions and basic facts on differential manifolds, vector fields and differential forms, with which we assume the reader to be acquainted. We refer to it for the notations used below.

1.1 Generalities on manifolds

Recall that a differentiable manifold of (real) dimension $n$ is a (paracompact) topological space $M$ provided with an atlas, i.e., a collection of charts $(U_j, \phi_j)$, where the domains $U_j$ form a (locally finite) open covering $\mathcal{U}$ of $M$, each $\phi_j : U_j \to \mathbb{R}^m$ is a homeomorphism, and the “transition maps” $\phi_i \circ \phi_j^{-1}$ are smooth functions on each $\phi_j(U_i \cap U_j)$. If $n = 2m$, we can regard $\mathbb{R}^{2m}$ as $\mathbb{C}^m$; we say that $M$ is a complex manifold if the transition maps are holomorphic (as multi-variable vector-valued complex functions); it suffices that they satisfy the Cauchy–Riemann equations.

We shall be mainly interested in the case where $M$ is compact; then its differentiable structure is defined by a finite atlas.

A useful property of the chart domains $U_j$ is that they be contractible, that is, that there exist $x_0 \in U_j$ and a smooth function $f : [0, 1] \times U_j \to U_j$ with $f(0, x) = x_0$, $f(1, x) = x$; informally, $U_j$ may be “deformed” to a point $\{x_0\}$. The intersection of two such domains need not be contractible: just think of the sphere $S^2$ covered by two large polar caps which overlap at the equator. It is good to know, however, that by using another (equivalent) atlas, we can avoid this difficulty. What we need is that the chart domains form a “good covering”
of $M$ [12].

**Definition 1.1.** An open covering $\mathcal{U} := \{U_j\}$ of a topological space is a **good covering**\(^1\) if every nonempty finite intersection $U_{j_1} \cap \cdots \cap U_{j_r}$ is contractible. A differentiable manifold $M$ always has a good covering: take a Euclidean metric on $M$ and cover $M$ by an atlas for which each $U_j$ is “geodesically convex” (that is, any two of its points can be connected by a minimal geodesic lying within $U_j$).\(^2\) All finite intersections are also geodesically convex, and hence are contractible (since one can deform to a point $x_0$ by retreating along geodesics emanating from that point). So we can replace the original atlas of $M$ by a (possibly larger) atlas whose domains form a good covering. From now on, we will assume that this has been done.

**Exercise 1.1.** Show that the following recipe defines a good covering $\{U_1, U_2, U_3, U_4\}$ of the sphere $S^2$. Take four points on $S^2$ that are vertices of a regular tetrahedron (e.g., if $a = 1/\sqrt{3}$, take $(\pm a, \pm a, \pm a)$ with either one or three positive signs). Connect these points by six great-circle arcs, which determine four spherical triangles $F_j$. Finally, let $U_j := \{ x \in S^2 : d(x, F_j) < \epsilon \}$ for some small $\epsilon$ (say, $\epsilon = 1/4$).

### 1.2 Principal bundles and vector bundles

Recall that a **fibre bundle** $E \xrightarrow{\pi} M$ or simply $E \rightarrow M$ (see Appendix A for notation) is a triple $(E, M, \pi)$, where the manifolds $E$ and $M$ are its **total space** and **base space** respectively, and $\pi: E \rightarrow M$ is a surjective submersion, subject to two conditions: (a) that each fibre $E_x := \pi^{-1}(\{x\})$ is diffeomorphic to a fixed manifold $F$ (the “typical fibre”), and (b) for some atlas $\{(U_j, \phi_j)\}$ of $M$ there is a family of local trivializations of $E$, i.e., diffeomorphisms

$$\psi_j: \pi^{-1}(U_j) \rightarrow U_j \times F$$

(1.1)

such that $\pi(\psi_j^{-1}(x, v)) = x$ for all $x \in U_j$, $v \in F$.

**Definition 1.2.** Let $G$ be a Lie group. A **principal $G$-bundle** $P \xrightarrow{\eta} M$ is a fibre bundle whose fibres are diffeomorphic to $G$, together with a free right action of $G$ on the total space $P$, whose orbits are the fibres $P_x = \eta^{-1}(x)$. We use the notation $\chi_j: \eta^{-1}(U_j) \rightarrow U_j \times G$ for the local trivializations.

**Exercise 1.2.** Suppose that $G$ is a Lie group and that $H$ is a closed subgroup of $G$ (so that $H$ is also a Lie group); then $H$ acts freely on $G$ by right multiplication $g \cdot h := gh$. If $\eta: G \rightarrow G/H$ is the quotient map, check that $G \xrightarrow{\eta} G/H$ is a principal $H$-bundle.

Before examining where principal bundles come from, let us describe a general recipe for manufacturing new fibre bundles using a given principal bundle. Indeed, this recipe is the main reason why principal bundles are used at all.

---

\(^1\)This is also known as a “contractible covering” [38], or a “Leray covering” [58].

\(^2\)It is a basic proposition of Riemannian geometry that each point has a geodesically convex neighbourhood; see [32] or [36] for the proof.
**Definition 1.3.** Let \( P \xrightarrow{\eta} M \) be principal \( G \)-bundle, and let \( F \) be a manifold on which the Lie group \( G \) acts on the left. The product manifold \( P \times F \) carries a right action of \( G \), given by

\[
(p, v) \cdot g := (p \cdot g, g^{-1} \cdot v).
\]

(1.2)

Let \( E := P \times_G F \) be the set of orbits of this action; we denote the orbit of \((p, v)\) by \([p, v]\).

Notice that \([p \cdot g, v] = [p, g \cdot v]\) on account of (1.2).

Define \( \pi : E \to M : [p, v] \mapsto \eta(p) \). For each \( p \in P \), the map \( v \mapsto [p, v] \) is a diffeomorphism from \( F \) to \( E_{\eta(p)} \), and \( E \xrightarrow{\pi} M \) is a fibre bundle with typical fibre \( F \), which is said to be associated to the given principal \( G \)-bundle.

**Definition 1.4.** A vector bundle \( E \xrightarrow{\pi} M \) is a fibre bundle whose “typical fibre” is a (real or complex) vector space \( V \), such that each fibre \( E_x \) is a vector space of dimension \( \dim V \), and such that for each \( x \in U_j \), the map \( v \mapsto \psi_j^{-1}(x, v) \) is a linear isomorphism from \( V \) onto \( E_x \).

A real vector bundle with typical fibre \( V = \mathbb{R} \) is called a real line bundle; a complex vector bundle with typical fibre \( V = \mathbb{C} \) is called a complex line bundle. When we say simply “line bundle”, we shall usually mean a complex line bundle.

Examples of (real) vector bundles over \( M \) are the tangent bundle \( TM \xrightarrow{\pi} M \), the cotangent bundle \( T^*M \xrightarrow{\pi} M \), and its exterior powers \( \Lambda^r T^*M \xrightarrow{\pi} M \): see Appendix A.

Given a representation \( \rho : G \to GL(V) \) of a Lie group, and a principal \( G \)-bundle \( P \xrightarrow{\eta} M \), we can form the associated bundle by taking \( g \cdot v \equiv \rho(g)v \); thus

\[
[p \cdot g, v] = [p, \rho(g)v] \quad \text{for} \quad p \in P, \ v \in V, \ g \in G.
\]

(1.3)

The fibres \( E_x \) become vector spaces by taking \([p, u] + [p, v] := [p, u + v]\), and \( \lambda[p, v] := [p, \lambda v]\); and the linearity of each \( \rho(g) \) shows that \( E \xrightarrow{\pi} M \) is a vector bundle with typical fibre \( V \).

We can now reverse the recipe of Definition 1.3 in order to associate a principal bundle to a given vector bundle:

**Definition 1.5.** Let \( E \xrightarrow{\pi} M \) be a vector bundle\(^3\) with typical fibre \( V \). Let \( P_x \), for \( x \in M \), denote the set of linear isomorphisms \( p : V \to E_x \) (each such \( p \) is called a frame for \( E_x \)). If \( g \in GL(V) \), then \( p \circ g \) is again a frame, so that \( GL(V) \) acts on the right (freely and transitively) on \( P_x \) by composition \( p \mapsto p \circ g \). The disjoint union \( P = \biguplus_{x \in M} P_x \), with \( \eta(p) := x \) for \( p \in P_x \), defines a principal \( GL(V) \)-bundle \( P \xrightarrow{\eta} M \), which is called the frame bundle of \( E \xrightarrow{\pi} M \).

**Exercise 1.3.** Using the identity representation of the group \( GL(V) \) on \( V \), check that the vector bundle associated to the frame bundle \( P \xrightarrow{\eta} M \) by the recipe (1.2) is the original vector bundle \( E \xrightarrow{\pi} M \).

---

\(^3\)We use a notation which covers the real and complex cases simultaneously; thus \( GL(V) \) denotes either \( GL_{\mathbb{R}}(V) \) or \( GL_{\mathbb{C}}(V) \), according as \( V \) is a real or complex vector space.
1.3 Hermitian vector bundles

If $V$ carries some extra structure (e.g., an inner product or an orientation), we can restrict to the subgroup $G \leq GL(V)$ which preserves that structure. We need to impose a corresponding structure on the fibres $E_x$ and consider only those frames $p: V \to E_x$ which are structure-preserving. Under $p \mapsto p \circ g$, these comprise a principal $G$-bundle associated to $E \overset{\pi}{\to} M$.

More generally, if we are given a Lie group $G$ with a representation $\rho: G \to GL(V)$, we define a right action of $G$ on frames by $p \cdot g := p \circ \rho(g)$. If, and sometimes this is "a big if", we can select a subset of frames for each $E_x$ for which this action is transitive and free (due to some extra structure given on the vector bundle), these will form the fibres $Q_x$ of a principal $G$-bundle $Q \overset{\pi}{\to} M$. Since $[p \cdot g, v] = [p, \rho(g)v]$ for $p \in Q$, $v \in V$, the vector bundle associated, via $\rho$, to the new principal bundle is still $E \overset{\pi}{\to} M$.

For instance, we could ask that a real vector bundle $E \to M$ be Euclidean, i.e., that each fibre $E_x$ carry a real inner product $g_x(\cdot, \cdot)$, which depends smoothly on $x$. If $V$ is a real vector space with a positive-definite inner product $q(\cdot, \cdot)$, we consider "orthogonal frames" $p$ satisfying $g_x(p(u), p(v)) = q(u, v)$ for all $u, v \in V$. It is clear that such frames form a principal $O(n)$-bundle associated to $E \overset{\pi}{\to} M$. Such an isomorphism $p$ is determined by choosing an orthonormal basis in $(E_x, g_x)$ (as the image under $p$ of a fixed orthonormal basis in $(V, q)$), so that this bundle is often called the "orthonormal frame bundle".

Alternatively, if $E \overset{\pi}{\to} M$ is a complex vector bundle, we could ask that it be Hermitian, i.e., that each fibre $E_x$ carry a (sesquilinear) inner product $h_x(\cdot, \cdot)$, depending smoothly on $x$. Then if $V$ is a complex Hilbert space with inner product $(\cdot \mid \cdot)$, we consider "unitary frames" $p$ satisfying $h_x(p(u), p(v)) = (u \mid v)$ for $u, v \in V$. Such frames form a principal $U(n)$-bundle associated to $E \overset{\pi}{\to} M$.

1.4 Operations on vector bundles

**Definition 1.6.** Given any (real or complex) vector bundle $E \to M$, one can form a dual vector bundle $E^* \to M$ whose fibre $E_x^*$ is the dual vector space\(^4\) of $E_x$. Fix a basis $\{v_1, \ldots, v_n\}$ for the typical fibre $V$. A frame $p: V \to E_x$ is defined by selecting a basis $\{e_1, \ldots, e_n\}$ for $E_x$ and setting $p(v_j) := e_j$ for each $j$; matching the dual bases $\{v'_1, \ldots, v'_n\}$, $\{e'_1, \ldots, e'_n\}$ gives a frame $p': V^* \to E_x^*$ for the dual bundle. Notice that $(p \circ g)' = p' \circ g^{-t}$ by change-of-basis formulae, where $g^{-t} = (g^{-1})^t$ is the contragredient matrix to $g \in GL(V)$.

**Exercise 1.4.** Verify that $E^* \to M$ is the vector bundle associated to the frame bundle of $E \to M$, via the representation $\rho(g) := g^{-t}$ of $GL(V)$ on $V^*$.

**Exercise 1.5.** Check that the cotangent bundle $T^*M \to M$ is the dual bundle to the tangent bundle $TM \to M$.

**Definition 1.7.** Given any two vector bundles $E \to M$, $E' \to M$ over the same base space, we can form two new vector bundles over $M$: their Whitney sum $E \oplus E' \to M$ and their

\(^4\)The dual space of $V$ is $V^* := \text{Hom}_\mathbb{R}(V, \mathbb{R})$ or $V^* := \text{Hom}_\mathbb{C}(V, \mathbb{C})$, the space of $\mathbb{R}$-linear or $\mathbb{C}$-linear forms on $V$, according as $V$ is a real or complex vector space.
tensor product $E \otimes E' \to M$, whose fibres at $x \in M$ are respectively the direct sum $E_x \oplus E'_x$ and the algebraic tensor product $E_x \otimes E'_x$.  

The $k$-th exterior power of $E \to M$ is the vector bundle $\Lambda^k E \to M$ whose fibre at $x$ is $\Lambda^k E_x$. (For $k = 0$, we take $\Lambda^0 E := M \times \mathbb{R}$.)

These operations can be combined; for instance, the exterior algebra bundle $\Lambda^\bullet E \to M$ is the Whitney sum of all exterior powers, for $k = 0, 1, \ldots, r$, where $r = \dim E_x$ is the rank of $E \to M$.

**Definition 1.8.** The complexification of a real vector bundle $E \to M$ is the complex vector bundle $E^C \to M$ with $E^C := E \otimes \mathbb{C}$, i.e., $(E_x)^C := E_x \otimes \mathbb{C} = E_x \oplus iE_x$ for each $x \in M$.

We write $T^C M$ and $T^* C M$ for the total spaces of the complexified tangent and cotangent bundles of $M$.

**1.5 Equivalent bundles**

**Definition 1.9.** A morphism of two fibre bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ is a pair of smooth maps $(\tau, \sigma)$, with $\tau : E \to E'$ and $\sigma : M \to M'$, such that $\pi' \circ \tau = \sigma \circ \pi$, i.e., such that the following diagram commutes:

$$
\begin{array}{ccc}
E & \xrightarrow{\tau} & E' \\
\downarrow & & \downarrow \pi' \\
M & \xrightarrow{\sigma} & M'
\end{array}
$$

and, in particular, $\tau(E_x) \subseteq E'_{\sigma(x)}$ for each $x \in M$.

A morphism of vector bundles is a bundle morphism for which $\tau : E_x \to E'_{\sigma(x)}$ is linear. When $M' = M$, we usually take $\sigma = \text{id}_M$.

**Definition 1.10.** Let $E \xrightarrow{\pi} M$ be a vector bundle and let $\phi : N \to M$ be a smooth map. Write $\phi^* E := \{(u, y) \in E \times N : \pi(u) = \phi(y)\}$ and define $\tilde{\pi} : \phi^* E \to N$ and $\tilde{\phi} : \phi^* E \to E$ by $\tilde{\pi}(u, y) := y$ and $\tilde{\phi}(u, y) := u$. Then $\tilde{\pi}^{-1}(y) = E_{\phi(y)}$, and so $\tilde{\phi}^* E \xrightarrow{\tilde{\pi}} N$ is a vector bundle, called the pullback bundle of $E \xrightarrow{\pi} M$ via $\phi$. Moreover, $(\tilde{\phi}, \phi)$ is a bundle morphism; in other words, we have a commutative diagram of smooth maps:

$$
\begin{array}{ccc}
\phi^* E & \xrightarrow{\tilde{\phi}} & E \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
N & \xrightarrow{\phi} & M
\end{array}
$$

**Exercise 1.6.** Show that the pullback bundle has the following universal property: if $E' \xrightarrow{\pi'} N$ is a vector bundle over $N$ and if $\rho : E' \to E$ is a map such that $(\rho, \phi)$ is a bundle morphism from this bundle to $E \xrightarrow{\pi} M$, then there is a unique bundle morphism $(\tau, \text{id}_N)$, with $\tau : E' \to \phi^* E$, so that $\rho = \phi \circ \tau$. (In other words, any bundle morphism with base map $\phi$ factors through the pullback bundle.)
Definition 1.11. Let \( E \xrightarrow{\pi} M, E' \xrightarrow{\pi'} M \) be two vector bundles over the same base space. A vector bundle equivalence between them is an invertible vector bundle morphism \( (\tau, \text{id}) \), which is given by a diffeomorphism \( \tau: E \to E' \) satisfying \( \pi' \circ \tau = \pi \) and such that \( \tau: E_x \to E'_x \) is a linear isomorphism for each \( x \).

We shall denote by \([E]\) the equivalence class of the vector bundle \( E \xrightarrow{\pi} M \); for the set of equivalence classes with typical fibre \( V \), we write \( \text{Vect}(M; V) \).

A vector bundle \( E \to M \) is trivial if it is equivalent to the product bundle \( M \times F \xrightarrow{\text{pr}_1} M \).

Note, from Exercise 1.6, that the pullback bundle is unique up to equivalence, and so defines a unique \([\phi^*E] \in \text{Vect}(N; V)\).

Exercise 1.7. Show that the tangent bundle \( T\mathbb{S}^1 \) of the unit circle \( \mathbb{S}^1 \) is trivial, by producing a pair of local trivializations which together form a cylinder.

Exercise 1.8. Let \( E \to M \) be an arbitrary vector bundle over \( M \); write \( E_r = M \times V \) with \( \dim V = r \), so that \( E_r \to M \) denotes the trivial vector bundle of rank \( r \). Show that \( E \otimes E_r \to M \) is equivalent to the Whitney sum \( E \oplus \cdots \oplus E \to M \) of \( r \) copies of \( E \to M \).

Definition 1.12. Let \( P \xrightarrow{\eta} M, P' \xrightarrow{\eta'} M \) be two principal \( G \)-bundles over the same base space. An equivalence between them is an invertible bundle morphism \( (\chi, \text{id}) \) which is \( G \)-equivariant, that is, \( \chi(p \cdot g) = \chi(p) \cdot g \) for any \( p \in P \). Note that \( \chi: P \to P' \) is a diffeomorphism.

We shall denote by \([P]\) the equivalence class of \( P \xrightarrow{\eta} M \); for the set of such equivalence classes we write \( \text{Prin}(M; G) \).

Proposition 1.1. Two vector bundles over \( M \) with the same typical fibre \( V \) are equivalent if and only if their frame bundles are equivalent as principal \( \text{GL}(V) \)-bundles. The association recipe thus yields a bijection \([E] \leftrightarrow [P]\) between \( \text{Vect}(M; V) \) and \( \text{Prin}(M; \text{GL}(V)) \).

Exercise 1.9. Prove this, using the defining relation (1.3) of an associated vector bundle.

1.6 Sections of vector bundles

Definition 1.13. A smooth section of a vector bundle \( E \xrightarrow{\pi} M \) is a smooth map \( s: M \to E \) such that \( \pi \circ s = \text{id}_M \), i.e., \( s(x) \in E_x \) for each \( x \in M \). The totality of smooth sections will be denoted by \( \Gamma(E) \), or by \( \Gamma(M, E) \) if it is necessary to specify the base space \( M \). Notice that \( \Gamma(E) \) is a module for the commutative algebra of functions \( C^\infty(M) \); the action of \( C^\infty(M) \) is just scalar multiplication in each fibre:\(^5\)

\[(fs)(x) := f(x)s(x),\]

for \( s \in \Gamma(E), f \in C^\infty(M) \).

\(^5\)It would perhaps be more convenient, in view of an eventual translation to the language of noncommutative geometry, to write the multiplication on the right: \((sf)(x) = s(x)f(x)\); but as this conflicts with traditional habits, and could be confusing in the case that \( s \) is a vector field, we will for the moment retain the usual notation of multiplying by scalars \( f(x) \) on the left.
“Global” smooth sections $s : M \to E$ can sometimes be hard to find. For instance, a line bundle admits a nonvanishing global section only if it is trivial. To see this, notice that any $v \in E_x$ is of the form $\lambda s(x)$, for a unique $\lambda \in \mathbb{C}$, since $s(x) \neq 0$; so $\lambda s(x) \mapsto (x, \lambda)$ is a vector bundle equivalence between $E$ and the trivial line bundle $M \times \mathbb{C}$.\footnote{Here we are falling into the sloppy habit of referring to a fibre bundle by naming its total space only. If the base space $M$ is fixed, this does no harm.}

**Lemma 1.2.** If $L \to M$ is a line bundle, then its tensor product with its dual line bundle, namely $L \otimes L^* \to M$, is a trivial line bundle.

**Proof.** Indeed, $L_x \otimes L_x^* \simeq \text{End}(L_x)$, so the tensor product bundle has an obvious nonvanishing section $s_0$, whose value at each $x$ is the identity operator on the line $L_x$. Since each $\text{End}(L_x)$ is a one-dimensional vector space, any $L \otimes L^* \to M$ is a line bundle, with a nonvanishing global section. \hfill \Box

**Corollary 1.3.** The set of equivalence classes of line bundles over $M$ has the structure of an abelian group.

**Proof.** Let $L \to M$, $L' \to M$ be any two line bundles over $M$, and define:

$$[L] [L'] := [L \otimes L'], \quad [L]^{-1} := [L^*]. \quad (1.4)$$

Let $L_0 := M \times \mathbb{C}$ so that $[L_0]$ is the trivial bundle class,\footnote{We use complex line bundles only to be specific; the argument for real line bundles is identical.} and note that $[L \otimes L_0] = [L]$ by Exercise 1.8. Since $[L \otimes L^*] = [L_0]$ by the previous Lemma, the inverse of $[L]$ is $[L^*]$. Moreover, the flip map $u \otimes v \mapsto v \otimes u$ from $L_x \otimes L_x^*$ to $L_x' \otimes L_x$ determines a bundle equivalence between $L \otimes L'$ and $L' \otimes L$, so that the product (1.4) is commutative. \hfill \Box

We shall soon identify this group with a cohomology group of $M$.

### 1.7 Local sections and transition functions

The question that now arises is how to deal with bundles that are not trivial, and how to give an effective description of such bundles. Since there are no nonvanishing global sections, we must make use of nonvanishing local sections $s_j \in \Gamma(U_j, E)$, where $\mathcal{U} := \{U_j\}$ is an open covering of $M$ by chart domains which admit local trivializations $\psi_j$ as in (1.1). Thus

$$\psi_j(s_j(x)) \equiv (x, f_j(x)) \quad (1.5)$$

where $f_j : U_j \to V \setminus \{0\}$ is a smooth nonvanishing function. (Indeed, to say that $s_j$ is smooth is the same as saying that its local representative $f_j$ is a smooth function.)

A global section $s \in \Gamma(M, E)$ is determined, via (1.5), by such a family of local representatives $f_j : U_j \to V$ (which may now take zero values).

Let $r = \dim V$ be the rank of the vector bundle $E \to M$. Suppose we can find a set $s_j = (s_{j1}, \ldots, s_{jr})$ with each $s_{jk} \in \Gamma(U_j, E)$ so that $\{s_{j1}(x), \ldots, s_{jr}(x)\}$ is a basis for the
fibre $E_x$ at each $x \in U_j$. Then if $t_j \in \Gamma(U_j, E)$ is any smooth local section, we can find smooth functions $h_j^1, \ldots, h_j^r \in C^\infty(U_j)$ such that $t_j = \sum_{k=1}^r h_j^k s_{jk}$ on $U_j$. Thus $\Gamma(U_j, E)$ is determined by the set of local sections $s_j$; and globally, the module $\Gamma(M, E)$ is determined by a family $\{(U_j, s_j)\}$ of such sets, one for each local chart of $M$. Such a family is sometimes called a local system of sections [38] for the vector bundle $E \to M$.

Note that $s_j$ is a local frame over $U_j$ and may be regarded as a local section of the frame bundle $P \to M$. To be precise, choose and fix a basis $\{v_1, \ldots, v_r\}$ for $V$, and let $p_j \in P_x$, for $x \in U_j$, be the linear isomorphism from $V$ to $E_x$ determined by $p_j(v_k) := s_{jk}(x)$ for $k = 1, \ldots, r$. Then $p_j \in \Gamma(U_j, P)$. Conversely, any such local section $p_j$ determines the local frame $s_j$.

Suppose that $(U_i, s_i)$ is another local frame and that the chart domains $U_i$ and $U_j$ overlap: $U_i \cap U_j \neq \emptyset$. Then for $x \in U_i \cap U_j$, the isomorphisms $p_i, p_j$ from $V$ to $E_x$ determined by $p_i(v_k) := s_{ik}(x)$, $p_j(v_k) := s_{jk}(x)$ are related by $p_j = p_i \circ g_{ij}(x)$ for some $g_{ij}(x) \in GL(V)$. In the notation of associated bundles, we have

$$[p_j, v] = [p_i \circ g_{ij}(x), v] = [p_i, g_{ij}(x)v]$$

(1.6)

for $v \in V$; or equivalently,

$$\psi_i \circ \psi_j^{-1}(x, v) = (x, g_{ij}(x)v),$$

(1.7)

i.e., $g_{ij}$ is the expression in local coordinates of the transition between the local trivializations $\psi_i, \psi_j$ of the vector bundle. Thus each $g_{ij} : U_i \cap U_j \to GL(V)$ is a smooth function.

Exercise 1.10. Show that a principal bundle $P \to M$ is trivial if and only if it admits a global smooth section $q : M \to P$.

Definition 1.14. Let $E \to M$ be a vector bundle, with typical fibre $V$, for which $\{(U_j, s_j) : j \in J\}$ is a local system of pointwise linearly independent local sections. The family of smooth functions $g_{ij} : U_i \cap U_j \to GL(V)$, defined whenever $U_i \cap U_j \neq \emptyset$, such that $s_i = g_{ij} \cdot s_j$ on $U_i \cap U_j$, satisfies the consistency conditions

$$g_{ii} = id\ \text{on}\ U_i, \quad g_{ij}g_{jk} = g_{ik}\ \text{on}\ U_i \cap U_j \cap U_k.$$  

(1.8)

(The notation $g_{ij} \cdot s_j$ denotes the natural action of $GL(V)$ on $E$, as expressed by (1.6) or (1.7).) The set $\{g_{ij}\}$ is called a family of transition functions for the vector bundle $E \to M$.

Suppose now that the vector bundle $E \to M$ carries extra structure, for instance a Hermitian metric. Then we can assume that the basis $s_j(x)$ for $E_x$ respects this structure (continuing the Hermitian-metric example, we may take it to be an orthonormal basis, with $s_{jk}(x) = p_j(v_k)$ for a fixed orthonormal basis of $V$). Then the transition functions $g_{ij}$ map $U_i \cap U_j$ into the subgroup $G$ of $GL(V)$ which preserves the appropriate structure (in our example, the unitary group of $V$). In summary, every $g_{ij}(x)$ belongs to the structure group $G$ of the principal bundle $P \to M$ to which the vector bundle is associated.

This suggests that we should study vector bundles by first classifying the corresponding frame bundles, and then invoking Proposition 1.1 to pass the classification to vector bundles.
This procedure works because the frame bundle is entirely determined by the transition functions. Specifically, we have the following “patching-together” construction.

Lemma 1.4. Let \( M \) be a manifold with an atlas of local charts \( \{(U_j, \phi_j) : j \in J\} \); let \( G \) be a Lie group and suppose we are given a family of smooth functions \( g_{ij} : U_i \cap U_j \to G \), defined for \( U_i \cap U_j \neq \emptyset \), that satisfies (1.8). Then there exists a principal \( G \)-bundle \( P \overset{\pi}{\to} M \) and a family of local sections \( p_j \in \Gamma(U_j, P) \) such that \( p_i(x) \cdot g_{ij}(x) = p_j(x) \) whenever \( x \in U_i \cap U_j \).

Proof. Let \( Q \) be the disjoint union \( \bigsqcup_{j \in J} U_j \times G \) and let \( P \) be the quotient space of \( Q \) formed by identifying \( (x, h)_i \in U_i \times G \) with \( (x, g_{ij}(x)h)_j \in U_j \times G \) whenever \( x \in U_i \cap U_j \). The condition (1.8) simply says that \( (x, h)_i \sim (x, g_{ij}(x)h)_j \) is an equivalence relation on \( Q \), so the quotient space is well defined. Write \( \eta[(x, h)_i] := x \); then it may be checked that \( P \) inherits from \( Q \) the structure of a differential manifold, of dimension \( \dim M + \dim G \), such that \( \eta : P \to M \) is a submersion; it remains to check that \( G \) acts freely and transitively on the right on each fibre \( \eta^{-1}(x) \); but it is obvious that the right action of \( G \) on \( Q \) given by

\[
(x, h)_i \cdot g := (x, hg)_i
\]

preserves equivalence classes and drops to a right action on \( P \) whose orbits are the fibres of \( \eta \).

If we are also given a representation \( \rho : G \to GL(V) \), we may then create a vector bundle with transition functions \( \{g_{ij}\} \) by association, using (1.3). Conclusion: one can always patch together a vector bundle with base \( M \) and structure group \( G \) from a set of transition functions satisfying (1.8) and a representation of \( G \).

The remaining question is how can one describe the equivalence of vector bundles in terms of the transition functions, in order to obtain a manageable classification. The answer is provided by the theory of Čech cohomology.

1.8 Čech cocycles

A family of transition functions forms what is called a “Čech 1-cocycle” with values in the structure group \( G \). Some difficulties arise in the cohomology theory of these objects for noncommutative structure groups, so we shall assume for the present that this group is abelian. This still covers many cases of interest, such as the groups \( \mathbb{C}, \mathbb{C}^\times, \mathbb{R}, U(1), \mathbb{Z}, \) and \( \mathbb{Z}_2 \).

Definition 1.15. Let \( \mathcal{U} = \{U_j : j \in J\} \) be an open covering of a manifold \( M \) and let \( A \) be an abelian group; we will write the group operation additively. For \( r \in \mathbb{N} \), a Čech \( r \)-cochain over \( \mathcal{U} \) with coefficients in \( A \) is a family of elements \( c_{j_0j_1...j_r} \in A \), indexed by the collections of \( (r + 1) \) sets \( \{U_{j_0}, U_{j_1}, \ldots, U_{j_r}\} \subset \mathcal{U} \) for which \( U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r} \neq \emptyset \). The set of all \( r \)-cochains is denoted \( C^r(\mathcal{U}, A) \); it is an abelian group.

One often needs a broader definition, where \( A \) is replaced by a family of abelian groups \( A := \{A_{j_0j_1...j_r}\} \) satisfying certain compatibility relations.\(^8\) We will always take \( A_{j_0j_1...j_r} \) to be

\(^8\)To be precise, \( A \) should be a sheaf of abelian groups over \( M \). The interested reader may consult [14] or [58] for the full story.
the collection of smooth functions from $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r}$ into a fixed abelian group $A$. For $r \in \mathbb{N}$, a Čech $r$-cochain over $\mathcal{U}$ with coefficients in $A$ is then a family of smooth functions $f_{j_0j_1,\ldots,j_r}: U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r} \to A$, defined whenever $U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r} \neq \emptyset$. The set of all such families is denoted $C^r(\mathcal{U},A)$.

For instance, the chart maps $\phi_j$ of a manifold $M$ form a 0-cochain in $C^0(\mathcal{U},\mathbb{R}^n)$ (or in $C^0(\mathcal{U},\mathbb{C}^n)$, if $M$ is a complex manifold). The transition functions of a complex line bundle $g_{ij}$ form a 1-cochain in $C^1(\mathcal{U},\mathbb{C}^\times)$; the transition functions of a Hermitian line bundle form a 1-cochain in $C^1(\mathcal{U},U(1))$.

**Definition 1.16.** The Čech complex is the cochain complex \( C^\bullet(\mathcal{U},A) \), where \( \delta \) is defined as follows. The coboundary operator $\delta_r: C^r(\mathcal{U},A) \to C^{r+1}(\mathcal{U},A)$ is given by

\[
(\delta a)_{ij} := a_i - a_j, \quad (\delta b)_{ijk} := b_{ij} - b_{ik} + b_{jk}
\]

for $a \in C^0(\mathcal{U},A)$, $b \in C^1(\mathcal{U},A)$, and $(\delta c)_{j_0\ldots j_r} := \sum_{k=0}^r (-1)^k c_{j_0\ldots j_{k-1} j_{k+1} \ldots j_r}$ in general. It is immediate that $\delta^2 = 0$ by cancellation of terms, so we have a complex

\[
C^0(\mathcal{U},A) \xrightarrow{\delta} C^1(\mathcal{U},A) \xrightarrow{\delta} C^2(\mathcal{U},A) \xrightarrow{\delta} \cdots
\]

which gives rise to cohomology groups in the standard way: $Z^r(\mathcal{U},A) := \{ c \in C^r(\mathcal{U},A) : \delta c = 0 \}$ are the Čech cocycles, $B^r(\mathcal{U},A) := \{ \delta b : b \in C^{r-1}(\mathcal{U},A) \}$ are the Čech coboundaries, and the quotient group $H^r(\mathcal{U},A) := Z^r(\mathcal{U},A)/B^r(\mathcal{U},A)$ is the $r$th Čech cohomology group of the covering $\mathcal{U}$ with coefficients in $A$.

Open coverings of $M$ form a directed set (under refinement); we can eliminate $\mathcal{U}$ by taking a direct limit: the $r$th Čech cohomology group of the manifold $M$ (with coefficients in $A$) is defined as \(^{10}\)

\[
\check{H}^r(M,A) := \lim_{\mathcal{U}} H^r(\mathcal{U},A).
\]

An essential result from topology [14] is that this limit is already attained when $\mathcal{U}$ is a good covering: $\check{H}^r(M,A) = H^r(\mathcal{U},A)$ in this case.

### 1.9 The Čech cohomology of $\mathbb{S}^2$  

Let us compute the Čech cohomology (with real coefficients) of the sphere $\mathbb{S}^2$. We know (Exercise 1.1) that it has a good covering $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ by open neighbourhoods of the four spherical triangles $F_i$ obtained by projecting an inscribed regular tetrahedron outward from the centre. Each $U_i \cap U_j$ is a neighbourhood of the edge $F_i \cap F_j$, and each $U_i \cap U_j \cap U_k$ is a neighbourhood of the vertex $F_i \cap F_j \cap F_k$; all are contractible. Thus 0-cochains are labelled by the faces of the tetrahedron, 1-cochains are labelled by its edges, and 2-cochains are labelled by its vertices. Therefore

\[
C^0(\mathcal{U},\mathbb{R}) \simeq \mathbb{R}^4, \quad C^1(\mathcal{U},\mathbb{R}) \simeq \mathbb{R}^6, \quad C^2(\mathcal{U},\mathbb{R}) \simeq \mathbb{R}^4;
\]

---

\(^9\)See Appendix A for generalities on cochain complexes and their cohomology groups.  
\(^{10}\)We use the notation $\check{H}$ to distinguish Čech cohomology from singular or de Rham cohomology; although we shall see that this is often unnecessary.
and since $U_1 \cap U_2 \cap U_3 \cap U_4 = \emptyset$, we have $C^r(U, \mathbb{R}) = 0$ for $r \geq 3$. Thus the Čech complex reduces to
\[
C^0(U, \mathbb{R}) \xrightarrow{\delta_0} C^1(U, \mathbb{R}) \xrightarrow{\delta_1} C^2(U, \mathbb{R}) \xrightarrow{\delta_2} 0. \tag{1.9}
\]
Now $a \in \ker \delta_0$ iff $a_1 = a_2 = a_3 = a_4$, so $\tilde{H}^0(S^2, \mathbb{R}) \simeq \mathbb{Z}^0(U, \mathbb{R}) \simeq \mathbb{R}$; and (by linearity of $\delta_0$) $B^1(U, \mathbb{R}) = \ker \delta_0 \simeq \mathbb{R}^3$.

The elements $b \in \ker \delta_1 = Z^1(U, \mathbb{R})$ satisfy $b_{ij} - b_{ik} + b_{jk} = 0$ for $\{i, j, k\} \subset \{1, 2, 3, 4\}$; these 4 linear equations for the six $b_{ij}$ form a system of rank 3; and hence $Z^1(U, \mathbb{R}) \simeq \mathbb{R}^3$. Thus the sequence (1.9) is exact at $C^1(U, \mathbb{R})$, and so $\tilde{H}^1(S^2, \mathbb{R}) = 0$.

Finally, $B^2(U, \mathbb{R}) = \ker \delta_1 \simeq \mathbb{R}^6 / \ker \delta_1 \simeq \mathbb{R}^3$. Since $Z^2(U, \mathbb{R}) = C^2(U, \mathbb{R}) \simeq \mathbb{R}^4$, we conclude that $\tilde{H}^2(S^2, \mathbb{R}) \simeq \mathbb{R}$.

**Exercise 1.11.** Compute the de Rham cohomology (Appendix A) of $S^2$. Show that closed 0-forms on $S^2$ are constant functions, that the volume form $\Omega = \sin \theta \, d\theta \, d\phi$ is a closed 2-form with nonzero integral, and that if $\beta$ is another 2-form with $\int_{S^2} \beta = \int_{S^2} \Omega$ then $\beta - \Omega$ is exact. If $\alpha = f(\theta, \phi) \, d\theta + g(\theta, \phi) \sin \theta \, d\phi$ is a closed 1-form, use the Poincaré lemma to show that, away from either the north or the south pole, $\alpha$ is of the form $d(\sin \theta \, h(\theta, \phi))$, and hence show that $\alpha$ is exact. Conclude that $\tilde{H}^r(S^2, \mathbb{R}) \simeq H^r_{\text{DR}}(S^2)$ for every $r$. Is that just a coincidence?

### 1.10 Line bundle classification

Let $L \rightarrow M$ be a complex line bundle with a local system $\{(U_j, s_j)\}$ of nonvanishing local sections and with transition functions $g_{ij}: U_i \cap U_j \rightarrow \mathbb{C}^\times$ such that $s_i = g_{ij} s_j$ on $U_i \cap U_j$. (Since the fibres are one-dimensional, we get $G = GL(1, \mathbb{C}) = \mathbb{C}^\times$ and $s_j \mapsto g_{ij} s_j$ is just the module action of functions on sections; this simplifies the notation considerably.) We may and shall assume that $U = \{U_j\}$ is a good covering. The fundamental result we need is the following [57].

**Proposition 1.5.** The family of transition functions $\mathbf{g} := \{g_{ij}\}$ is a Čech 1-cocycle for the good covering $U$; its cohomology class $[\mathbf{g}] \in \tilde{H}^1(M, \mathbb{C}^\times)$ is independent of the local system of sections $s_j$, and depends only on the equivalence class $[L]$ of the complex line bundle $L \rightarrow M$. Moreover, the correspondence $[L] \mapsto [\mathbf{g}] \in \tilde{H}^1(M, \mathbb{C}^\times)$ is an isomorphism of abelian groups.

**Proof.** From its definition, $\mathbf{g}$ is clearly a Čech 1-cochain in $C^1(U, \mathbb{C}^\times)$. The consistency condition (1.8) says that $(\delta \mathbf{g})_{ijk} := g_{ij} g_{jk} g_{ik} \equiv 1$ on $U_i \cap U_j \cap U_k$, which means that $\mathbf{g}$ is a Čech 1-cocycle. If a different local system $\{(U_j, t_j)\}$ is chosen, then $t_i = h_i s_i$, where $h_i: U_i \rightarrow \mathbb{C}^\times$ is smooth, i.e., $\mathbf{h} \in C^0(U, \mathbb{C}^\times)$. Clearly $t_i = (h_i / h_j) g_{ij} t_j$, so that the transition functions for the new local system form the 1-cocycle $\mathbf{g} + \delta \mathbf{h}$ (in additive notation). Thus the line bundle determines the class $[\mathbf{g}]$ in $\tilde{H}^1(M, \mathbb{C}^\times)$.

The frame bundle $P \rightarrow M$ of a line bundle is formed simply by deleting the zero section from $L$, i.e., by taking $P_x = L_x \setminus \{0\}$ for each $x \in M$. A nonvanishing local section $s_j$ of $L \rightarrow M$ may thus also be regarded as a section of the frame bundle $P \rightarrow M$. From Exercise 1.4, we see that the transition functions for the dual line bundle $L^* \rightarrow M$ are

---

11 The notation $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers; we revert to multiplicative notation when working with this group.
1/g_{ij}. On passing to additive notation, we conclude that the dual bundle determines the class $-\delta \Pi_1 \in H^1(\mathcal{M}, \mathbb{C}^\times)$.

Similarly, if $g_{ij}$ are transition functions for another line bundle $L' \to \mathcal{M}$, then the tensor product bundle $L \otimes L' \to \mathcal{M}$ has transition functions $g_i g_j'$, and so determines the class $[g] + [g'] \in \tilde{H}^1(\mathcal{M}, \mathbb{C}^\times)$. Therefore $[L] \mapsto [g]$ is a homomorphism from the group of line bundle classes to the group $\tilde{H}^1(\mathcal{M}, \mathbb{C}^\times)$.

If $[g] = 0$ in $\tilde{H}^1(\mathcal{M}, \mathbb{C}^\times)$, then $g$ is a coboundary $\delta f$, i.e., $g_{ij} = f_i / f_j$ where each $f_i : U_i \to \mathbb{C}^\times$ is a smooth function. But then $f_i^{-1} s_i = f_j^{-1} s_j$ whenever $U_i \cap U_j \neq \emptyset$, and so there is a global nonvanishing section $s \in \Gamma(L)$ given by $s := f_j^{-1} s_j$ on $U_j$; and therefore $L \to \mathcal{M}$ is a trivial line bundle. Hence the kernel of homomorphism $[L] \mapsto [g]$ is zero. From Lemma 1.4, a line bundle can always be patched together from a given Čech 1-cocycle in $C^1(\mathcal{U}, \mathbb{C}^\times)$, having this cocycle as its set of transition functions; this shows that $[L] \mapsto [g]$ is surjective. □

1.11 The second cohomology group

Consider the short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{\epsilon} \mathbb{C}^\times \to 0,$$

(1.10)

where $\iota$ is inclusion and $\epsilon(z) := e^{2\pi iz}$. One may form Čech cochains with values in any of these groups. If $c \in Z^r(\mathcal{U}, \mathbb{C}^\times)$, then $c = \epsilon(b)$ with $b \in C^r(\mathcal{U}, \mathbb{C})$; since $\epsilon(\delta b) = \delta c = 0$, we can find $a \in C^{r+1}(\mathbb{Z})$ with $\iota(a) = \delta b$; since $\iota(\delta a) = \delta(\iota a) = 0$, we have $\delta a = 0$ and so $a \in Z^{r+1}(\mathcal{U}, \mathbb{Z})$. One checks that $[c] \mapsto [a]$ is a well-defined homomorphism\(^\text{12}\) $\partial : H^r(\mathcal{M}, \mathbb{C}^\times) \to H^{r+1}(\mathcal{M}, \mathbb{Z})$, called the Bockstein homomorphism [23, 58]. We therefore get a long exact sequence in cohomology:

$$\cdots \to \tilde{H}^1(\mathcal{M}, \mathbb{C}) \to \tilde{H}^1(\mathcal{M}, \mathbb{C}^\times) \xrightarrow{\partial} \tilde{H}^2(\mathcal{M}, \mathbb{Z}) \to \tilde{H}^2(\mathcal{M}, \mathbb{C}) \to \cdots$$

Notice that the discrete group $\mathbb{Z}$ need not be underlined: if $\mathcal{U}$ is a good covering, an element of $C^r(\mathcal{U}, \mathbb{Z})$ is a family of $\mathbb{Z}$-valued smooth functions with connected domains, i.e., a family of constant functions; thus $C^r(\mathcal{U}, \mathbb{Z}) = C^r(\mathcal{U}, \mathbb{Z})$ for all $r$, and so $\tilde{H}^r(\mathcal{M}, \mathbb{Z}) = H^r(\mathcal{M}, \mathbb{Z})$.

Exercise 1.12. Verify that $\partial$ does not depend on the choices of $a$, $b$ and $c$ within their cohomology classes, and that $\text{ker} \partial = \text{im} \epsilon$ and $\text{im} \iota = \ker \partial$.

Exercise 1.13. If $\mathcal{U}$ is a good covering of $\mathcal{M}$ and $\{\psi_j\}$ is a partition of unity subordinate to $\mathcal{U}$, define, for $c \in Z^{r+1}(\mathcal{U}, \mathbb{C})$, an element $b \in C^r(\mathcal{U}, \mathbb{C})$ by $b_{j_{0}...j_{r-1}} := \sum_{j} c_{j_{0}...j_{r-1}} \psi_j$ (this is a locally finite sum). Show that $\delta b = c$ and conclude that $\tilde{H}^k(\mathcal{M}, \mathbb{C}) = 0$ for $k > 0$.

Proposition 1.6. The Bockstein homomorphism $\partial$ is an isomorphism of abelian groups between $\tilde{H}^1(\mathcal{M}, \mathbb{C}^\times)$ and $\tilde{H}^2(\mathcal{M}, \mathbb{Z})$.

Proof. This follows from the preceding exercise, but it is instructive to produce the isomorphism explicitly. Let $g \in Z^1(\mathcal{U}, \mathbb{C}^\times)$; since each $U_i \cap U_j$ is contractible, we can find a smooth

\(^{12}\)This is just the standard construction of the “connecting homomorphism” in homological algebra.
function \( f_{ij} : U_i \cap U_j \to \mathbb{C} \) such that \( \epsilon \circ f_{ij} = g_{ij} \); we could write \( f_{ij} = (2\pi i)^{-1} \log g_{ij} \), but this is not quite correct since the complex logarithm is “multi-valued”. Define
\[
a_{ijk} := f_{ij} - f_{ik} + f_{jk} : U_i \cap U_j \cap U_k \to \mathbb{C}
\]
whenever \( U_i \cap U_j \cap U_k \neq \emptyset \). Then \( \exp(2\pi i a_{ijk}) = g_{ij}g_{jk}/g_{ik} = 1 \), and so \( a_{ijk} \) is \( \mathbb{Z} \)-valued. Thus \( a \) is an element of \( C^2(U, \mathbb{Z}) \). Now (1.11) says that \( a = \delta f \) in \( C^2(U, \mathbb{C}) \), and so \( (\delta a)_{ijkl} := a_{ijk} - a_{ikl} + a_{ijl} - a_{jkl} = 0 \) on \( U_i \cap U_j \cap U_k \cap U_l \); these are algebraic relations among \( \mathbb{Z} \)-valued functions, and so \( \delta a = 0 \) in \( C^2(U, \mathbb{Z}) \); hence \( a \in Z^2(U, \mathbb{Z}) \).

If we take \( g'_{ij} := (h_i/h_j)g_{ij} \) with \( h \in C^0(U, \mathbb{C}) \), we can find a smooth function \( k_i : U_i \to \mathbb{C} \) such that \( \epsilon \circ k_i = h_i \); then \( \epsilon \circ (f_{ij} + k_i - k_j) = g'_{ij} \). In other words, we modify \( f \) to \( f + \delta k \) in \( C^1(U, \mathbb{C}) \), and \( a = \delta f \) is unchanged. Therefore we obtain a well-defined homomorphism \( \partial : [g] \mapsto [a] \) from \( \check{H}^1(M, \mathbb{C}^\times) \) to \( \check{H}^2(M, \mathbb{Z}) \).

To see that this is an isomorphism, let \( \{\psi_j\} \) be a smooth partition of unity subordinate to \( U \). Suppose \( a \in Z^2(U, \mathbb{Z}) \) is any Čech 2-cocycle; define \( f \in C^1(U, \mathbb{C}) \) by \( f_{ij} := \sum_r a_{ijr} \psi_r \), and define \( g \in C^1(U, \mathbb{C}^\times) \) by \( g_{ij} := \exp(2\pi i f_{ij}) \). Then
\[
f_{ij} - f_{ik} + f_{jk} = \sum_r (a_{ijr} - a_{ikr} + a_{jkr}) \psi_r = \sum_r a_{ijk} \psi_r = a_{ijk}
\]
on using \( \delta a = 0 \); hence \( \delta f = a \). This shows that \( \partial \) is onto.

If \( \partial [g] = 0 \) in \( \check{H}^2(U, \mathbb{Z}) \), we can arrange, by suitably choosing a representative \( g \) in its class, that \( f_{ij} - f_{ik} + f_{jk} \equiv 0 \) on each \( U_i \cap U_j \cap U_k \). Define \( k \in C^0(U, \mathbb{C}) \) by \( k_i := \sum_r f_{ir} \psi_r \); then \( k_i - k_j = \sum_r (f_{ir} - f_{jr}) \psi_r = \sum_r f_{ij} \psi_r = f_{ij} \), and so \( f = \delta k \). If \( h_i := \exp(2\pi i k_i) \), then \( g = \delta h \) in \( C^1(U, \mathbb{C}^\times) \), and so \( [g] = 0 \) in \( \check{H}^2(U, \mathbb{C}^\times) \). Hence \( \partial \) is one-to-one, which establishes that \( \partial : \check{H}^1(M, \mathbb{C}^\times) \to \check{H}^2(M, \mathbb{Z}) \) is an isomorphism.

The upshot is that complex line bundles over \( M \) are classified by the integral Čech cohomology group \( \check{H}^2(M, \mathbb{Z}) \), at the very small price of going up to the second level in cohomology. This is a first taste of “quantization”, that is, an unexpected discreteness which appears in a seemingly continuous family of objects. The culprit here is the exact sequence (1.10), due to the periodicity of the exponential function.

1.12 Classification of Hermitian line bundles

We are particularly interested in Hermitian line bundles \( L \to M \), which have an inner product in each fibre (and therefore the local sections \( s_j \) can be chosen so that \( s_j(x) \in L_x \) is a unit vector). These have structure group \( U(1) \), and the previous arguments show that the group of equivalence classes of Hermitian line bundles is isomorphic to the Čech cohomology group \( \check{H}^1(M, U(1)) \).


We have a short exact sequence of abelian groups:
\[
0 \to \mathbb{Z} \xrightarrow{\cdot} \mathbb{R} \xrightarrow{\epsilon} U(1) \to 0,
\]
where \( \epsilon(t) := e^{2\pi i t} \) for \( t \in \mathbb{R} \). The corresponding long exact sequence is
\[
\cdots \to \check{H}^1(M, \mathbb{R}) \to \check{H}^1(M, U(1)) \overset{\partial}{\to} \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathbb{R}) \to \cdots
\]
and again \( \check{H}^1(M, \mathbb{R}) = \check{H}^2(M, \mathbb{R}) = 0 \) by the partition-of-unity construction. Here also, the Bockstein homomorphism is an isomorphism between \( \check{H}^1(M, U(1)) \) and \( \check{H}^2(M, \mathbb{Z}) \). In this case, the image of each \( g_{ij} \) is not the whole of the unit circle \( U(1) \), since \( U_i \cap U_j \) is contractible; by passing a half-line from the origin through an omitted point of \( U(1) \), we can choose a branch of the logarithm for which \( f_{ij} := (2\pi i)^{-1} \log g_{ij} \) is well-defined. We then have
\[
a_{ijk} := \frac{1}{2\pi i} (\log g_{ij} - \log g_{ik} + \log g_{jk}) \in \mathbb{Z}
\] (1.13)
since \( \exp(2\pi i a_{ijk}) \equiv 1 \) (note that the three branches of the logarithm on the right hand side of (1.13) need not be the same). Once again, we get \( a \in C^2(\mathcal{U}, \mathbb{Z}) \) with \( \delta a = 0 \), so \( a \in Z^2(\mathcal{U}, \mathbb{Z}) \). This leads to the following result.

**Proposition 1.7.** The Bockstein homomorphism \( \partial \) is an isomorphism of abelian groups between \( \check{H}^1(M, U(1)) \) and \( \check{H}^2(M, \mathbb{Z}) \). \( \square \)

**Exercise 1.15.** Write out the proof, using the arguments of Proposition 1.6.

The Čech cohomology groups obviously depend only on the topology of \( M \), and are computed by elementary topological arguments. However, it is desirable to replace them by de Rham cohomology groups, so that one can work with differential forms. This we do in Section 3.

### 1.13 Classification of vector bundles

In order to classify vector bundles of rank higher than 1 (up to equivalence), we face the obstacle that the group structure on the line bundles does not extend to the higher-rank case. In fact, it turns out that the most useful classification of vector bundles relies on a weaker equivalence relation, called *stable equivalence*, which does not distinguish between a vector bundle \( E \to M \) and the Whitney sum \( E \oplus E_0 \to M \) whenever the second summand \( E_0 \to M \) is a *trivial* bundle. This notion arises from the following fundamental property of vector bundles.

**Proposition 1.8.** Let \( E \to M \) be a vector bundle over a compact manifold \( M \). Then we can find another vector bundle \( E' \to M \) such that the Whitney sum \( E \oplus E' \to M \) is a trivial vector bundle.

**Proof.** Let \( \mathcal{U} = \{U_1, \ldots, U_m\} \) be a finite open covering of \( M \) by chart domains, and let \( \{s_j^1, \ldots, s_j^k\} \) be linearly independent sections in \( \Gamma(U_j, E) \) (where \( k \) is the rank of \( E \)). Let \( \{\psi_j\}_{1 \leq j \leq m} \) be a smooth partition of unity subordinate to \( \mathcal{U} \). Let \( \sigma_j \in \Gamma(E) \) be defined as \( \psi_j s_j^i \) on \( U_j \) and as 0 on the complement of \( U_j \); notice that the vectors \( \sigma_j^i(x) \) span the fibre \( E_x \), for any \( x \in M \).
Write $n = km$, and define $f : M \times \mathbb{C}^n \to E$ (in the case of complex fibres; the real-fibre case is analogous) by $f(x, t) := \sum_{j,t} t_j \sigma_j^x(x)$; then $f$ is a surjective bundle map, i.e., $(f, \text{id}_M)$ is a bundle morphism. Write $E'_x := \{(x, t) : t \in \mathbb{C}^n, f(x, t) = 0\}$. Choose some hermitian (or Riemannian) metric on $M$, and let $F_x$ be the orthogonal complement of $E'_x$ in $\{x\} \times \mathbb{C}^n$; one checks that the $F_x$ form the fibres of a vector bundle $F \to M$, and that $(f, \text{id}_M)$ is a bundle equivalence between this bundle and $E \to M$. Since $F \oplus E' = M \times \mathbb{C}^n$, we thereby obtain an invertible bundle map from $M \times \mathbb{C}^n$ to $E \oplus E'$.

**Definition 1.17.** Let $M$ be a compact manifold. We say that two vector bundles $E \to M$ and $F \to M$ are stably equivalent if there exists a trivial bundle $E_0 \to M$ such that $E \oplus E_0$ and $F \oplus E_0$ are equivalent. We denote by $[E]$ the stable equivalence class of $E$.

**Exercise 1.16.** Show that $E \to M$ and $F \to M$ are stably equivalent iff $[E \oplus G] = [F \oplus G]$ for some third vector bundle $G \to M$ (which need not be trivial).

The equivalence classes of vector bundles over $M$ form an abelian semigroup with identity, under the operation $[E] + [F] := [E \oplus F]$. One would like to embed this in an abelian group in some canonical way. In fact there is a standard construction of such a group, by abstract nonsense. For any abelian semigroup $A$ with identity, let $K(A)$ be the abelian group with the following universal property: there is a unital semigroup homomorphism $\theta : A \to K(A)$ such that, whenever $G$ is a group and $\gamma : A \to G$ is a unital semigroup homomorphism, there is a unique group homomorphism $\kappa : K(A) \to G$ for which $\gamma = \kappa \circ \theta$. Clearly, $K(A)$ is unique up to isomorphism, and it is called the Grothendieck group of $A$.

**Exercise 1.17.** Check that a group with the desired universal property is given by the following construction. Define an equivalence relation on $A \times A$ by $(a, b) \sim (a', b')$ iff $a + b' + c = a' + b + c$ for some $c \in A$, and let $K(A)$ be the set of equivalence classes with the obvious sum operation, where the class of $(b, a)$ is inverse to the class of $(a, b)$; let $\theta(a)$ be the class of $(a, 0)$. Check also that each element of $K(A)$ is of the form $\theta(a) - \theta(b)$ for some elements $a, b \in A$.

**Exercise 1.18.** What is $K(\mathbb{N})$? What is $K(A)$ if $A$ is the multiplicative semigroup $\mathbb{Z} \setminus \{0\}$?

An abelian semigroup $A$ is said to “allow cancellation” if $a + c = b + c$ implies $a = b$, for any $c \in A$. Regrettably, the semigroup $\text{Vect}(M)$ of (ordinary) equivalence classes of vector bundles over $M$ does not usually allow cancellation. For example, the tangent bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ is not trivial, whereas the normal bundle (of lines from the origin of $\mathbb{R}^3$ through the sphere $\mathbb{S}^2$) is trivial, and their Whitney sum is the trivial bundle $\mathbb{S}^2 \times \mathbb{R}^3 \to \mathbb{S}^2$; thus $[T\mathbb{S}^2] + [\mathbb{S}^2 \times \mathbb{R}] = [\mathbb{S}^2 \times \mathbb{R}^2] + [\mathbb{S}^2 \times \mathbb{R}]$, so that cancellation is not allowed. In this case, the homomorphism $\theta$ is not injective.

**Definition 1.18.** The $K$-theory of a compact manifold $M$ is the Grothendieck group $K^0(M) := K(\text{Vect}(M))$. Two vector bundles $E \to M$ and $F \to M$ have the same image in $K^0(M)$ iff they are stably equivalent; thus any element of $K^0(M)$ can be written as a difference of two stable equivalence classes, $[E] - [F]$. 

18
Exercise 1.19. Prove the assertion that \( \theta([E]) = \theta([F]) \iff [E] = [F] \). Show also that any element of \( K^0(M) \) can be written as \( [E] - [O_k] \) for some \( k \), where \( O_k \rightarrow M \) denotes the trivial bundle of rank \( k \).

It turns out that \( K^0(M) \) carries important topological information about the manifold \( M \); in that context, vector bundles may be viewed as auxiliary tools to study topological spaces. There is a companion group, called \( K^1(M) \), which together with \( K^0(M) \) forms a cohomology theory distinct from the Čech and de Rham cohomologies. This theory is fully developed in the monograph of Atiyah [3].

2 Complex projective spaces

We illustrate the general theory with a look at a particular class of manifolds, namely the complex projective spaces. These are complex manifolds, that is, differentiable manifolds whose transition maps are holomorphic; thus we may use complex local coordinates to describe them. They possess three important features: (a) a Hermitian metric; (b) a distinguished nondegenerate closed 2-form, which is known as a “symplectic structure”; (c) a “complex structure”, which at each point specifies a linear automorphism of the tangent space whose square is \(-1\). Moreover, any two of these three structures determine the third; a complex manifold with such a triple structure is called a Kähler manifold. The complex projective spaces are perhaps the simplest examples of compact Kähler manifolds.

2.1 Complex manifolds

Definition 2.1. A complex manifold of complex dimension \( m \) is a differentiable manifold of real dimension \( n = 2m \) which has an atlas of local charts \((U_j, \phi_j)\), with \( \phi_j: U_j \rightarrow \mathbb{C}^m \), such that the transition maps \( \phi_i \circ \phi_j^{-1} \) are holomorphic. If \((x^1, \ldots, x^m, y^1, \ldots, y^m)\) are local (real) coordinates on \( U_j \), write \( z^k := x^k + iy^k, \bar{z}^k := x^k - iy^k \); then \((z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m)\) is an alternative system of local (real) coordinates on \( U_j \).

Complex-valued 1-forms\(^1\) are locally generated by \( dz^1, \ldots, dz^m, d\bar{z}^1, \ldots, d\bar{z}^m \), where we write \( dz^k := dx^k + idy^k, d\bar{z}^k := dx^k - idy^k \). Complex-valued vector fields in \( \mathfrak{X}(M, \mathbb{C}) \) are locally generated by \( \partial/\partial z^1, \ldots, \partial/\partial z^m, \partial/\partial \bar{z}^1, \ldots, \partial/\partial \bar{z}^m \), where \( \partial/\partial z^k := \frac{i}{2}(\partial/\partial x^k - i\partial/\partial y^k) \) and \( \partial/\partial \bar{z}^k := \frac{i}{2}(\partial/\partial x^k + i\partial/\partial y^k) \); more precisely, \( \mathfrak{X}(U_j, \mathbb{C}) \) is a \( C^\infty(U_j, \mathbb{C}) \)-module with these generators. (The notation is chosen so that \( dz^j(\partial/\partial \bar{z}^k) = \delta^j_k \) and \( d\bar{z}^j(\partial/\partial z^k) = 0 \).)

In particular, we may write the differential of \( f \in C^\infty(M, \mathbb{C}) \) as
\[
df = \frac{\partial f}{\partial z^k} dz^k + \frac{\partial f}{\partial \bar{z}^k} d\bar{z}^k \equiv \partial f + \bar{\partial} f, \tag{2.1}\]
where, here and in the future, we use the Einstein summation convention of summing over repeated upper and lower indices.\(^2\)

\(^1\)In this section, all vector fields and differential forms will be taken complex-valued unless stated otherwise.

\(^2\)To avoid possible misunderstandings, we emphasize that we do not sum over repeated upper or repeated lower indices, unless a summation sign appears explicitly.
Exercise 2.1. Check that the decomposition \( df = \partial f + \bar{\partial} f \) does not depend on the local coordinate system, on account of the holomorphicity of the transition maps.

Exercise 2.2. A smooth map \( f : M \to N \) is called holomorphic iff all its local expressions \( \psi_i \circ f \circ \phi_j^{-1} \) have holomorphic Cartesian components. Verify that \( f \in C^\infty(M, \mathbb{C}) \) is holomorphic iff \( \bar{\partial} f = 0 \).

Definition 2.2. In view of (2.1), we have a splitting \( A^1(M) = A^{1,0}(M) \oplus A^{0,1}(M) \) of \( C^\infty(M, \mathbb{C}) \)-modules, and more generally,

\[
A^r(M) = \bigoplus_{p+q=r} A^{p,q}(M),
\]

where each \( A^{p,q}(M) \) is locally spanned by \( r \)-forms of the type

\[
f(z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m) \, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}.
\]

Thus the algebra of differential forms is bigraded: \( A^\bullet(M) = \bigoplus_{p+q} A^{p,q}(M) \). Again by (2.1), the exterior derivative splits as \( d = \partial + \bar{\partial} \), where \( \partial : A^{p,q}(M) \to A^{p+1,q}(M) \), \( \bar{\partial} : A^{p,q}(M) \to A^{p,q+1}(M) \). The identity \( d^2 = 0 \) yields the three identities

\[
\partial^2 = 0, \quad \bar{\partial} \partial = -\bar{\partial} \partial, \quad \bar{\partial}^2 = 0,
\]

on taking account of the grading degrees. (These identities say that the spaces \( A^{p,q}(M) \) form the vertices of a “double complex”.)

Exercise 2.3. Verify that the conjugation \( \omega \mapsto \bar{\omega} \) on \( A^\bullet(M, \mathbb{C}) = A^\bullet(M, \mathbb{R}) \otimes_\mathbb{R} \mathbb{C} \) is such that \( \bar{\partial} \bar{\omega} = \bar{\partial} \bar{\omega} \) and that \( \bar{\omega} \in A^{q,p}(M) \) whenever \( \omega \in A^{p,q}(M) \).

2.2 Local charts for complex projective spaces

Definition 2.3. The \( m \)-dimensional complex projective space \( \mathbb{CP}^m \) is the set of complex lines through the origin in \( \mathbb{C}^{m+1} \), i.e., the one-dimensional complex subspaces of \( \mathbb{C}^{m+1} \). For any nonzero \( v \in \mathbb{C}^{m+1} \), the line \( \langle v \rangle \equiv \mathbb{C} v \) lies in \( \mathbb{CP}^m \), and \( \eta : \mathbb{C}^{m+1} \setminus \{0\} \to \mathbb{CP}^m : v \mapsto \langle v \rangle \) is a quotient map.\(^3\) If \((z^0, z^1, \ldots, z^m)\) denotes coordinates in \( \mathbb{C}^{m+1} \) (with respect to the standard orthonormal basis), we may regard each \( z^j \) as a linear form on \( \mathbb{C}^{m+1} \); these cannot vanish simultaneously on \( \mathbb{C}^{m+1} \setminus \{0\} \), so we get the following chart domains for \( \mathbb{CP}^m \):

\[
U_j := \{ \langle v \rangle \in \mathbb{CP}^m : z^j(v) \neq 0 \}, \quad j = 0, 1, \ldots, m.
\]

Since \( \langle v \rangle \notin U_j \) iff \( \langle v \rangle \subset \ker z^j \), we can identify the complement of \( U_j \) with the set of lines in the hyperplane \( \ker z^j \), which is homeomorphic to \( \mathbb{CP}^{m-1} \). In particular, each \( U_j \) is open and dense in \( \mathbb{CP}^m \), since its complement is a lower-dimensional submanifold.

\(^3\)This quotient map actually defines the topology of \( \mathbb{CP}^m \).
For \( v \in U_j \) and \( k \neq j \), we define
\[
w_k^j(v) := \frac{z_k(v)}{z_j(v)}.
\]
This is well-defined since the fraction is unchanged under \( v \mapsto \lambda v \) with \( \lambda \in \mathbb{C}^\times \). Now
\[
\phi(v) := (w_0^j, \ldots, w_m^j)
\]
is a homeomorphism from \( U_j \) onto \( \mathbb{C}^m \), and \((w_0^j, \ldots, w_m^j)\) is a system of local coordinates for the chart \((U_j, \phi_j)\).

The \( w_k^j \) are Cartesian coordinates corresponding to the “homogeneous” coordinates \( z_k \), that is, \((w_0^j, w_1^j, \ldots, w_m^j) = [z^0 : z^1 : \ldots : z^m]\) in the standard notations.

On the overlap \( U_i \cap U_j \), we have
\[
w_k^i(v) = w_k^j(v) z_j(v) / z_i(v) = w_k^j(v) w_i^j(v)
\]
for \( k \notin \{i, j\} \), so the transition map \( \phi_i \circ \phi_j^{-1} \), when written as \( u \mapsto v \), is given by rational functions: \( v^j = 1/u^i \) and \( v^k = u^k / u^i \) for other \( k \), and hence is holomorphic. Thus the atlas \( \{(U_j, \phi_j) : j = 0, 1, \ldots, m\} \) makes \( \mathbb{CP}^m \) a complex manifold.

Since \( \mathbb{CP}^m \setminus U_m \approx \mathbb{CP}^{m-1} \) and \( U_m \approx \mathbb{C}^m \approx \mathbb{R}^{2m} \), we have (by induction) that \( \mathbb{CP}^m \) is topologically a cell complex with one \( k \)-dimensional cell in every even dimension \( k = 0, 2, \ldots, 2m \) and no odd-dimensional cells. This allows us to compute the singular homology groups of the complex projective spaces by standard topological techniques [23]: we get \( H_k(\mathbb{CP}^m, \mathbb{Z}) = \mathbb{Z} \) if \( k = 0, 2, \ldots, 2m \); \( H_k(\mathbb{CP}^m, \mathbb{Z}) = 0 \) for all other \( k \). Thus \( \mathbb{CP}^m \) is a “torsion-free” space, i.e., all its integral homology groups are free; it is known that then the singular cohomology groups with integer coefficients can be computed by duality:
\[
H^k(\mathbb{CP}^m, \mathbb{Z}) = \text{Hom}(H_k(\mathbb{CP}^m, \mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, \ldots, 2m, \\ 0 & \text{otherwise.} \end{cases}
\]

2.3 The Kähler form

Definition 2.4. For \( v = (z^0, z^1, \ldots, z^m) \in \mathbb{C}^{m+1} \), we write \( \|v\|^2 = \sum_k |z^k|^2 = \sum_k z^k \bar{z}^k \), and define
\[
\Phi_0(v) := i \partial \bar{\partial} \log \|v\|^2 = i \partial \left( \|v\|^{-2} \sum_k z^k d \bar{z}^k \right)
\]
\[
= \frac{i}{\|v\|^4} \left( \|v\|^2 \sum_k d z^k \wedge d \bar{z}^k - \sum_{r,s \neq j} \bar{z}^r z^s d z^r \wedge d \bar{z}^s \right),
\]

\(^4\)The notation \( \hat{j} \) indicates that the index \( j \) is omitted from the sequence \( 0, 1, \ldots, m \).
which is a 2-form in $A^{1,1}(\mathbb{C}^{m+1} \setminus \{0\})$. It is clear that $\Phi_0$ is homogeneous of degree 0, i.e., $\Phi_0(\lambda v) = \Phi_0(v)$ for $\lambda \in \mathbb{C}^\times$. This means that there is a 2-form $\Phi \in A^{1,1}(\mathbb{C}^m)$ such that $\Phi_0 = \eta^* \Phi$.

To find an expression for $\Phi$ in local coordinates on $U_j$, we may identify $U_j$ with a subset of the hyperplane $z^j = 1$ in $\mathbb{C}^{m+1}$, by using (2.3) with denominator 1. With this convention, the norm squared of a vector $v$ in this hyperplane defines a positive function $Q_j$ on $U_j$ (which tends to infinity at the boundary); in fact,

$$\|v\|^2 = Q_j(\langle v \rangle) := 1 + \sum_{k\neq j} w_j^k \bar{w}_j^k$$

(2.6) for $\langle v \rangle \in U_j$ with local coordinates (2.4). Thus

$$\Phi = i \partial \bar{\partial} \log Q_j = \frac{i}{Q_j^2} \left( Q_j \sum_{k \neq j} dw_j^k \wedge \bar{d}w_j^k - \sum_{r,s \neq j} \bar{w}_j^r w_j^s d\bar{w}_j^r \wedge d\bar{w}_j^s \right).$$

(2.7)

The 2-form $\Phi$ is in fact real-valued, as is evident from its local expression (2.7), or alternatively by noting that $\bar{\Phi} = -i \partial \bar{\partial} \log Q_j = \Phi$ on account of (2.2); this $\Phi \in A^2(\mathbb{C}^m, \mathbb{R})$ is called the Kähler form on $\mathbb{CP}^m$.

It is important to note that $\Phi = -\partial \bar{\partial} Q_j$ is a closed 2-form. This follows at once from (2.2), since $d\Phi = (\partial + \bar{\partial}) \Phi = -\partial^2(\partial Q_j) + \bar{\partial}^2(\partial Q_j) = 0$.

Exercise 2.4. A real-valued 2-form $\beta \in A^2(M)$ is called nondegenerate if each $\beta_x \in \Lambda^2 T_x^* M$ is nondegenerate as an alternating $\mathbb{R}$-bilinear form, i.e., $\beta_x(u, v) = 0$ for all $v \in T_x M$ implies $u = 0$ in $T_x M$. Show that $\beta$ is nondegenerate iff for any local expression $\beta|_U = b_{rs} dx^r \wedge dx^s$, the matrix of local coefficients $[b_{rs}]$ is nonsingular at each point of $U$. Verify that the Kähler form $\Phi$ is nondegenerate on $\mathbb{CP}^m$ by showing that the matrix $A$ with entries $Q_j \delta_{rs} - \bar{w}_j^r w_j^s$ is positive definite (apply the Schwarz inequality to show that $\varepsilon^t A \varepsilon > 0$ for all $\varepsilon \in \mathbb{C}^{m+1} \setminus \{0\}$).

Exercise 2.5. Show that $\Phi^{*m}$, the $m$-fold exterior power of $\Phi$, is a volume form on $\mathbb{CP}^m$, i.e., a 2m-form which is nonzero at each point of $\mathbb{CP}^m$ when regarded as a section of the line bundle $\Lambda^{2m} T^* \mathbb{CP}^m \to \mathbb{CP}^m$.

2.4 The Fubini–Study metric

Definition 2.5. Let $M$ be a differential manifold of even (real) dimension $n = 2m$. An almost complex structure on $M$ is an operator $J: \mathfrak{X}(M) \to \mathfrak{X}(M)$ which is $C^\infty(M, \mathbb{R})$-linear,\(^5\) and which satisfies $J^2 = -\mathrm{id}$. Thus $(JX)_x = J_x(X_x)$ where $J_x \in \mathrm{End}_\mathbb{R}(T_x M)$ with $(J_x)^2 = -\mathrm{id}$ for each $x \in M$.\(^6\) Almost complex structures need not exist; if one does exist, we say that $(M, J)$ is an almost complex manifold.

\(^5\)In other words, $J$ can be identified with a tensor on $M$, of bidegree $(1, 1)$, given by $(X, \alpha) \mapsto \alpha(J(X))$. One also says that $J$ is a “tensorial operator”.

\(^6\)An almost complex structure may therefore also be defined as a bundle automorphism $(J, \mathrm{id})$ of the tangent bundle $TM \to M$ for which $J^2 = -\mathrm{id}$.
It turns out that \( M \) need not be a complex manifold in order to possess an almost complex structure; a known example is the sphere \( S^6 \) which has an almost complex structure related to the Cayley numbers. See [15, 58] for some discussion of this. Here we will stick to complex manifolds.

A complex structure can always be defined locally if \( \dim M \) is even; it suffices to take local coordinates \((x^1, \ldots, x^m, y^1, \ldots, y^m)\) on a chart domain \( U \) and let

\[
J\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \quad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}.
\]  

(2.8)

Now \( J \) may be amplified to a \( C^\infty(U, \mathbb{C}) \)-linear operator on \( \mathfrak{X}(U, \mathbb{C}) \) in the natural way; from (2.8) we derive

\[
J\left(\frac{\partial}{\partial z^k}\right) = i \frac{\partial}{\partial z^k}, \quad J\left(\frac{\partial}{\partial \bar{z}^k}\right) = -i \frac{\partial}{\partial \bar{z}^k}.
\]  

(2.9)

Exercise 2.6. Verify this independence on an overlap \( U_i \cap U_j \) of charts of \( \mathbb{CP}^m \).

Lemma 2.1. Let \( M \) be a manifold with an almost complex structure \( J \), and let \( \Phi \in \mathcal{A}^2(M) \) be a real-valued nondegenerate 2-form on \( M \) which is invariant under \( J \), i.e., \( \Phi(JX, JY) = \Phi(X, Y) \) for \( X, Y \in \mathfrak{X}(M) \). Then the recipe \( g(X, Y) := \Phi(X, JY) \) defines a symmetric tensor on \( M \) which is also \( J \)-invariant.

Proof. Clearly \( g \) is a tensor of bidegree \((2, 0)\); symmetry follows from invariance, since \( g(Y, X) = \Phi(Y, JX) = -\Phi(JX, Y) = \Phi(JX, \bar{J}Y) = \Phi(X, JY) = g(X, Y) \).

Exercise 2.7. Check that \( g \) is positive definite, and thus defines a Riemannian metric on \( \mathbb{CP}^m \). (Recall that the matrix with entries \( Q_j \delta^s_r - \bar{w}_j^s w^s_j \) is positive definite, by Exercise 2.4.)

Definition 2.6. The metric (2.10) is the Fubini–Study metric on \( \mathbb{CP}^m \).
2.5 The Riemann sphere

When \( m = 1 \), the complex projective space \( \mathbb{C}P^1 \) is identified with the Riemann sphere \( \mathbb{C}_\infty = \mathbb{C} \cup \{ \infty \} \), by identifying \([z^0 : z^1] \) with \( z = z^1/z^0 \in \mathbb{C} \) if \( z^0 \neq 0 \), and \([1 : 0] \) with \( \infty \). Notice that \( z \equiv u_0^3 \) is the local complex coordinate in \( U_0 = \mathbb{C}_\infty \setminus \{ \infty \} \). By writing \( z = x + iy \), we can also identify \( \mathbb{C}_\infty \) with the two-sphere \( S^2 \), regarding the latter as a submanifold of \( \mathbb{R}^3 \), via the stereographic projection

\[
f(z) := \left( \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)
\]

and \( f([1 : 0]) := (0, 0, 1) \).

The Kähler form on \( \mathbb{C}_\infty \) is easily found. We have \( Q_0((v)) = 1 + |v|^2 \) for \( v = (z^0, z^1) \in \mathbb{C}^2 \), so (2.7) reduces to

\[
\Phi = \frac{i \, dz \wedge d\bar{z}}{(1 + |z|^2)^2} = -\frac{2 \, dx \wedge dy}{(1 + x^2 + y^2)^2}.
\]

**Exercise 2.8.** Write \((u^1, u^2, u^3)\) to denote the right hand side of (2.11). Check that \( \Phi = f^* \Omega \), where \( \Omega \in \mathcal{A}^2(\mathbb{R}^3) \) is given by \( \Omega = (du^1 \wedge du^2)/2u^3 \). The usual spherical coordinates on \( S^2 \) are defined by the map \( h : \mathbb{R}^2 \to S^2 \) where \( h(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). Check that \( h^* \Omega = -\frac{1}{2} \sin \theta \, d\theta \wedge d\phi \).

The Riemannian metric (2.10) reduces to \( g = 2(1 + |z|^2)^{-2} \, dz \cdot d\bar{z} \) or equivalently \( g = 2(1 + x^2 + y^2)^{-2} \, (dx^2 + dy^2) \), where \( dx^2 \) denotes the symmetric product \( dx \cdot dx \).

**Exercise 2.9.** Show that \( g = f^* (\frac{1}{2} G) \) where \( G = (du^1)^2 + (du^2)^2 + (du^3)^2 \) is the standard Riemannian metric on \( \mathbb{R}^3 \), and that \( h^* G = d\theta^2 + \sin^2 \theta \, d\phi^2 \).

We may regard \((\theta, \phi)\) as local coordinates on the Riemann sphere. Thus we write simply:

\[
\Phi = -\frac{1}{2} (\sin \theta \, d\theta \wedge d\phi), \quad g = \frac{1}{2} (d\theta^2 + \sin^2 \theta \, d\phi^2).
\]

It may prove useful to have available some relations between the local coordinate systems \((z, \bar{z})\) and \((\theta, \phi)\). We list a few identities that follow from the previous formulæ:

\[
z = \frac{\sin \theta}{1 - \cos \theta} e^{i\phi}, \quad dz = \frac{e^{i\phi}}{1 - \cos \theta} (-d\theta + i \sin \theta \, d\phi), \quad Q_0 = \frac{2}{1 - \cos \theta}.
\]

**Exercise 2.10.** Verify the formulæ (2.14) and derive (2.13) directly.

**Exercise 2.11.** Show that the complex structure \( \mathcal{J} \) on \( S^2 \) satisfying \( g(X, Y) = \Phi(X, \mathcal{J} Y) \) is given by \( \mathcal{J}(\partial/\partial \theta) = -\csc \theta \, \partial/\partial \phi \) and \( \mathcal{J}(\csc \theta \, \partial/\partial \phi) = \partial/\partial \theta \) in spherical coordinates.

3 The de Rham complex and Hodge duality

3.1 The de Rham complex

**Definition 3.1.** If \( M \) is an \( n \)-dimensional differential manifold, its **de Rham complex** is the cochain complex

\[
\mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \rightarrow \cdots \rightarrow \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \rightarrow \cdots \rightarrow \mathcal{A}^n(M)
\]
where $\mathcal{A}^k(M) = \mathcal{A}^k(M, \mathbb{R})$ denotes the real-valued differential forms on $M$, and $d$ is the exterior derivation. We recall (see Appendix A) that its $k$-cocycles are the closed differential $k$-forms $Z^k_{\text{dr}}(M) := \{ \omega \in \mathcal{A}^k(M) : d\omega = 0 \}$, and its $k$-coboundaries are the exact differential $k$-forms $B^k_{\text{dr}}(M) := \{ d\beta : \beta \in \mathcal{A}^{k-1}(M) \}$. The $k$-th de Rham cohomology group $H^k_{\text{dr}}(M) := \ker(d : \mathcal{A}^k(M)) / \operatorname{im}(\delta : \mathcal{A}^{k-1}(M))$ is a real vector space.

The 0-cocycles in $Z^0_{\text{dr}}(M)$ are locally constant smooth functions, so $H^0_{\text{dr}}(M) = \mathbb{R}^m$ if $M$ has exactly $m$ connected components. In particular, $H^0_{\text{dr}}(M) \neq 0$. If $U$ is a contractible manifold, then by the Poincaré lemma, $H^k_{\text{dr}}(U) = 0$ for each $k = 1, 2, \ldots, n$.

If $M$ is an orientable compact manifold (without boundary), with volume form $\nu \in \mathcal{A}^n(M)$, then $d\nu = 0$ and $\nu$ is not exact since $\int_M \nu \neq 0$. If $\mu \in \mathcal{A}^n(M)$ is exact, with $\mu = d\beta$, say, then $\int_M \mu = \int_M d\beta = 0$ by Stokes’ theorem, since $\partial M = \emptyset$; thus, a volume form cannot be exact.) Therefore $H^n_{\text{dr}}(M) \neq 0$. Moreover, since the integral vanishes on exact $(n-1)$-forms, again by Stokes’ theorem, we see that $[\omega] \mapsto \int_M \omega$ is a well-defined linear form on $H^n_{\text{dr}}(M)$.

### 3.2 The Riemannian volume form

**Definition 3.2.** Let $g$ be a Riemannian metric on the compact oriented manifold $M$. Then $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is a symmetric $C^\infty(M)$-bilinear form such that each $g_x : T_x M \times T_x M \to \mathbb{R}$ is positive definite. The pair $(M, g)$ is called a Riemannian manifold.

If $U \subseteq M$ is a chart domain with local coordinates $x^1, \ldots, x^n$, let $g_{ij} := g(\partial/\partial x^i, \partial/\partial x^j) \in C^\infty(U)$; then $[g_{ij}]$ is a symmetric matrix, whose determinant we write as $\det g$ (by a slight abuse of notation, since this determinant is coordinate-dependent), and we have $g = g_{ij} \, dx^i \wedge dx^j$ on $U$.

For each $x \in M$, there is a vector space isomorphism $\hat{g}_x : T_x M \to T^*_x M$ of $\nu \mapsto (\nu \mapsto g_x(u, v))$; these determine a diffeomorphism $\hat{g} : TM \to T^*M$ such that $(\hat{g}, \operatorname{id}_M)$ is an equivalence between the tangent and cotangent bundles. They also determine tensorial operators $X \mapsto X^\flat : \mathfrak{X}(M) \to \mathcal{A}^1(M)$, given by $X^\flat(Y) := g(X, Y)$, and its inverse $\alpha \mapsto \alpha^\flat : \mathcal{A}^1(M) \to \mathfrak{X}(M)$, given by $\alpha(Y) := g(\alpha^\flat, Y)$. These “musical isomorphisms” [8] allow us to identify vector fields and 1-forms and to use them interchangeably, as the occasion demands.

The metric $g$ defines thus defines bilinear pairings, not only of vector fields but also of 1-forms and indeed of differential forms of any degree; we will denote all these pairings by $\langle \cdot, \cdot \rangle$ whenever a fixed $g$ is given. Thus $\langle X \mid Y \rangle := g(X, Y)$ for $X, Y \in \mathfrak{X}(M)$; $(\alpha \mid \beta) := g(\alpha^\flat, \beta^\flat)$ for $\alpha, \beta \in \mathcal{A}^1(M)$; and the pairing on $\mathcal{A}^k(M)$ (for $k > 1$) is determined by

$$
(\alpha^1 \wedge \cdots \wedge \alpha^k \mid \beta^1 \wedge \cdots \wedge \beta^k) := \det[(\alpha^i \mid \beta^j)]
$$

for $\alpha^1, \ldots, \alpha^k, \beta^1, \ldots, \beta^k \in \mathcal{A}^1(M)$.

**Definition 3.3.** Let $(M, g)$ be an oriented Riemannian manifold. Choose a local orthonormal frame $X_1, \ldots, X_n$ for $\mathfrak{X}(U)$, i.e., vector fields such that $g(X_r, X_s) = \delta_{rs}$, which is oriented.

---

1 A complex $(C^\bullet, d)$ is called “acyclic” if $H^k(C^\bullet, d) = 0$ for $k > 0$. Thus the Poincaré lemma says that the de Rham complex of a contractible manifold is acyclic.
that is, \( \nu(X_1, \ldots, X_n) > 0 \) where \( \nu \) is a volume form on \( M \) which defines the orientation. Let \( \hat{\theta}^k := X_k^i \in \mathcal{A}^1(U) \), so that \( \hat{\theta}^1, \ldots, \hat{\theta}^n \) is a local (oriented) orthonormal frame for \( \mathcal{A}^1(U) \). Then
\[
\Omega := \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \cdots \wedge \hat{\theta}^n \in \mathcal{A}^n(M)
\] (3.1)
is a volume form on \( M \), since \( (\Omega \mid \Omega) = 1 \) and in particular \( \Omega_x \neq 0 \) in \( \Lambda^n T_x^* M \) for all \( x \in M \). This \( \Omega \) is called the **Riemannian volume form** on \((M, g)\).

To see that \( \Omega \) is in fact independent of the choice of oriented orthonormal frame, we argue as follows. If \( Y_1, \ldots, Y_n \) is another oriented orthonormal frame for \( \mathfrak{X}(V) \), where \( U \cap V \not= \emptyset \), then \( Y_i := a_i^j X_j \) where \( a = [a_i^j] \) is a smooth function on \( U \cap V \) with values in \( SO(n) \), the group of orthogonal \( n \times n \) matrices of determinant 1. (That is to say, \( a \) is a local section of the principal \( SO(n) \)-bundle of oriented orthonormal frames on \( M \).) Now \( \hat{\theta}^k := Y_k^a \) gives the corresponding oriented orthonormal basis for \( \mathcal{A}^1(V) \), and thus \( \hat{\theta}^k = a^k_i \hat{\theta}_i \), from which it follows that \( \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^n = (\det a) \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^n = \Omega \) on \( U \cap V \). Therefore, \( \Omega \) is defined globally by (3.1).

One can express \( \Omega \) in terms of local coordinates \( x^1, \ldots, x^n \) on \( U \) for which the \( \partial / \partial x^i \) need not be orthonormal, i.e., \( g_{ij} \neq \delta_{ij} \) in general. The local coordinates should, however, be compatible with the orientation, which means that \( \det g > 0 \). If \( y^1, \ldots, y^n \) are other local coordinates on \( V \), and if \( \tilde{g}_{ij} := g(\partial / \partial y^i, \partial / \partial y^j) \in C^\infty(V) \), then \( \det \tilde{g} = J^2 \det g \), where \( J := \det(\partial y^i / \partial x^j) \) is the Jacobian of the transition function (which is positive). Hence
\[
\sqrt{\det \tilde{g}} \, dy^1 \wedge \cdots \wedge dy^n = \sqrt{\det g} \, dx^1 \wedge \cdots \wedge dx^n
\] (3.2)
on \( U \cap V \), and thus the coordinate-independent expression \( \sqrt{\det g} \, dx^1 \wedge \cdots \wedge dx^n \) defines a volume on \( M \). In particular, one may choose \( y^1, \ldots, y^n \) so that \( \tilde{g}_{ij} = \delta_{ij} \), so \( \det \tilde{g} = 1 \) and \( dy^1, \ldots, dy^n \) is an orthonormal frame: so the volume form (3.2) coincides with \( \Omega \) of (3.1). In other words,
\[
\Omega = \sqrt{\det g} \, dx^1 \wedge \cdots \wedge dx^n
\] (3.3)
in any oriented local coordinate system.

### 3.3 The Hodge star operator

**Definition 3.4.** Let \((M, g)\) be a compact oriented Riemannian manifold of dimension \( n \), and write \( m := \lfloor n/2 \rfloor \) (so \( n = 2m \) or else \( n = 2m - 1 \)). Define the **Hodge star operator** \( \ast : \mathcal{A}^\bullet(M, \mathbb{C}) \to \mathcal{A}^\bullet(M, \mathbb{C}) \) as follows. Choose a local orthonormal frame \( X_1, \ldots, X_n \) for \( \mathfrak{X}(U) \) on some chart domain \( U \subset M \), and let \( \hat{\theta}^1, \ldots, \hat{\theta}^n \) be the corresponding local orthonormal frame for \( \mathcal{A}^1(U) \), determined by \( \hat{\theta}^i(X_j) := \delta^i_j \). Define \( \ast \) on \( \mathcal{A}^\bullet(U, \mathbb{C}) \) by
\[
\ast := i^m(\epsilon(\hat{\theta}^1) - \iota(X_1)) \cdots (\epsilon(\hat{\theta}^n) - \iota(X_n)),
\] (3.4)
where \( \iota(X) \) denotes contraction with the vector field \( X \) and \( \epsilon(\alpha) : \omega \mapsto \alpha \wedge \omega \) denotes exterior product with the 1-form \( \alpha \). It is readily checked that the right hand side of (3.4) is independent of these local orthonormal frames, and therefore defines an operator on all of \( \mathcal{A}^\bullet(M, \mathbb{C}) \).
To simplify products of noncommuting quantities as in (3.4), we use the following notation: if \( a_{j_1}, a_{j_2}, \ldots, a_{j_k} \) are elements of some algebra, where the indices \( j_r \) are in increasing order, write
\[
\prod_{1 \leq r \leq k} a_{j_r} := a_{j_1}a_{j_2} \ldots a_{j_k}.
\] (3.5)

Thus \( \star = i^m \prod_{1 \leq r \leq k} \epsilon(\hat{\theta}^r) - \iota(X_r) \). It is also convenient to write \( J := \{ j_1, \ldots, j_k \} \) and denote the right hand side of (3.5) by \( \prod_{j \in J} a_j \), where it is understood that the indices \( j \in J \) are arranged in increasing order.

**Exercise 3.1.** Let \( V \) be an \( n \)-dimensional oriented real vector space with a positive definite symmetric bilinear form \( g \). For \( u \in V \), \( \alpha \in V^* \), define operators \( \iota(u) : \Lambda^k V^* \to \Lambda^{k-1} V^* \) and \( \epsilon(\alpha) : \Lambda^k V^* \to \Lambda^{k+1} V^* \) by \( \iota(u)\eta(v_1, \ldots, v_{k-1}) := \eta(u, v_1, \ldots, v_{k-1}) \) and \( \epsilon(\alpha)\eta := \alpha \wedge \eta \).

Let \( \{ e_1, \ldots, e_n \}, \{ e'_1, \ldots, e'_n \} \) be oriented orthonormal bases for \((V, g)\) (so the change-of-basis matrix \([g(e_i, e'_j)]\) has determinant +1) and let \( \{ \zeta^1, \ldots, \zeta^n \}, \{ \zeta'^1, \ldots, \zeta'^n \} \) be the respective dual bases for \( V^* \). Show that \( \prod_{1 \leq r \leq n} \epsilon(\zeta'^r) - \iota(e'_r) = \prod_{1 \leq r \leq n} \epsilon(\zeta^r) - \iota(e_r) \) as operators on \( \Lambda^* V^* \).

**Exercise 3.2.** Use the previous exercise to show that the right hand side of (3.4) is independent of the given local orthonormal frames \{\(X_j\)} and \{\(\hat{\theta}^k\)}.

Notice that when \( m \) is even, i.e., when \( n \) is of the form \( 4k \) or \( 4k + 3 \) for some integer \( k \), then it is not necessary to use complex-valued forms since \( \star \) takes \( \mathcal{A}^*(M, \mathbb{R}) \) to \( \mathcal{A}^*(M, \mathbb{R}) \). However, for \( n \) of the form \( 4k + 1 \) or \( 4k + 2 \), we have \( i^m = \pm i \) and complex forms are needed. More traditional treatments of Hodge duality [1, 28, 33, 57] use different sign conventions, so that \( \star \) operates on \( \mathcal{A}^*(M, \mathbb{R}) \) in all cases, but the important involutivity property (see Lemma 3.3 below) is then lost.

**Lemma 3.1.** If \( J := \{ j_1, \ldots, j_k \} \subseteq \{ 1, \ldots, n \} \) and if \( J' := \{ 1, \ldots, n \} \setminus J = \{ i_1, \ldots, i_{n-k} \} \) is its complement, with indices written in increasing order in both cases, let \( \eta(J, J') = \pm 1 \) denote the sign of the shuffle permutation which reorders \( 1, 2, \ldots, n \) as \( (j_1, \ldots, j_k, i_1, \ldots, i_{n-k}) \).

Write, for brevity, \( \hat{\theta}^J := \hat{\theta}^{j_1} \wedge \cdots \wedge \hat{\theta}^{j_k} \), where \( \hat{\theta}^1, \ldots, \hat{\theta}^n \) is a local orthonormal basis of \( \mathcal{A}^1(M) \).

Then the Hodge star operator is given explicitly on this basis by the recipe:
\[
\star \hat{\theta}^J = i^m (-1)^{nk+k(k-1)/2} \eta(J, J') \hat{\theta}^{J'}.
\] (3.6)
Proof. This identity results from the following calculation:

\[ \star \hat{\theta}^J = i^m \prod_{1 \leq j \leq n} (\epsilon(\hat{\theta}^j) - \iota(X_j)) \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^k \]

\[ = i^m (-1)^{k(k-1)/2} \prod_{1 \leq j \leq n} (\epsilon(\hat{\theta}^j) - \iota(X_j)) \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^j \]

\[ = i^m (-1)^{k(k-1)/2} \eta(J', J) \prod_{i \in J'} \epsilon(\hat{\theta}^i) \prod_{j \in J} (-\iota(X_j)) \hat{\theta}^j \wedge \cdots \wedge \hat{\theta}^j \]

\[ = i^m (-1)^{k(k-1)/2} (-1)^k \eta(J', J) \prod_{i \in J'} \epsilon(\hat{\theta}^i) 1 \]

\[ = i^m (-1)^{k(k-1)/2} (-1)^k (-1)^k (-1)^{k(n-k)} \eta(J, J') \hat{\theta}^J'. \]

Here, one first reverses the order of the exterior product of 1-forms \( \hat{\theta}^1 \wedge \cdots \wedge \hat{\theta}^k \), by a permutation of sign \((-1)^{k(k-1)/2}\). Then the operators \( \epsilon(\hat{\theta}^j) - \iota(X_j) \) act successively on forms of type \( \hat{\theta}^j \wedge \beta^j \), yielding \(-\beta^j\) at each stage, until only the constant \((-1)^k\) remains; next, the operators \( \epsilon(\hat{\theta}^j) - \iota(X_i) \), with \( i \in J' \), act successively on forms of type \( \gamma^j \), yielding \( \hat{\theta}^j \wedge \gamma^j \) at each stage, until \( \hat{\theta}^J' \) is created. Lastly, the identity \( \eta(J, J') \eta(J', J) = (-1)^{k(n-k)} \) is used: this is just the observation that the permutation which interchanges the blocks of indices \( J \) and \( J' \) —while preserving the order within each block— is a product of \( k(n-k) \) transpositions. Now (3.6) follows on noticing that \((-1)^{k}\) \((-1)^{k(n-k)} = (-1)^n(-1)^{k(k+1)} = (-1)^{nk}. \)

\[ \square \]

Corollary 3.2. The Hodge star operator maps \( \mathcal{A}^k(M, \mathbb{C}) \) into \( \mathcal{A}^{n-k}(M, \mathbb{C}) \). \[ \square \]

The next calculation shows that the presence of the factor \( i^m \) is what ensures that \( \star \) is an involution, i.e., an operator whose square is the identity.

Lemma 3.3. \( \star \star = \text{id} \).

Proof. From (3.6) it is clear that \( \star \star \hat{\theta}^J = C_J \hat{\theta}^J \) for some constant \( C_J \). One computes:

\[ C_J = i^{2m}(-1)^{nk+k(k-1)/2}(-1)^{n(n-k)+(n-k)(n-k-1)/2} \eta(J, J') \eta(J', J) \]

\[ = (-1)^m(-1)^{n^2}(-1)^{k(k-1)+(n-k)(n-k-1)/2}(-1)^{k(n-k)} \]

\[ = (-1)^m(-1)^{n^2}(-1)^{(n^2-n(2k+1)+2k^2)/2}(-1)^{nk-k^2} \]

\[ = (-1)^m(-1)^{(3n^2-n)/2} = (-1)^m(-1)^{(n^2+n)/2} = (-1)^m(-1)^{(2m+1)} = +1. \]

This establishes that \( \star \star = \text{id} \), which also implies that \( \star \) maps \( \mathcal{A}^k(M, \mathbb{C}) \) onto \( \mathcal{A}^{n-k}(M, \mathbb{C}) \). \( \square \)

Exercise 3.3. If \( X \in \mathfrak{X}(M) \), show that \( \star \epsilon(X^\flat) \star = (-1)^n \iota(X) \) as operators from \( \mathcal{A}^k(M) \) to \( \mathcal{A}^{k-1}(M) \). (Reduce to the special case \( X = X_j \).)

28
Consider the case $M = \mathbb{R}^3$; though not compact, it is an oriented Riemannian manifold with the usual euclidean metric; here $\hat{\theta}^k = dx^k$, and $\star$ is determined by

$$
\star 1 = -\Omega = -dx^1 \wedge dx^2 \wedge dx^3,
\star dx^1 = dx^2 \wedge dx^3, \quad \star dx^2 = -dx^1 \wedge dx^3, \quad \star dx^3 = dx^1 \wedge dx^2,
$$
on account of (3.6).

When $M = \mathbb{C}$ with local coordinates $x, y$, take $z = x + iy$. For the usual metric, we have $\hat{\theta}^1 = dx, \hat{\theta}^2 = dy$, and so $\star dx = i dy, \star dy = -i dx$. Using the coordinates $z, \bar{z}$, we then have $\star dz = dz, \star d\bar{z} = -d\bar{z}$. Thus, $\star df = \star (\partial f + \bar{\partial} f) = \bar{\partial} f - \partial f$, so that $f \in C^\infty(\mathbb{C}, \mathbb{C})$ is holomorphic iff $\bar{\partial} f = 0$ (by the Cauchy–Riemann equations) iff $\star df = df$. More generally, any $\alpha \in A^\bullet(M, \mathbb{C})$ may be written as $\alpha = \alpha^+ + \alpha^-$ where $\alpha^+$ is selfdual, i.e., $\star \alpha^+ = \alpha^+$, and $\alpha^-$ is antselfdual, i.e., $\star \alpha^- = -\alpha^-$. When $n = 2m$ and $\alpha \in A^m(M)$, we have $\alpha^+, \alpha^- \in A^m(M)$ also. When $n = 2$ and $M$ is a compact complex manifold (i.e., a Riemann surface), $f \in C^\infty(M, \mathbb{C})$ is holomorphic iff $df$ is selfdual.

An important example is $M = S^2$ with the Riemannian metric $g = d\theta^2 + \sin^2 \theta d\phi^2$. In spherical coordinates, a local orthonormal basis is given by

$$
\hat{\theta}^1 = d\theta, \quad \hat{\theta}^2 = \sin \theta d\phi.
$$

The star operator on $S^2$ is determined by

$$
\star 1 = i\Omega = i \sin \theta \, d\theta \wedge d\phi, \quad \star d\theta = i \sin \theta \, d\phi,
$$

and thus $\star \Omega = -i$ and $\star (\sin \theta \, d\phi) = -i \, d\theta$.

For a general system of coordinates, the star operator can be described as follows. On a chart domain $U$, a $k$-form $\omega \in A^k(U)$ can be written as $\omega = \omega_{j_1 \ldots j_k} \, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$. New coefficients with "raised indices" are defined by $\omega^{r_1 \ldots r_k} = g_r^{i_1 j_1} \cdots g_r^{i_k j_k} \omega_{j_1 \ldots j_k}$ where $[g^r_s]$ denotes the matrix inverse to the matrix $[g_{ij}]$. Let $\epsilon_{t_1 \ldots t_n} := 0$ if $t_1, \ldots, t_n$ contains a repeated index, and otherwise let $\epsilon_{t_1 \ldots t_n} := \pm 1$ be the sign of the permutation $(1, \ldots, n) \mapsto (t_1, \ldots, t_n)$. Then $\star \omega$ is given by

$$
\star (\omega_{j_1 \ldots j_k} \, dx^{j_1} \wedge \cdots \wedge dx^{j_k}) = i^n (-1)^{nk+k(k-1)/2} \frac{\sqrt{\det g}}{(n-k)!} \epsilon_{t_1 \ldots t_n \, r_1 \ldots r_n} \omega^{r_1 \ldots r_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_{n-k}}.
$$

(3.7)

**Exercise 3.4.** Check the validity of (3.7) by showing that its right hand side is invariant under a general change of (oriented) frame, and that it reduces to (3.6) for an orthonormal frame.

**Exercise 3.5.** Work out the action of $\star$ for the Fubini–Study metric on $\mathbb{C}P^n$.

The Hodge star operator relates the exterior product of forms to the Riemannian volume form, as follows.
Lemma 3.4. For $\alpha, \beta \in \mathcal{A}^k(M, \mathbb{C})$, 
\[
\alpha \wedge * \beta = i^m(-1)^{nk+k(k-1)/2}(\alpha \mid \beta) \Omega.
\] (3.8)

Proof. Since both sides of (3.8) are $C^\infty(M)$-bilinear in $(\alpha, \beta)$, it suffices to verify equality for a local basis; thus we may take $\alpha = \hat{\theta}^I$, $\beta = \hat{\theta}^J$ where $I$ and $J$ are subsets of \{1, \ldots, n\}.

Now $(\hat{\theta}^I \mid \hat{\theta}^J) = 1$ or 0 according as $I = J$ or not. If $I \neq J$ then $\hat{\theta}^I \wedge * \hat{\theta}^J = 0$ from (3.6), since $I \cap J' \neq \emptyset$. Also, $\hat{\theta}^I \wedge * \hat{\theta}^J = i^m(-1)^{nk+k(k-1)/2} \eta(J, J') \hat{\theta}^I \wedge J = i^m(-1)^{nk+k(k-1)/2} \Omega$. $\square$

Definition 3.5. The complex vector space $\mathcal{A}^k(M, \mathbb{C})$ becomes a prehilbert space under the positive definite Hermitian form
\[
\langle \alpha \mid \beta \rangle := \int_M (\bar{\alpha} \mid \beta) \Omega,
\]
where the integral over $M$ is normalized by the requirement that $\int_M \Omega = 1$. Its completion with respect to this “integrated inner product” \(^2\) is a Hilbert space which may be denoted $L^{2,k}(M)$. Notice that $\langle \alpha \mid \beta \rangle = \langle \beta \mid \bar{\alpha} \rangle$, i.e., the conjugation of forms is antilinear. The inner product may be extended to all of $\mathcal{A}^\bullet(M, \mathbb{C})$ by declaring that $k$-forms and $l$-forms be orthogonal for $k \neq l$; the completion of $\mathcal{A}^\bullet(M, \mathbb{C})$ is the Hilbert-space direct sum
\[
L^{2,\bullet}(M) := \bigoplus_{k=0}^n L^{2,k}(M).
\]

From (3.8), it follows that
\[
\langle \star \alpha \mid \star \beta \rangle = (-1)^m \bar{\alpha} \wedge \star \beta.
\]

Lemma 3.5. The Hodge star operator is isometric, i.e., if $\alpha, \beta \in \mathcal{A}^k(M, \mathbb{C})$ then
\[
\langle \star \alpha \mid \star \beta \rangle = \langle \alpha \mid \beta \rangle.
\]

Proof. From (3.6) it follows that $\star \bar{\alpha} = (-1)^m \bar{\alpha}$ for any $\alpha \in \mathcal{A}^\bullet(M, \mathbb{C})$. Applying (3.8) twice,
\[
\langle \star \alpha \mid \star \beta \rangle = i^m(-1)^{n(n-k)+(n-k)(n-k-1)/2} \int_M \bar{\alpha} \wedge \beta
\]
\[
= i^m(-1)^{n^2+n(n-1)/2+k(k+1)/2} \int_M \bar{\alpha} \wedge \beta
\]
\[
= i^m(-1)^n(n+1)/2+k(k+1)/2(-1)^k(n-k) \int_M \beta \wedge \star \bar{\alpha}
\]
\[
= (-1)^m(-1)^{n(n+1)/2} \langle \bar{\beta} \mid \bar{\alpha} \rangle = \langle \alpha \mid \beta \rangle
\]

since $\frac{1}{2}n(n+1) = m(2m \pm 1) \equiv m \mod 2$. \hfill $\Box$

Since $\star \star = id$ on $L^{2,k}(M)$, this isometry also satisfies $\langle \star \alpha \mid \gamma \rangle = \langle \alpha \mid \star \gamma \rangle$ for $\alpha \in \mathcal{A}^k(M, \mathbb{C})$, $\gamma \in \mathcal{A}^{n-k}(M, \mathbb{C})$; thus the Hodge star operator extends to a selfadjoint unitary operator on $L^{2,k}(M) \oplus L^{2,n-k}(M)$.

\(^2\)The double brackets are intended to distinguished the integrated inner product from the $C^\infty(M)$-valued form $(\cdot \mid \cdot)$.
\section{The Hodge Laplacian}

\textbf{Definition 3.6.} The codifferential $\delta : \mathcal{A}^k(M, \mathbb{C}) \to \mathcal{A}^{k-1}(M, \mathbb{C})$ is the adjoint of the exterior derivative with respect to the integrated inner product. In other words, if $\alpha \in \mathcal{A}^k(M, \mathbb{C})$, $\beta \in \mathcal{A}^{k-1}(M, \mathbb{C})$, $\delta \alpha$ is determined by the relation

$$\langle \langle \delta \alpha \mid \beta \rangle \rangle := \langle \langle \alpha \mid d\beta \rangle \rangle.$$ \hspace{1cm} (3.9)

The Riesz theorem “guarantees” that there is a unique element $\delta \alpha \in L^2_{k-1}(M)$ satisfying (3.9), though it is not immediately clear that $\delta \alpha \in \mathcal{A}^{k-1}(M, \mathbb{C})$. That this is indeed the case, and that $\delta$ takes $\mathcal{A}^k(M, \mathbb{R})$ to $\mathcal{A}^{k-1}(M, \mathbb{R})$, is a consequence of the following basic identity.

\textbf{Lemma 3.6.} $\delta = (-1)^{n+1} \star d \star$. \hspace{1cm} \Box

\textit{Proof.} If $\alpha \in \mathcal{A}^k(M, \mathbb{C})$ and $\beta \in \mathcal{A}^{k-1}(M, \mathbb{C})$, then $d\bar{\beta} \wedge \star \alpha + (-1)^{k-1} \bar{\beta} \wedge d(\star \alpha) = d(\bar{\beta} \wedge \star \alpha)$ is an exact $n$-form, whose integral is zero by Stokes’ theorem. Thus

$$\langle \langle \beta \mid \delta \alpha \rangle \rangle = \langle \langle d\bar{\beta} \wedge \star \alpha \rangle \rangle = i^{-m}(-1)^{nk+k(k-1)/2} \int_M d\bar{\beta} \wedge \star \alpha$$
$$= i^{-m}(-1)^{nk+k(k-1)/2} \int_M (-1)^k \bar{\beta} \wedge d(\star \alpha)$$
$$= (-1)^{nk+k(k-1)/2}(-1)^k (-1)^{n(k-1)+(k-1)(k-2)/2} \langle \langle \bar{\beta} \mid \star d \star \alpha \rangle \rangle$$
$$= (-1)^{n+1} \langle \langle \beta \mid \star d \star \alpha \rangle \rangle,$$

and so $\delta \alpha = (-1)^{n+1} \star d \star \alpha$ since $\beta$ is arbitrary. \hspace{1cm} \Box

Clearly $\delta^2 = 0$, and in particular $(\mathcal{A}^\bullet(M), \delta)$ is a chain complex. We say that $\omega \in \mathcal{A}^k(M)$ is \textbf{coclosed} if $\delta \omega = 0$, or \textbf{coexact} if $\omega = \delta \zeta$ for some $\zeta \in \mathcal{A}^{k+1}(M)$.

\textbf{Definition 3.7.} The \textbf{Hodge Laplacian} is the operator $\Delta$ on $\mathcal{A}^\bullet(M)$ defined as

$$\Delta := (d + \delta)^2 = d\delta + \delta d.$$

Notice that $\Delta$ takes $\mathcal{A}^k(M)$ to $\mathcal{A}^k(M)$, since $d$ raises and $\delta$ lowers degrees by one. [The restriction of $\Delta$ to 0-forms is called the \textit{Laplace–Beltrami operator} on $C^\infty(M)$.

A $k$-form $\gamma$ is called \textbf{harmonic} if $\Delta \gamma = 0$. We denote the vector space of harmonic $k$-forms by $\text{Harm}^k(M)$.

\textbf{Lemma 3.7.} A $k$-form $\gamma$ is harmonic iff $d\gamma = \delta \gamma = 0$.

\textit{Proof.} If $\gamma$ is both closed and coclosed, then clearly $\Delta \gamma = d(\delta \gamma) + \delta(d \gamma) = 0$. On the other hand, for any $\omega \in \mathcal{A}^\bullet(M)$,

$$\langle \langle \omega \mid \Delta \omega \rangle \rangle = \langle \langle \omega \mid d \delta \omega \rangle \rangle + \langle \langle \omega \mid \delta d \omega \rangle \rangle = \langle \langle \delta \omega \mid \delta \omega \rangle \rangle + \langle \langle d \omega \mid d \omega \rangle \rangle \geq 0,$$

so that $\Delta \gamma = 0$ implies $d\gamma = \delta \gamma = 0$. \hspace{1cm} \Box

\textbf{31}
The Laplacian $\Delta$ commutes with the operators $\star$, $d$ and $\delta$. Indeed, $\star d\delta = (-1)^{n+1}d\star d\delta = \delta d\star$ and $\star d\delta = (-1)^{n+1}d\star d = d\delta\star$, so $\Delta = \star d\delta + \delta d\star = \Delta\star$. Also, $d\Delta = d\delta d = \Delta d$ and $\delta\Delta = d\delta = \Delta\delta$ since $d^2 = \delta^2 = 0$.

On the Hilbert space $L^2(M)$, $\Delta = d\delta + \delta d = dd^* + d^*d$ is a formally selfadjoint positive operator with domain $\mathcal{A}^*(M, \mathbb{C})$; however, it is generally an unbounded operator. It can be made bounded by defining a larger Hilbert space norm on $\mathcal{A}^*(M, \mathbb{C})$ and completing to obtain a smaller Hilbert space $H^{2n,*}(M)$, called a Sobolev space. We shall not go into this matter here; we refer to [28] for the details. It turns out that $\Delta$ is then bounded as an operator from $\sigma$ invertible after deleting a neighbourhood of the zero section of the cotangent bundle. In fact, the principal symbol $\sigma$ of $\Delta$ has closed range in $L^2$ Fredholm operators. This implies both that $\ker \Delta$ is finite-dimensional, and that $\Delta$ has closed range in $L^2(M)$.

Equation $\Delta u = 0$. For the original treatment of Hodge, one can examine his book [33]; for more modern proofs, in the somewhat more general setting of an “elliptic complex”, one may consult [23, 28]. The inverse operator $\Delta^{-1}$, defined on the orthogonal projector on the Hilbert space $L^{2,k}(M)$ whose range is $\text{Harm}^k(M)$, then if $\omega \in \mathcal{A}^k(M)$ the form $\omega - P\omega$ lies in the range of the selfadjoint operator $\Delta$; it turns out that this range is closed, even if one restricts $\Delta$ to its original domain $\mathcal{A}^k(M)$. Hence, $\omega - P\omega = \Delta\eta$ for some $\eta \in \mathcal{A}^k(M)$; and therefore $\omega = d(\delta\eta) + \delta(d\eta) + P\omega$.

The full proof of Hodge’s theorem depends on verifying that $\Delta$ is elliptic and identifying carefully the range of $\Delta$. For the original treatment of Hodge, one can examine his book [33]; for more modern proofs, in the somewhat more general setting of an “elliptic complex”, one may consult [23, 28]. The inverse operator $\Delta^{-1}$, defined on the orthogonal projector on the Hilbert space $L^{2,n}(M)$ whose range is $\text{Harm}^n(M)$, is an integral operator, namely the “Green operator” for the partial differential equation $\Delta\alpha = 0$.

Corollary 3.9. Each de Rham class $[\omega] \in H_{\text{DR}}(M)$ contains exactly one harmonic form; thus $\gamma \mapsto [\gamma]$ is an isomorphism of $\text{Harm}^k(M, \mathbb{R})$ onto $H^k_{\text{DR}}(M)$, and therefore every de Rham cohomology space of $M$ is finite dimensional.

---

This means that the “principal symbol” of $\Delta$, which is a certain matrix-valued function on $T^*M$, becomes invertible after deleting a neighbourhood of the zero section of the cotangent bundle. In fact, the principal symbol $\sigma_{\Delta}$ of the Laplacian is given by $\sigma_{\Delta}(\xi) = -g(x)(\xi^1, \xi^2)$ for $\xi \in T_x M$.

4A differential operator whose generalized solutions are automatically smooth is called “hypoelliptic”. Any elliptic operator is hypoelliptic.

32
\( Proof. \) If \( \omega = d\alpha + \delta\beta + \gamma \) in \( \mathcal{A}^k(M, \mathbb{R}) \), with \( \gamma \) harmonic, then \( \omega \) is closed iff \( d\delta\beta = 0 \) iff \( (\beta | d\delta\beta) = (\delta\beta | d\beta) = 0 \) iff \( \delta\beta = 0 \). For \( \omega = d\alpha + \gamma \in Z^{k}_{\text{dR}}(M) \), we clearly have \( [\omega] = [\gamma] \). Moreover, if \( \omega^\prime \in Z^{k}_{\text{dR}}(M) \) with \( [\omega^\prime] = [\omega] \), then \( \omega^\prime = \omega + d\zeta \) for some \( \zeta \in \mathcal{A}^{k-1}(M) \), so \( \omega^\prime = d(\alpha + \zeta) + \gamma \); hence \( \omega^\prime \) and \( \omega \) have the same harmonic component \( \gamma \). \( \square \)

Finally, notice that \( \star d = (-1)^{n-1}\delta \star \), so that whenever \( \omega = d\alpha + \delta\beta + \gamma \) in \( \mathcal{A}^k(M, \mathbb{C}) \), we also have \( \star\omega = d((-1)^{n-1}\star\beta) + \delta((-1)^{n-1}\star\alpha) + \star\gamma \) in \( \mathcal{A}^{n-k}(M, \mathbb{C}) \). The uniqueness of the Hodge decomposition says that the star operator is a linear isomorphism of \( \text{Harm}^k(M, \mathbb{C}) \) onto \( \text{Harm}^{n-k}(M, \mathbb{C}) \); if desired, one can match real harmonic forms by \( \gamma \mapsto i^{-m}\gamma \). Passing to cohomology, this yields a well-defined \( \mathbb{R} \)-linear isomorphism

\[
[\gamma] \mapsto [i^{-m}\gamma] : H^k_{\text{dR}}(M) \overset{\sim}{\longrightarrow} H^{n-k}_{\text{dR}}(M).
\]

This isomorphism is called Poincaré duality.\(^5\)

4 The Hodge Laplacian on the 2-sphere

In this section, we investigate in detail the Hodge Laplacian on the 2-sphere \( S^2 \), both as an illustration of the Hodge theory in general, and in order to introduce a first example of a Dirac operator. The sphere is, of course, an oriented Riemannian manifold, but it is also round, i.e., it is a homogeneous space under the group of rotations of \( \mathbb{R}^3 \). The rotation invariance of the Laplacian \( \Delta \) facilitates a complete description of all its eigenvalues and eigenvectors, which in turn leads to a corresponding spectral description of the Dirac operator \( \slashed{D} = d + \delta \).

4.1 The rotation group in three dimensions

**Definition 4.1.** The rotation group \( SO(3) \) consists of all \( 3 \times 3 \) real matrices \( A \) satisfying \( A^tA = I_3 \) and \( \det A = 1 \). Any rotation belongs to a one-parameter subgroup \( \{ \exp tN : t \in \mathbb{R} \} \), where \( N \) belongs to the Lie algebra \( \mathfrak{so}(3) \) of the rotation group, i.e., the \( 3 \times 3 \) real matrices satisfying \( N^t + N = 0 \), \( \text{Tr} N = 0 \). Write

\[
N = \begin{bmatrix}
0 & -n^3 & n^2 \\
n^3 & 0 & -n^1 \\
-n^2 & n^1 & 0
\end{bmatrix}, \tag{4.1}
\]

and suppose \( (n^1)^2 + (n^2)^2 + (n^3)^2 = 1 \). Then if \( n = (n^1, n^2, n^3) \in \mathbb{R}^3 \), the matrix identities \( N^2 = nn^t - I_3 \), \( N^3 = -N \) hold, and it follows that \( \exp tN = I + N \sin t + N^2(1 - \cos t) \in SO(3) \). Now the rotation action of \( SO(3) \) on \( \mathbb{R}^3 \) (or on \( S^2 \)) is given explicitly by

\[
(\exp tN) x = (I + N \sin t + N^2(1 - \cos t)) x = x + (n \times x) \sin t + (n(n \cdot x) - x)(1 - \cos t) = x \cos t + (n \times x) \sin t + n(n \cdot x)(1 - \cos t). \tag{4.2}
\]

\(^5\)For noncompact Riemannian manifolds, one can define “compactly supported de Rham cohomology” \( H^*_{\text{dR},c}(M) \) by starting from the cochain complex of differential forms with compact support. Then Poincaré duality is a family of isomorphisms between \( H^*_{\text{dR}}(M) \) and \( H^*_{\text{dR},c}(M) \). In the compact case, both cohomologies coincide.
Let $L_1, L_2, L_3$ be the generators of the rotation group, i.e., the elements of $\mathfrak{so}(3)$ given by replacing $n$ in (4.1) by the standard orthonormal basis vectors; so $N = n^1 L_1 + n^2 L_2 + n^3 L_3$. It is immediate that $[L_1, L_2] = L_3$, $[L_2, L_3] = L_1$ and $[L_3, L_1] = L_2$, or more compactly, $[L_i, L_j] = \epsilon_{ij}^k L_k$.

It is very convenient to introduce the matrices $L_{\pm} := L_1 \pm i L_2$ (which belong to the complexified Lie algebra $\mathfrak{so}(3, \mathbb{C})$). Then

$$L_1^2 + L_2^2 + L_3^2 = -L_1 + L_2^2 - iL_3$$

as $3 \times 3$ complex matrices.\footnote{Any Lie algebra $\mathfrak{g}$ gives rise to an associative algebra, called its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, whose elements are polynomial combinations of elements of $\mathfrak{g}$, reduced by the commutation relations among such elements. The Poincaré–Birkhoff–Witt theorem [35] proves that the natural map from $\mathfrak{g}$ to $\mathcal{U}(\mathfrak{g})$ is injective, so that $\mathfrak{g}$ may be regarded as a subspace of $\mathcal{U}(\mathfrak{g})$. Now (4.3) may be regarded as an identity in $\mathcal{U}(\mathfrak{so}(3, \mathbb{C}))$.}

**Definition 4.2.** The action of $SO(3)$ on $\mathbb{S}^2$ induces an action of $C^\infty(\mathbb{S}^2)$ by $(R \cdot f)(x) := f(R^{-1}x)$; more generally, $SO(3)$ acts on $\mathcal{A}^*(\mathbb{S}^2)$ by $R \cdot \omega := (R^{-1})^* \omega$. For the corresponding action of the Lie algebra $\mathfrak{so}(3)$ on $C^\infty(\mathbb{S}^2)$, the homomorphism property of the group action induces a Leibniz rule, so that $N \in \mathfrak{so}(3)$ acts on $C^\infty(\mathbb{S}^2)$ as a vector field $\tilde{N}$, called the fundamental vector field of $N$, which is given explicitly by

$$\tilde{N}f(x) := \left. \frac{d}{dt} \right|_{t=0} f(\exp(-tN)x).$$

Since the rotation action on $SO(3)$ is a restriction of an action on $\mathbb{R}^3$, the same formula yields fundamental vector fields in $\mathcal{X}(\mathbb{R}^3)$. In particular, since $\exp(-tL_1)x = (x^1, x^2 - tx^3, x^3 + tx^2) + O(t^2)$ and similarly for $\exp(-tL_2)x$ and $\exp(-tL_3)x$, on account of (4.2), the corresponding fundamental vector fields on $\mathbb{R}^3$ are

$$\tilde{L}_1 = x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3}, \quad \tilde{L}_2 = x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1}, \quad \tilde{L}_3 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}. \quad (4.4)$$

On transforming (4.4) to spherical coordinates $(r, \theta, \phi)$, one finds that the $\tilde{L}_j$ are independent of $r$ and of $\partial/\partial r$ (as expected). Indeed,

$$\tilde{L}_{\pm} = e^{\pm i\phi} \left[ \mp i \frac{\partial}{\partial \theta} + \cot \theta \frac{\partial}{\partial \phi} \right], \quad \tilde{L}_3 = -\frac{\partial}{\partial \phi}, \quad (4.5)$$

which may be regarded as vector fields either in $\mathcal{X}(\mathbb{R}^3 \setminus \{0\})$ or in $\mathcal{X}(\mathbb{S}^2)$.

**Exercise 4.1.** Verify the formulae (4.5) for the fundamental vector fields.

The sphere $\mathbb{S}^2$ is a homogeneous space for the rotation group, and may be identified with the coset space $SO(3)/SO(2)$, where $SO(2)$ is the subgroup of east-west rotations which fix the north pole. One may therefore consider a system of local coordinates for $\mathbb{S}^2$ which
privileges the cartesian coordinate $x^3$. We adopt the following local coordinates for the remainder of this chapter:

$$\zeta := x^1 + ix^2 = e^{i\phi} \sin \theta, \quad x^3 = \cos \theta.$$  

We also write $\bar{\zeta} := x^1 - ix^2 = e^{-i\phi} \sin \theta$, which supplies a third local coordinate for $\mathbb{R}^3$. Let us abbreviate $\partial_3 := \partial/\partial x^3$, $\partial_{\zeta} := \partial/\partial \zeta$, $\bar{\partial}_{\zeta} := \partial/\partial \bar{\zeta}$. In these coordinates, the fundamental vector fields are given by

$$\tilde{L}_+ = -2ix^3 \bar{\partial}_{\zeta} + i\zeta \partial_3, \quad \tilde{L}_- = 2ix^3 \partial_{\zeta} - i\bar{\zeta} \partial_3, \quad \tilde{L}_3 = i\bar{\zeta} \bar{\partial}_{\zeta} - i\zeta \partial_3. \quad (4.6)$$

A polynomial in the variables $\tilde{L}_j$ gives a differential operator on $S^2$. For instance, the operator corresponding to (4.3) satisfies

$$\tilde{L}_- \tilde{L}_+ + \tilde{L}_3^2 - i \tilde{L}_3 \zeta = (\tilde{L}_3 - i)(\tilde{L}_3 - i)(-i\zeta) = -2\zeta, \quad (4.7)$$

since $\bar{\partial}_{\zeta}$, and therefore also $\tilde{L}_+$, vanishes on holomorphic functions of $\zeta$.

### 4.2 The Hodge operators on the sphere

**Lemma 4.1.** The Hodge star operator on $\mathcal{A}^\bullet(S^2)$ is determined by the relations

$$\star (d\zeta \wedge dx^3) = \zeta, \quad \star (d\zeta) = x^3 d\zeta - \zeta dx^3,$$

in the $(\zeta, x^3)$ coordinates.

**Proof.** First observe that

$$d\zeta \wedge dx^3 = (ie^{i\phi} \sin \theta \, d\phi + e^{i\phi} \cos \theta \, d\theta) \wedge (-\sin \theta \, d\theta) = ie^{i\phi} \sin^2 \theta \, d\theta \wedge d\phi = i\zeta \Omega,$$

so that $\star 1 = i\Omega = \zeta^{-1} d\zeta \wedge dx^3$, and $\star (\zeta) = d\zeta \wedge dx^3$; reciprocally, $\star (d\zeta \wedge dx^3) = \zeta$. Hodge duals of 1-forms are given by

$$\star d\zeta = e^{i\phi} \star (i \sin \theta \, d\phi) + e^{i\phi} \cos \theta \star (d\theta) = e^{i\phi} d\theta + e^{i\phi} \cos \theta (i \sin \theta \, d\phi),$$

which simplifies to $\star (d\zeta) = x^3 d\zeta - \zeta dx^3$, and reciprocally, $\star (x^3 d\zeta - \zeta dx^3) = d\zeta$. \hfill \Box

The codifferential $\delta = -\star d\star$ is now easily found. For instance, $\delta (d\zeta) = \star d(\zeta \, dx^3 - x^3 \, d\zeta) = \star (2d\zeta \wedge dx^3) = 2\zeta$, and since $\delta (\zeta) = 0$, it follows that

$$\Delta (\zeta) = (d\delta + \delta d)\zeta = \delta (d\zeta) = 2\zeta. \quad (4.8)$$
Definition 4.3. Since rotations act on differential forms through $R \cdot \omega := (R^{-1})^*\omega$, their generators act as Lie derivatives:

$$L_j \omega := \left. \frac{d}{dt} \right|_{t=0} (\exp(-tL_j)^*\omega) = L_{L_j} \omega$$

for $j = 1, 2, 3$. Lie derivatives commute with exterior derivation: $L_X d = dL_X$ as operators on $\mathcal{A}^*(S^2)$, for any $X \in \mathfrak{X}(S^2)$; and, in particular, $L_j d = dL_j$ for $j = 1, 2, 3$.

An operator $T$ on $\mathcal{A}^*(S^2)$ is rotation-invariant if $T(R \cdot \omega) = R \cdot (T\omega)$ for any $R \in SO(3)$. Since the three one-parameter subgroups $\{ \exp(tL_j) : t \in \mathbb{R} \}$ generate $SO(3)$, this holds iff $T_{L_j} = L_j T$ for $j = 1, 2, 3$. For instance, the Hodge star operator $\ast = i(\hat{\theta}^1) - \iota(\hat{\theta}^1 \cdot (\hat{\theta}^2) - \iota(\hat{\theta}^2))$ is unchanged if the local oriented orthonormal frame $\{\hat{\theta}^1, \hat{\theta}^2\}$ is replaced by $\{R \cdot \hat{\theta}^1, R \cdot \hat{\theta}^2\}$; this says that $\ast$ is invariant under rotations, and therefore $\ast L_j = L_j \ast$ for $j = 1, 2, 3$.

Since $\delta = -\ast d \ast$, $\Delta = d\delta + \delta d$, it follows that $\delta L_j = L_j \delta$ and $\Delta L_j = L_j \Delta$ for $j = 1, 2, 3$. In particular, the Hodge Laplacian is rotation-invariant.

Exercise 4.2. Use the Cartan identity $L_X = \iota_X d + d\iota_X$ to show that $L_j \Omega = 0$ for $j = 1, 2, 3$. What can one conclude from this?

The Laplace–Beltrami operator $\Delta_0$ (the restriction of $\Delta$ to $C^\infty(S^2)$) is thus a second-order differential operator which commutes with rotations. It therefore represents a quadratic element in the centre of the algebra $\mathfrak{u}(\mathfrak{so}(3))$. Now it is known [34, 35] that this centre is the polynomial algebra $\mathbb{R}[C]$ generated by the “Casimir element” $C = L_1^2 + L_2^2 + L_3^2$. Thus $\Delta_0 = a(\tilde{L}_1^2 + \tilde{L}_2^2 + \tilde{L}_3^2)$ for some constant $a$; $\Delta$ and each $L_j$ commutes with $d$, and so $\Delta = a(L_1^2 + L_2^2 + L_3^2) = a(L_- L_+ + L_3^2 - iL_3)$. From (4.7) and (4.8) one sees that $a = -1$, and therefore

$$\Delta = -L_- L_+ - L_3^2 + iL_3.$$  \hspace{1cm} \text{(4.9)}$$

Exercise 4.3. Show directly, using only the commutation relations $[L_i, L_j] = \epsilon_{ij}^k L_k$, that any quadratic polynomial in $L_1, L_2, L_3$ commuting with each $L_3$ must be a multiple of $L_1^2 + L_2^2 + L_3^2$.

4.3 Eigenvectors for the Laplacian

Since $\Delta$ is invariant under rotations, any subspace of differential forms which is stable under the $SO(3)$ action is mapped by $\Delta$ into another such subspace. The search for eigenvectors under $\Delta$ should therefore start with the irreducible subspaces of the representation $R \mapsto (R^{-1})^*$ of the compact group $SO(3)$ on $\mathcal{A}^*(S^2)$. This is the point of view adopted by Folland [26], who obtained a complete spectral decomposition of the Hodge Laplacian on a sphere of any dimension. Here we show how this works for the 2-sphere.

It is useful to recall some facts about representations of compact Lie groups, which may be found in many places, e.g. [13, 37, 53]. Any irreducible unitary representation of a compact group $G$ acts on a finite dimensional Hilbert space, and the Hilbert space of any unitary representation may be written as a (possibly infinite) direct sum of irreducible subrepresentations. Moreover, by the Peter–Weyl theorem, all such irreducible representations already
occur in the decomposition of the regular representation of \( G \) on \( L^2(G) \), and their carrier spaces are in fact subspaces of \( C^\infty(G) \).

In the case \( G = SO(3) \), the identification \( SO(3)/SO(2) \approx S^2 \) goes as follows: elements of \( SO(3) \) are parametrized by three local coordinates \((\phi, \theta, \psi)\), called “Euler angles”, and \( SO(2) \) is regarded as the one-parameter subgroup \( \{ \exp \psi L_3 : \psi \in \mathbb{R} \} \); the remaining angles \((\theta, \phi)\) as the spherical coordinates on \( S^2 \). Thus functions on \( S^2 \) are identified with functions on \( SO(3) \) which are constant on \( SO(2) \)-cosets, i.e., functions which do not depend on the variable \( \psi \); in this way the representation of \( SO(3) \) on \( L^2(S^2) \) —or on its completion \( L^2(S^2) \)— becomes a subrepresentation of the regular representation on \( L^2(SO(3)) \). Its irreducible subspaces are spanned by the spherical harmonics \( Y_{lm}(\theta, \phi) \), where \( l = 0, 1, 2, \ldots \) and \( m = -l, \ldots, l-1, l \); indeed \( \mathfrak{H}_l := \text{span}\{ Y_{lm} : m = -l, \ldots, l \} \) is an irreducible subspace of dimension \( 2l + 1 \), and the \( Y_{lm} \) form an orthonormal basis for \( L^2(S^2) \). It turns out, again by the Peter–Weyl theorem, that \( SO(3) \) has (up to equivalence) exactly one irreducible unitary representation of each odd dimension, so the orthonormality of the spherical harmonics shows that in the decomposition of the rotation action on \( C^\infty(S^2) \), each such representation occurs once only, and that there are no other irreducible subrepresentations.\(^2\)

There is a general principle for finding irreducible representations of a compact Lie group \( G \), called the “theorem of the highest weight” \([13, 35, 37]\). One finds a maximal torus\(^3\) in \( G \), that is, a subgroup \( T \leq G \) which is a torus, i.e., isomorphic to \( \mathbb{T}^k \) for some \( k \), with \( k \) maximal; if \( G = SO(3) \), then \( k = 1 \) and \( T := SO(2) \cong \mathbb{T} \) will do. In a representation space for \( G \), one looks for a joint eigenvector for the torus \( T \); when \( G = SO(3) \), this is just an eigenvector \( v \) for \( L_3 \), whose eigenvalue is called a “weight” of \( T \). Within \( \mathfrak{g}_C \) one finds \( k \) “raising elements” which annihilate \( v \) (since the weight is “highest”) and \( k \) “lowering elements” which, applied successively to \( v \), generate a basis for an irreducible representation space \( V \); the commutation relations in \( \mathfrak{g}_C \) ensure that applying other generators does not enlarge the space \( V \). When \( G = SO(3) \), there is one raising element, namely \( L_+ \), and one lowering element, namely \( L_- \).

The upshot of the general theory is this. Within each space of \( k \)-forms on the sphere \((k = 0, 1, 2)\), we must find forms \( \alpha \) satisfying \( \mathcal{L}_+ \alpha = 0 \) and \( \mathcal{L}_3 \alpha = c \alpha \) for some eigenvalue \( c \), and such that the vector space \( \text{span}\{ \mathcal{L}_- \alpha : r \in \mathbb{N} \} \) is finite dimensional. The identity (4.9) guarantees that \( \alpha \) is also an eigenvector for the Laplacian \( \Delta \).

**Definition 4.4.** From (4.6), the identity \( \mathcal{L}_+ f = 0 \) is satisfied whenever \( f \in C^\infty(S^2) \) is holomorphic in \( \zeta \) and independent of \( x^3 \). Since \((\mathcal{L}_3 f)(\zeta) = -i \zeta f'(\zeta)\), this \( f \) is an eigenvector for \( \mathcal{L}_3 \) iff it is of the form \( f(\zeta) = c^l \) for some \( l \); since \( c^l = e^{i\theta} \sin^l \theta \), the smoothness of \( f \) forces the condition \( l \in \mathbb{N} \). Define \( \phi_0 \in \mathcal{A}^0(S^2) \) by \( \phi_0(\zeta, x^3) : = \zeta^l \), for \( l = 0, 1, 2, \ldots \).

**Exercise 4.4.** Show that \( \mathcal{L}_-^k \phi_0 \) is a linear combination of terms \((x^3)^{k-2r} \zeta^r \zeta^{l-k+r} \) and check that \( \mathcal{L}_-^k \phi_0 \neq 0 \) for \( k = 0, 1, \ldots, 2l \) but \( \mathcal{L}_-^{2l+1} \phi_0 = 0 \). Conclude that the functions \( \{ R : \phi_0 : \)
Since $\Delta$ commutes with the exterior derivative and the Hodge star operator, the spectrum of the Riemannian manifold $S^2$ has multiplicity 2 for each $l \in \mathbb{N}$.

**Exercise 4.5.** Prove that the functions $\{ \mathcal{L}_-^r \phi_{0l} : l \in \mathbb{N}, r = 0, \ldots, 2l \}$ span a dense subset of $L^2(S^2)$ by showing that any homogeneous polynomial in the variables $\zeta, \overline{\zeta}, x^3$ is a linear combination of these.

**Exercise 4.6.** Check that $\langle \langle \phi_{0l} \mid \phi_{0m} \rangle \rangle = 0$ for $l \neq m$, and that $\langle \langle \alpha \mid \mathcal{L}_- \beta \rangle \rangle = \langle \langle \mathcal{L}_+ \alpha \mid \beta \rangle \rangle$ for $\alpha, \beta \in \mathcal{A}^0(S^2)$, using (4.6). Show that $[L_+, L_-] = -2iL_3$ and $[L_3, L_-] = iL_-$ and conclude that $\mathcal{L}_+ \mathcal{L}_-^r \phi_{0l} = a_{lr} \mathcal{L}_-^{l-1} \phi_{0l}$ for some constant $a_{lr}$. Deduce that the functions $\mathcal{L}_-^r \phi_{0l}$, when suitably normalized, yield an orthonormal basis for $L^2(S^2)$.

We have thus identified a complete set of irreducible subrepresentations of the rotation action on $\mathcal{A}^0(S^2)$. Note that $\mathcal{L}_3(\zeta^l) = -il\zeta^l$, so the $\mathcal{L}_3$-eigenvalue is $-il$. It is immediate from (4.9) that $\Delta(\zeta^l) = (l^2 + l) \zeta^l$. We take stock that

$$
\mathcal{L}_+ \phi_{0l} = 0, \quad \mathcal{L}_3 \phi_{0l} = -il \phi_{0l}, \quad \Delta \phi_{0l} = l(l + 1) \phi_{0l}. \tag{4.10}
$$

It is now clear that the Laplace–Beltrami operator $\Delta_0$ is a formally selfadjoint, positive operator on $L^2(S^2)$, with spectrum $\text{sp}(\Delta_0) = \{ l(l + 1) : l \in \mathbb{N} \}$. Since $\Delta_0$ commutes with $\mathcal{L}_-$, each $\mathcal{L}_-^r \phi_{0l}$ ($r = 0, \ldots, 2l$) is an eigenvector for the eigenvalue $l(l + 1)$, which therefore has multiplicity $2l + 1$. The set $\text{sp}(\Delta_0)$, with these multiplicities, is sometimes referred to as the spectrum of the Riemannian manifold $S^2$ [8].

Note, in particular, that $\ker \Delta_0 = \text{Harm}^0(S^2)$ is the one-dimensional space spanned by $\phi_{00}$, i.e., the space of constant functions. Thus, the only harmonic functions on $S^2$ are the constants.

### 4.4 Spectrum of the Hodge Laplacian

**Definition 4.5.** Since $\Delta$ commutes with the exterior derivative and the Hodge star operator, we can manufacture more eigenvectors by applying these operators to the $\phi_{0l}$. Since $d\phi_{00} = 0$ because $\phi_{00}$ is constant, $d\phi_{0l}$ is an eigenvector only for $l \geq 1$. Define

$$
\psi_{1l} := l^{-1} d\phi_{0l} = \zeta^{l-1} d\zeta, \\
\phi_{1l} := -\star \psi_{1l} = -\zeta^{l-1} \star d\zeta = \zeta^l dx^3 - \zeta^{l-1} x^3 d\zeta,
$$

for $l = 1, 2, 3, \ldots$. Since $\mathcal{L}_+, \mathcal{L}_3$ and $\Delta$ each commutes with $d$ and $\star$, we obtain from (4.10):

$$
\mathcal{L}_+ \psi_{1l} = 0, \quad \mathcal{L}_3 \psi_{1l} = -il \psi_{1l}, \quad \Delta \psi_{1l} = l(l + 1) \psi_{1l}, \\
\mathcal{L}_+ \phi_{1l} = 0, \quad \mathcal{L}_3 \phi_{1l} = -il \phi_{1l}, \quad \Delta \phi_{1l} = l(l + 1) \phi_{1l}.
$$

Thus the 1-forms $\psi_{1l}$ and $\phi_{1l}$ are highest-weight vectors for irreducible representations of $SO(3)$, of dimension $2l + 1$; in fact, the representation spaces are spanned respectively by $\{ \mathcal{L}_- \psi_{1l} : r = 0, \ldots, 2l \}$ and $\{ \mathcal{L}_-^r \phi_{1l} : r = 0, \ldots, 2l \}$.

**Exercise 4.7.** Show that $\langle \langle \psi_{1l} \mid \psi_{1m} \rangle \rangle = 0$ for $l \neq m$, and that $\langle \langle \psi_{1l} \mid \phi_{1m} \rangle \rangle = 0$ for all $l, m \geq 1$. 

38
Exercise 4.8. Check that $\langle \alpha \mid L_{-} \beta \rangle = \langle L_{+} \alpha \mid \beta \rangle$ for $\alpha, \beta \in \mathcal{A}^1(S^2)$, and conclude that the 1-forms $\{ L_{-}^l \psi_{1l}, L_{+}^l \phi_{1l} : l \geq 1, r = 0, \ldots, 2l \}$ are orthogonal.

The 1-forms $\psi_{1l}$ are exact, so applying $d$ to them yields only zero. However, the 1-forms $\phi_{1l}$ are coexact, since $\phi_{1l} = -\ast d \phi_{0l} = \delta(\ast \phi_{0l})$, so we may define

$$\psi_{2l} := (l + 1)^{-1} d \phi_{1l} = \zeta^{l-1} dx^3 \wedge d \zeta = i \zeta^l \Omega$$

for $l = 1, 2, 3, \ldots$. Since the expression $i \zeta^l \Omega$ also makes sense for $l = 0$, we also define $\psi_{20} := i \Omega$. Then

$$L_{+} \psi_{2l} = 0, \quad L_{+} \psi_{2l} = -il \psi_{2l}, \quad \Delta \psi_{2l} = l(l + 1) \psi_{2l}$$

for $l = 0, 1, 2, \ldots$. Furthermore, since $\ast 1 = i \Omega$, we have $\ast \psi_{2l} = \phi_{0l}$ for $l \in \mathbb{N}$.

We may summarize this information with two commutative diagrams:

\[
\begin{array}{ccc}
\phi_{0l} & \xleftarrow{\ast} & \psi_{2l} \\
\downarrow{l^{-1}d} & & \uparrow{(l+1)^{-1}d} \\
\psi_{1l} & \xrightarrow{\ast \delta} & \phi_{1l} \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\phi_{0l} & \xrightarrow{\ast} & \psi_{2l} \\
\downarrow{(l+1)^{-1}d} & & \uparrow{l^{-1}d} \\
\psi_{1l} & \xleftarrow{\ast \delta} & \phi_{1l} \\
\end{array}
\quad (4.11)
\]

where $l \geq 1$. The first diagram takes stock of the foregoing definitions, and the vertical arrows in the second diagram are formed by composing three arrows from the first, using the identity $\delta = -\ast d \ast$. From this it is evident that the $\psi_{kl}$ are exact forms and the $\phi_{kl}$ are coexact forms. A complete circuit of four arrows in either diagram corresponds to applying the operator $(l(l + 1))^{-1}(-\ast d \ast d - d \ast d \ast) = (l(l + 1))^{-1} \Delta$, which acts as the identity at each vertex.

We now have a complete set of eigenforms for $\Delta$. Indeed, we have shown already that the $L_{-}^l \phi_{0l}$ are a complete set of eigenforms of degree 0. Since $\ast$ is a bijection between 0-forms and 2-forms, the $L_{-}^l \psi_{2l}$, for $l \in \mathbb{N}$, $r = 0, \ldots, 2l$ densely span $\mathcal{A}^2(S^2)$. For the 1-forms, we may use the Hodge decomposition (3.10); the diagrams (4.11) show that $d : \delta \mathcal{A}^1(S^2) \rightarrow \mathcal{A}^0(S^2)$ and $\delta : \mathcal{A}^1(S^2) \rightarrow \delta \mathcal{A}^2(S^2)$ are bijections, that the $\psi_{1l}$ densely span $\mathcal{A}^0(S^2)$, and that the $\phi_{1l}$ densely span $\delta \mathcal{A}^2(S^2)$. It remains only to observe that there are no nonzero harmonic 1-forms. This can be seen by noting that by writing any 1-form as $\alpha = f(\zeta) \, d \zeta + g(x^3) \, dx^3 + h(\zeta, x^3)(x^3 d \zeta - \zeta \, dx^3)$, since $f(\zeta) \, d \zeta + g(x^3) \, dx^3$ is exact, and $h(x^3 d \zeta - \zeta \, dx^3) = \ast(h \, d \zeta)$ is coexact. Thus, $\{ L_{-}^l \psi_{1l}, L_{-}^l \phi_{1l} : l \geq 1, r = 0, \ldots, 2l \}$ forms an orthogonal basis for $\mathcal{A}^1(S^2)$.

The spectrum of the Hodge Laplacian is therefore $\text{sp}(\Delta) = \{ l(l + 1) : l \in \mathbb{N} \}$, with multiplicities $4l(l + 1)$ for the eigenvalue $l(l + 1)$ when $l \geq 1$, and multiplicity 2 for the zero eigenvalue. Indeed, $\ker \Delta = \text{span}\{ \phi_{00}, \psi_{20} \} = \text{Harm}^*(S^2)$. If $K$ denotes the operator on $L^2(S^2)$ which is zero on $\text{Harm}^*(S^2)$ and inverts $\Delta$ on its orthogonal complement, then $K$ is a bounded selfadjoint operator and, moreover, is compact, since its spectrum $\text{sp}(K) = \{ (l(l + 1))^{-1} : l \in \mathbb{N} \}$ accumulates only at 0 and consists of eigenvalues with finite multiplicities. Also, $\Delta K = K \Delta = I - P$, where $P$ is the orthogonal projector of rank 2 with range $\text{Harm}^*(S^2)$. 

39
Corollary 4.2. The de Rham cohomology spaces of $S^2$ are given by

$$H^*_{dR}(S^2) \simeq \operatorname{Harm}^*(S^2) \simeq \mathbb{R} \oplus 0 \oplus \mathbb{R},$$

(4.12)
on decomposing $\ker \Delta$ by degrees of forms.

Notice that this coincides with the Čech cohomology $\check{H}^*(S^2, \mathbb{R})$, computed in subsection 1.9.

Definition 4.6. An important topological invariant of a manifold $M$ is its Euler characteristic

$$\chi(M) := \sum_{k=0}^{\dim M} (-1)^k \dim H^k_{dR}(M).$$

For the 2-sphere, (4.12) yields $\chi(S^2) = 1 - 0 + 1 = 2$.

4.5 The Hodge–Dirac operator

Definition 4.7. The Hodge–Dirac operator on the 2-sphere $S^2$ is the operator $\mathcal{D} := d + \delta$, whose square is the Hodge Laplacian: $\mathcal{D}^2 = \Delta$.

Let $A^\text{even}(S^2) := A^0(S^2) \oplus A^2(S^2)$ be the algebra of differential forms of even degree on $S^2$, and write $A^\text{odd}(S^2) := A^1(S^2)$ to denote the odd-degree forms. Then $\mathcal{D}$ is an odd operator in the sense that it interchanges forms of even and odd parities:

$$\mathcal{D}(A^\text{even}(S^2)) \subseteq A^\text{odd}(S^2),$$

$$\mathcal{D}(A^\text{odd}(S^2)) \subseteq A^\text{even}(S^2).$$

From (4.11), the action of $\mathcal{D}$ is given explicitly by

$$\mathcal{D}\phi_{0l} = l \psi_{1l}, \quad \mathcal{D}\psi_{1l} = (l + 1) \phi_{0l},$$

$$\mathcal{D}\psi_{2l} = l \phi_{1l}, \quad \mathcal{D}\phi_{1l} = (l + 1) \psi_{2l},$$

(4.13)

for $l = 0, 1, 2, \ldots$ on the left, and $l = 1, 2, 3, \ldots$ on the right. It follows that

$$\mathcal{D}(\sqrt{l + 1} \phi_{0l} \pm \sqrt{l} \psi_{1l}) = \pm \sqrt{l(l + 1)} \left( \sqrt{l + 1} \phi_{0l} \pm \sqrt{l} \psi_{1l} \right),$$

$$\mathcal{D}(\sqrt{l + 1} \psi_{2l} \pm \sqrt{l} \phi_{1l}) = \pm \sqrt{l(l + 1)} \left( \sqrt{l + 1} \psi_{2l} \pm \sqrt{l} \phi_{1l} \right).$$

Since these eigenforms for $\mathcal{D}$ densely span $L^2\ast(S^2)$, one concludes that $\mathcal{D}$ is a formally selfadjoint\footnote{The Euler characteristic may be computed as the integral of a certain differential form over $M$, as we shall see later.} unbounded linear operator on $L^2\ast(S^2)$, with spectrum $\operatorname{sp}(\mathcal{D}) = \{ \pm \sqrt{l(l + 1)} : l \in \mathbb{N} \}$.

Exercise 4.9. What are the multiplicities of the eigenvalues of $\mathcal{D}$?

\footnote{The term “formally selfadjoint” means that $\langle \mathcal{D}\omega | \eta \rangle = \langle \omega | \mathcal{D}\eta \rangle$ for $\omega, \eta \in A^\ast(S^2)$; operator theorists would say that $\mathcal{D}$ is “symmetric”. With some more work, one can check that $\mathcal{D}$ is essentially selfadjoint, which means that it has an extension to a larger domain in the Hilbert space $L^2\ast(S^2)$ which is a closed, unbounded selfadjoint operator. See [39] for a proof of this.}
It also follows from (4.13) that \( \ker \mathcal{D} = \text{span}\{\phi_{00}, \psi_{20}\} \), and that \( \mathcal{D} \) has dense range, so that \( \text{coker} \mathcal{D} = 0 \); moreover, the restriction of \( \mathcal{D} \) to \( L^2(\mathbb{S}^2) \) has a compact inverse. Thus \( \mathcal{D} \) is a Fredholm operator.\(^6\) Its index is given by

\[
\text{ind} \mathcal{D} := \dim(\ker \mathcal{D}) - \dim(\text{coker} \mathcal{D}) = 2 - 0 = 2.
\]

**Corollary 4.3.** \( \text{ind} \mathcal{D} = \chi(\mathbb{S}^2) \).

This is a first example of an *index theorem*, wherein a certain integer obtained by an integral over the manifold (namely, the Euler characteristic) turns out to be equal to the Fredholm index of an operator (namely, the Hodge–Dirac operator) which is bound up with the geometric structure of the manifold. For a wider discussion of index theorems in modern geometry and topology, we refer to [9, 28, 39, 42].

There is an equivalent method of computing the index from the grading of \( \mathcal{A}^\bullet(\mathbb{S}^2) \) into forms of even and odd degree. Write \( L^2(\mathbb{S}^2) = L^2(\text{even})(\mathbb{S}^2) \oplus L^2(\text{odd})(\mathbb{S}^2) \) where \( L^2(\text{even})(\mathbb{S}^2) \) and \( L^2(\text{odd})(\mathbb{S}^2) \) are the respective completions of \( \mathcal{A}(\text{even})(\mathbb{S}^2) \) and \( \mathcal{A}(\text{odd})(\mathbb{S}^2) \); this is then a graded Hilbert space. The odd operator may be written as

\[
\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_1 \\ \mathcal{D}_0 & 0 \end{pmatrix},
\]

where \( \mathcal{D}_0 : L^2(\text{even})(\mathbb{S}^2) \to L^2(\text{odd})(\mathbb{S}^2) \) and \( \mathcal{D}_1 : L^2(\text{odd})(\mathbb{S}^2) \to L^2(\text{even})(\mathbb{S}^2) \). The selfadjointness of \( \mathcal{D} \) says that \( \mathcal{D}_1 = \mathcal{D}_1^\dagger \), and this may be verified directly from (4.13) also. Now \( \text{coker} \mathcal{D}_1 \simeq (\ker \mathcal{D}_0)^\perp \) and \( \text{coker} \mathcal{D}_0 \simeq (\ker \mathcal{D}_1)^\perp \), so the index of \( \mathcal{D} \) equals

\[
\text{ind} \mathcal{D} = \dim(\ker \mathcal{D}_0) - \dim(\ker \mathcal{D}_1).
\]

From (4.13), the right hand side of (4.14) equals \( 2 - 0 = 2 \), as expected.

**Exercise 4.10.** Show that \( l \left\langle \psi_I | \psi_I \right\rangle = (l+1) \left\langle \phi_{0l} | \phi_{0l} \right\rangle \) and that \( l \left\langle \phi_{II} | \phi_{II} \right\rangle = (l+1) \left\langle \psi_{2l} | \psi_{2l} \right\rangle \). Deduce from (4.13) that \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) are adjoints of each other.

The Hodge–Dirac operator is one of many operators which are collectively known as *Dirac operators*. Some common properties are: (i) they are unbounded selfadjoint Fredholm operators; (ii) the Hilbert spaces on which they act are graded into “even” and “odd” subspaces, which the Dirac operators interchange; (iii) their squares are “generalized Laplacians” (of which we shall have more to say later) acting on Riemannian manifolds.

**Definition 4.8.** As a second example, let us redefine the grading of differential forms by splitting \( \mathcal{A}^\bullet(\mathbb{S}^2) \) into the \((\pm 1)\)-eigenspaces for the Hodge star operator: thus \( \mathcal{A}^\pm(\mathbb{S}^2) := \{ \omega : \ast \omega = \omega \} \) is the space of *selfdual forms*, and \( \mathcal{A}^\mp(\mathbb{S}^2) := \{ \eta : \ast \eta = -\eta \} \) is the set of *antiselfdual forms*. Now \( \delta \ast = -\ast d \) implies that \( \mathcal{D} \ast = -\ast \mathcal{D} \), so the *same* operator \( \mathcal{D} = d + \delta \) interchanges selfdual and antiselfdual forms: \( \mathcal{D}(\mathcal{A}^\pm(\mathbb{S}^2)) \subseteq \mathcal{A}^\mp(\mathbb{S}^2) \).

----

\(^6\)Conventionally, Fredholm operators are taken to be bounded, whereas \( \mathcal{D} \) and \( \Delta \) are not. One could remedy this by redefining the norm on the domain space, but then the domain and range would lie in different Hilbert spaces. Instead, we use the alternative definition [22] of a Fredholm operator as a closed, possibly unbounded, operator between Hilbert spaces which has dense domain, finite-dimensional kernel and finite-codimensional (therefore closed) range. In this sense, the operator closures of \( \mathcal{D} \) and \( \Delta \) are unbounded Fredholm operators on \( L^2(\mathbb{S}^2) \).
The diagrams (4.11) show that the forms $\psi_{2l} + \phi_{0l} \ (l \geq 0)$ and $\phi_{1l} - \psi_{1l} \ (l \geq 1)$ are selfdual, whereas $\psi_{2l} - \phi_{0l} \ (l \geq 0)$ and $\phi_{1l} + \psi_{1l} \ (l \geq 1)$ are antiselfdual. Since $\nabla$ commutes with $L_-$, applying powers to $L_-$ to these forms generates a complete set [i.e., a set which spans a dense subspace of $L^{2,*}(S^2)$]. Thus we may rewrite (4.13) as

$$\nabla(\psi_{2l} + \phi_{0l}) = l(\phi_{1l} + \psi_{1l}), \quad \nabla(\phi_{1l} - \psi_{1l}) = (l + 1)(\psi_{2l} - \phi_{0l}),$$

$$\nabla(\phi_{1l} + \psi_{1l}) = (l + 1)(\psi_{2l} + \phi_{0l}), \quad \nabla(\psi_{2l} - \phi_{0l}) = l(\phi_{1l} - \psi_{1l}),$$

where $l \geq 1$ in all cases; and $\nabla(\psi_{20} \pm \phi_{00}) = 0$.

Observe that, in the new grading of $L^{2,*}(S^2)$, the odd operator $\nabla$ can be written as

$$\nabla = \begin{pmatrix} 0 & \nabla_- \\ \nabla_+ & 0 \end{pmatrix},$$

where $\nabla_\pm : L^{2,\pm}(S^2) \to L^{2,\mp}(S^2)$. Now we define its index as

$$\text{ind } \nabla := \dim(\ker \nabla_+) - \dim(\ker \nabla_-),$$

which equals $1 - 1 = 0$ since the ker $\nabla_\pm$ are one-dimensional spaces, spanned by $(\psi_{20} \pm \phi_{00})$.

The corresponding topological quantity \cite{9} arises from the bilinear form $s([\alpha], [\beta]) := \int_{S^2} \alpha \wedge \beta$ on $H^1_{\text{dR}}(S^2)$. The integral depends only on the cohomology classes $[\alpha]$ and $[\beta]$, and $s$ is antisymmetric; we assign to $s$ a “signature” of zero.\footnote{A less trivial example arises on considering a Riemannian manifold $M$ of dimension $n = 4k$; the corresponding formula yields a symmetric bilinear form $q([\alpha], [\beta]) := \int_M \alpha \wedge \beta$ on $H^2_{\text{dR}}(M)$, and the topological invariant is the signature of this bilinear form.}

The index theorem for this case is the equality of this zero signature with the zero index for $\nabla$ as defined by (4.15).

## 5 Connections on vector bundles

Line bundles are classified by integral Čech 2-cocycles, according to the theory developed in Section 1. In this chapter, we show how to produce, for a given Hermitian line bundle $E \to M$, a Čech 2-cocycle which is associated to its class. This 2-cocycle comes from the curvature form of a connection on the line bundle. A connection, or covariant derivative, is a general structure which supplements that of a vector bundle with a notion of “parallel displacement” among neighbouring fibres. There are several possible ways to introduce connections; we shall adopt here an algebraic approach, in the spirit of Cartan calculus of vector fields and forms.

### 5.1 Modules of vector-valued forms

**Lemma 5.1.** Suppose that $E \to M$ and $E' \to M$ are two vector bundles over the (compact) manifold $M$. If $\tau : E \to E'$ is a bundle map, i.e., a smooth map such that $(\tau, \text{id}_M)$ is a vector bundle morphism, there is an $C^\infty(M)$-linear map $\tau_* : \Gamma(E) \to \Gamma(E')$ given by $\tau_* s := \tau \circ s$; and the correspondence $\tau \mapsto \tau_*$ satisfies $(\text{id}_E)_* := \text{id}_{\Gamma(E)}$ and $(\tau \circ \sigma)_* := \tau_* \circ \sigma_*$.\footnote{A less trivial example arises on considering a Riemannian manifold $M$ of dimension $n = 4k$; the corresponding formula yields a symmetric bilinear form $q([\alpha], [\beta]) := \int_M \alpha \wedge \beta$ on $H^2_{\text{dR}}(M)$, and the topological invariant is the signature of this bilinear form.}
Proof. The $C^\infty(M)$-linearity of $\tau$ means that $\tau_s(fs) = f\tau_s(s)$ for $s \in \Gamma(E)$, $f \in C^\infty(M)$, which follows from the linearity of $\tau : E_x \to E'_x$ for each $x \in M$, since $\tau_s(fs)(x) := \tau(f(x)s(x)) = f(x)\tau(s(x)) = (f \tau_s)(x)$. The remaining assertions are obvious. \hfill $\square$

Let $A := C^\infty(M)$ be the algebra of smooth functions on $M$. One way to restate the preceding Lemma is to say that $E \mapsto \Gamma(E)$, $\tau \mapsto \tau_s$ is a covariant functor $\Gamma : \text{Vect}(M) \to \text{Mod}(A)$ from the category of vector bundles over $M$ to the category of $A$-modules.

**Exercise 5.1.** Verify that $\Gamma(E^*) = \Gamma(E)^*$, where $E^* \to M$ is the dual vector bundle to $E \to M$, and the notation $E^*$, for an $A$-module $E$, denotes the module of $A$-linear maps from $E$ to $A$.

**Definition 5.1.** If $A$ is any (real or complex) algebra, a bimodule $E$ over $A$ is a vector space with bilinear operations $A \times E \to E$ and $E \times A \to E$, usually simply as $(a, s) \mapsto as$ and $(s, a) \mapsto sa$, satisfying $1s = s1 = s$, $a(as') = (a'a)s$, and $(sa')a = s(a'a)$, for $s \in E$, $a, a' \in A$. The tensor product $E \otimes_A E'$ of two such bimodules is the bimodule Suppose whose elements are finite sums $\sum j s_j \otimes s'_j$ with $s_j \in E$ and $s'_j \in E'$, subject only to the relations $(sa) \otimes s' - s \otimes (as') = 0$, for $s \in E$, $s' \in E'$, $a \in A$. When $A$ is commutative, the identification $sa = as$ makes each $A$-module into an $A$-bimodule; the foregoing recipe defines the tensor product of $A$-modules in that case.

**Proposition 5.2.** Let $A := C^\infty(M)$ be the algebra of smooth functions on a compact manifold $M$, and let $E \to M$, $E' \to M$ be vector bundles over $M$. Then there is a canonical isomorphism of $A$-modules:

$$\Gamma(E) \otimes_A \Gamma(E') \simeq \Gamma(E \otimes E').$$

**Proof.** If $s \in \Gamma(E)$, $s' \in \Gamma(E')$, let $s \otimes s'$ denote the section $x \mapsto s(x) \otimes s'(x)$ of the tensor product bundle $E \otimes F \to M$. We provisionally denote by $s \otimes s'$ the element of $\Gamma(E) \otimes_A \Gamma(E')$ given by the definition of the tensor product of $A$-modules. Let $\theta : \Gamma(E) \otimes_A \Gamma(E') \to \Gamma(E \otimes E')$ be the $A$-linear map determined by $\theta(s \otimes s') := s \otimes s'$; the claim is that $\theta$ is an isomorphism.

If $U$ is a chart domain in $M$ over which the bundles $E \to M$ and $E' \to M$ are trivial, then $E \otimes E' \to M$ is also trivial over $U$. Indeed, we have seen in subsection 1.7 that any local section $t \in \Gamma(U, E)$ is of the form $t = \sum_{k=1}^r h^k s_k$, where $\{s_1, \ldots, s_r\}$ is a local system of sections for $E$ over $U$, and $h_1, \ldots, h_r \in C^\infty(U)$; in other words, the sections $\{s_1, \ldots, s_r\}$ generate $\Gamma(U, E)$ freely over $C^\infty(U)$. If $\{s'_1, \ldots, s'_r\}$ is a local system of sections for $E'$ over $U$, they generate $\Gamma(U, E')$ freely as a $C^\infty(U)$-module, and it is clear that $\{s_j \otimes s'_k : 1 = j = 1, \ldots, r; k = 1, \ldots, l\}$ generate $\Gamma(U, E \otimes E')$ as a free $C^\infty(U)$-module, since $\{s_j(x) \otimes s'_k(x) : 1 = j = 1, \ldots, r; k = 1, \ldots, l\}$ is a basis for $E_x \otimes E'_x$ for each $x \in U$. In summary, $\theta$ is an isomorphism whenever the bundles $E \to M$ and $E' \to M$ (and consequently $E \otimes E' \to M$) are trivial.

In the general case, there are vector bundles $F \to M$ and $F' \to M$ such that $E \oplus F \to M$ and $E' \oplus F' \to M$ are trivial,2 by Proposition 1.8. Let $\iota : E \to E \oplus F$ and

---

1 The bimodule operations on $E \otimes_A E'$ are, of course, defined by $a(s \otimes s') := (as) \otimes s'$ and $(s \otimes s')a := s \otimes (sa)$.

2 The use of Proposition 1.8 (existence of the supplementary bundle) is the only point in this proof where the compactness of $M$ is used. It should be said that, with some work to establish the existence of a finite trivialising open covering of $M$, the compactness assumption can be dropped: we refer to [17] for a proof.

43
σ: E ⊕ F → E be the extension and restriction maps: \( τ(u) := (u, 0) \), σ\( (u, v) := u \), and let \( τ': E' → E' ⊕ F' \), \( σ': E' ⊕ F' → E' \) be similarly defined. Then \( σ \circ τ = id_E \), so \( σ \circ τ = id_{Γ(E)} \) and also \( σ' \circ τ' = id_{Γ(E')} \); thus, \( τ \) and \( τ' \) are injective, whereas \( σ \) and \( σ' \) are surjective.

Now \( E_x ⊗ E'_x \) is a direct summand of the vector space \( (E_x ⊗ F_x) ⊗ (E'_x ⊗ F'_x) \) for each \( x \in M \); this yields bundle maps \( τ': E ⊗ E' → (E ⊕ F) ⊗ (E' ⊕ F') \), \( σ': (E ⊕ F) ⊗ (E' ⊕ F') → E ⊗ E' \) satisfying \( σ' \circ τ' = id_{Γ(E ⊗ E')} \). Finally, let

\[
τ_\ast \circ τ'_\ast: Γ(E) ⊗_A Γ(E') → Γ(E ⊕ F) ⊗_A Γ(E' ⊕ F')
\]

be defined by \( (τ_\ast \circ τ'_\ast)(s ⊗ s') := τ_\ast s ⊗ τ'_\ast s' \), and let \( σ_\ast \circ σ'_\ast: Γ(E ⊕ F) ⊗_A Γ(E' ⊕ F') → Γ(E) ⊗_A Γ(E') \) be defined similarly. We thus have two commutative diagrams:

\[
\begin{array}{ccc}
Γ(E) ⊗_A Γ(E') & \xrightarrow{θ} & Γ(E ⊗ E') \\
\downarrow{τ_\ast \circ τ'_\ast} & & \downarrow{τ'_\ast} \\
Γ(E ⊕ F) ⊗_A Γ(E' ⊕ F') & \xrightarrow{θ} & Γ((E ⊕ F) ⊗ (E' ⊕ F'))
\end{array}
\]

(5.1)

where \( Θ \) is the isomorphism of free \( A \)-modules already obtained, and

\[
\begin{array}{ccc}
Γ(E) ⊗_A Γ(E') & \xrightarrow{θ} & Γ(E ⊗ E') \\
\downarrow{σ_\ast \circ σ'_\ast} & & \downarrow{σ'_\ast} \\
Γ(E ⊕ F) ⊗_A Γ(E' ⊕ F') & \xrightarrow{θ} & Γ((E ⊕ F) ⊗ (E' ⊕ F'))
\end{array}
\]

(5.2)

From (5.1), \( θ \) is injective, since \( τ_\ast \circ τ'_\ast \) and \( τ'_\ast \) are injective and \( Θ \) is bijective; and (5.2) shows analogously that \( θ \) is surjective. Thus \( θ \) is an \( A \)-linear isomorphism in the general case. □

**Corollary 5.3.** Each \( A \)-linear map from \( Γ(E) \) to \( Γ(E') \) is of the form \( τ_\ast \) for some bundle map \( τ: E → E' \).

**Proof.** The maps \( τ: E → E' \) form the total space of the vector bundle \( Hom(E, E') → M \), whose fibres are \( Hom(E_x, E'_x) \). Thus \( τ \) can be identified with a section of the vector bundle \( E^* ⊗ E' → M \). On the other hand, an \( A \)-linear map from \( Γ(E) \) to \( Γ(E') \) belongs to \( Hom_A(Γ(E), Γ(E')) \). Thus \( τ_\ast \) corresponds to \( θ^{-1}(τ) \) under these identifications. Since \( θ \) is bijective, these account for all \( A \)-linear maps from \( Γ(E) \) to \( Γ(E') \). □

**Definition 5.2.** Consider the vector bundle \( E' = Λ^r T^* M \), whose smooth sections are the \( r \)-forms over \( M \): \( A^r(M) = Γ(Λ^r T^* M) \). Define

\[
A^r(M, E) := Γ(E) ⊗_A A^r(M) \simeq Γ(E ⊕ Λ^r T^* M),
\]

(5.3)

using Proposition 5.2. The elements of the \( A \)-module \( A^r(M, E) \) are finite sums of the form \( \sum_k s_k \otimes ω_k \), where \( s_k ∈ Γ(E), ω_k ∈ A^r(M) \); we shall refer to them as \( "E\)-valued \( r \)-forms over \( M" \).

On a complex manifold, we may also define \( A^{p,q}(M, E) := Γ(E) ⊗_A A^{p,q}(M) \). 44
5.2 Connections

**Definition 5.3.** A **connection** on a complex vector bundle \( E \to M \) is a \( \mathbb{C} \)-linear\(^3\) map \( \nabla: \Gamma(E) \to \mathcal{A}^1(M, E) \) such that

\[
\nabla(sf) = (\nabla s)f + s \otimes df \quad \text{for all } s \in \Gamma(E), \, f \in C^\infty(M).
\]

(5.4)

This **Leibniz rule**\(^4\) shows that the definition is local, i.e., that \( \nabla s \) is determined by its restrictions to chart domains \( U_j \). To see that, let \( \{f_j\} \) be a smooth partition of unity subordinate to \( \{U_j\} \), and notice that \( \sum_j df_j = d(\sum_j f_j) = d(1) = 0 \), and thus \( \nabla s = \sum_j \nabla(s f_j) \) by (5.4).

This locality is immediately useful in showing that connections exist on any vector bundle. First, if \( E \to M \) is a trivial bundle, \( E \cong M \times \mathbb{C}^r \), so that \( \Gamma(E) \cong \mathcal{A}^r \), the exterior derivative \( d: \mathcal{A}^r \to \mathcal{A}^{r+1} \) gives an isomorphism \( \psi: \Gamma(U_j, E) \to \mathcal{A}^r(\mathbb{C}^r) \) for \( j \in U_j \). This is just the pullback to sections of the local trivialization of \( E \). Since \( \mathcal{A}^1(U_j, E) \cong \Gamma(U_j, E) \otimes C^\infty(U_j) \mathcal{A}^1(U_j) \), we can choose a local system of sections \( s_j = (s_j^1, \ldots, s_j^r) \) for \( E \to M \); the expansion \( t = \sum s_j h^k j^k \) gives an isomorphism \( \psi_j^*: \Gamma(U_j, E) \to \mathbb{C}^r(\mathbb{C}^r) \otimes \mathcal{A}^1(U_j) \otimes \mathcal{A}^1(U_j) \). Now define \( \nabla_j := (\psi_j^* \otimes \text{id})^{-1} \circ d \circ \psi_j^* \), so that \( \nabla_j(\cdot): \Gamma(U_j, E) \to \mathcal{A}^1(U_j, E) \) is a connection on \( \pi^{-1}(U_j) \to U_j \). Finally, take any smooth partition of unity \( \{f_j\} \) subordinate to the cover \( \{U_j\} \), and define \( \nabla := \sum_j \nabla_j(s f_j) \); one checks that \( \nabla \) is a connection on the vector bundle \( E \to M \).

**Exercise 5.2.** Carry out this check.

The Leibniz rule (5.4) means that \( \nabla \) is not itself \( \mathcal{A} \)-linear. However, if \( \nabla_0 \) and \( \nabla_1 \) are two connections on \( E \to M \), it is immediate from (5.4) that \( (\nabla_1 - \nabla_0)(sf) = (\nabla_1 s - \nabla_0 s)f \) for \( s \in \Gamma(E) \) and \( f \in \mathcal{A} \). By Corollary 5.3, \( \nabla_1 - \nabla_0 = \alpha \) for some \( \alpha \in \Gamma(\text{End} E \otimes T^*M) = \mathcal{A}^1(M, \text{End} E) \). Therefore,

\[
\nabla_1 s = \nabla_0 s + \alpha \circ s \quad \text{for } s \in \Gamma(E);
\]

(5.5)

conversely, given \( \nabla_0 \), this equation defines a connection \( \nabla_1 \) when \( \alpha \in \mathcal{A}^1(M, \text{End} E) \). This says that the set of all connections on \( E \to M \) is an **affine space**, based on the vector space \( \mathcal{A}^1(M, \text{End} E) \).

**Definition 5.4.** If \( X \in \mathfrak{X}(M) \) is a vector field, the **contraction** \( \iota_X: \mathcal{A}^1(M) \to \mathcal{A} \) is \( \mathcal{A} \)-linear, since \( \iota_X(f \beta) = f \beta(X) = f \iota_X \beta \) for \( f \in \mathcal{A} \), \( \beta \in \mathcal{A}^1(M) \). Thus it extends to an \( \mathcal{A} \)-linear map from \( \mathcal{A}^1(M, E) = \Gamma(E) \otimes_{\mathcal{A}} \mathcal{A}^1(M) \) to \( \Gamma(E) \otimes_{\mathcal{A}} \mathcal{A} = \Gamma(E) \) by \( \iota_X(s \otimes \beta) := s(\iota_X \beta) = s \beta(X) \).

If \( \nabla: \Gamma(E) \to \mathcal{A}^1(M, E) \) is a connection, we write \( \nabla_X := \iota_X \circ \nabla: \Gamma(E) \to \Gamma(E) \). This is a \( \mathbb{C} \)-linear map satisfying the Leibniz rule:

\[
\nabla_X(sf) = (\nabla_X s)f + s(Xf).
\]

(5.6)

---

\(^3\)For real vector bundles, we require only that \( \nabla \) be an \( \mathbb{R} \)-linear map.

\(^4\)It is convenient to regard \( \Gamma(E) \) and \( \mathcal{A}^1(M, E) \) as right \( \mathcal{A} \)-modules; if one wishes to regard them as left \( \mathcal{A} \)-modules by identifying \( fs = sf \), the Leibniz rule can equivalently be written as \( \nabla(fs) = f(\nabla s) + df \otimes s \).
Definition 5.5. If $E \to M$ is a Hermitian vector bundle, the metric $h$ on $E$ defines a $\mathcal{A}$-sesquilinear map $\Gamma(E) \times \Gamma(E) \to \mathcal{A}$ by $(s|t) : x \mapsto h_x(s(x), t(x))$, for $s, t \in \Gamma(E).$ This extends to a sesquilinear form $\Gamma(E) \times \mathcal{A}^k(M, E) \to \mathcal{A}^k(M)$ by tensoring, i.e., $(s|t \odot \omega) := (s|t)\omega$ when $\omega \in \mathcal{A}^k(M)$.

A connection $\nabla$ on $E \to M$ is compatible with the metric if
\[
(s|t) + (s|\nabla t) = d(s|t) \quad \text{for all} \quad s, t \in \Gamma(E).
\] Notice that this is yet another variant of the Leibniz rule. If $X \in \mathcal{X}(M)$, a compatible connection satisfies $\nabla_X(s|t) = X(s|t)$.

Exercise 5.3. In general, for any connection $\nabla$ on a vector bundle $E \to M$, one may define a dual connection $\nabla^*$ on the dual bundle $E^* \to M$ by stipulating that the following Leibniz rule should hold: $\langle \nabla^*\xi, s \rangle + \langle \xi, \nabla s \rangle = d\langle \xi, s \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the evaluation map $\Gamma(E^*) \times \Gamma(E) \to \mathcal{A}.$ Verify that, for given $\nabla$ and $\xi \in \Gamma(E^*)$, this Leibniz rule determines a well-defined element $\nabla^*\xi$ of $\mathcal{A}^1(M, E^*)$, and that the operator $\nabla^*$ thus obtained is a connection.

Exercise 5.4. Suppose that $E \to M$ and $E' \to M$ are equivalent vector bundles and that $\tau : E \to E'$ is an invertible bundle map. Extend $\tau_*$ to an operator from $\mathcal{A}^k(M, E)$ to $\mathcal{A}^k(M, E')$ by tensoring: $\tau_*(s \otimes \omega) := \tau_* s \otimes \omega$ for $s \in \Gamma(E), \omega \in \mathcal{A}^k(M).$ If $\nabla$ is a connection on $E \to M$, show that $\nabla' := \tau_* \circ \nabla \circ \tau_*^{-1}$ is a connection on $E' \to M$.

5.3 Curvature of a connection

Definition 5.6. If $\nabla : \Gamma(E) \to \mathcal{A}^1(M, E)$ is a connection on a vector bundle $E \to M$, there is a canonical way to extend $\nabla$ to a linear map $\nabla : \mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E).$ Since $\mathcal{A}^k(M, E)$ is a vector space generated by elements of the form $s \otimes \omega$, for $s \in \Gamma(E), \omega \in \mathcal{A}^k(M)$, it suffices to define
\[
\nabla(s \otimes \omega) := (\nabla s) \wedge \omega + s \otimes d\omega \quad \text{for} \quad s \in \Gamma(E), \omega \in \mathcal{A}^k(M).
\] If $\eta \in \mathcal{A}^*(M)$, it follows that
\[
\nabla(s \otimes (\omega \wedge \eta)) = (\nabla s) \wedge \omega \wedge \eta + s \otimes d\omega \wedge \eta + (-1)^k s \otimes \omega \wedge d\eta,
\] from which one obtains the “graded Leibniz rule”:
\[
\nabla(\zeta \wedge \eta) = (\nabla \zeta) \wedge \eta + (-1)^k \zeta \wedge d\eta \quad \text{if} \quad \zeta \in \mathcal{A}^k(M, E), \eta \in \mathcal{A}^*(M).
\] Exercise 5.5. Verify that the extended map $\nabla : \mathcal{A}^k(M, E) \to \mathcal{A}^{k+1}(M, E)$ is well-defined, i.e., if $\sum_j s_j \otimes \omega_j = \sum_r t_r \otimes \eta_r$, with $s_j, t_r \in \Gamma(E)$ and $\omega_j, \eta_r \in \mathcal{A}^k(M)$, then $\nabla(s_j) \otimes \omega_j = \sum_r \nabla(t_r) \otimes \eta_r$.

The iterated map $\nabla^2 : \Gamma(E) \to \mathcal{A}^2(M, E)$ satisfies
\[
\nabla^2(sf) = \nabla(\nabla(s)f) + \nabla(s \otimes df)
\]\[= (\nabla^2 s)f - \nabla s \wedge df + \nabla s \wedge df + s \otimes d(df) = (\nabla^2 s)f,
\] for $s \in \Gamma(E), f \in \mathcal{A}$. Thus $\nabla^2$ is an $\mathcal{A}$-linear map from $\Gamma(E)$ to $\mathcal{A}^2(M, E) = \Gamma(E \otimes \Lambda^2 T^*M)$; by Corollary 5.3, $\nabla^2 s = \omega s$ with $\omega \in \Gamma(End E \otimes \Lambda^2 T^*M) = \mathcal{A}^2(M, End E)$. This “matrix-valued 2-form” $\omega$ is called the curvature of the connection $\nabla$. 

46
Exercise 5.6. If \( \tau : E \to E' \) is an invertible bundle map between equivalent vector bundles over \( M \), and if \( \nabla, \nabla' \) are connections on \( E \) and \( E' \) related by \( \nabla' = \tau_* \circ \nabla \circ \tau^{-1} \), show that their curvatures are likewise related by \( \varpi' = \tau_* \circ \varpi \circ \tau^{-1} \).

On the trivial bundle \( E = M \times \mathbb{C}^r \), let \( d \) be the connection \( (h^1, \ldots, h^r) \mapsto (dh^1, \ldots, dh^r) \) given by the exterior derivative. From (5.5), we can write \( \nabla s = ds + \alpha s \) where \( \alpha \in \mathcal{A}^1(M, \text{End} E) = \mathcal{A}^1(M)^{r \times r} \) is a matrix of 1-forms on \( M \). Explicitly, if \( s \in \mathcal{A}^r \) has \( k \)-th component \( h^k \), then \( \alpha s \) has \( k \)-th component \( \alpha^k_i h^i \), so \( \alpha = [\alpha^k_i] \in \mathcal{A}^1(M)^{r \times r} \). In this case, the curvature \( \varpi \) is a matrix of 2-forms:

\[
\varpi s = \nabla^2 s = d(ds + \alpha s) + \alpha \wedge (ds + \alpha s) = (\alpha ds - \alpha \wedge ds) + (\alpha \wedge ds + (\alpha \wedge \alpha)s) = (d\alpha + \alpha \wedge \alpha)s, \tag{5.10}
\]

so that \( \varpi = d\alpha + \alpha \wedge \alpha \) in \( \mathcal{A}^2(M)^{r \times r} \). In components, \( \varpi^k_i = d\alpha^k_i + \alpha^k_m \wedge \alpha^m_i \).

5.4 A curvature formula

Lemma 5.4. The curvature \( \varpi \) of a connection \( \nabla \) on a line bundle \( E \to M \) satisfies

\[
\varpi(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]} s \tag{5.11}
\]

for all \( X, Y \in \mathfrak{X}(M) \) and \( s \in \Gamma(E) \).

Proof. Since \( \varpi \in \mathcal{A}^2(M, \text{End} E) = \Gamma(\text{End} E) \otimes \mathcal{A}^2(M) \), the evaluation on the pair of vector fields \( X, Y \) yields \( \varpi(X, Y) \in \Gamma(\text{End} E) \), so that \( \varpi(X, Y)s \in \Gamma(E) \). Let the right hand side of (5.11) be denoted provisionally by \( F(X, Y)s \). We claim that \( s \mapsto F(X, Y)s \) is \( \mathcal{A} \)-linear. To see that, denote by \( \tilde{f} \) the operator on \( \Gamma(E) \) of right multiplication by \( f \in \mathcal{A} \); then the Leibniz rule (5.6) can be rewritten as \( [\nabla_X, \tilde{f}] = \tilde{Xf} \). The desired \( \mathcal{A} \)-linearity of \( F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \) follows from

\[
[F(X, Y), \tilde{f}] = [[\nabla_X, \nabla_Y], \tilde{f}] - [\nabla_{[X, Y]}, \tilde{f}]
= [\nabla_X, [\nabla_Y, \tilde{f}]] + [[\nabla_X, \tilde{f}], \nabla_Y] - [\nabla_{[X, Y]}, \tilde{f}]
= [\nabla_X, \tilde{Yf}] + [\tilde{Xf}, \nabla_Y] - ([X, Y]f^-)^\tau
= (X(Yf) - Y(Xf) - [X, Y]f^-)\tau = 0,
\]
on using the Jacobi identity. Thus, by Corollary 5.3, \( F(X, Y) \) lies in \( \Gamma(\text{End} E) \), for each \( X, Y \in \mathfrak{X}(M) \). Furthermore, the formula \( \varpi_h = \tilde{h} \varpi \) for \( h \in \mathcal{A} \), which entails \( \nabla_h X = \tilde{h} \nabla X \), shows that \( F \) is \( \mathcal{A} \)-bilinear in \((X, Y)\), and so \( F \in \mathcal{A}^2(M, \text{End} E) \).

We must now show that \( \varpi \) and \( F \) coincide. It is enough to check this locally, since if \( \{f_j\} \) is a partition of unity on \( M \), the identities \( \varpi(X, Y)s = \sum_j \varpi(X, Y)s f_j \), \( F(X, Y)s = \sum_j F(X, Y)s f_j \) show that it suffices to prove \( \varpi(X, Y)s = F(X, Y)s \) when \( X, Y \) and \( s \) vanish outside a chart domain \( U_j \). Thus we may suppose that \( E \to M \) is a trivial bundle and indeed that \( E = M \times \mathbb{C}^r \), whereupon \( \varpi = d\alpha + \alpha \wedge \alpha \) if \( \alpha \in \mathcal{A}^2(M)^{r \times r} \) is the matrix of 1-forms satisfying \( \nabla = d + \alpha \).

47
Let us denote by \( h \) the element \((h_i, \ldots, h_r)\) of \( \mathcal{A}^r = \Gamma(E) \); then \( \nabla_X h = \iota_X (dh + ah) \) has \( k \)-th component \( X h^k + \alpha_i^k(X) h^l \). Thus \( F(X, Y) h \) has \( k \)-th component

\[
X(Y h^k + \alpha_i^k(Y) h^l) + \alpha_i^k(X) Y h^l + \alpha_m^k(X) \alpha_l^m(Y) h^l - Y(X h^k + \alpha_i^k(X) h^l) \\
- \alpha_i^k(Y) X h^l - \alpha_m^k(Y) \alpha_l^m(X) h^l - [X, Y] h^k - \alpha_i^k([X, Y]) h^l
\]

that is, \( F(X, Y) \) is a matrix of 2-forms with components \( (d \alpha^k + \alpha_m^k \wedge \alpha_l^m)(X, Y) \). Hence \( F = d \alpha + \alpha \wedge \alpha = \omega \) in \( \mathcal{A}^2(M, End E) \).

**Exercise 5.7.** Verify that \( F(gX, hY) s = F(X, Y) s \) \( g h \) for \( g, h \in \mathcal{A} \).

The connection \( \nabla \) is determined by local 1-forms \( \alpha_j \in \mathcal{A}^1(U_j, End E) \), and by a local system \( s_j \) in \( \Gamma(U_j, E) \) with respect to which the identification \( h \leftrightarrow s_j h^k \) is made. It should be carefully noted that the local 1-forms \( \alpha_j \) do not patch together to give a global 1-form \( \alpha \in \mathcal{A}^1(M, End E) \) unless \( E \to M \) is a trivial bundle, since there is no global \( d \) for which \( \nabla = d + \alpha \) unless the vector bundle is trivial. However, the local 2-forms \( \omega_j = d \alpha_j + \alpha_j \wedge \alpha_j \in \mathcal{A}^2(U_j, End E) \) do patch together, since they are restrictions to the \( U_j \) of the global 2-form \( \omega \).

When \( L \to M \) is a line bundle, there is a simplification: by Lemma 1.2, the line bundle \( End L \to M \) is trivial, so the curvature form \( \omega \) belongs to \( \mathcal{A}^2(M) \), since \( \mathcal{A}^2(M, End L) \simeq \mathcal{A}^2(M) \otimes \mathcal{A} \Gamma(End L) \simeq \mathcal{A}^2(M) \otimes \mathcal{A} = \mathcal{A}^2(M) \) via canonical isomorphisms. Furthermore, there are local 1-forms \( \alpha_j \in \mathcal{A}^1(M) \) such that \( \omega = d \alpha_j \) on each chart domain \( U_j \), so that the curvature \( \omega \) is locally exact and hence is a closed 2-form on \( M \).

The curvature form \( \omega \) depends on the connection \( \nabla \), but its de Rham class \([\omega] \in H^2_{dR}(M)\) does not. To see that, recall from (5.5) that if \( \nabla_0 \) and \( \nabla_1 \) are two connections on \( L \), with respective curvatures \( \omega_0 \) and \( \omega_1 \), then \( \nabla_1 s - \nabla_0 s = \alpha s \) for some \( \alpha \in \mathcal{A}^1(M) \), and hence \( \omega_1 - \omega_0 = d \alpha \), an exact 2-form. Thus the class \([\omega] \) depends only on the line bundle \( L \to M \).

**Exercise 5.8.** Suppose \( L \to M \) and \( L' \to M \) are equivalent line bundles and that \( \tau : L \to L' \) is an invertible bundle map. Show that \( \tau_* \in \Gamma(\text{Hom}(L, L')) \) intertwines the canonical isomorphisms \( \Gamma(End L) \simeq \mathcal{A} \) and \( \Gamma(End L') \simeq \mathcal{A} \), and deduce that the connections \( \nabla \) and \( \tau_* \circ \nabla \circ \tau_*^{-1} \) have the same curvature. Conclude that the class \([\omega] \) depends only on the equivalence class \([L] \) of the line bundle.

### 5.5 From de Rham cohomology to Čech cohomology

We have now associated to any line bundle \( L \to M \), by very different procedures, two second-degree cohomology classes, namely the Čech class obtained directly from its transition
functions and the de Rham class of the curvature of an arbitrary connection. We are led to suspect that these two cohomologies are related in some underlying fashion. This is indeed the case: one way of proving the de Rham theorem, which says that the cohomology of real-valued differential forms on the manifold $M$ depends only on the topology of $M$ (i.e., not on its differential structure) is to show that the de Rham cohomology is isomorphic to Čech cohomology with constant real coefficients.\footnote{Another way is to establish an isomorphism between de Rham cohomology and singular cohomology. This is thoroughly dealt with in [23].} For the full proof, we refer to [12] or [17, Appendix E]. We need only the second-degree case of this isomorphism; however, its proof illustrates the general method.

**Proposition 5.5.** Let $M$ be a compact manifold. Then $H^2_{dR}(M) \simeq H^2(M, \mathbb{R})$ by a canonical isomorphism.

**Proof.** Select a finite good covering\footnote{This is the only point at which compactness is invoked; and the compactness assumption may be removed whenever the existence of a finite good covering can be established independently.} $\mathcal{U} = \{U_j\}$ for $M$. We must show that $H^2_{dR}(M) \simeq H^2(\mathcal{U}, \mathbb{R})$. We first define maps between $Z^2_{dR}(M)$ and $Z^2(\mathcal{U}, \mathbb{R})$, which have some ambiguities that can be removed by passing to cohomology, thereby yielding well-defined $\mathbb{R}$-linear maps between $H^2_{dR}(M)$ and $H^2(\mathcal{U}, \mathbb{R})$.

Start with a closed 2-form $\omega$ in $\mathcal{A}^2(M)$; denote by $\omega := \{\omega_j\}$ the set of restrictions of $\omega$ to each set $U_j$ of the good covering; this is an element of $C^0(\mathcal{U}, \mathcal{A}^2)$, the set of Čech 0-cochains with 2-form coefficients. Since $d\omega = 0$ and each $U_j$ is contractible, the Poincaré lemma shows that there is $\alpha_j \in \mathcal{A}^1(U_j)$ with $d\alpha_j = \omega_j$ for each $j$; write $\alpha := \{\alpha_j\} \in C^0(\mathcal{U}, \mathcal{A}^1)$. Now on each nonvoid overlap $U_i \cap U_j$, we have $d(\alpha_i - \alpha_j) = \omega_i - \omega_j = 0$, so that $\alpha_i - \alpha_j = df_{ij}$ with $f := \{f_{ij}\} \in C^1(\mathcal{U}, \mathcal{A}^0)$. On each nonvoid $U_i \cap U_j \cap U_k$ we have $0 = d(f_{ij} - f_{ik} + f_{jk}) = 0$ by cancellation, so that $a_{ijk} := f_{ij} - f_{ik} + f_{jk}$ is a constant real-valued function (since $U_i \cap U_j \cap U_k$ is connected); since $a_{ijk} - a_{ijl} + a_{kl} - a_{jk} = 0$ on $U_i \cap U_j \cap U_k \cap U_l$ by cancellation, we have $\delta a = 0$, i.e., $a \in Z^2(\mathcal{U}, \mathbb{R})$.

There are some ambiguities in the choices of $\alpha_j$ and $f_{ij}$, so this process does not give a well-defined map $\omega \mapsto a$. Firstly, we could replace each $\alpha_j$ by $\alpha_j + dg_j$, where $g := \{g_j\} \in C^0(\mathcal{U}, \mathcal{A}^0)$; then $f_{ij}$ becomes $f_{ij} + g_i - g_j$, which leaves $a_{ijk}$ unchanged. Secondly, we could replace each $f_{ij}$ by $f_{ij} + c_{ij}$, where the $c_{ij}$ are constant functions, i.e., $c := \{c_{ij}\} \in C^1(\mathcal{U}, \mathbb{R})$; this changes $a$ to $a + \delta c$, and $[a] \in H^2(\mathcal{U}, \mathbb{R})$ is left unchanged. Therefore $\omega \mapsto [a]$ is well-defined. Thirdly, we could replace $\omega$ by $\omega + d\beta$, adding an exact form; then $\alpha_j$ becomes $\alpha_j + \beta_j$, and $f_{ij} = (\alpha_i + \beta_i) - (\alpha_j + \beta_j)$ is unchanged because $\beta_i$ and $\beta_j$ agree on $U_i \cap U_j$. Therefore, $[a]$ depends only on the de Rham class of $\omega$, so $[\omega] \mapsto [a]$ is a well-defined $\mathbb{R}$-linear map from $H^2_{dR}(M)$ to $H^2(\mathcal{U}, \mathbb{R})$.

To go the other way, start from $a \in Z^2(\mathcal{U}, \mathbb{R})$, and take a smooth partition of unity $\{\psi_j\}$ subordinate to the covering $\mathcal{U}$. As in the proof of Proposition 1.6, define $f \in C^1(\mathcal{U}, \mathcal{A}^0)$ by $f_{ij} := \sum a_{ijr}\psi_j$; then $f_{ij} - f_{ik} + f_{jk} = a_{ijk}$ just as in (1.12), since $\delta a = 0$; moreover, $df_{ij} - df_{ik} + df_{jk} = 0$ since the $a_{ijk}$ are constant. Define $\alpha \in C^0(\mathcal{U}, \mathcal{A}^1)$ by $\alpha := \sum k \psi_k f_{jk}$; now $\alpha_i - \alpha_j = \sum_k \psi_k f_{ij} = f_{ij}$ on $U_i \cap U_j$, and so $\alpha_i - \alpha_j = 0$ there, which says that the
local 2-forms \( \omega_j := d\alpha_j \) patch together to give a global 2-form \( \omega \) on \( M \). Since \( d\omega_j = 0 \), \( \omega \) is closed, i.e., \( \omega \in Z^2_{\text{dR}}(M) \).

The correspondence \( a \mapsto \omega \) depends on the partition of unity \( \{ \psi_j \} \), but it is clear that \( a \mapsto f \mapsto \alpha \mapsto \omega \) retracts the earlier path \( \alpha \mapsto f \mapsto a \) thus we have shown that the linear map \( [\omega] \mapsto [a] \) is surjective. Moreover, if \( [a] = 0 \), then \( a = \delta c \) with \( c \in C^1(\mathcal{U}, \mathbb{R}) \), so that \( f_{ij} := \sum_{r}(c_{ij} - c_{ir} + c_{jr})\psi_r = c_{ij} - h_i + h_j \) where \( h_j := \sum r c_{jr}\psi_r \) and thus \( df_{ij} = dh_j - dh_i \); but then \( \alpha_j = \sum_k \psi_k(dh_k - dh_j) = \beta_j - dh_j \), where \( \beta_j \) is the restriction to \( U_j \) of the 1-form \( \beta := \sum_k \psi_k dh_k \in \Lambda^1(M) \); thus \( \omega = d\beta \) is exact, i.e., \( [\omega] = 0 \). Hence \( [\omega] \mapsto [a] \) is injective, and is therefore an isomorphism (or real vector spaces) between \( H^2_{\text{dR}}(M) \) and \( H^2(\mathcal{U}, \mathbb{R}) \). \( \square \)

The foregoing proof can be cast in a more algebraic framework, as follows. The abelian groups (actually, vector spaces) \( C^p_q := C^p(\mathcal{U}, \Lambda_q^\mathbb{R}) \) of \( q \)-form-valued 

\( \check{C}ech \) \( p \)-cochains are related by two coboundary operators, \( \delta : C^p_q \to C^{p+1,q} \) and \( d : C^p_q \to C^{p,q+1} \), satisfying \( \delta d - d\delta = 0 \); if we introduce \( \partial := (-1)^p d \) on \( C^p_q \), we get \( \delta \partial + \partial \delta = 0 \), so we obtain a "double complex". By introducing \( D := \delta + \partial : C^p_q \to C^{p+1,q} \oplus C^{p,q+1} \), and \( E^m := \bigoplus_{p+q=m} C^p_q \), we obtain a new cochain complex \( (E^\bullet, D) \), sometimes called the \( \check{C}ech-\text{de Rham} \) complex. (For instance, a 1-cochain for this complex is of the form \( \alpha \oplus f \) where \( \alpha \) and \( f \) are as in the proof of Proposition 5.5, and \( D(\alpha \oplus f) = d\alpha \oplus (\delta\alpha - df) \oplus \delta f \), which evaluates to \( \omega \oplus 0 \oplus a \).

A general theorem \([12, 17]\) asserts that the \( k \)-th cohomology groups for the complexes \( (\Lambda^\bullet(M), d) \), \( (E^\bullet, D) \), and \( (C^\bullet(\mathcal{U}, M), \delta) \) are isomorphic, for any \( k \in \mathbb{N} \).

**Definition 5.7.** The standard inclusion \( \iota : \mathbb{Z} \to \mathbb{R} \) induces an injection of \( \check{C}ech \) homology groups \( \iota_* : H^2(\mathbb{Z}, \mathbb{Z}) \to H^2(\mathbb{R}, \mathbb{R}) \). We regard \( H^2(\mathbb{Z}, \mathbb{Z}) \) as a subset of \( H^2(\mathbb{R}, \mathbb{R}) \) by identifying it with its image under \( \iota_* \). We say that a de Rham cohomology class \( [\omega] \in H^2_{\text{dR}}(M) \) is integral if the corresponding \( \check{C}ech \) class \( [a] \) lies in \( H^2(\mathbb{Z}, \mathbb{Z}) \).

**Theorem 5.6.** Let \( L \longrightarrow M \) be a Hermitian line bundle over a compact manifold, and let \( \nabla \) be a connection on it, compatible with the metric, with curvature \( \omega \). Then \([2\pi i)^{-1}\omega] \) is an integral de Rham class, corresponding to the class \([L]\) of the line bundle in \( H^2(\mathbb{Z}, \mathbb{Z}) \).

**Proof.** Let \( \mathcal{U} = \{ U_j \} \) be a covering of \( M \) by chart domains, and let \( \{ s_j \} \) be a local system of nonvanishing sections for the line bundle \( L \longrightarrow M \). Then \( s_j \in \Gamma(U_j, L) \) for each \( j \). By the locality property of \( \nabla \), we have \( \nabla s_j \in \Lambda^1(U_j, L) = \Gamma(U_j, L) \otimes_{C^\infty(U_j)} \Lambda^1(U_j) \), so \( \nabla s_j = s_j \otimes \alpha_j \) for some \( \alpha_j \in \Lambda^1(U_j) \) (because \( s_j \) generates \( \Gamma(U_j, L) \) as a \( C^\infty(U_j)-\)module). We may take \( s_j \) to be normalized with respect to the metric, i.e., \( (s_j | s_j) = 1 \) on \( U_j \). Since \( \nabla \) is compatible, we find that \( \alpha_j \) is a purely imaginary 1-form, because \( \partial \alpha_j = \omega_j \), so \( \nabla s_j = (s_j | \nabla s_j) = 0 \).

By \((5.8), \nabla^2 s_j = (s_j \otimes \alpha_j) \wedge \alpha_j + s_j \wedge d\alpha_j = s_j \wedge d\alpha_j \), so that \( \omega = d\alpha_j \) on \( U_j \). Let us write \( \beta_j := (2\pi i)^{-1}\alpha_j \); then \( \beta_j \) are real-valued 1-forms, satisfying \( d\beta_j = (2\pi i)^{-1}\omega \). Thus \([2\pi i)^{-1}\omega] \in H^2_{\text{dR}}(M) \) is a real de Rham class.

The local sections \( s_j \) are related by \( s_i = g_{ij}s_j = s_jg_{ij} \) on \( U_i \cap U_j \), where the \( g_{ij} \) are the \( U(1)-\)valued transition functions of the line bundle (see Definition 1.14). Then

\[
 s_j \otimes g_{ij}\alpha_i = s_i \otimes \alpha_i = \nabla s_i = \nabla(s_jg_{ij}) = (s_j \otimes \alpha_j)g_{ij} + s_j \otimes dg_{ij}
\]
on \( U_i \cap U_j \); since \( s_j \) does not vanish there, we get

\[
\alpha_i = g_{ij}^{-1} \alpha_j g_{ij} + g_{ij}^{-1} dg_{ij} = \alpha_j + g_{ij}^{-1} dg_{ij} \quad \text{on} \quad U_i \cap U_j.
\]

This gives

\[
\beta_i - \beta_j = (2\pi i)^{-1} g_{ij}^{-1} dg_{ij} = (2\pi i)^{-1} d(\log g_{ij})
\]

for a suitable branch of the logarithm, and \( f_{ij} := (2\pi i)^{-1} \log g_{ij} \) is a real-valued function on \( U_i \cap U_j \) satisfying \( \beta_i - \beta_j = df_{ij} \). Now \( a_{ijk} := f_{ij} - f_{ik} + f_{jk} \) gives the Čech 2-cocycle \( a \in C^2(U, \mathbb{R}) \) such that \([a]\) corresponds to \([(2\pi i)^{-1} \omega]\) under the isomorphism of Proposition 5.5. On the other hand, the proof of Proposition 1.6 constructs this very same Čech class \([a]\) as the element of \( \bar{H}^2(M, \mathbb{Z}) \) which corresponds to \([L]\) under the Bockstein isomorphism between \( \bar{H}^1(M, U(1)) \) and \( \bar{H}^2(M, \mathbb{Z}) \). Hence, \([(2\pi i)^{-1} \omega]\) is integral and corresponds to \([L]\). \( \square \)

In particular, we see that a closed 2-form on \( M \) is the curvature of some compatible connection only if it equals \( 2\pi i \) times an integral 2-form.\(^8\)

### 5.6 Line bundles over \( \mathbb{C}P^m \)

**Definition 5.8.** Any element of \( \mathbb{C}P^m \) is a line through the origin in \( \mathbb{C}^{m+1} \), and two such lines intersect only at the origin. The **tautological line bundle** \( L \rightarrow \mathbb{C}P^m \) is given by taking the disjoint union of these lines; the fibre at any element of \( \mathbb{C}P^m \) is the very same line. More explicitly, we can take \( L \) to be

\[
L := \{ (\ell, v) \in \mathbb{C}P^m \times \mathbb{C}^{m+1} : v \in \ell \},
\]

and define \( \pi(\ell, v) := \ell \).

Here \( L \rightarrow \mathbb{C}P^m \) is manifestly a subbundle of a trivial vector bundle. Indeed, let \( \ell^\perp \) be the subspace of \( \mathbb{C}^{m+1} \) orthogonal to \( \ell \) (with respect to the usual inner product on \( \mathbb{C}^{m+1} \)), and if \( E := \{ (\ell, u) \in \mathbb{C}P^m \times \mathbb{C}^{m+1} : u \in \ell^\perp \} \), then the Whitney sum \( L \oplus E = \mathbb{C}P^m \times \mathbb{C}^{m+1} \) is trivial.

**Exercise 5.9.** If \( \eta : \mathbb{C}^{m+1} \setminus \{0\} \rightarrow \mathbb{C}P^m \) is the quotient map, \( \mathbb{C}^{m+1} \setminus \{0\} \xrightarrow{\eta} \mathbb{C}P^m \) is a principal \( \mathbb{C}^\times \)-bundle. Show that the tautological line bundle is associated to this principal bundle via the representation \( \rho \) of \( \mathbb{C}^\times \) on \( \mathbb{C} \) given by the multiplication \( \rho(\lambda)(\mu) := \lambda \mu \).

**Definition 5.9.** The dual of the tautological line bundle on \( \mathbb{C}P^m \) is the **hyperplane bundle** \( H \rightarrow \mathbb{C}P^m \), where \( H_\ell = \ell^\ast \) (the one-dimensional space of linear functionals on \( \ell \)).

--

\(^8\)This observation is the launching point for the theory of geometric quantization, which seeks to represent certain functions on a symplectic manifold \((M, \Omega)\), i.e., a manifold equipped with a closed nondegenerate 2-form, by operators on a Hilbert space. The elements of this Hilbert space come from sections of a certain Hermitian line bundle over \( M \), equipped with a compatible connection whose curvature is \((2\pi i \hbar)^{-1} \Omega\) (where \( \hbar \) is a positive constant, identified with Planck’s constant). Theorem 5.6 shows that such a line bundle exists if \([\hbar^{-1} \Omega]\) is integral; this is a discreteness condition on the original symplectic form \( \Omega \), hence the use of the word “quantization”.

51
The tensor product bundles $L^\otimes k \to \mathbb{CP}^m$ and $H^\otimes l \to \mathbb{CP}^m$, for $k,l \in \mathbb{N}$, give more examples of line bundles over $\mathbb{CP}^m$. (We take $L^\otimes 0 = H^\otimes 0 := \mathbb{CP}^m \times \mathbb{C}$ by convention.) We aim to show that these are distinct, and that any line bundle over $\mathbb{CP}^m$ is equivalent to one on this list. Since $[L]^{-1} = [H]$ in the group of line bundle classes, and since $H^2_{dR}(\mathbb{CP}^m) = \mathbb{R}$, we have $\mathbb{H}^2(\mathbb{CP}^m, \mathbb{Z}) \cong \mathbb{Z}$, so it suffices to verify that $[H]$ corresponds to a generator of the infinite cyclic group $H^2(\mathbb{CP}^m, \mathbb{Z})$.

**Definition 5.10.** A complex vector bundle $E \to M$ on a complex manifold $M$ is a **holomorphic vector bundle** if its transition functions $g_{ij} : U_i \cap U_j \to GL(m, \mathbb{C})$ are holomorphic. The space of holomorphic sections of this line bundle is denoted by $\mathcal{O}(M, E)$ or simply by $\mathcal{O}(E)$.

**Exercise 5.10.** Compute the transition functions for the tautological and hyperplane bundles over $\mathbb{CP}^m$ and hence show that these are holomorphic line bundles.

Consider first the trivial bundle $\mathbb{CP}^m \times \mathbb{C} \to \mathbb{CP}^m$; a holomorphic section of this bundle is of the form $s(x) = (x, f(x))$, where $f : \mathbb{CP}^m \to \mathbb{C}$ is a holomorphic function. If $(U, \phi)$ is a chart with $\phi(U) = \mathbb{C}^m$, then $f \circ \phi^{-1} : \mathbb{C}^m \to \mathbb{C}$ is an entire holomorphic function on $\mathbb{C}^m$, which is bounded since $\mathbb{CP}^m$ is compact and $f$ is continuous; thus, by Liouville’s theorem, $f$ is constant. More generally, any holomorphic function on a compact complex manifold is constant.

It turns out that the tautological line bundle has no global holomorphic sections, other than the zero section. The hyperplane bundle $H \to \mathbb{CP}^m$, by contrast, has nontrivial global holomorphic sections. For any $f \in (\mathbb{C}^{m+1})^*$, define $s_f \in \Gamma(\mathbb{CP}^m, H)$ by $s_f(\ell) := f|\ell$. Conversely, any holomorphic section of $H \to \mathbb{CP}^m$ is of the form $s(\ell) := g|\ell$, where $g : \mathbb{C}^{m+1} \to \mathbb{C}$ is a holomorphic function whose restriction to each one-dimensional subspace is linear, i.e., $g$ is homogeneous of degree one. Since only first-degree terms can then occur in the Taylor series of $g$, the function $g$ is itself linear, i.e., $g \in (\mathbb{C}^{m+1})^*$ and $s = s_g$. Thus, $f \mapsto s_f$ is a linear bijection between $(\mathbb{C}^{m+1})^*$ and $\mathcal{O}(H)$.

In particular, $\mathcal{O}(H)$ is finite-dimensional, with dim $\mathcal{O}(H) = m + 1$. A basis is given by $\sigma_0, \ldots, \sigma_m$, where $\sigma_j = s_{z^j}$ and $z^j \in (\mathbb{C}^{m+1})^*$ be the $j$-th coordinate function, for $j = 0, 1, \ldots, m$.

The standard inner product $\langle \cdot | \cdot \rangle$ on $\mathbb{C}^{m+1}$ gives metrics on the line bundles $L$ and $H$. We may write $(u \mid v)_{\ell} := \langle u \mid v \rangle^0$ for $u, v \in L_{\ell}$, i.e., $u, v \in \ell \subset \mathbb{C}^{m+1}$. Now if $\phi, \psi \in L^*$ are given by $\phi(v) = \langle u \mid v \rangle$, $\psi(v) = \langle w \mid v \rangle$ for $v \in \ell$, then

$$\langle \phi \mid \psi \rangle_{\ell} := \frac{\langle u \mid v \rangle^0 \langle w \mid v \rangle}{\langle v \mid v \rangle} = \frac{\langle w \mid v \rangle \langle v \mid u \rangle}{\langle v \mid v \rangle},$$

which is independent of any nonzero $v \in \ell$. When $\ell \in U_j$, we have the equality

$$\langle v \mid v \rangle = z^0 \bar{z}^0 + \cdots + z^m \bar{z}^m = z^j \bar{z}^j \left( 1 + \sum_{k \neq j} w_j^k \bar{w}_j^k \right) = |z^j(v)|^2 Q_j(\ell),$$

52
and in particular
\[
(\sigma_j | \sigma_j) = Q_j^{-1} = \left(1 + \sum_{k \neq j} w_j^k \bar{w}_j^k\right)^{-1} \quad \text{on } U_j.
\]
(5.13)

By continuity, \( (\sigma_j | \sigma_j) = 0 \) and thus \( \sigma_j \) vanishes on the complement of \( U_j \).

### 5.7 Connections on the hyperplane bundle

**Definition 5.11.** When \( M \) is a complex manifold and \( \nabla \) is a connection on a holomorphic vector bundle \( E \to M \), we say that \( \nabla \) is compatible with the holomorphic structure if \( \nabla s \in A^{1,0}(E) \) whenever \( s \in \mathcal{O}(E) \). If the vector bundle is also Hermitian, we say that \( \nabla \) is a canonical connection if it is compatible with both the metric and the holomorphic structure.

**Proposition 5.7.** On the hyperplane bundle \( H \to \mathbb{C}P^m \), there is a unique canonical connection.

**Proof.** Let \( \{U_j, \sigma_j\} \) be the local system of holomorphic sections of the previous Section. Since \( \sigma_j \) is nonvanishing on \( U_j \) (e.g., on account of (5.13)), the argument of the proof of Theorem 5.6 shows that any connection \( \nabla \) on the hyperplane bundle is given on each \( U_j \) by

\[
\nabla \sigma_j = \sigma_j \otimes \alpha_j, \quad \text{for some } \alpha_j \in A^1(U_j).
\]

Thus \( \nabla \) is compatible with the holomorphic structure iff each \( \alpha_j \) lies in \( A^{1,0}(U_j) \).

We may write \( \alpha_j = a_{jk} dw_j^k \) with each \( a_{jk} \in C^\infty(U_j) \), so that \( \nabla \) is also compatible with the metric iff

\[
Q_j^{-1} \left( a_{jk} dw_j^k + \bar{a}_{jk} d\bar{w}_j^k \right) = (\sigma_j | \nabla \sigma_j) + (\nabla \sigma_j | \sigma_j) = d(\sigma_j | \sigma_j)
\]

\[
= d(Q_j^{-1}) = -Q_j^{-2} d\left( \sum_{k \neq j} w_j^k \bar{w}_j^k \right)
\]

iff \( a_{jk} = -Q_j^{-1} \bar{a}_j^k \) for all \( k \neq j \). Thus \( \nabla \) satisfies

\[
\nabla \sigma_j = -\sigma_j \otimes Q_j^{-1} \sum_{k \neq j} \bar{w}_j^k dw_j^k,
\]

so this calculation establishes existence and uniqueness of the canonical connection. \( \square \)

**Proposition 5.8.** The curvature of the canonical connection on the hyperplane bundle is \(-i\Phi\), where \( \Phi \) is the Kähler form on \( \mathbb{C}P^m \).

**Proof.** The curvature \( \omega \) satisfies \( \omega = d\alpha_j \) on \( U_j \), so

\[
\omega = d\alpha_j = Q_j^{-1} \sum_{k \neq j} dw_j^k \wedge d\bar{w}_j^k + Q_j^{-2} dQ_j \wedge \sum_{k \neq j} \bar{w}_j^k dw_j^k
\]

\[
= Q_j^{-2} \left( Q_j \sum_{k \neq j} dw_j^k \wedge d\bar{w}_j^k - \sum_{r,s \neq j} \bar{w}_r^s w_j^r dw_j^r \wedge d\bar{w}_s^r \right) = -i\Phi,
\]

by comparison with (2.7). \( \square \)
The element of $H^2_{2\mathbb{R}}(\mathbb{CP}^n)$ which corresponds to the equivalence class of the hyperplane bundle is therefore $[-(2\pi)^{-1}\Phi]$.

5.8 Characteristic classes

We have seen that the curvature of a Hermitian line bundles over $M$ is a closed 2-form $\omega$ on $M$ and $[(2\pi)^{-1}\omega]$ is an integral cohomology class. For vector bundles of higher rank, it is possible to obtain integral cohomology classes of even degree from the matrix-valued curvature $\omega$ by taking suitable traces of its exterior powers. These classes are topological invariants of the manifold $M$.

Definition 5.12. Let $\tilde{q}: (\mathbb{C}^{r\times r})^k \to \mathbb{C}$ be a symmetric $k$-linear map on the vector space of complex $r \times r$ matrices. One says that $\tilde{q}$ is invariant (under the adjoint representation $\text{Ad}(g)A := gAg^{-1}$ of $GL(r, \mathbb{C})$) if

$$\tilde{q}(gA_1g^{-1}, \ldots, gA_kg^{-1}) = \tilde{q}(A_1, \ldots, A_k)$$

(5.14)

for all $g \in GL(r, \mathbb{C})$, $A \in \mathbb{C}^{r\times r}$.

The function $q: \mathbb{C}^{r\times r} \to \mathbb{C}$ defined by $q(A) := \tilde{q}(A, A, \ldots, A)$ is called a homogeneous polynomial of degree $k$; when $\tilde{q}$ is invariant, we call it an invariant polynomial on $\mathbb{C}^{r\times r}$. With the “polarization formula” $\tilde{q}(A_1, \ldots, A_k) := (k!)^{-1}\sum_{|J|=s}(-1)^{k-s}q(A_{j_1} + \cdots + A_{j_s})$, the map $\tilde{q}$ may be recovered from $q$.

More generally, an invariant polynomial on $\mathbb{C}^{r\times r}$ is a finite sum of homogeneous invariant polynomials of various degrees. (A constant function is an invariant polynomial of degree zero.)

For instance, in the expansion $\det(t - A) = \sum_{k=0}^r (-1)^k q_k(A)t^{r-k}$, each $q_k$ is an invariant homogeneous polynomial; here $q_r(A) = \det A$, $q_1(A) = \text{tr} A$, and $q_2(A) = \sum_{i<j} \lambda_i(A)\lambda_j(A)$, where the $\lambda_j(A)$ are the eigenvalues of $A$. Notice that $2q_2(A) = (\text{tr} A)^2 - \text{tr}(A^2)$.

Lemma 5.9. If $q: \mathbb{C}^{r\times r} \to \mathbb{C}$ is an invariant polynomial on $\mathbb{C}^{r\times r}$, then for $A_1, \ldots, A_k, B \in \mathbb{C}^{r\times r}$ we have $\sum_{j=1}^k \tilde{q}(A_1, \ldots, [B, A_j], \ldots, A_k) = 0$.

Proof. Set $g = e^{tB}$ in (5.14), and differentiate with respect to $t$ at $t = 0$; the result follows on noting that $(d/dt) |_{t=0} e^{tB}Ae^{-tB} = BA - AB = [B, A]$. \qed

If $E_\tau = M \times \mathbb{C}^r$, so that $E_\tau \to M$ is the trivial bundle of rank $r$, then $q$ yields a polynomial map from $\Gamma(\text{End} E_\tau) = A^{r\times r} = C^\infty(M, \mathbb{C}^{r\times r})$ to $A = C^\infty(M)$ by writing $q(A)(x) := q(A(x))$; and $\tilde{q}$ similarly defines an $A$-valued symmetric $k$-linear map on $A^{r\times r}$. The invariance property is $q(gAg^{-1}) := q(A)$, where $g \in C^\infty(M, GL(r, \mathbb{C}))$. Since $\tau(v):= gv$ for $v \in A^r = \Gamma(E^r)$ defines a bundle automorphism $(\tau, \text{id}_M)$ of $E_\tau \to M$, the invariance condition may be reexpressed as $q \circ \text{Ad} \tau = q$ for every invertible bundle map $\tau: E_\tau \to E_\tau$.

Exercise 5.11. Verify that for any vector bundle $E \to M$ of rank $r$, the recipe $q(A)(x) := q(A(x))$ yields a map $q: \Gamma(\text{End} E) \to A$ (and by polarization, a $k$-linear map $\tilde{q}: \Gamma(\text{End} E)^k \to A$), such that $q(\tau \circ A \circ \tau^{-1}) = q(A)$ for any invertible bundle map $\tau: E \to E$. 

54
These may be extended to maps $q, \tilde{q}$ on $\mathcal{A}^*(M, \text{End } E)$ with values in $\mathcal{A}^*(M)$ by defining

$$\tilde{q}(A_1 \otimes \eta_1, \ldots, A_k \otimes \eta_k) := \tilde{q}(A_1, \ldots, A_k) \eta_1 \land \cdots \land \eta_k. \quad (5.15)$$

The invariance property of Lemma 5.9 may be expressed in this context as

$$\sum_{j=1}^{k} (-1)^{s_0(s_1 + \cdots + s_{j-1})} \tilde{q}(\omega_1, \ldots, [\beta, \omega_j], \ldots, \omega_k) = 0 \quad (5.16)$$

whenever $\beta \in \mathcal{A}^{s_0}(M, \text{End } E)$ and $\omega_j \in \mathcal{A}^{s_j}(M, \text{End } E)$ for $j = 1, \ldots, k$. The bracket on $\mathcal{A}^*(M, \text{End } E)$ is defined as the linear extension of the recipe:

$$[A \otimes \eta, B \otimes \zeta] := [A, B] \eta \land \zeta \quad (5.17)$$

for $A, B \in \Gamma(\text{End } E), \eta, \zeta \in \mathcal{A}^*(M)$.

Exercise 5.12. Check the invariance formula (5.16).

Exercise 5.13. If $\alpha \in \mathcal{A}^k(M, \text{End } E_r), \beta \in \mathcal{A}^l(M, \text{End } E_r)$ are matrix-valued forms, deduce from (5.17) that

$$[\alpha, \beta] = \alpha \land \beta - (-1)^{kl} \beta \land \alpha$$

in $\mathcal{A}^{k+l}(M, \text{End } E_r)$.

Exercise 5.14. By taking the exterior derivative of the right hand side of (5.15), show that

$$d(\tilde{q}(\omega_1, \ldots, \omega_k)) = \sum_{j=1}^{k} (-1)^{s_1 + \cdots + s_{j-1}} \tilde{q}(\omega_1, \ldots, d\omega_j, \ldots, \omega_k) \quad (5.18)$$

when each $\omega_j \in \mathcal{A}^{s_j}(M, \text{End } E_r)$ is a matrix-valued form.

It is convenient to introduce the notation $q'(\omega; \theta) := \sum_{j=1}^{k} \tilde{q}(\omega, \ldots, \theta, \ldots, \omega)$, where $\theta$ appears in the $j$-th place and $\omega$ elsewhere; if $\omega$ has even degree, it follows from (5.16) that $q'(\omega; [\beta, \omega]) = 0$ for any $\beta \in \mathcal{A}^*(M, \text{End } E)$. Also, if $\omega \in \mathcal{A}^{\text{even}}(M, \text{End } E_r)$ is a matrix-valued form of even degree, then (5.18) gives $d(q(\omega)) = q'(\omega; d\omega) \in \mathcal{A}^{\text{odd}}(M, \text{End } E_r)$.

Proposition 5.10. Let $\nabla$ be a connection on a vector bundle $E \rightarrow M$, with curvature $\omega \in \mathcal{A}^2(M, \text{End } E)$. If $q$ is an invariant polynomial on $\mathcal{O}^{r \times r}$, then $q(\omega)$ is closed in $\mathcal{A}^{\text{even}}(M)$.

Proof. To show that $d(q(\omega)) = 0$ on $M$, it is enough to show that $d(q(\omega)) = 0$ on each chart domain $U$. Thus we can suppose that $E \rightarrow M$ is a trivial bundle, that $\nabla = d + \alpha$ with $\alpha \in \mathcal{A}^1(M, \text{End } E)$ and that $\omega = d\alpha + \alpha \land \alpha$.

Now $\mathcal{A}^k(M, \text{End } E) \simeq (\mathcal{A}^k(M))^{r \times r}$ by the triviality of the bundle, and the exterior derivative $d: \mathcal{A}^*(M, \text{End } E) \rightarrow \mathcal{A}^*(M, \text{End } E)$ is an antiderivation, so

$$d\omega = d(\alpha \land \alpha) = d\alpha \land \alpha - \alpha \land d\alpha = \omega \land \alpha - \alpha \land \omega = [\omega, \alpha], \quad (5.19)$$

which is known as the Bianchi identity for the curvature. Now

$$d(q(\omega)) = q'(\omega; d\omega) = -q'(\omega; [\alpha, \omega]) = 0$$

by the invariance of $q$. 

55
Exercise 5.15. A connection $\nabla$ on $E \to M$, with curvature $\omega$, yields a connection on the vector bundle $\text{End} E \to M$, also denoted $\nabla$, by adopting the Leibniz rule $\nabla(as) = (\nabla a)s + a(\nabla s)$ as a definition, i.e., by setting $(\nabla a)s := \nabla(\nabla a) - a(\nabla s)$ for $a \in \Gamma(\text{End} E)$, $s \in \Gamma(E)$. Check that $a \mapsto \nabla a$ is indeed a connection. This extends to a linear map $\nabla : \mathcal{A}^k(M, \text{End} E) \to \mathcal{A}^{k+1}(M, \text{End} E)$ by (5.8). Use (5.8) and (5.9) to show that $\nabla^2 \beta = \omega \wedge \beta$ for any $\beta \in \mathcal{A}^1(M, E)$, and deduce that $\nabla^2 s = \nabla(\nabla^2 s) - \nabla^2(\nabla s) = 0$ for any $s \in \Gamma(E)$. Finally, show that the equation $\nabla \omega = 0$ reduces to the Bianchi identity (5.19) locally,\(^9\) when $\omega$ is of the form $\omega = d\alpha + \alpha \wedge \alpha$.

Proposition 5.11. Let $\nabla$ be a connection on a vector bundle $E \to M$, with curvature $\omega$, and let $q$ be an invariant polynomial. Then the cohomology class $[q(\omega)]$ is independent of $\nabla$.

Proof. Let $\nabla_0$, $\nabla_1$ be two connections on $E \to M$. Then, by (5.5), $\nabla_1 = \nabla_0 + \beta$ with $\beta \in \mathcal{A}^1(M, \text{End} E)$. Set $\nabla_t := (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t\beta$ for $0 \leq t \leq 1$; this is a connection on $E \to M$, whose curvature we denote by $\omega_t$. The relation $[q(\omega_0)] = [q(\omega_1)]$ follows from the transgression formula:

$$q(\omega_t) - q(\omega_0) = \int_0^1 q'(\omega_t; \beta) \, dt.$$  

To verify this relation, it suffices to show that $(d/dt)q(\omega_t) = (d/dt)q(\omega_t; \beta)$ for $0 < t < 1$. By $k$-linearity of $\nabla_t$, we have

$$\frac{d}{dt} q(\omega_t) = \frac{d}{dt} q(\omega_t, \ldots, \omega_t) = \sum_{j=1}^k q(\omega_t, \ldots, (d/dt)\omega_t, \ldots, \omega_t) = q'(\omega_t; (d/dt)\omega_t).$$  

To see that $q'(\omega_t; (d/dt)\omega_t) = (d/dt)q(\omega_t; \beta)$, we may compare these two $2k$-forms over any chart domain $U \subset M$; or equivalently, we may assume that the bundle $E \to M$ is trivial, that $\nabla_t = d + \alpha_t$ and $\omega_t = d\alpha_t + \alpha_t \wedge \alpha_t$. In that case, $\alpha_t = \alpha_0 + t\beta$, and

$$\frac{d}{dt} \omega_t = \frac{d}{dt} \left(d\alpha_0 + t d\beta + (\alpha_0 + t\beta) \wedge (\alpha_0 + t\beta)\right) = d\beta + \alpha_0 \wedge \beta + \beta \wedge \alpha_0 + 2t(\beta \wedge \beta) = d\beta + \alpha_t \wedge \beta + \beta \wedge \alpha_t = d\beta + [\alpha_t, \beta].$$  

\(^9\)The identity $\nabla \omega = 0$ for any connection is therefore called the Bianchi identity.
It remains to compute
\[
d(q' (\omega; \beta)) = \sum_{j=1}^{k} d(\tilde{q}(\omega_1, \ldots, \omega_j))
\]
\[
= \sum_{i<j} \tilde{q}(\omega_1, \ldots, d\omega_i, \beta, \ldots, \omega_j) + q' (\omega_i; d\beta) - \sum_{i>j} \tilde{q}(\omega_1, \ldots, \beta, \ldots, d\omega_i, \ldots, \omega_j)
\]
\[
= - \sum_{i<j} \tilde{q}(\omega_1, \ldots, [\alpha_i, \omega_i], \ldots, \beta, \ldots, \omega_j) + q' (\omega_i; d\beta)
\]
\[
+ \sum_{i>j} \tilde{q}(\omega_1, \ldots, \beta, \ldots, [\alpha_i, \omega_i], \ldots, \omega_j)
\]
\[
= \sum_{j=1}^{k} \tilde{q}(\omega_1, \ldots, [\alpha_1, \beta_1], \ldots, \omega_j) + q' (\omega_i; d\beta) = q' (\omega_i; d\beta + [\alpha_i, \beta])
\]
\[
= q' (\omega_i; (d/\text{dt})\omega_i),
\]
(with \(d\omega_i\) in the \(i\)-th position and \(\beta\) in the \(j\)-th position in the double summations). Here we have used the Bianchi identity \(d\omega_i + [\alpha_i, \omega_i] = 0\) and the invariance property (5.16) applied to the form \(\tilde{q}(\omega_1, \ldots, d\omega_i, \ldots, \beta, \ldots, \omega_j)\) and also to \(\tilde{q}(\omega_1, \ldots, \beta, \ldots, d\omega_i, \ldots, \omega_j)\).

If \(E' \to M\) is another vector bundle equivalent to \(E \to M\), and it \(\tau: E \to E'\) is an invertible bundle map, then by Exercises 5.4 and 5.6, the recipes \(\nabla' := \tau_* \circ \nabla \circ \tau_*^{-1}\) and \(\omega' := \tau_* \circ \omega \circ \tau_*^{-1}\) match connections and curvatures on both vector bundles; by the invariance property (5.14) of the polynomial \(q\), we have \(q(\omega') = q(\omega)\) in \(\mathcal{A}^{2k}(M)\). Thus the cohomology class \([q(\omega)]\) depends only on the equivalence class of the vector bundle \(E \to M\), and we may denote it by \(q([E])\), or more simply by \(q(E) \in H_{\text{dR}}^{2k}(M) \otimes \mathbb{R} \mathbb{C}\).

If \(E \to M\) is a Hermitian vector bundle, we consider only bundle maps \(\tau: E \to E'\) which preserve the fibre metrics. Thus we may use polynomials \(q\) which are only invariant under the unitary group \(U(r)\) rather than \(GL(r, \mathbb{C})\); in (5.16) we may only use forms \(\beta\) with either \(i/\beta\) or \(\beta\) itself real-valued. If a connection \(\nabla\) on \(E \to M\) is compatible with the metric, then \(-i \omega\) is real-valued, and the cohomology class \([q(-i \omega)]\) belongs to \(H_{\text{dR}}^{2k}(M)\).

**5.9 Chern classes and the Chern character**

**Definition 5.13.** Let \(E \to M\) be a Hermitian vector bundle, together with a compatible connection \(\nabla\) whose curvature is \(\omega \in \mathcal{A}^2(M, \text{End } E)\). Define a \(U(r)\)-invariant polynomial on \(\mathbb{C}^{r \times r}\) by
\[
c(A) := \text{det}(1_r - \frac{1}{2\pi i} A),
\]
which may be written as a sum of homogeneous polynomials
\[
c(A) = 1 + c_1(A) + c_2(A) + \cdots + c_r(A)
\]
where \(c_1(A) = (i/2\pi) \text{tr } A\), \(c_r(A) = (i/2\pi)^r \text{det } A\); and if \(\{i \lambda_1, \ldots, i \lambda_r\}\) are the eigenvalues of \(A\), then \(c_k(A) = (-1/2\pi)^k \sum \lambda_j \lambda_j \cdots \lambda_j\); the invariant homogeneous polynomials \(c_k\) are real-valued on the Lie algebra \(u(r) = i\mathbb{R}^{r \times r}\) of the unitary group \(U(r)\).
The class \(c_k(E) \in H^{2k}_{\text{dR}}(M)\) is called the \textbf{\(k\)-th Chern class}, and \(c(E) \in H^\text{even}_{\text{dR}}(M)\) is called the \textbf{total Chern class} of the vector bundle.

**Exercise 5.16.** If \(E^* \to M\) is the dual vector bundle to \(E \to M\), and if \(\nabla^*\) is the dual connection to \(\nabla\) (see Exercise 5.3), show that \(\nabla^*\) has curvature \(-\omega^t \in \mathcal{A}^2(M, \text{End}(E^*))\), and conclude that \(c_k(\nabla^*) = (-1)^kc_k\)

**Exercise 5.17.** If \(\phi: N \to M\) is a smooth map, and if \(\nabla\) is a connection on a Hermitian vector bundle \(E \to M\) with curvature \(\omega\), find a connection \(\nabla'\) on the pullback bundle \(\phi^*E \to N\) whose curvature is \(\phi^*\omega \in \mathcal{A}^2(N, \text{End}(\phi^*E))\). Conclude that \(c_k(\phi^*E) = \phi^*c_k(E) \in H^k_{\text{dR}}(N)\).

For Hermitian line bundles \((r = 1)\), the total Chern class is just \(c(L) := 1 + c_1(L)\). From Theorem 5.6, we know that \(c_1(L)\) is an \textit{integral} cohomology class, so \(c(L)\) is also integral. This integrality property holds for Chern classes of any Hermitian vector bundle. Indeed, if \(E \to M\) and \(E' \to M\) are two Hermitian vector bundles with compatible connections \(\nabla\) and \(\nabla'\) and respective curvatures \(\omega\) and \(\omega'\), then \(\nabla \oplus \nabla'\) is a connection on \(E \oplus E' \to M\), with curvature \(\omega \oplus \omega' \in \mathcal{A}^2(M, \text{End}(E \oplus E'))\). Clearly

\[
c(\omega \oplus \omega') = \det\left(1 - \frac{\omega}{2\pi i}\right) \wedge \det\left(1r - \frac{\omega'}{2\pi i}\right) = c(\omega) \wedge c(\omega'),
\]

on account of (5.15). Now the wedge product of closed forms induces a product of cohomology classes, since \([\eta + d\zeta] \wedge (\eta' + d\zeta') = \eta \wedge \eta' + d(\eta \wedge \zeta' + \zeta \wedge (\eta' + d\zeta))\) if \(d\eta = 0\) and \(d\eta' = 0\), so \([\eta] [\eta']\) is not affected by adding an exact form to either \(\eta\) or \(\eta'\); in other words, the recipe \([\eta][\eta'] := [\eta \wedge \eta']\) is a well-defined product\(^{10}\) making \(H^\bullet_{\text{dR}}(M)\) into a ring. The even-degree classes form a \textit{commutative} subring \(H^{\text{even}}_{\text{dR}}(M)\). On passing to cohomology, (5.20) becomes

\[
c(E \oplus E') = c(E)c(E') \quad \text{in} \quad H^\text{even}_{\text{dR}}(M).
\]

The integral 2-forms generate an integral subring \(H^\text{even}(M, \mathbb{Z})\). By (5.21), \(c(E)\) is integral, i.e., lies in \(H^\text{even}(M, \mathbb{Z})\), whenever \(E \to M\) is a Whitney sum of Hermitian line bundles. An important \textbf{splitting principle} (see [12] for a proof) asserts that some pullback bundle \(\phi^*E \to N\) can be split (into a sum of line bundles) in such a way that \(c(E) \mapsto \phi^*(c(E)) = c(\phi^*E)\) is injective, and therefore \(c(E)\) is integral since \(c(\phi^*E)\) is.

**Definition 5.14.** The \textbf{Chern character} \(\text{ch}(E)\) of the Hermitian vector bundle \(E \to M\) of rank \(r\) is given by the invariant power series

\[
\text{ch}(A) := \text{tr}(\exp((2\pi)^{-1}iA)) = 1 + \text{ch}_1(A) + \text{ch}_2(A) + \cdots,
\]

or equivalently by the invariant polynomial obtained by discarding the terms \(c_k\) with \(2k > \dim M\), since \(\omega^k \in \mathcal{A}^{2k}(M, \text{End} E)\) and hence \(c_k(\omega) = 0\) for \(2k > \dim M\). This polynomial is found explicitly by writing the eigenvalues of \(A\) as \(\{2\pi i\mu_1, \ldots, 2\pi i\mu_r\}\), expanding the function \(e^{-\mu_1} + \cdots + e^{-\mu_r}\) in a Taylor series, and discarding high-degree terms.

\[^{10}\text{This product is usually called the} \textit{cup product} \text{in de Rham cohomology.}\]
Exercise 5.18. Check that $\text{ch}_1(E) = c_1(E)$, $\text{ch}_2(E) = \frac{1}{2}(c_1(E)^2 - 2c_2(E))$, and $\text{ch}_3(E) = \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E))$.

In general, the $\text{ch}_k(E)$ are polynomial combinations of the Chern classes $c_j(E)$ with rational coefficients, on account of the $1/k!$ terms in the Taylor series expansion; thus they might not be integral classes, but rational classes, i.e., elements of $H^{\text{even}}(M, \mathbb{Q}) = H^{\text{even}}(M, \mathbb{Z}) \otimes \mathbb{Q}$.

**Proposition 5.12.** The Chern character satisfies the homomorphism properties:

\[
\begin{align*}
\text{ch}(E \oplus E') &= \text{ch}(E) + \text{ch}(E'), \\
\text{ch}(E \otimes E') &= \text{ch}(E) \text{ ch}(E').
\end{align*}
\]

**Proof.** The invariant power series $\text{ch}(A)$ of (5.22) satisfies $\text{ch}(A \oplus A') = \text{ch}(A) + \text{ch}(A')$ and $\text{ch}(A \otimes 1 + 1 \otimes A') = \text{ch}(A) \text{ ch}(A')$; where $\oplus$ and $\otimes$ denote the usual direct sum and tensor product of matrices. These identities are simple to check for diagonal matrices, therefore hold for diagonalizable matrices by invariance, and hence hold generally, since the set of diagonalizable $r \times r$ matrices is dense in $\mathbb{C}^{r \times r}$.

Given connections $\nabla$, $\nabla'$ with curvatures $\omega$, $\omega'$ on the respective vector bundles $E \rightarrow M$ and $E' \rightarrow M$, the curvatures of $\nabla \oplus \nabla'$ and $\nabla \otimes \nabla'$ are $\omega \oplus \omega' \in \mathbb{A}^2(M, \text{End}(E \oplus E'))$ and $\omega \otimes 1 + 1 \otimes \omega' \in \mathbb{A}^2(M, \text{End}(E \otimes E'))$. On replacing $A, A'$ by $\omega, \omega'$ in the foregoing matrix identities, bearing in mind (5.15), and on passing to cohomology, one obtains the desired formulae (5.23) for $\text{ch}(E \oplus E')$ and $\text{ch}(E \otimes E')$. \qed

Exercise 5.19. Explain how the tensor product connection $\nabla \otimes \nabla'$ is defined, and check the given formula for its curvature.

A trivial bundle $E_r = M \times \mathbb{C}^r \rightarrow M$ has a “flat” connection (i.e., a connection with zero curvature) namely $d$, and thus $\text{ch}(E_r) = 1$. Thus, if two vector bundles $E \rightarrow M$ and $F \rightarrow M$ are stably equivalent, then $\text{ch}(E) - \text{ch}(F)$ is an integral multiple of 1 in $H^{\text{even}}(M, \mathbb{Q})$. If one defines the “reduced $K$-theory” $\tilde{K}^0(M)$ of $M$ as the quotient of $K^0(M)$ by $\mathbb{Z}$ (on identifying $r \in \mathbb{N}$ with $[E_r] \in K^0(M)$), the tensor product of vector bundles makes $\tilde{K}^0(M)$ a commutative ring, and thus the Chern character defines a ring homomorphism $\text{ch}: \tilde{K}^0(M) \rightarrow H^{\text{even}}(M, \mathbb{Q})$.

For Hermitian line bundles $L \rightarrow M$ and $L' \rightarrow M$, Proposition 5.12 yields the identity

\[
c_1(L \otimes L') = c_1(L) + c_1(L'),
\]

(5.24)

by extracting the component in $H^2_{\text{dR}}(M)$ from the formula $\text{ch}(L \otimes L') = \text{ch}(L) \text{ ch}(L)$, using $\text{ch}_1(L) = c_1(L)$. Thus $c_1$ determines a homomorphism from the group of line bundle classes to the additive group $H^2(M, \mathbb{Z})$ of integral de Rham classes. Since $c_1(L) = [(i/2\pi)\omega]$ when $\omega$ is the curvature of a compatible connection on $L \rightarrow M$, and $c_1(L^*) = -c_1(L)$, we conclude that $[L] \mapsto c_1(L^*)$ is the isomorphism described in Theorem 5.6.

To show that $[L] \neq [L']$, it is enough to show that $c_1(L) \neq c_1(L')$. Moreover, if $\dim M = 2m$, then $c_1(L)^m = [(i/2\pi)^m \omega^m]$ is an integral $2m$-form, and so $\int_M (i/2\pi)^m \omega^m \in \mathbb{Z}$, since the identification of $H^2_{\text{dR}}(M)$ with $\mathbb{R}$ is precisely the map $[\nu] \mapsto \int_M \nu$.  

59
5.10 Classification of line bundles over $\mathbb{CP}^m$

**Proposition 5.13.** The group of classes of line bundles over $\mathbb{CP}^m$ is an infinite cyclic group, generated by the class $[H]$ of the hyperplane bundle.

**Proof.** Since we already know that $\tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}) \equiv \mathbb{Z}$, we need only check that $[H]$ is a generator. Equivalently, we must check that $c_1(H) = [(2\pi)^{-1}\Phi]$ is a generator for $H^2(M, \mathbb{Z})$. For this, it is enough to show that $\int_{\mathbb{CP}^m}(2\pi)^{-m}\Phi^m = 1$.

Since the complement of the chart domain $U_0$ is a lower-dimensional submanifold (diffeomorphic to $\mathbb{CP}^{m-1}$), we need only show that the integral over $U_0$ equals 1. We may therefore use the formula (2.7) (with $j = 0$) for the Kähler form $\Phi$. Then

$$\int_{U_0} \Phi^m = \int_{\mathbb{C}^m} \left(iQ_0^{-1} \sum_{k=1}^m dw_0^k \wedge \bar{dw}_0^k - iQ_0^{-2} \sum_{k=1}^m \bar{w}_0^k w_0^l d\bar{w}_0^k \wedge dw_l^0\right)^m.$$  (5.25)

At the point $(r, 0, \ldots, 0) \in \mathbb{C}^m$, $Q_0$ simplifies to $1 + r^2$, and the integrand on the right hand side of (5.24) becomes

$$\left(i(1 + r^2)^{-2} dw_0^1 \wedge \bar{dw}_0^1 + i(1 + r^2)^{-1} \sum_{k=2}^m dw_0^k \wedge \bar{dw}_0^k \right)^m = \frac{i^m m!}{(1 + r^2)^{m+1}} \prod_{k=1}^m dw_0^k \wedge \bar{dw}_0^k = \frac{2^m m!}{(1 + r^2)^{m+1}} \prod_{k=1}^m dx^k \wedge dy^k$$

$$= \frac{2^m m!}{(1 + r^2)^{m+1}} \lambda,$$  (5.26)

where $\lambda$ is Lebesgue measure on $\mathbb{C}^m$, and $w_0^k = x^k + iy^k$ give Cartesian coordinates on $\mathbb{C}^m$. Interpreting $r$ as a polar coordinate on $\mathbb{C}^m$, one can write $\lambda = r^{2m-1} dr d^{2m-1}\theta$, with $\theta \in S^{2m-1}$ being the angular part. The right hand side of (5.25) is invariant under the unitary group $U(m)$, as is the Kähler form, so it represents the integrand at all points, not just at $(r, 0, \ldots, 0)$.

The volume of the sphere $S^{2m-1}$ is $\Omega_{2m} = 2 \pi^m / (m-1)!$ (see [1, 8, 28], for instance), so the desired integral is

$$\int_{\mathbb{CP}^m} (2\pi)^{-m}\Phi^m = \frac{m!}{\pi^m} \int_{\mathbb{C}^m} \frac{r^{2m-1}}{(1 + r^2)^{m+1}} dr d^{2m-1}\theta$$

$$= \frac{m!}{\pi^m} \Omega_{2m} \int_0^{\infty} \frac{r^{2m-1}}{(1 + r^2)^{m+1}} dr = \int_0^{\infty} \frac{2mr^{2m-1}}{(1 + r^2)^{m+1}} dr$$

$$= \int_0^{\infty} \frac{mr^{m-1}}{(1 + t)^{m+1}} dt = \int_0^1 m u^{m-1} du = 1,$$

on substituting $t = r^2$, $u = t / (1 + t)$.

**Corollary 5.14.** The hyperplane bundle is not trivial, since $c_1(H) \neq 0$. \(\square\)
This completes the classification of Hermitian line bundles over $\mathbb{CP}^m$, since any line bundle $L' \to \mathbb{CP}^m$ satisfies $c_1(L') = k c_1(H)$ for some $k \in \mathbb{Z}$; thus $L' \sim H^{\otimes k}$ if $k > 0$, $L' \sim L^{\otimes (-k)}$ if $k < 0$, and $L'$ is trivial if $k = 0$. Furthermore, $k$ is precisely the integral over $\mathbb{CP}^m$ of $(-2\pi i)^{-m} \omega_L^m$ where $\omega_L$ is the curvature of any compatible connection on $L' \to \mathbb{CP}^m$.

5.11 The Levi-Civita connection on the tangent bundle

**Definition 5.15.** Let $M$ be a Riemannian manifold, and let $\nabla$ be a connection on the tangent bundle $TM \to M$. The **fundamental 1-form** $\theta$ is the unique element of $A^1(M, TM)$ satisfying $\iota_X \theta = X$ for all $X \in \mathfrak{X}(M) = \Gamma(TM)$. The **torsion** of $\nabla$ is $T := \nabla \theta \in A^2(M, TM)$.

**Exercise 5.20.** Show that the contraction map $\iota_X : A^1(M, E) \to \Gamma(E)$ of Definition 5.4 extends to an $\mathcal{A}$-linear map $\iota_X : A^k(M, E) \to A^{k-1}(E)$ such that $\iota_X(\zeta \wedge \eta) = (\iota_X \zeta) \wedge \eta + (-1)^k \zeta \wedge \iota_X \eta$ for $\zeta \in A^k(M, E)$, $\eta \in A^*(M)$, provided we define $\iota_X s := 0$ for $s \in \Gamma(E) = A^0(M, E)$.

We can then define $T(X, Y) := \iota_Y(\iota_X T) \in \mathfrak{X}(M)$.

**Lemma 5.15.** If $X, Y \in \mathfrak{X}(M)$, then $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

**Proof.** It is not hard to show that $\nabla_X \zeta = \nabla(\iota_X \zeta) + \iota_X (\nabla \zeta)$ and $\nabla_X (\iota_Y \zeta) = \iota_Y (\nabla_X \zeta) + \iota_{[X, Y]} \zeta$ for $\zeta \in A^1(M, TM)$. For the particular case $\zeta = \theta$, these identities give

$$T(X, Y) = \iota_Y (\iota_X \nabla \theta) = \iota_Y (\nabla_X \theta - \nabla (\iota_X \theta)) = \iota_Y (\nabla_X \theta - \nabla X)$$

$$= \nabla_X (\iota_Y \theta) - \iota_{[X, Y]} \theta - \nabla_Y X = \nabla_X Y - [X, Y] - \nabla_Y X, \quad (5.27)$$

as claimed. \qed

**Exercise 5.21.** Verify the aforementioned formulae for $\nabla_X \zeta$ and $\nabla_X (\iota_Y \zeta)$ by applying (5.8) and (5.9) (and their analogues for $\iota_X$) in the case $\zeta = Z \otimes \alpha$ with $Z \in \mathfrak{X}(M)$, $\alpha \in A^1(M)$.

**Proposition 5.16.** If $M$ is a Riemannian manifold, there is a unique connection $\nabla$ on the tangent bundle $TM \to M$, which is compatible with the Euclidean metric on $TM$ and is torsion-free.

**Proof.** Compatibility with the metric demands that $(\nabla X | Y) + (X | \nabla Y) = d(X | Y)$, as in (5.7), where $(X | Y) = g(X, Y)$ denotes the bilinear pairing on $\mathfrak{X}(M) = \Gamma(TM)$ determined by the Euclidean metric $g$. By Lemma 5.15, $\nabla$ is torsion-free iff $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in \mathfrak{X}(M)$.

Thus $Z(X | Y) = \iota_Z (\nabla X | Y) + \iota_Z (X | \nabla Y) = (\nabla_Z X | Y) + (X | \nabla_Z Y)$. A short calculation then shows that

$$2(\nabla_Z X | Y) = X(Y | Z) - Y(Z | X) + Z(X | Y)$$

$$+ (X | [Y, Z]) + (Y | [Z, X]) - (Z | [X, Y]), \quad (5.28)$$

which establishes the uniqueness of $\nabla$. On the other hand, it is easy to show that the right hand side is $\mathcal{A}$-linear in $Y$ and $Z$, hence is of the form $\iota_Z (DX | Y)$, where $D : \mathfrak{X}(M) \to \mathcal{A}^1(M, TM)$. By replacing $X$ by $hX$ (with $h \in \mathcal{A}$) on the right hand side of (5.27) and then simplifying, one verifies the Leibniz rule for $D$, which proves the existence of the desired connection. \qed
Definition 5.16. The unique metric-compatible torsion-free connection on the tangent bundle of a Riemannian manifold $M$ is called its **Levi-Civita connection**. Its curvature, in $\mathcal{A}^2(M, TM)$, is usually denoted by $R$, and is also called the **Riemannian curvature tensor**\(^\text{11}\) of $M$.

**Exercise 5.22.** Verify the following property of the Riemannian curvature tensor:

\[
R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \quad \text{for} \quad X, Y, Z \in \mathcal{X}(M),
\]

(5.29)

using only Lemma 5.4, Lemma 5.15, and the Jacobi identity.

**Exercise 5.23.** Verify the following symmetry properties of the Riemannian curvature tensor:

\[
(W \mid R(X, Y)Z) = -(R(X, Y)W \mid Z) = (X \mid R(W, Z)Y),
\]

for $W, X, Y, Z \in \mathcal{X}(M)$.

### 6 Clifford algebras

A Clifford algebra is an associative algebra which is generated by starting with a real vector space and defining a product of vectors in such a way that the square of any vector is a scalar. This can be done consistently if the generating vector space is Euclidean, i.e., if it carries a symmetric bilinear form. For the zero bilinear form, the corresponding algebra is just the exterior algebra on the given vector space; otherwise, it has the same underlying vector space as the exterior algebra, but with a modified product operation. The Clifford algebra has interesting matrix representations, whose representation spaces are called “Clifford modules”. All of these spaces are graded into an “even” part and an “odd” part, so we begin with a general discussion of vector spaces and algebras which are $\mathbb{Z}_2$-graded.

### 6.1 Superspaces and superalgebras

**Definition 6.1.** A **superspace** is just a vector space with a given $\mathbb{Z}_2$-grading: $V = V^+ \oplus V^-$; here $V^+$ is called the **even subspace** and $V^-$ is called the **odd subspace**\(^1\).

A **superalgebra** is an algebra whose underlying vector space is a superspace: $A = A^+ \oplus A^-$, where the product respects the grading,\(^2\) i.e., $A^+ \cdot A^+ \subseteq A^+$, $A^- \cdot A^- \subseteq A^-$, $A^+ \cdot A^- \subseteq A^-$, and $A^- \cdot A^+ \subseteq A^-$.\(^2\)

---

\(^1\)Since $\mathcal{A}^2(M, TM) = \mathcal{X}(M) \otimes \mathcal{A}^2(M)$, the map $(X, Y, \alpha) \mapsto \alpha(R(X, Y))$ is a tensor of bidegree $(2, 1)$ on $M$.

\(^2\)The unfortunate prefix “super”, which is simply a synonym for “$\mathbb{Z}_2$-graded”, was introduced by Feliks Berezin [7] about 30 years ago, and has since become fashionable. Berezin wished to extend the calculus of Gaussian integrals by regarding an exterior algebra as a “space of functions of anticommuting variables”. This crazy idea works astonishingly well.

If we regard the exponents $+$ and $-$ as the elements of the additive group $\mathbb{Z}_2$, these four inclusions may be collected as the formula $A^+ \cdot A^+ \subseteq A^{++}$. In this way we can define a $G$-graded algebra for any abelian group $G$, although only the cases $G = \mathbb{Z}_2$ and $G = \mathbb{Z}$ are commonly used.
Definition 6.2. The exterior algebra $\Lambda^*V$ of a vector space $V$ is a $\mathbb{Z}$-graded algebra, whose subspace of degree $k$ is $\Lambda^k(V)$ for $k = 0, 1, \ldots, \dim V$ (and for $k < 0$ or $k > \dim V$, one sets $\Lambda^k(V) := \{0\}$), since $\Lambda^k(V) \wedge \Lambda^l(V) \subseteq \Lambda^{k+l}(V)$ for $k, l \in \mathbb{Z}$.

But $\Lambda^*V$ is also a superalgebra, since we may define

$$\Lambda^+(V) := \bigoplus_{k \text{ even}} \Lambda^k(V), \quad \Lambda^-(V) := \bigoplus_{k \text{ odd}} \Lambda^k(V).$$

Indeed, any $\mathbb{Z}$-graded algebra becomes a superalgebra, by collecting the subspaces of even degree and of odd degree in this manner.

Definition 6.3. If $V = V^+ \oplus V^-$ is a superspace, then $\text{End} V$ is a superalgebra, with

$$\text{End}^+ V := \text{End}(V^+) \oplus \text{End}(V^-),$$
$$\text{End}^- V := \text{Hom}(V^+, V^-) \oplus \text{Hom}(V^-, V^+),$$

Exercise 6.1. Guess the definition of a supermodule. Any superspace $V$ is a supermodule for the superalgebra $\text{End} V$.

Definition 6.4. We say an element $a$ of a superalgebra $A$ is homogeneous if either $a \in A^+$ or $a \in A^-$; its parity $\#a$ is defined as $\#a := 0$ if $a \in A^+$, $\#a := 1$ if $a \in A^-$. Analogously, in a $\mathbb{Z}$-graded algebra, we define the degree of a homogeneous element as $\#a := k$ if $a \in A^k$.

There is an important, if somewhat informal, sign rule in superalgebra, which says that in any calculation in which the order of multiplication of homogeneous elements is reversed (i.e., $ab$ is changed to $ba$), a sign factor of $(-1)^{\#a\#b}$ must be inserted. Thus, for example, we say that a superalgebra is “supercommutative” if $ab = (-1)^{\#a\#b}ba$ for homogeneous elements $a, b \in A$; in other words, even elements commute with both even and odd elements, but two odd elements anticommute. Notice that the exterior algebra $\Lambda^*V$ is supercommutative.

Definition 6.5. The superbracket in a superalgebra is the bilinear operation $A \times A \to A$ defined, for $a, b$ homogeneous, by

$$\{[a, b]\} := ab - (-1)^{\#a\#b}ba.$$

A superalgebra is supercommutative iff all supercommutators $[a, b]$ vanish, i.e., if the superbracket is trivial. The superbracket satisfies the properties:

$$[[a, b], b] + (-1)^{\#a\#b} [b, a] = 0,$$
$$[[a, [b, c]], a] = [[a, b], c] + (-1)^{\#a\#b}[[b, [a, c]], a]. \quad (6.1)$$

A vector space with a bilinear operation (of any kind) which satisfies (6.1) is called a Lie superalgebra.

Several bracket notations are in general use. Usually one writes $[a, b] = ab - ba$, and anticommutators $ab + ba$ are denoted $\{a, b\}$ or sometimes $[a, b]_+$; thus $[[a, b], a] = [a, b]$ if either $a$ or $b$ is even, and $[[a, b], b] = \{a, b\}$ if both $a$ and $b$ are odd. Warning: many superalgebraists use $[a, b]$ to denote a supercommutator, even when $a$ and $b$ are odd.
Definition 6.6. A supertrace on a superalgebra $A$ is a linear form $\tau$ which vanishes on supercommutators, i.e., $\tau([a, b]) = 0$ for all $a, b \in A$.

When $A = \text{End} E$ for $E$ a superspace, we may write $a \in A^+$ and $b \in A^-$ as
\[
a = \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix}, \quad b = \begin{pmatrix} 0 & b^- \\ b^+ & 0 \end{pmatrix},
\]
and we define a supertrace on $\text{End} E$ by
\[\text{Str}(a + b) := \text{Tr}(a^+) - \text{Tr}(a^-).\]

Exercise 6.2. Write the supercommutator $[a, b]$ for $a, b$ homogeneous elements of $\text{End} E$ in the matrix notation (6.2) —there are four cases— and thus verify that (6.3) defines a supertrace.

The tensor product $U \otimes V$ of two superspaces $U$ and $V$ is a superspace, with the grading:
\[
(U \otimes V)^+ := (U^+ \otimes V^+) \oplus (U^- \otimes V^-),
\]
\[
(U \otimes V)^- := (U^+ \otimes V^-) \oplus (U^- \otimes V^+).
\]
The tensor product of two superalgebras $A$ and $B$, with the standard product recipe $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$ is not, however, a superalgebra, since this product does not respect the grading (6.4). However, one defines a graded tensor product, denoted $A \boxtimes B$, whose underlying superspace is $A \otimes B$, by using the multiplication rule:
\[
(a \otimes b)(a' \otimes b') := (-1)^{\#a' \#b} aa' \otimes bb'
\]
(for $a, a', b, b'$ homogeneous) in view of the passage of $a'$ to the left of $b$.

Exercise 6.3. Verify that the product (6.5) is compatible with the grading (6.4) on the superspace $A \otimes B$.

Exercise 6.4. If $U$ and $V$ are (ungraded) vector spaces, show that $\Lambda^\bullet(U \oplus V) \simeq \Lambda^\bullet U \boxtimes \Lambda^\bullet V$ as superalgebras.

Exercise 6.5. If $U$ and $V$ are superspaces, define an action of $\text{End} U \boxtimes \text{End} V$ on the superspace $U \otimes V$ in such a way that $\text{End} U \boxtimes \text{End} V \simeq \text{End}(U \otimes V)$ as superalgebras.

Exercise 6.6. Let $A$ be a supercommutative superalgebra, and let $V$ be a superspace. Establish the following identity for homogeneous elements of $A \boxtimes \text{End} V$:
\[\{a \otimes T, b \otimes S\} = (-1)^{\#T \#b} ab \otimes [T, S].\]
Show that the linear map $\text{Str}_A: A \boxtimes \text{End} V \to A$ given by $\text{Str}_A(a \otimes T) := a \text{Str}(T)$ is an $A$-valued supertrace, in the sense that it vanishes on supercommutators.

Definition 6.7. A superbundle over a smooth manifold $M$ is a vector bundle $E \to M$ which is a Whitney sum of two vector bundles: $E = E^+ \oplus E^-$. The fibres $E_x = E^+_x \oplus E^-_x$ are superspaces.
The space of sections $\Gamma(E)$ is a $\mathbb{Z}_2$-graded $\mathbb{C}^\infty(M)$-module: $\Gamma(E) = \Gamma(E^+) \oplus \Gamma(E^-)$. The $E$-valued forms on $M$ may also be graded by degree, and so, in accordance with (6.4), $\mathcal{A}^\ast(M, E)$ carries the total $\mathbb{Z}_2$-grading:

$$\mathcal{A}^+ (M, E) := \mathcal{A}^{\text{even}} (M, E^+) \oplus \mathcal{A}^{\text{odd}} (M, E^-),$$

$$\mathcal{A}^- (M, E) := \mathcal{A}^{\text{even}} (M, E^-) \oplus \mathcal{A}^{\text{odd}} (M, E^+).$$

### 6.2 Clifford algebras

In this section we treat briefly the algebraic theory of Clifford algebras over Euclidean vector spaces. The presentation follows the treatment in [9], and the appendix to [54]. For a more detailed exposition of the algebraic theory, see [27] or [39]. All these rely on the fundamental paper of Atiyah, Bott and Shapiro [4].

**Definition 6.8.** Let $V$ be a real vector space, with $\dim \mathbb{R} V = n$, and $q$ a positive definite symmetric bilinear form on $V$; we shall call the pair $(V, q)$ a Euclidean vector space. The **Clifford algebra** $\mathbb{C}l(V) = \mathbb{C}l(V,q)$ is an associative algebra generated by the elements of $V$ subject (only) to the relation $v^2 = -q(v,v)1$. More precisely, one defines $\mathbb{C}l(V,q) := \mathcal{T}(V)/I(q)$, where $\mathcal{T}(V)$ is the tensor algebra over $V$ and $I(q)$ is the ideal generated by $\{ v \otimes v + q(v,v)1 : v \in E \}$. The canonical mapping of $V$ into $\mathbb{C}l(V,q)$ is injective, so $V$ may be regarded as a subspace of $\mathbb{C}l(V,q)$; we then have the fundamental relation

$$uv + vu = -2q(u,v) \quad \text{for } u, v \in V. \quad (6.6)$$

**Exercise 6.7.** Check that (6.6) is a consequence of $v^2 = -q(v,v)$.

**Proposition 6.1.** The algebra $\mathbb{C}l(V,q)$ satisfies the following universal property: if $A$ is a real algebra with identity and $f : V \to A$ is a linear mapping such that $f(v)^2 = -q(v,v)1$, then there is a unique algebra homomorphism $\tilde{f} : \mathbb{C}l(V,q) \to A$ such that $f = \tilde{f}|_V$. \hfill \square

**Exercise 6.8.** Give the proof of Proposition 6.1.

A Clifford algebra is $\mathbb{Z}_2$-graded, as follows. By taking $f(v) := -v$ in the previous proposition, with $A = \mathbb{C}l(V,q)$, we see that there is a unique automorphism $\alpha$ of $\mathbb{C}l(V,q)$ extending $f$, such that $\alpha^2 = \text{id}$. From the definition, each element of $\mathbb{C}l(V,q)$ is a linear combination of products $v_1 v_2 \ldots v_k$ with $v_1, \ldots, v_k \in V$; clearly we have $\alpha(v_1 v_2 \ldots v_k) = (-1)^k v_1 v_2 \ldots v_k$. We write $\mathbb{C}l^\pm (V,q)$ to denote the $(\pm 1)$-eigenspaces of $\alpha$. Thus $\mathbb{C}l(V,q)$ is a superalgebra. Notice that $V \subset \mathbb{C}l^-(V,q)$.

Now take $A$ to be the opposite algebra of $\mathbb{C}l(V,q)$ (the same vector space with the product reversed), and $f(v) := v$; we get an involutive antiautomorphism of $\mathbb{C}l(V,q)$ given by $(v_1 \ldots v_k)^! := v_k \ldots v_1$. We define another antiautomorphism $a \mapsto \bar{a}$, called conjugation, by $\bar{a} := \alpha(a^!) = \alpha(a^!)$.

Many properties of the Clifford algebra come from the following simple proposition, due to Chevalley [16].

---

3A Clifford algebra can be defined if $q$ is a symmetric bilinear form of any signature. However, we shall use only forms which are either positive definite or (occasionally) negative definite.
Proposition 6.2. Let $(V, q)$ and $(W, r)$ be two Euclidean vector spaces, and let $q \oplus r$ be the symmetric bilinear form on $V \oplus W$ given by $(q \oplus r)(v_1 + w_1, v_2 + w_2) := q(v_1, v_2) + r(w_1, w_2)$. Then there is an isomorphism of superalgebras $\text{Cl}(V \oplus W, q \oplus r) \cong \text{Cl}(V, q) \otimes \text{Cl}(W, r)$.

Proof. Define $f: V \oplus W \to \text{Cl}(V, q) \otimes \text{Cl}(W, r)$ by $f(v + w) := v \otimes 1 + 1 \otimes w$. Using (6.5), we see that $f(v + w)^2 = -(q \oplus r)(v + w, v + w) 1 \otimes 1$; therefore $f$ extends to an algebra homomorphism $\hat{f}: \text{Cl}(V \oplus W, q \oplus r) \to \text{Cl}(V, q) \otimes \text{Cl}(W, r)$. Since the elements $v \otimes 1 + 1 \otimes w$ generate $\text{Cl}(V, q) \otimes \text{Cl}(W, r)$ as an algebra, $\hat{f}$ is surjective; to see that it is injective, it suffices to compute $\hat{f}$ on a basis for $\text{Cl}(V \oplus W, q \oplus r)$ generated by bases for $V$ and $W$. \hfill \Box

If $e_1, \ldots, e_n$ is an orthonormal basis of $(V, q)$, write $e_J := e_{j_1} \cdots e_{j_k}$ for $J = \{j_1, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ with $1 \leq j_1 < \cdots < j_k \leq n$; and let $e_0 := 1$. Then $\{e_J: J \subseteq \{1, 2, \ldots, n\}\}$ is a basis of $\text{Cl}(V, q)$. In particular, $\dim \text{Cl}(V, q) = 2^n$ and $\dim \text{Cl}^+(V, q) = \dim \text{Cl}^-(V, q) = 2^{n-1}$.

Exercise 6.9. Use the previous Proposition to verify this basis, by induction on $n$.

Suppose $V = \mathbb{R}^n$ and that $q_n$ is the standard Euclidean inner product on $\mathbb{R}^n$; in that case, we abbreviate $\text{Cl}(n) := \text{Cl}(\mathbb{R}^n, q_n)$. Since $\text{Cl}(1) = \text{span}\{1, e_1\}$ with $e_1^2 = -1$, we have $\text{Cl}(1) \cong \mathbb{C}$ (as real algebras); with $\text{Cl}^+(1) = \mathbb{R}$, $\text{Cl}^-(1) = i\mathbb{R}$. Next, $\text{Cl}(2) = \text{span}\{1, e_1, e_2, e_1 e_2\} \cong \mathbb{H}$, the algebra of quaternions. In both these cases, $a \mapsto \bar{a}$ denotes the usual conjugation.

Since $\text{Cl}(\mathbb{R}, -q_1) = \text{span}\{1, e_1\}$ with $e_1^2 = +1$, we have $\text{Cl}(\mathbb{R}, -q_1) \cong \mathbb{R} \oplus \mathbb{R}$, so $\text{Cl}(V, q)$ depends on the signature of the form $q$. Also, $\text{Cl}(\mathbb{R}^2, -q_2) \cong \mathbb{R}^{2 \times 2}$, by taking $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. A complete list of the algebras $\text{Cl}(\mathbb{R}^n, \pm q_n)$ is given in [4], and reproduced in [27, 39]. A basic fact is what is called Bott periodicity: $\text{Cl}(n + 8) \cong \text{Cl}(n) \otimes \text{Cl}(8)$ (ungraded tensor product). Since $\text{Cl}(8) \cong \mathbb{R}^{16 \times 16}$ is a real matrix algebra, this means that $\text{Cl}(n + 8) \cong \text{Cl}(n)^{16 \times 16}$, so one only need determine $\text{Cl}(n)$ for $n = 1, 2, \ldots, 8$.

Exercise 6.10. Show that $\text{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}$. The remaining cases are $\text{Cl}(4) \cong \mathbb{H}^{2 \times 2}$, $\text{Cl}(5) \cong \mathbb{C}^{4 \times 4}$, $\text{Cl}(6) \cong \mathbb{R}^{8 \times 8}$, $\text{Cl}(7) \cong \mathbb{R}^{8 \times 8} \oplus \mathbb{R}^{8 \times 8}$.

6.3 Clifford actions

Definition 6.9. A Clifford module for the Clifford algebra $\text{Cl}(V, q)$ is a superspace $F = F^+ \oplus F^-$ together with an even homomorphism (called a Clifford action) $c: \text{Cl}(V, q) \to \text{End} F$. In other words, $c(a)F^+ \subseteq F^+$ if $a \in \text{Cl}^+(V, q)$, and $c(b)F^- \subseteq F^-$ if $b \in \text{Cl}^-(V, q)$.

Suppose that $F$ has a (Euclidean or hermitian) inner product $(\cdot \mid \cdot)$, and denote the adjoint of $A \in \text{End} F$ by $A^\dagger$, that is, $(x \mid A^\dagger y) := (Ax \mid y)$ for $x, y \in F$. We say that $F$ is a selfadjoint Clifford module if $c(a)^\dagger = c(\bar{a})$ for all $a \in A$. Notice that it is enough to see that $c(v)^\dagger = -c(v)$ for all $v \in V$, i.e., that each $c(v)$ is skewadjoint.

Write $\text{End}_{\text{Cl}(V, q)} F := \{R \in \text{End} F : [c(a), R] = 0, a \in \text{Cl}(V, q)\}$ to denote the subalgebra of $\text{End} F$ consisting of operators which supercommute with the Clifford action.

Exercise 6.11. Check that $[RS, c(v)] = R[S, c(v)] + (-1)^q S[R, c(v)]S$ for $R, S$ homogeneous elements of $\text{End} F$, and conclude that $\text{End}_{\text{Cl}(V, q)} F$ is indeed an algebra.
Definition 6.10. Let $(V, q)$ be a finite-dimensional real vector space with a nondegenerate bilinear form. The dual space $V^*$ of $\mathbb{R}$-linear forms on $V$ is a real vector space of the same dimension; we may define the so-called musical isomorphisms [8], namely $♭ : V \to V^*$ and $♯ : V^* \to V$, by

$$u^{♭}(v) := q(u, v), \quad q(λ^{♯}, v) := λ(v).$$

They are mutually inverse: $(u^{♭})^{♯} = u$, $(λ^{♯})^{♭} = λ$.

If $W$ is a complex vector space with a hermitian inner product $⟨⟨· | ·⟩⟩$ (or a Hilbert space, not necessarily finite-dimensional), we define $♭ : W \to W^*$ and $♯ : W^* \to W$ similarly by $u^{♭}(v) := ⟨⟨u | v⟩⟩$, $⟨⟨λ^{♯} | v⟩⟩ := λ(v)$. In the complex case, the musical isomorphisms are antilinear.

Definition 6.11. Let $(V, q)$ be an Euclidean vector space, and let $Λ•V$ be its exterior algebra. For $v \in V$, define the exterior multiplication $ϵ(v)$ and the contraction $ι(v^{♭})$ in $\text{End}(Λ•V)$ by

$$ϵ(v)(v_1 \land \cdots \land v_k) := v \land v_1 \land \cdots \land v_k,$$

$$ι(v^{♭})(v_1 \land \cdots \land v_k) := \sum_{j=1}^{k} (-1)^{j-1} q(v, v_j) v_1 \land \cdots \land \hat{v_j} \land \cdots \land v_k. \quad (6.7)$$

(The notation $\hat{v_j}$ indicates that the term with index $j$ is missing from the exterior product.) For $k = 0$, we define $ϵ(v)1 := v$, $ι(v^{♭})1 := 0$. Note that $ι(v^{♭})$ is a graded derivation:

$$ι(v^{♭})(α \land β) = ι(v^{♭})α \land β + (-1)^{#α} α \land ι(v^{♭})β \quad \text{for} \quad α, β \in Λ•V.$$

On $Λ•V$ we use the real inner product

$$(u_1 \land \cdots \land u_m, v_1 \land \cdots \land v_n) := δ_{mn} \det[q(u_k, v_l)]$$

It is easily seen that $ϵ(v)^{♭} = ι(v^{♭})$ for $v \in V$. Define

$$c(v)α := ϵ(v)α - ι(v^{♭})α, \quad \text{for} \quad α \in Λ•V. \quad (6.8)$$

Exercise 6.12. Check that $ϵ(v)^{♭} = ι(v^{♭})$ and that

$$ϵ(u)ι(v^{♭}) + ι(v^{♭})ϵ(u) = q(v, u).$$

Conclude that $c(v)^{2} = -q(v, v) 1$ for $v \in V$.

By Proposition 6.1, $c$ extends to a Clifford action of $Cℓ(V, q)$ on $Λ•V$. Since $c(v)^{♭} = -c(v)$, this action is selfadjoint.

---

4 This definition applies to both symmetric and antisymmetric bilinear forms.

5 In the infinite-dimensional case, the bijectivity of the musical isomorphisms is a restatement of the Riesz theorem.

6 It could have been defined as the unique graded derivation that takes $u$ to $q(v, u)$.  

67
Definition 6.12. The \( \mathbb{R} \)-linear map \( \sigma : \text{Cl}(V, q) \rightarrow \Lambda^* V \) given by
\[
\sigma(a) := c(a) 1 \quad (6.9)
\]
for \( a \in \text{Cl}(V, q) \), is a vector-space isomorphism (but not an algebra isomorphism) between the Clifford algebra and the exterior algebra, called [9] the symbol map. (It is easy to see that \( \sigma \) is surjective; since \( \text{Cl}(V, q) \) and \( \Lambda^* V \) have the same dimension \( 2^n \), it is also injective.) Note that \( \sigma(1) = 1 \) and \( \sigma(v) = v \) for \( v \in V \).

The inverse isomorphism \( Q : \Lambda^* V \rightarrow \text{Cl}(V, q) \) may be called [9] a quantization map, since it maps a supercommutative algebra to an algebra which is no longer supercommutative.\(^7\)

Exercise 6.13. Show that \( \sigma(uv) = u \wedge v - q(u, v) 1 \), so that \( Q(u \wedge v) = uv + q(u, v) 1 \). Establish the general formula:
\[
Q(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\tau \in S_k} (-1)^\tau v_{\tau(1)} \cdots v_{\tau(k)}, \quad (6.10)
\]
where \( (-1)^\tau \) denotes the sign of the permutation \( \tau \). In particular, \( Q(e_{j_1} \wedge \cdots \wedge e_{j_k}) = e_{j_1} \cdots e_{j_k} \) if \( \{e_1, \ldots, e_n\} \) is an orthonormal basis for \( V \) and \( j_1 < \cdots < j_k \).

Exercise 6.14. The map \( Q \) does not preserve the \( \mathbb{Z} \)-grading of \( \Lambda^* V \), but does preserve its filtration. A filtration of an algebra \( A \) is a system of subspaces \( A_j \) with \( A_j \subseteq A_{j+1} \) and \( A_j A_j \subseteq A_{j+j} \). For \( A = \text{Cl}(V, q) \), take \( A_j \) to be the subspace generated by products at most \( j \) vectors in \( V \). There is an associated graded algebra, namely \( \text{gr} A := \bigoplus_j A_j/A_{j-1} \) and a canonical “symbol map” \( \sigma : A \rightarrow \text{gr} A \). Check that for \( A = \text{Cl}(V, q) \), the associated graded algebra is none other than \( \Lambda^* V \) with the symbol map given by (6.9).

6.4 Complex Clifford algebras

Definition 6.13. The complexification \( \text{Cl}(V, q) := \text{Cl}(V, q) \otimes_{\mathbb{R}} \mathbb{C} \) can be regarded as the Clifford algebra over \( \mathbb{C} \) of the complexified vector space \( V_{\mathbb{C}} \) with the same quadratic form \( q \) amplified to \( V_{\mathbb{C}} \); any vector in \( V_{\mathbb{C}} \) can be written uniquely as \( w = u + iv \) with \( u, v \in V \), and one defines \( q(u, v) := q(u, u') - q(v, v') + iq(u, v') - iq(v, u') \); notice that the amplified bilinear form \( q \) is not positive definite on \( V_{\mathbb{C}} \). On \( V_{\mathbb{C}} \) all nondegenerate symmetric bilinear forms are equivalent, so we may abbreviate \( \text{Cl}(V, q) \) to \( \text{Cl}(V) \); in particular, \( \text{Cl}(V, q) \simeq \text{Cl}(\ell(n)) := \text{Cl}(\mathbb{C}^n, q_n) \) if \( \dim_{\mathbb{R}} V = n \).

Exercise 6.15. Prove that \( \text{Cl}(1) \simeq \mathbb{C} \oplus \mathbb{C} \) and that \( \text{Cl}(2) \simeq \mathbb{C}^{2 \times 2} \).

\(^7\)Quantum field theory deals with Fermi fields, which are systems of operators \( \{ \phi(v) : v \in V \} \) satisfying an “anticommutation relation” akin to (6.6). It helps to imagine an analogous situation in which the anticommutators vanish, i.e., the algebra generated by the Fermi fields is replaced by a supercommutative algebra; restoration of the scalar terms \( q(u, v) \) then corresponds to quantizing the latter system.
Exercise 6.16. Prove that $\mathcal{C}l(n+2) \cong \mathcal{C}l(n) \otimes \mathcal{C}l(2)$ (ungraded tensor product)\(^8\) by showing that the $\mathbb{C}$-linear map $f: \mathbb{C}^{n+2} \rightarrow \mathcal{C}l(n) \otimes \mathcal{C}l(2)$ given by $f(e_1) := 1 \otimes e_1, f(e_2) := 1 \otimes e_2, f(e_j) := i e_{j-2} \otimes e_1 e_2$ for $j \geq 3$, satisfies $\{f(e_j), f(e_k)\} = -2 \delta_{jk} (1 \otimes 1)$, and hence extends to the desired isomorphism.

From this we find that $\mathcal{C}l(2m) \cong \mathbb{C}^{N \times N}$ with $N = 2^m$; and $\mathcal{C}l(2m+1) \cong \mathbb{C}^{N \times N} \oplus \mathbb{C}^{N \times N}$. In particular, $\mathcal{C}l(V,q)$ is a simple matrix algebra iff $\dim_{\mathbb{R}} V$ is even. From now on, we consider only the case that $V$ has even dimension $n = 2m$.

Definition 6.14. Any finite-dimensional module for the matrix algebra $A = \mathbb{C}^{N \times N}$ is of the form $F = S_1 \oplus \cdots \oplus S_r$ where each $S_k$ has dimension $N$ and $A$ acts irreducibly on each $S_k$. Another way to express this is to say that $F = W \otimes S$, where $\dim_{\mathbb{C}} S = N$ and $A$ acts trivially on $W$, that is, $a \cdot (w \otimes s) = w \otimes (a \cdot s)$. Therefore, if $\dim V$ even, any Clifford module for $\mathcal{C}l(V)$ is of the form $F = W \otimes S$, with $c(a)(w \otimes s) := w \otimes c(a)s$, where $S$ is an irreducible Clifford module, unique up to equivalence, called the spinor module (or “spinor space”) for $\mathcal{C}l(V)$. As algebras, $\mathcal{C}l(V)$ and $\text{End}_{\mathbb{C}}(S)$ are isomorphic.

Exercise 6.17. Define $\text{Hom}_{\mathcal{C}l(V)}(S,F)$ to be the vector space of $\mathbb{C}$-linear maps $T: S \rightarrow F$ which intertwine the Clifford actions of $\mathcal{C}l(V)$ on $S$ and on $F$, i.e., $c(a)(Ts) := T(c(a)s)$ for all $s \in S$, $a \in \mathcal{C}l(V)$. Check that the map $T \otimes s \mapsto Ts$ gives an isomorphism $\text{Hom}_{\mathcal{C}l(V)}(S,F) \otimes S \cong F$, and deduce that $\text{Hom}_{\mathcal{C}l(V)}(S,F) \cong W$ and $\text{End}_{\mathcal{C}l(V)}(F) \cong \text{End}_{\mathbb{C}}(W)$.

Definition 6.15. Let $(V,q)$ be an oriented Euclidean vector space of even dimension $n = 2m$ (over $\mathbb{R}$), and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $(V,q)$ which is compatible with the given orientation.\(^9\) We define the chirality element of $\mathcal{C}l(V)$ as

$$
\gamma := i^{m} e_1 e_2 \cdots e_n.
$$

If $\{e_1', \ldots, e_n'\}$ is another oriented orthonormal basis, then $e_j' = \sum_{k=1}^{n} g_{jk} e_k$ for $g \in SO(n)$ an orthogonal matrix of determinant $+1$, and so $i^{m} e_1' e_2' \cdots e_n' = \det(g) \gamma = \gamma$: thus the recipe (6.11) is independent of the chosen basis.

The chirality element $\gamma$ satisfies $\gamma^2 = 1$ (that is why the factor $i^m$ belongs in the definition), and it anticommutes with the copy of $V$ within $\mathcal{C}l(V)$. Indeed, $e_j \gamma = -\gamma e_j$ since $e_j$ anticommutes with every $e_k$ except $e_j$ itself; by linearity, $v\gamma = -\gamma v$ for all $v \in V$. Therefore $\gamma \nu \gamma = -\nu$ for $v \in V$, so by Proposition 6.1 $\gamma a \gamma = \alpha(a)$ for all $a \in \mathcal{C}l(V)$. The $\mathbb{Z}_2$-grading on $\mathcal{C}l(V)$ induced by $\gamma$ (by declaring an element even or odd according as it commutes or anticommutes with $\gamma$) coincides, fortunately, with the original grading given by $\alpha$.

Notice that if $n = 4k$ is a multiple of 4, then $\gamma$ belongs to the real Clifford algebra $\mathcal{C}l(V,q)$, and it is not necessary to use the complexification $\mathcal{C}l(V)$.\(^9\)

---

\(^8\) This is the “Bott periodicity” identity for complex Clifford algebras.

\(^9\) Recall that an orientation of $V$ is a choice of a positive direction in the real line $\Lambda^n V$; a basis $\{v_1, \ldots, v_n\}$ for $V$ is compatible with the orientation iff $v_1 \wedge \cdots \wedge v_n$ is positive.
6.5 The Fock space of spinors

The availability of $\gamma$ means that any (ordinary) complex module $F$ under the Clifford algebra can be given a $\mathbb{Z}_2$-grading by taking $F^\pm$ to be the $(\pm 1)$-eigenspaces of $c(\gamma) \in \text{End}_C F$. In particular, the spinor module is a superspace $S = S^+ \oplus S^-$, with $\mathbb{C}I(V) \simeq \text{End}_C S$ as superalgebras. We now give an explicit construction of the spinor module.

**Definition 6.16.** A Euclidean vector space $(V, q)$ of even real dimension $n = 2m$ can be made into a complex Hilbert space by choosing an orthogonal complex structure $J$ on $(V, q)$; this is an $\mathbb{R}$-linear operator on $V$ satisfying:

$$J^2 = -1 \quad \text{and} \quad q(Ju, Jv) = q(u, v) \quad \text{for } u, v \in V. \quad (6.12)$$

Now we regard $V$ as a complex vector space via the rule $(\alpha + i\beta)v = \alpha v + \beta Jv$ for $\alpha, \beta \in \mathbb{R}$. The hermitian form

$$\langle [u \mid v] \rangle := q(u, v) + iq(Ju, v)$$

is positive definite since $q$ is. We shall denote the complex Hilbert space thus obtained by $(V, q, J)$. Notice that $i\langle [u \mid v] \rangle = \langle [u \mid Jv] \rangle = -\langle [Ju \mid v] \rangle$ for $u, v \in V$.

**Exercise 6.18.** For $(V, q) = (\mathbb{R}^4, q_1)$, find all $J \in \text{End}_\mathbb{R} V = \mathbb{R}^{4 \times 4}$ satisfying (6.12). Show that the set of such $J$ has two connected components, each homeomorphic to a 2-sphere $S^2$.

**Exercise 6.19.** If $g \in \text{End}_\mathbb{R} (V)$, check that $gJ = Jg$. Conclude that $\{ g \in \text{GL}_\mathbb{R}(V) : gJ = Jg \} = \text{GL}_C(V, q, J)$. If $O(V, q)$ denotes the orthogonal group of those $g \in \text{GL}_\mathbb{R}(V)$ for which $q(gu, gv) = q(u, v)$ for all $u, v \in V$, show that $U_J(V) := O(V, q) \cap \text{GL}_C(V, q, J)$ is the unitary group of the Hilbert space $(V, q, J)$, i.e., the group of complex automorphisms satisfying $\langle [gu \mid gv] \rangle = \langle [u \mid v] \rangle$ for $u, v \in V$.

**Exercise 6.20.** Let $\mathcal{J}(V, q)$ denote the set of all orthogonal complex structures on a Euclidean vector space $(V, q)$. Show that $\mathcal{J}(V, q)$ is empty if $\dim_\mathbb{R} V$ is odd, and nonempty if $\dim_\mathbb{R} V$ is even (produce an example). Show that the orthogonal group $O(V, q)$ acts transitively on $\mathcal{J}(V, q)$ by $J \mapsto gJg^{-1}$, and that the isotropy subgroup of any $J$ is the unitary group $U_J(V)$.

**Exercise 6.21.** Check that $\{ u_1, Ju_1, \ldots, u_m, JU_m \}$ is an orthonormal basis for $(V, q)$ when $\{ u_1, \ldots, u_m \}$ is an orthonormal basis for $(V, q, J)$, and deduce that $O(V, q) \simeq O(\mathbb{R}^{2m}, q_{2m}) \equiv O(2m)$ and that $U_J(V) \simeq U(\mathbb{C}^m) \equiv U(m)$. Conclude that $\mathcal{J}(V, q)$ is diffeomorphic to the quotient manifold $O(2m)/U(m)$ (which, as it happens, has two connected components).\(^{10}\)

**Definition 6.17.** Choose and fix an orthogonal complex structure $J$ on $(V, q)$. Denote by $\mathcal{F}_J(V)$ or simply by $\mathcal{F}(V)$ the complex exterior algebra $\Lambda^\bullet(V, q, J)$ over the Hilbert space $(V, q, J)$. Then $\mathcal{F}(V)$ is a complex vector space of dimension $2^n$, called the Fock space.\(^{11}\)

\(^{10}\) The identity component $SO(V, q)$ of the group $O(V, q)$, which consists of those $g$ with determinant $+1$, satisfies $SO(V, q) \simeq SO(2m)$, so one of the two components of $\mathcal{J}(V, q)$ is the set of “orientation-preserving” $J$, which is diffeomorphic to $SO(2m)/U(m)$.

\(^{11}\) This is sometimes called a fermion Fock space to distinguish it from the Hilbert space formed from a symmetric algebra over $V$ (suitably completed), which is known as a “boson Fock space”; these spaces describe many-particle states of fermions and bosons in quantum field theory.
over \((V, q, J)\). The Fock space is itself a complex Hilbert space, with the inner product determined by
\[
\langle u_1 \wedge \cdots \wedge u_m | v_1 \wedge \cdots \wedge v_n \rangle := \delta_{mn} \det [\langle u_k | v_l \rangle].
\]
We can choose (by induction on \(\dim V\)) an orthonormal basis \(\{e_1, \ldots, e_n\}\) for \(V\) so that \(Je_{2r-1} = e_{2r}\) for \(r = 1, \ldots, m\).

It is useful to identify \((V, q, J)\) with an \(m\)-dimensional complex subspace \(W\) of the complexification \(V_C = V \otimes _R \mathbb{C}\); take \(W := \{v - iJv \in V_C : v \in V\}\). Notice that \(W\) is isotropic for the amplified bilinear form \(q\) on \(V_C\), i.e., \(q(u - iJu, v - iJv) = 0\) for \(u, v \in V\), on account of \((6.12)\); and moreover, \(\dim_C W = m = \frac{1}{2} \dim_C V_C\), so \(W\) is a maximally isotropic subspace.\(^{12}\)

The conjugate subspace \(\overline{W} := \{v + iJv \in V_C : v \in V\}\) satisfies \(W \cap \overline{W} = 0\), \(W \oplus \overline{W} = V_C\), and indeed \(\overline{W}\) is the orthogonal complement of \(W\) under the inner product on \(V_C\) given by
\[
\langle w | z \rangle := 2q(\bar{w}, z).
\]
The operator \(P_+ := \frac{1}{2}(I - iJ)\) is then a unitary isomorphism between \((V, q, J)\) and \(W\), so we may identify these Hilbert spaces. The Riesz theorem allows us to identify the dual space \(V^*\) with \(\overline{W}\), and if \(w = P_+ v\), we identify \(v^*\) with \(\bar{w} := \frac{1}{2}(I + iJ)v\). An orthonormal basis for \(W\) is \(\{z_1, \ldots, z_m\}\), where \(z_k := P_+ e_{2k} = P_+(Je_{2k-1}) = iz_k\).

We may also identify \(\mathcal{F}(V)\) with the complex exterior algebra \(\Lambda^* W\). Now define \(c(v)\) on \(\Lambda^* W\), for \(v \in V\), by
\[
c(v)\alpha := \epsilon(P_+ v)\alpha - \epsilon(P_- v)\alpha,
\]
where \(\epsilon(z)(w_1 \wedge \cdots \wedge w_k) = \sum_{j=1}^k (-1)^{j-1} \langle z | w_j \rangle w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_k\), in accordance with \((6.7)\), and extend to \(V_C\) by \(c(v_1 + iv_2) := c(v_1) + ic(v_2)\).

**Exercise 6.22.** Show that \(\{c(w), \epsilon(z)\} = \langle z | w \rangle\) for \(w, z \in W\).

**Exercise 6.23.** Show that \(c(w) = c(w)\) and \(c(\bar{z}) = -\epsilon(z)\) for \(w, z \in W\). Conclude that \(c\) extends to a self-adjoint Clifford action on \(\Lambda^* (W)\).

**Proposition 6.3.** The operator \(c(\gamma)\) on the Fock space is the grading operator on the superspace \(\Lambda^k W\).

**Proof.** The assertion is that the \((\pm 1)\)-eigenspaces of \(c(\gamma)\) are the even and odd subspaces \(\Lambda^k W\); in other words, \(c(\gamma)\alpha = (-1)^k \alpha\) for all \(\alpha \in \Lambda^k W\).

First of all, notice that \(z_k \bar{z}_k - \bar{z}_k z_k = i e_{2k-1} e_{2k}\), so that \(\gamma = \prod_{1 \leq k \leq m} (z_k \bar{z}_k - \bar{z}_k z_k)\). Moreover, \(c(z_k \bar{z}_k - \bar{z}_k z_k) = [c(z_k), c(\bar{z}_k)] = [\epsilon(z_k), \epsilon(\bar{z}_k)]\). Now a direct calculation shows that, for \(\alpha = z_{j_1} \wedge \cdots \wedge z_{j_k}\), we have \(\{\epsilon(\bar{z}_r), c(z_r)\} = \pm \alpha\), with negative sign iff \(r \in \{j_1, \ldots, j_k\}\).

Since \(\Lambda^* W\) is generated (as a complex vector space) by such \(\alpha\), we obtain \(c(\gamma)\alpha = (-1)^k \alpha\).

\(^{12}\)If \(Z \leq V_C\) has dimension \(k\), the subspace of vectors \(w\) with \(q(w, z) = 0\) for all \(z \in Z\) has dimension \(2m - k\), since \(q\) is nondegenerate; thus an isotropic subspace can have dimension at most \(m\). A maximally isotropic subspace is also called a polarization for \(q\).
6.6 The Pin and Spin groups

Definition 6.18. Let \((V,q)\) be a Euclidean vector space with \(q\) positive definite. The invertible elements of \(\mathcal{C}\ell(V,q)\) form a group which includes all nonzero scalars and all nonzero \(v \in V\) (since \(v^{-1} = -v/q(v,v)\)); and the “twisted conjugation” \(\phi(a)b := \alpha(a)ba^{-1}\) provides a linear action of this group on \(\mathcal{C}\ell(V,q)\). The invertible elements which preserve the subspace \(V\) under this action form the Clifford group \(\Gamma(V,q) := \{ a : \phi(a)(V) \subset V \}\). The nonzero scalars \(t \in \mathbb{R}^\times\) lie in \(\Gamma(V,q)\), and so do the nonzero vectors \(u \in V \setminus \{0\}\), since if \(v \in V\),

\[
\phi(u)v = -uvu^{-1} = (vu - 2q(u,v))u^{-1} = v - 2\frac{q(u,v)}{q(u,u)}u,
\]

so \(u \in \Gamma(V,q)\). Geometrically, (6.14) is the reflection in the hyperplane orthogonal to \(u\), since \(\phi(u)u = -u\) and \(\phi(u)v = v\) iff \(q(u,v) = 0\).

Lemma 6.4. The kernel of \(\phi : \Gamma(V,q) \to GL_{\mathbb{R}}(V)\) is the subgroup \(\mathbb{R}^\times\) of nonzero scalars.

Proof. If \(a \in \ker \phi\), let \(a = a^+ + a^-\) with \(a^\pm \in \mathcal{C}\ell^\pm(V,q)\); taking even and odd parts of the equation \(\alpha(a)v = va\) gives \(a^\pm v = \pm va^\pm\) for \(v \in V\). If \(\{e_1, \ldots, e_n\}\) is an orthonormal basis for \(V\), and if \(V_1\) denotes the hyperplane in \(V\) orthogonal to \(e_1\), then \(a^+ = b^+ + e_1b^-\) with \(b^\pm \in \mathcal{C}\ell^\pm(V_1,q)\). Now the equation \(a^+e_1 = e_1a^+\) yields \(b^+e_1 + b^- = e_1b^+ - b^-\), so \(b^- = 0\); hence \(a^+ \in \mathcal{C}\ell(V_1,q)\). A similar argument shows that \(a^- \in \mathcal{C}\ell(V_1,q)\). An obvious induction argument now shows that \(a\) has only a scalar component when expanded in the basis \(\{e_j : J \subseteq \{1, \ldots, n\}\}\) of \(\mathcal{C}\ell(V,q)\), and so \(a \in \mathbb{R}^\times\).

If \(v \in V\), then \(v\bar{v} = v\alpha(v) = -v^2 = q(v,v)1\), so the quadratic form \(q(\cdot, \cdot)\) extends to the map \(a \mapsto \bar{a}a\) of \(\mathcal{C}\ell(V,q)\) into itself. If \(a = \sum_j a_je_j\) is the expansion of \(a\) with respect to the basis \(\{e_j : J \subseteq \{1, 2, \ldots, n\}\}\) of \(\mathcal{C}\ell(V,q)\), then \(a\bar{a}\) and \(\bar{a}a\) have the same scalar component \(\sum_j a^2\). If \(a \in \Gamma(V,q)\) and \(v \in V\), then \(\phi(a)v = u\) implies \(u = \alpha(a)^{-1}va\), and so \(u = u^\dagger = \bar{a}v\bar{a}^{-1} = \phi(\bar{a})v\); in consequence, \(\phi(a\bar{a})v = \phi(a)\phi(\bar{a})v = v\), so that \(a\bar{a} \in \ker \phi = \mathbb{R}^\times\). Thus \(a \mapsto a\bar{a} = \bar{a}a\) yields a group homomorphism of \(\Gamma(V,q)\) into \(\mathbb{R}^\times\). Its kernel is \(\text{Pin}(V,q) := \{ a \in \Gamma(V,q) : a\bar{a} = 1 \}\).

If \(a \in \Gamma(V,q)\) and \(v \in V\), then

\[
q(\phi(a)v, \phi(a)v) = \alpha(a)va^{-1}\bar{a}^{-1}\bar{v}\alpha(\bar{a}) = \alpha(\bar{a}a)(a\bar{a})^{-1}v\bar{v} = -v^2 = q(v,v),
\]

since \(a\bar{a} = \bar{a}a\) is a scalar, and so \(\phi(a)\) lies in the orthogonal group \(O(V,q)\). Restriction to \(\text{Pin}(V,q)\) yields a homomorphism \(\phi : \text{Pin}(V,q) \to O(V,q)\) which is surjective since the orthogonal group is generated by the reflections \(13\) (6.14), and \(v \in \text{Pin}(V,q)\) whenever \(q(v,v) = 1\).

Definition 6.19. The Spin group of the Euclidean vector space \((V,q)\) is defined as the even part of \(\text{Pin}(V,q)\), i.e.,

\[
\text{Spin}(V,q) := \mathcal{C}\ell^\times(V,q) \cap \text{Pin}(V,q) := \{ a \in \Gamma(V,q) \cap \mathcal{C}\ell^\times(V,q) : a\bar{a} = 1 \}.
\]

\[13\]This is the Cartan–Dieudonné theorem, which can be proved by writing an orthogonal matrix in normal form as a direct of \(k\) plane rotation matrices and possibly a \(-1\) diagonal entry, and by recalling that a plane rotation is a product of two reflections in lines.
The image \( \phi(\text{Spin}(V,q)) \) is the subgroup \( SO(V,q) \) of \( O(V,q) \) generated by products of an even number of reflections. This is the rotation group, also called the special orthogonal group, since \( g \in O(V,q) \) lies in \( SO(V,q) \) iff \( \det g = 1 \).

The homomorphisms \( \phi: \text{Pin}(V,q) \to O(V,q) \) and \( \phi: \text{Spin}(V,q) \to SO(V,q) \) have the same kernel \( \{ \pm 1 \} \cong \mathbb{Z}_2 \), by Lemma 6.4. In particular, we have a short exact sequence of Lie groups:

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(V,q) \longrightarrow SO(V,q) \longrightarrow 1,
\]

so that \( \text{Spin}(V,q) \) is a double covering of the rotation group.

**Exercise 6.24.** Show that \( \text{Pin}(V,q) \) is the set of products \( \{v_1 \ldots v_k\} \) of unit vectors (i.e., vectors \( v_j \in V \) such that \( q(v_j, v_j) = 1 \) for each \( j \)), by showing that these products form a normal subgroup which \( \phi \) maps onto \( O(V,q) \). Deduce that \( \text{Spin}(V,q) \) is the set of products \( \{v_1 \ldots v_{2n}\} \) of an even number of unit vectors.

**Exercise 6.25.** If \( u, v \in V \) are unit vectors in \( V \), with \( u \neq \pm v \), and if \( a(t) := 1 + q(u, v) \sin 2t \) for \( 0 \leq t \leq \frac{\pi}{2} \), define \( w(t) := a(t)^{-1/2}(\cos tu + \sin tv), z(t) := a(t)^{-1/2}(\cos tv + \sin tu) \), and \( b := (1 - q(u, v)^2)^{-1/2}(uw + q(u, v) 1) \). Show that \( t \mapsto w(t)z(-t) \) is a continuous path in \( \text{Spin}(V,q) \) from 1 to -1 through \( b \), and that \( t \mapsto w(t)z(-t) \) is a continuous loop in \( \text{Spin}(V,q) \) from \( uv \) to \( b \) and back. Deduce that \( \text{Spin}(V,q) \) is a connected group.\(^{14}\)

On passing to the complexification, we may regard \( \text{Spin}(V,q) \) as a subgroup of the invertible elements of the complex Clifford algebra \( \mathbb{C}l(V) \); the circle group \( U(1) \), regarded as complex scalars is another such subgroup. Now if \( \lambda, \mu \in U(1) \) and \( a, b \in \text{Spin}(V,q) \), then \( \lambda a = \mu b \) in \( \mathbb{C}l(V) \) iff \( (\lambda, a) \sim (\lambda, -a) \); thus \( \text{Spin}(V,q) \) and \( U(1) \) generate the subgroup

\[
\text{Spin}^c(V) \simeq (\text{Spin}(V,q) \times U(1))/\mathbb{Z}_2,
\]

where the quotient map is defined by the relation \( (\lambda, a) \sim (\lambda, -a) \). Define \( \phi^c(\lambda a)v := \lambda^2 \phi(a)v \) for \( \lambda a \in \text{Spin}^c(V) \); then \( \phi^c \) maps \( \text{Spin}^c(V) \) onto \( SO(V,q) \times U(1) \) with kernel \( \{ \pm 1 \} \), so there is another short exact sequence of Lie groups:

\[
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}^c(V) \longrightarrow SO(V,q) \times U(1) \longrightarrow 1.
\]

The Lie algebra of the spin group \( \text{Spin}(V,q) \) is readily identified as a subspace of the Clifford algebra \( \mathbb{C}l(V,q) \). Indeed, let \( A^+_2 \) be the subspace of \( \mathbb{C}l^+(V,q) \) with basis \( \{1\} \cup \{e_i e_j : i \neq j \} \) with \( \{e_1, \ldots, e_n\} \) an (arbitrary) orthonormal basis of \( V \) (compare with Exercise 6.14), and let \( C^2(V,q) \) be the image of \( \Lambda^2 V \) under the quantization map \( Q \).

**Lemma 6.5.** \( C^2(V,q) = \{ \ b \in A^+_2 : \bar{b} = -b \} \).

**Proof.** Since \( Q(u \wedge v) = uv + q(u, v) \), it follows that \( C^2(V,q) \subseteq A^+_2 \). Counting dimensions, \( \dim C^2(V,q) = \dim \Lambda^2 V = \frac{1}{2}n(n-1) = \dim A^+_2 - 1 \), so \( C^2(V,q) \) is a hyperplane in \( A^+_2 \).

\(^{14}\)Since the fundamental group of \( SO(V,q) \) is \( \mathbb{Z}_2 \), this shows that the covering map \( \phi \) is nontrivial and hence that \( \text{Spin}(V,q) \) is the universal covering group of \( SO(V,q) \).
\( e_i e_j = e_j e_i = -e_i e_j \) for \( i < j \), so the map \( b \mapsto \tilde{b} + b \) is a scalar-valued \( \mathbb{R} \)-linear form on \( A^+ \), whose kernel is a hyperplane; and since
\[
Q(u \wedge v) = vu + q(u, v) 1 = -uv - q(u, v) 1 = -Q(u \wedge v)
\]
for \( u, v \in V \), it coincides with \( C^2(V, q) \).

If \( b \in C^2(V, q) \) and \( w \in V \), then \( [b, w] = [b, w] \) lies in \( V \) also; to see that, it suffices to take \( b = uv + q(u, v) 1 \), whereupon
\[
[b, w] = uvw - wuv = uvw + uvw - wuv - wuv = 2q(u, w)v - 2q(v, w)u \in V.
\]
The only elements of \( A^+_2 \) which commute with \( V \) are scalars, so \( [b, w] \) vanishes for all \( w \in V \) iff \( b = 0 \) in \( C^2(V, q) \). Hence \( \tau(b)v := [b, v] \) defines an injective \( \mathbb{R} \)-linear map \( \tau: C^2(V, q) \to \text{End} \ V \).

**Proposition 6.6.** The map \( \tau \) is a Lie algebra isomorphism from \( C^2(V, q) \) onto the Lie algebra of antisymmetric operators \( \mathfrak{so}(V, q) := \{ A \in \text{End} \ V : q(Au, v) \equiv -q(u, Av) \} \).

**Proof.** First notice that \( C^2(V, q) \) is a Lie algebra under the bracket \( [b, d] := bd - db \) of \( \text{Cl}^+(V, q) \). Indeed, on taking \( b = uv + q(u, v) 1 \), \( d = wz + q(w, z) 1 \), one finds that
\[
[b, d] = [uv, w]z + w[uv, z] \in A^+_2
\]
using (6.15), and \( [b, d] = [\tilde{d}, \tilde{b}] = [-d, -b] = -[b, d] \), so \( [b, d] \in C^2(V, q) \). Also from (6.15),
\[
q([b, w], z) = 2q(u, w)q(v, z) - 2q(v, w)q(u, z) = -q(w, [b, z]),
\]
which shows that \( \tau(b) \in \mathfrak{so}(V, q) \). Since \( \dim \mathfrak{so}(V, q) = \frac{1}{2} n(n - 1) = \dim C^2(V, q) \) and \( \tau \) is injective, its image is all of \( \mathfrak{so}(V, q) \). Finally, the Jacobi identity yields
\[
\tau([b, d])v = [[b, d], v] = [[b, v], d] + [b, [d, v]] = -\tau(d)\tau(b)v + \tau(b)\tau(d)v,
\]
so \( \tau([b, d]) = [\tau(b), \tau(d)] \) where the latter bracket is the commutator bracket in \( \text{End} \ V \) (which preserves \( \mathfrak{so}(V, q) \)); thus, \( \tau \) is a Lie algebra isomorphism.

**Lemma 6.7.** If \( A \in \mathfrak{so}(V, q) \) and \( \{ e_1, \ldots, e_n \} \) is an orthonormal basis for \( V \), then
\[
\tau^{-1}(A) = \frac{1}{2} \sum_{j<k} q(Ae_j, e_k) e_j e_k = \frac{1}{4} \sum_{j,k=1}^n q(Ae_j, e_k) e_j e_k. \tag{6.16}
\]

**Proof.** The series in (6.16) are equal since \( A \) is antisymmetric, and they lie in \( C^2(V, q) \) by Lemma 6.5; if \( b \) denotes either series, we show that \( q(e_r, \tau(b)e_s) = q(e_r, Ae_s) \) for all \( r, s \). Indeed,
\[
\tau(b)e_s = \frac{1}{2} \sum_{j<k} q(Ae_j, e_k) [e_j e_k, e_s] = \frac{1}{2} \sum_{j<k} q(Ae_j, e_k) (2\delta_{js}e_k - 2\delta_{ks}e_j)
\]
\[
= \sum_{k>s} q(Ae_s, e_k) e_k - \sum_{j<s} q(Ae_j, e_s) e_j = \sum_{k=1}^n q(Ae_s, e_k) e_k,
\]
by antisymmetry of \( A \), so \( q(e_r, \tau(b)e_s) = q(Ae_s, e_r) = q(e_r, Ae_s) \) as required.

74
The Lie algebra $C^2(V,q)$ gives rise to a Lie group, namely, the group of invertible elements of $\text{Cl}^+(V,q)$ generated by the elements $\exp b := 1 + \sum_{k \geq 1} b^k/k!$, for $b \in C^2(V,q)$, i.e., by the image of $C^2(V,q)$ under the exponential map in the Clifford algebra. Notice that if $a = \exp b$ and $w \in V$, then

$$awa^{-1} = (\exp b) w (\exp(-b)) = \sum_{k,l \geq 0} \frac{1}{k!l!} b^k w (-b)^l$$

$$= \sum_{r \geq 0} \frac{1}{r!} \sum_{k=0}^r \left( \begin{array}{c} r \\ k \end{array} \right) b^k w (-b)^{r-k} = \sum_{r \geq 0} \frac{1}{r!} [b, [b, \ldots, [b, w] \ldots]]$$

$$= \sum_{r \geq 0} \frac{1}{r!} \tau(b)^r w,$$  \hspace{1cm} (6.17)

so that $awa^{-1} \in V$, and hence $a \in \Gamma(V,q)$. From the relation $\bar{b} = -b$ it follows that $a\bar{a} = (\exp b)(\exp(-b)) = 1$, and so $a \in \text{Spin}(V,q)$.

**Exercise 6.26.** If $b = uv + q(u,v) 1$ with $u,v \in V$, show that

$$q(\tau(b)w,\tau(b)w) \leq 16 q(u,u)q(v,v)q(w,w).$$

If $\Sigma$ denotes the series on the right hand side of (6.17), verify its convergence by obtaining the estimate $q(\Sigma,\Sigma) \leq e^8 s(b)q(w,w)$ where $s(b)^2 = q(u,u)q(v,v)$.

**Proposition 6.8.** The exponential map $\exp: C^2(V,q) \to \text{Spin}(V,q)$ is surjective.

**Proof.** The equation (6.17) reads

$$\phi(\exp b) = \exp(\tau(b)),$$  \hspace{1cm} (6.18)

where the second exponential map is the matrix exponential in $\text{End} V$. Now any rotation in $SO(V,q)$ is of the form $\exp A$ for some antisymmetric operator $A \in \mathfrak{so}(V,q)$, as is seen by expressing the matrix of the rotation in canonical form. Therefore, $\{\phi(\exp b) : b \in C^2(V,q)\} = SO(V,q)$.

Also, if $u,v$ are orthogonal unit vectors in $V$ and if $\theta \in \mathbb{R}$, then $\exp(\theta uv) = \cos \theta + \sin \theta \, uv$. In particular, $-1 = \exp(\pi uv)$ lies in $\exp(C^2(V,q))$. Now if $b = \tau^{-1}(A)$ with $A \in \mathfrak{so}(V,q)$, choose an orthonormal basis for $V$ such that $q(Ae_j, e_k) = 0$ unless $k = j \pm 1$. From (6.16), $b = \sum_{r=1}^m b_r e_{2r-1} e_{2r}$, with $b_r = \frac{1}{2} q(Ae_{2r-1}, e_{2r})$. Thus $b$ commutes with $\pi e_1 e_2$, and so $-\exp b = \exp(\pi e_1 e_2 + b)$. It follows that $\exp(C^2(V,q))$ is a subset of $\text{Spin}(V,q)$ doubly covering $SO(V,q)$, and hence is all of $\text{Spin}(V,q)$. \hfill $\square$

Thus we may regard $C^2(V,q)$ as the Lie algebra of $\text{Spin}(V,q)$. It should be noted that $\text{Spin}(V,q)$ and $\text{Pin}(V,q)$ are compact Lie groups, since they are closed subsets of the “unit sphere” in $\text{Cl}(V,q)$ determined by the condition $\sum_j a_j^2 = 1$. Therefore, Proposition 6.8 exemplifies the well-known fact that the exponential mapping from the Lie algebra to a compact connected Lie group is surjective [13].
6.7 The spin representation

Proposition 6.9. Let $S := \Lambda^\pm W$ be the spinor module for $\mathfrak{Cl}(V)$. Define $c: \text{Pin}(V,q) \to \text{End}_C S$ by restriction of the Clifford action $c: \mathfrak{Cl}(V) \to \text{End}_C S$ given by (6.13). Then $c$ is an irreducible unitary representation of the Lie group $\text{Pin}(V,q)$.

Proof. Since $c: \mathfrak{Cl}(V) \to \text{End}_C S$ is multiplicative, its restriction to $\text{Pin}(V,q)$ is a representation of this group on $S$. Since the Clifford action is self-adjoint (see Exercise 6.24), we have $c(a)^\dagger = c(\bar{a}) = c(a^{-1}) = c(a)^{-1}$ for $a \in \text{Pin}(V,q)$, because $a\bar{a} = 1$; thus $c(a)$ is a unitary operator on $S$.

The irreducibility of $c$ follows from Schur’s Lemma: we show that any $T \in \text{End}_C S$ commuting with $\{c(a) : a \in \text{Pin}(V,q)\}$ is a scalar operator. Indeed, $T$ must commute with $c(w) = \epsilon(w)$ for any $w \in W$, since $V \setminus \{0\} \subset \text{Pin}(V,q)$ and $W \subset \mathbb{V}_C$; and $T$ must also commute with $c(\bar{z}) = -\iota(z)$ for $z \in W$. If $\Omega$ is a unit vector in the one-dimensional space $\Lambda^0 W \simeq \mathbb{C}$, then $\iota(\bar{z}) T \Omega = T(\iota(\bar{z}) \Omega) = 0$, so $T \Omega = t \Omega \in \Lambda^0 W$ for some $t \in \mathbb{C}$. Now $T(w_1 \wedge \cdots \wedge w_r) = T(\epsilon(w_1) \cdots \epsilon(w_r) \Omega) = \epsilon(w_1) \cdots \epsilon(w_r) (t \Omega) = t w_1 \wedge \cdots \wedge w_r$, so $T = t \text{id}_S$. \hfill $\square$

Definition 6.20. The irreducible representation given by the previous Proposition is called the spin representation of the group $\text{Pin}(V,q)$. Its restriction to the subgroup $\text{Spin}(V,q)$ is called the spin representation.

Proposition 6.10. The spin representation is not irreducible, but it is the direct sum of two inequivalent irreducible representations on the subspaces $S^+ := \Lambda^+_C W$ and $S^- := \Lambda^-_C W$.

Proof. Recall Proposition 6.3: $c(\gamma)$ is the grading operator on $S = S^+ \oplus S^-$. Now the chirality element $\gamma$ anticommutes with every $v \in V$, and so $\gamma a = a \gamma$ for all $a \in \text{Spin}(V,q)$ by Exercise 6.24. Thus $c(\gamma)$ commutes with $\{c(a) : a \in \text{Spin}(V,q)\}$, so by Schur’s lemma the spin representation cannot be irreducible; in fact, it is the direct sum of two subrepresentations on the $\pm 1$-eigenspaces of $c(\gamma)$, which are $S^+$ and $S^-$. If $T \in \text{End}_C S^\pm$ commutes with $c(a)$ for all $a \in \text{Spin}(V,q)$, then $T$ must commute with $c(w_1)c(w_2) = \epsilon(w_1)\epsilon(w_2)$ and with $c(z_1)c(z_2) = \iota(z_1)\iota(z_2)$ for $w_1, w_2, z_1, z_2 \in W$, so $T$ is a scalar operator, by adapting the argument of the proof of Proposition 6.9. Thus both subrepresentations are irreducible.

Notice that $\dim S^+ = \dim S^- = 2^{m-1}$, so these “half-spin representations” have the same dimension. However, they are not equivalent, since if $R \in \text{Hom}(S^+, S^-)$ is an intertwining operator, then $R(c(e_2j-1)e_2j) = c(e_2j-1)e_2j)R$ for $j = 1, \ldots, m$ and hence $Rc(\gamma) = c(\gamma)R$; but then $R\alpha = Rc(\gamma)\alpha = c(\gamma)R\alpha = -R\alpha$ for all $\alpha \in S^+$, so $R = 0$; inequivalence follows from Schur’s lemma. \hfill $\square$

Exercise 6.27. Show that $\gamma \in \text{Spin}^+(V)$.

The whole theory of Clifford algebras and spin representations may be extended, with a little care, to infinite-dimensional vector spaces with a symmetric bilinear form. The Clifford algebra must be complete, so its complexification is defined as the unique $C^*$-algebra with the desired universal property. The orthogonal group must be restricted to the subgroup of operators $g \in \text{End}_K V$ whose antilinear part, with respect to a fixed complex structure, is
Hilbert–Schmidt; this is necessary to guarantee convergence of series similar to (6.16). For an exposition of the infinite-dimensional spin representation which extrapolates from the present treatment, we refer to [2]. An alternative method, wherein the spin representation is defined by permuting certain “Gaussian” elements in the Fock space, is developed in [31], based on ideas of [44].

7 Global Clifford modules

Given a compact Riemannian manifold \((M, g)\), we can form a Clifford bundle \(\mathbf{C}(M) \to M\) by gluing together the complex Clifford algebras \(\mathbf{C}(\mathbb{T}xM, g_x)\). The space of smooth sections \(\Gamma(\mathbf{C}M)\) is an algebra, and its completion to the space of continuous sections \(\Gamma_{cont}(\mathbf{C}M)\) is canonically a \(C^*\)-algebra. Naturally, we want to consider vector bundles of representation spaces for the algebras \(\mathbf{C}(\mathbb{T}xM, g_x)\), whose section spaces are modules over the algebra \(\Gamma(\mathbf{C}M)\). However, although the Clifford algebra bundle can be built over any Riemannian manifold, topological obstructions can prevent putting together the corresponding Clifford modules. We shall restrict our attention to oriented, even-dimensional Riemannian manifolds and we shall say that \((M, g)\) is a spin manifold if a Clifford module corresponding to the irreducible spinor representation exists. Such manifolds may have none, one, or several inequivalent spin structures; our first task is to sort out the possibilities.

7.1 Clifford algebra bundles

**Definition 7.1.** Let \(M\) be a compact manifold, and let \(E \to M\) be a real vector bundle of rank \(n\) with a Euclidean metric \(g\). For any \(x \in M\), \((E_x, g_x)\) is a Euclidean vector space, and we can form the Clifford algebra \(\mathbf{C}(E_x, g_x)\). These form the fibres of a real vector bundle \(\mathbf{C}E \to M\) of rank \(2n\), which extends the vector bundle \(E \to M\) in the sense that there is an injective bundle map \(j: E \to \mathbf{C}E\); since \(\mathbf{C}(E_x, g_x) \simeq \Lambda^\mathbf{\bullet} E_x\) as vector spaces, just take \(\mathbf{C}E\) to be the total space of the exterior algebra bundle \(\Lambda^\mathbf{\bullet} E \to M\), and let \(j\) be the map which identifies each \(v_x \in E_x\) with its canonical image in \(\Lambda^1 E_x\). The declaration \(j(v_x)^2 := -g_x(v_x, v_x)\) determines, by Proposition 6.1, a bilinear product in each fibre which depends smoothly on \(x \in M\), and also determines isomorphisms \(\mathbf{C}(E_x, g_x) \simeq \mathbf{C}(\mathbb{T}xM, g_x)\).

The complexification of \(\mathbf{C}E \to M\) is \(\mathbf{C}E \to M\), the complex Clifford algebra bundle; notice that \(\mathbf{C}E = \mathbf{C}E_\mathbb{C}\).

The module of smooth sections \(\Gamma(\mathbf{C}E)\) is thus an algebra under the pointwise Clifford product \((\kappa \xi)(x) := \kappa(x)\xi(x)\). If the rank of \(E \to M\) is even, the centre of this algebra is just \(A = C^\infty(M)\), by identifying smooth functions on \(M\) with scalar-valued sections of the Clifford algebra bundle.\(^1\)

\(^1\)The continuous sections \(\Gamma_{cont}(\mathbf{C}E)\) form a \(C^*\)-algebra whose centre is the algebra \(C(M)\) of continuous complex-valued functions on \(M\). This \(C^*\)-algebra is noncommutative, but in a fairly trivial way: it is “Morita-equivalent” to the commutative \(C^*\)-algebra \(C(M)\). This means [46] that \(\Gamma_{cont}(\mathbf{C}E) \otimes \mathcal{K} \simeq C(M) \otimes \mathcal{K}\), where \(\mathcal{K}\) is the elementary \(C^*\)-algebra of compact operators on a separable infinite-dimensional Hilbert space. In particular, since \(\mathcal{K}\) is a simple \(C^*\)-algebra, there is a bijective correspondence between the irreducible representations of these Morita-equivalent \(C^*\)-algebras.
Exercise 7.1. Let $Q \to M$ be the principal $O(n)$-bundle of orthonormal frames for $E \to M$. Check that any $g \in O(n)$ extends to an automorphism $\rho(g)$ of the Clifford algebra $\Cl(n)$, and that $\rho(g)\rho(g') = \rho(gg')$ for $g, g' \in O(n)$. Then show that $\Cl E \to M$ is the vector bundle associated to $Q \to M$ via the representation $\rho$ of $O(n)$.

Definition 7.2. The Clifford bundle over a Riemannian manifold $(M, g)$ is the bundle of Clifford algebras $\Cl(M) \to M$ obtained by taking the tangent bundle $TM \to M$ as the generating vector bundle, with the Riemannian metric $g$ defining the Euclidean structure on $TM$. It would be more consistent to write $\Cl TM$ rather than $\Cl(M)$, but this is not usually done, for the following reason. The cotangent bundle $T^*M \to M$ is equivalent to the tangent bundle via the bundle map $\hat{g}: TM \to T^*M$ induced by the metric (see the discussion after Definition 3.2), and this in turn defines a Euclidean structure $g^{-1}$ on the cotangent bundle by

$$g^{-1}(\alpha, \beta) := g(\alpha^2, \beta^2) \quad \text{for} \quad \alpha, \beta \in \mathcal{A}^1(M),$$

where the vector field $\alpha^2$ is determined by $g(\alpha^2, Y) := \alpha(Y)$. This extends to a bundle equivalence $\tilde{g}: \Cl TM \to \Cl T^*M$. Thus we may and shall regard sections of $\Cl(M) \to M$ as Clifford products of vector fields or of forms, according to momentary convenience. We shall, as before, write the scalar products in (7.1) as $(\alpha \mid \beta)$ and $(\alpha^2 \mid \beta^2)$ respectively.

Exercise 7.2. Verify that the map $\tilde{g}: TM \to T^*M$ extends to an equivalence of the Clifford algebra bundles $\Cl TM$ and $\Cl T^*M$ by applying Proposition 6.1 on the fibres.

Exercise 7.3. Let $g_{ij} := g(\partial/\partial x^i, \partial/\partial x^j)$ be the matrix entries of the metric $g$ in local coordinates, and let $[g^{rs}]$ be the inverse matrix to $[g_{ij}]$; show that $(dx^r)^t = g^{rs} \partial/\partial x^s$ and conclude that $g^{-1}(dx^r, dx^s) = g^{rs}$, thereby justifying the notation $g^{-1}$ for the metric in $T^*M$.

In view of Proposition 5.2, the sections of the Clifford algebra bundle may be constructed by an alternative method. Starting from a Euclidean vector bundle $E \to M$, one can define the tensor bundle $\mathcal{T}(E) \to M$ as the (infinite) Whitney sum of the tensor product bundles $E^\otimes k \to M$, where $E^\otimes 0 = M \times \mathbb{R}$, the trivial rank-one bundle. The ideals $I(g_x)$ of $\mathcal{T}(E_x)$ glue together to give a bundle of ideals $I(g) \to M$, and $\Cl E \to M$ may be defined as the quotient bundle $\mathcal{T}(E)/I(g) \to M$. The module of sections $\Gamma(I(g))$ is an $\mathcal{A}$-submodule of $\Gamma(\mathcal{T}(E))$, and the quotient $\mathcal{A}$-module may be identified with $\Gamma(\Cl E)$. Note, in particular, that $\Gamma(I(g))$ is generated as an $\mathcal{A}$-module by the elements $s \otimes s + g(s, s)$, for $s \in \Gamma(E)$.

Proposition 7.1. Any connection $\nabla$ on a Euclidean vector bundle $E \to M$ which is compatible with the metric on $E$ extends to a connection, also denoted by $\nabla$, on $\Cl E \to M$, such that

$$\nabla_X (\kappa \xi) = (\nabla_X \kappa) \xi + \kappa (\nabla_X \xi)$$

for $X \in \mathfrak{X}(M)$, $\kappa, \xi \in \Gamma(\Cl E)$; that is, each $\nabla_X$ is a derivation on the algebra $\Gamma(\Cl E)$.

Proof. Compatibility of $\nabla$ with the metric is expressed by (5.7), which is equivalent to the condition that $(\nabla_X s \mid t) + (s \mid \nabla_X t) = X(s \mid t)$ for $s, t \in \Gamma(E)$, $X \in \mathfrak{X}(M)$. The operators $\nabla_X$
from $\Gamma(C)$ are convenient, we shall denote any of these three connections by $\nabla$; it is clear that $X \mapsto \nabla_X (s_1 \otimes \cdots \otimes s_k)$ is a connection on the bundle $E^{\otimes k} \to M$ for each $k$. Moreover, each $\nabla_X : \Gamma(T(E)) \to \Gamma(T(E))$ is an $A$-linear derivation, i.e., it satisfies the Leibniz rule
\[
\nabla_X(\sigma \otimes \tau) = (\nabla_X \sigma) \otimes \tau + \sigma \otimes (\nabla_X \tau).
\]

On applying the extended $\nabla$ to a generator of the module of ideals $\Gamma(I(g))$, we get
\[
\nabla_X [s \otimes s + (s \mid s)] = (s + \nabla_X s) \otimes (s + \nabla_X s) + (s + \nabla_X s \mid s + \nabla_X s) - s \otimes s - (s \mid s) - \nabla_X s \otimes \nabla_X s - (\nabla_X s \mid \nabla_X s).
\]
Therefore, $\nabla_X$ preserves $\Gamma(I(g))$, and so drops to the quotient algebra $\Gamma(C \ell E)$. The derivation property (7.3) survives on passing to the quotient, and (7.2) follows.

One may eliminate the vector field $X$ in the usual way, by noting that both sides of (7.2) are $A$-linear in $X$, and by extending the Clifford product in $\Gamma(C \ell E)$ to an $A$-bilinear form from $\Gamma(C \ell E) \times A^1(M, C \ell E)$ to $A^1(M, C \ell E)$. Thus (7.2) may be rewritten as
\[
\nabla(\kappa \xi) = (\nabla \kappa) \xi + \kappa(\nabla \xi) \quad \text{for} \quad \kappa, \xi \in \Gamma(C \ell E),
\]
which is an equation in $A^1(M, C \ell E)$.

**Definition 7.3.** The **Levi-Civita connection** on the Clifford bundle over $M$ is the derivation $\nabla : \Gamma(C \ell(M)) \to A^1(M, C \ell(M))$ which extends the Levi-Civita connection on the tangent bundle (or equivalently, its dual connection on the cotangent bundle). Whenever convenient, we shall denote any of these three connections by $\nabla^{LC}$.

**Definition 7.4.** A **Clifford module** over a Riemannian manifold $M$ is a module of sections $\Gamma(F)$ of a superbundle $F \to M$, together with a **Clifford action** of the sections of the complexified Clifford bundle $\Gamma(\bar{C} \ell(M))$. Such an action is defined as an $A$-bilinear map $\Gamma(\bar{C} \ell(M)) \times \Gamma(F) \to \Gamma(F)$, written $(\kappa, s) \mapsto c(\kappa)s$, such that $c(1)s = s$ and $c(\kappa)c(\xi)s = c(\kappa \xi)s$ for all $s \in \Gamma(F)$ and $\kappa, \xi \in \Gamma(\bar{C} \ell(M))$, or else as a homomorphism $c : \Gamma(\bar{C} \ell(M)) \to \Gamma(End F)$, which is even in the sense that $c : \Gamma(\bar{C} \ell^\pm(M)) \to \Gamma(End^\pm F)$.

The evenness of the action is equivalent to the condition that for any $\alpha \in A^1(M)$, the operator $c(\alpha)$ interchanges $\Gamma(F^+)$ and $\Gamma(F^-)$.

If $F \to M$ is a Hermitian vector bundle, we say that the Clifford action is **self-adjoint** if $c(\kappa)^\dagger = c(\bar{\kappa})$ for $\kappa \in C \ell(M)$, or equivalently, if $c(\alpha)^\dagger = -c(\alpha)$ for any (real-valued) $\alpha \in A^1(M)$. More precisely, this condition reads
\[
(c(\alpha)s \mid t) + (s \mid c(\alpha)t) = 0,
\]
for $s, t \in \Gamma(F)$, using the scalar product obtained from the hermitian structure on $F$.

Obvious examples of Clifford modules are $\Gamma(\bar{C} \ell(M))$ itself, with the left multiplication $c_l(\kappa)\xi := \kappa \xi$; and $\Gamma(\bar{C} \ell(M))$ itself, with conjugated right multiplication $c_r(\kappa)\xi := \bar{\kappa} \bar{\xi}$. Another example is the module of complex-valued exterior forms $A^\bullet(M, \mathbb{C})$, with the Clifford action obtained by applying the recipe (6.8) to each fibre:
\[
c(\alpha)\omega := \alpha \wedge \omega - i(\alpha^\#:\omega).
\]

79
However, these examples are “too large”, since the Clifford actions on the fibres are reducible, and it is to be expected that the global module should also be decomposable into a direct sum of several submodules. Therefore we seek irreducible Clifford modules, where each fibre is linearly isomorphic to the Fock space $S = \Lambda^* W$ of subsection 6.5 (provided $n = \dim M$ is even). In that case, after choosing an oriented orthonormal basis for $T^*_x M$, the spin representation of the group $\text{Spin}(n) := \text{Spin}(\mathbb{R}^n, q_n)$ yields a homomorphism $c: \text{Spin}(n) \to GL_C(F_x)$; all such homomorphisms form a principal $\text{Spin}(n)$-bundle associated to the vector bundle $F \to M$ via the spin representation. There is, however, a topological obstruction to the existence of such a principal bundle. Before we can proceed, we must identify this obstruction, in order to know whether irreducible Clifford modules are available at all.

7.2 Existence of spin structures

**Definition 7.5.** A real vector bundle $E \to M$ is orientable if its transition functions $g_{ij}: U_i \cap U_j \to GL(r, \mathbb{R})$ can be chosen to satisfy $\det g_{ij} > 0$ for all $i, j$. A Euclidean vector bundle has transition functions with values in the orthogonal group $O(r)$; thus it is orientable iff one can choose the $g_{ij}$ to satisfy $\det g_{ij} = +1$. (This may be expressed by saying that the structure group of $E$ can be reduced from $O(r)$ to $SO(r)$.)

To see whether a Euclidean vector bundle is orientable or not, take a good covering $\mathcal{U} = \{U_j\}$ for $M$ and choose transition functions $g_{ij}$ for $\mathcal{U}$. Then $\det g_{ij}: U_i \cap U_j \to \{\pm 1\} = \mathbb{Z}_2$, and since $g_{ij}g_{jk} = g_{ik}$ on each nonempty $U_i \cap U_j \cap U_k$, we conclude that $\det g$ is a Čech 1-cocycle in $C^1(\mathcal{U}, \mathbb{Z}_2)$. Any set of transition functions is of the form $g'_{ij} = f_i g_{ij} g_j^{-1}$ with $f_j: U_j \to O(r)$, as follows from the proof of Proposition 1.5. (If a local system of sections $s_i$ for $E$ transforms as $s_i = g_{ij}s_j$, and an equivalent system of local sections is given by $s'_j := f_j s_j$, the new transition functions satisfy $g'_{ij} f_j = f_i g_{ij}$.) Hence $\det f \in C^0(\mathcal{U}, \mathbb{Z}_2)$ and $\det g' = \det g + \delta(\det f)$. Thus the class $w_1(E) := [\det g] \in H^1(M, \mathbb{Z}_2)$ depends only on the equivalence class $[E]$ of the vector bundle. It is usually called the first Stiefel–Whitney class of $E$ [23, 39]. It is also customary to write $w_1(M) := w_1(TM)$ when $(M, g)$ is a Riemannian manifold. We may summarize the situation as follows.

**Lemma 7.2.** A Euclidean vector bundle $E \to M$ is orientable iff $w_1(E) = 0$ in $H^1(M, \mathbb{Z}_2)$. A Riemannian manifold $(M, g)$ is orientable iff $w_1(M) = 0$.

**Proof.** The condition $w_1(E) = 0$ means that transition functions $g'_{ij}$ can be chosen to satisfy $\det g'_{ij} = +1$ identically. 

Suppose now that $E \to M$ is an oriented Euclidean vector bundle (i.e., that $w_1(E) = 0$), with transition functions $g_{ij}: U_i \cap U_j \to SO(r)$ for a good covering $\mathcal{U}$ of $M$. Since each $U_i \cap U_j$ is contractible, one can lift these maps to the double covering $\text{Spin}(r)$ of $SO(r)$, i.e., we can find smooth functions $h_{ij}: U_i \cap U_j \to \text{Spin}(r)$ such that $\phi(h_{ij}) = g_{ij}$ for all $i, j$. The relation $\phi(h_{ij}h_{jk}h_{ik}^{-1}) = g_{ij}g_{jk}g_{ik}^{-1} = 1$ in $SO(r)$ shows that

$$a_{ijk} := h_{ij}h_{jk}h_{ik}^{-1} = \pm 1$$

---

2We identify $\{\pm 1\}$ with $\mathbb{Z}_2$, in order to use additive notation when combining Čech cocycles.
in Spin(r). Thus \( a \) is a Čech 2-cochain in \( C^2(\mathcal{U}, \mathbb{Z}_2) \); indeed, \( a \) is a Čech 2-cocycle, as a short calculation shows.

**Exercise 7.4.** Compute \( \delta a \), to check that it is trivial in \( C^3(\mathcal{U}, \mathbb{Z}_2) \).

We can make random changes of signs of some of the \( h_{ij} \), i.e., we can let \( h'_{ij} := b_{ij} h_{ij} \) where \( b_{ij} : U_i \cap U_j \to \{ \pm 1 \} \) are arbitrary (but constant) sign functions. Then \( b \in C^1(\mathcal{U}, \mathbb{Z}_2) \), and \( a'_{ijk} := h'_{ij} h'_{jk} (h'_{ik})^{-1} \) yields \( a' \in C^2(\mathcal{U}, \mathbb{Z}_2) \) with \( a' = a + \delta b \). Therefore, the class \( w_2(E) := [a] \in \check{H}^2(\mathcal{U}, \mathbb{Z}_2) \) depends only on the equivalence class \([E]\) of the vector bundle; it is called the **second Stiefel–Whitney class** of \( E \). One writes \( w_2(M) := w_2(TM) \) when \((M, g)\) is an oriented Riemannian manifold.

**Definition 7.6.** A spin structure on an oriented Riemannian manifold \((M, g)\) of dimension \( n \) is a principal Spin\((n)\)-bundle \( P \to M \) together with a bundle map \( \tau : P \to Q \) where \( Q \to M \) is the principal \( SO(n)\)-bundle of oriented orthonormal frames for the tangent bundle \( TM \to M \), such that \( \tau(p \cdot g) = \tau(p) \circ \phi(g) \) for \( p \in P \), \( g \in \text{Spin}(n) \). We say that \( M \) is a **spin manifold** if it admits at least one spin structure.

By Lemma 1.4, such a principal Spin\((n)\)-bundle may be assembled from transition functions \( h_{ij} : U_i \cap U_j \to \text{Spin}(n) \) satisfying \( \phi(h_{ij}) = g_{ij} \), where the \( g_{ij} \) are the transition functions of the frame bundle, provided only that \( h_{ij} h_{jk} = h_{ik} \) on each nonempty \( U_i \cap U_j \cap U_k \). Thus \( M \) admits a spin structure iff \( w_2(M) = 0 \) in \( \check{H}^2(M, \mathbb{Z}_2) \).

Since \( \tau : P \to Q \) reproduces the double covering \( \phi : \text{Spin}(n) \to SO(n) \) on each fibre, the map \( \tau \) is two-to-one.

Inequivalent spin structures may arise from different choices of \( h_{ij} \) covering the same \( g_{ij} \). The only freedom here comes from the sign changes \( b_{ij} = \pm 1 \) mentioned above, where \( h'_{ij} = b_{ij} h_{ij} \) determine another spin structure on \( M \). Since \( h_{ij} h_{jk} = h_{ik} \) and similarly for the \( h'_{ij} \), it follows that \( b_{ij} b_{jk} = b_{ik} \) also; thus \( b \) is a Čech 1-cocycle. Now if \( b = \delta c \) with \( c \in C^0(\mathcal{U}, \mathbb{Z}_2) \), i.e., if \( b_{ij} = c_i / c_j \) with all \( c_j = \pm 1 \), then \( h'_{ij} = (c_i / c_j) h_{ij} \) as functions from \( U_i \cap U_j \to \text{Spin}(n) \), so the corresponding principal Spin\((n)\)-bundles are equivalent. Conversely, a principal bundle equivalence yields relations \( h'_{ij} = (a_i / a_j) h_{ij} \) for some smooth functions \( a_j : U_j \to \text{Spin}(n) \) satisfying \( a_i = \pm a_j \) on overlaps; thus, \( a_j \) can be chosen so that \( c_i a_i = c_j a_j \) on overlaps, and consequently \( h'_{ij} = (c_i / c_j) h_{ij} \). In summary, the two spin structures are equivalent iff \( b = \delta c \) for some \( c \), which gives the following result.

**Lemma 7.3.** If \( M \) is a Riemannian manifold with \( w_1(M) = 0 \) and \( w_2(M) = 0 \), the inequivalent spin structures on \( M \) are classified by the Čech cohomology group \( \check{H}^1(M, \mathbb{Z}_2) \). \( \square \)

The computation of the Stiefel–Whitney classes for particular manifolds involves either a good deal of combinatorial calculation (see [28], for instance) or general theorems from topology [23, 39, 41]. One such general theorem is particularly useful: if \( E \to M \) is a complex vector bundle and \( E_\mathbb{R} \to M \) is the underlying real vector bundle, then \( w_1(E_\mathbb{R}) = 0 \) and \( w_2(E_\mathbb{R}) \) is the image of the first Chern class \( c_1(E) \) under the canonical homomorphism from \( \check{H}^2(M, \mathbb{Z}) \) to \( \check{H}^2(M, \mathbb{Z}_2) \) obtained from the standard homomorphism \( j : \mathbb{Z} \to \mathbb{Z}_2 \) (namely,
reduction modulo 2).\(^3\) That \(w_1(E_R) = 0\) should be no surprise, because a complex vector bundle is always orientable. To see that, notice firstly that the unitary group \(U(r)\) can be regarded as a subgroup of \(SO(2r)\), since \(U(r)\) permutes orthonormal bases in \(\mathbb{C}^r\), and any such basis \(\{e_1, \ldots, e_r\}\) yields an orthonormal basis \(\{e_1, f_1, \ldots, e_r, f_r\}\) for \(\mathbb{R}^{2r}\), by setting \(f_r := ie_r\); and since \(U(r)\) is connected, it is therefore contained\(^4\) in the identity component \(SO(2r)\) of the orthogonal group of \(\mathbb{R}^{2r}\). Now, with respect to any Hermitian metric on \(E\), a local orthonormal basis of sections for \(\Gamma(U, E)\) gives a local orthonormal basis of sections for \(\Gamma(U, E_R)\), and the \(U(r)\)-valued transition functions of \(E\) may thus be regarded as \(SO(2r)\)-valued transition functions of \(E_R\).

Since the first Chern class \(c_1(H)\) of the hyperplane bundle \(H \to \mathbb{CP}^m\) is a generator of \(\tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}) \simeq \mathbb{Z}\), so that \(c_1(H) \mapsto 1\) under this group isomorphism, its modulo-2 reduction is not zero: therefore, \(w_2(H_R) \neq 0\) in \(\tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)\). Now it can be shown \([28]\) that the tangent bundle \(T\mathbb{CP}^m\) has the following property: if \(E_r \to \mathbb{CP}^m\) denotes the trivial real bundle of rank \(r\), then \(T\mathbb{CP}^m \oplus E_2\) and \(H_R \oplus \cdots \oplus H_R\) (with \(m+1\) summands) are equivalent real bundles over \(\mathbb{CP}^m\). These facts suffice to decide which of the \(\mathbb{CP}^m\) are spin manifolds.

**Proposition 7.4.** The complex projective space \(\mathbb{CP}^m\) is a spin manifold iff \(m\) is odd; and for odd \(m\), the spin structure on \(\mathbb{CP}^m\) is unique.

**Proof.** The isomorphism \(\text{Cl}(V, q) \otimes \text{Cl}(W, r) \simeq \text{Cl}(V \oplus W, q \oplus r)\) of Proposition 6.2, restricted to the spin subgroups, shows that \(\text{Spin}(V, q)\) and \(\text{Spin}(W, r)\) may be regarded as commuting subgroups of \(\text{Spin}(V \oplus W, q \oplus r)\). Thus we may embed the direct product \(\text{Spin}(k) \times \text{Spin}(l)\) as a subgroup of \(\text{Spin}(k+l)\); and \(\phi\) maps this to the usual embedding of \(SO(k) \times SO(l)\) in \(SO(k+l)\). Thus, whenever \(E \to M\) and \(F \to M\) are oriented Euclidean vector bundles, the transition functions \(g_{ij} + g'_{ij}\) of \(E \oplus F\) take values in \(SO(k) \times SO(l)\), and lift to \(h_{ij} + h'_{ij}\) in \(\text{Spin}(k) \times \text{Spin}(l)\). The second Stiefel–Whitney class therefore satisfies\(^5\) the relation \(w_2(E \oplus F) = w_2(E) + w_2(F)\).

When \(M = \mathbb{CP}^m\), we thereby obtain

\[
w_2(\mathbb{CP}^m) + 0 = w_2(T\mathbb{CP}^m \oplus E_2) = w_2(H_R \oplus \cdots \oplus H_R) = (m + 1) w_2(H_R). \tag{7.5}
\]

Any nontrivial element of the group \(\tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)\) is of order two,\(^6\) and \(w_2(H_R) \neq 0\), so the right hand side of (7.5) vanishes iff \(m\) is odd.

To get uniqueness, we must show that \(\tilde{H}^1(\mathbb{CP}^m, \mathbb{Z}_2) = 0\). This follows from the Bockstein homomorphism construction of subsection 1.11, applied to the exact sequence of abelian groups

\[0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_2 \to 0,\]

\(^3\)See Appendix B of [39] for a proof.

\(^4\)If one chooses an orthonormal basis in \(\mathbb{C}^r\) for which \(g \in U(r)\) is diagonal, i.e., \(g(e_j) = e^{i\alpha_j}e_j\), then \(g(e_j) = \cos \alpha_j e_j + \sin \alpha_j f_j\), \(g(f_j) = -\sin \alpha_j e_j + \cos \alpha_j f_j\), so the image of \(g\) in \(SO(2r)\) is a direct sum of \(2 \times 2\) rotation blocks.

\(^5\)When \(E\) and \(F\) are not orientable, this additivity breaks down; there is a “product formula” of Whitney \([41]\) which yields the relation \(w_2(E \oplus F) = w_2(E) + w_2(F) + w_1(E)w_1(F)\).

\(^6\)Indeed, \(\tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)\) is a vector space over the field \(\mathbb{Z}_2\).
where ‘2’ denotes multiplication by two, and $j : \mathbb{Z} \to \mathbb{Z}_2$ is reduction modulo 2. The Bockstein construction yields a long exact sequence in Čech cohomology

$$\cdots \to \check{H}^1(M, \mathbb{Z}_2) \xrightarrow{\partial} \check{H}^2(M, \mathbb{Z}) \xrightarrow{2_*} \check{H}^2(M, \mathbb{Z}) \xrightarrow{j_*} \check{H}^2(M, \mathbb{Z}_2) \xrightarrow{\partial} \check{H}^3(M, \mathbb{Z}) \to \cdots$$

for any compact manifold $M$. Let us chase this diagram backwards from $\check{H}^3(M, \mathbb{Z})$ in the case $M = \mathbb{C}P^m$. We know that $\check{H}^3(\mathbb{C}P^m, \mathbb{Z}) = 0$ from (2.5), so $j_*$ is surjective.\footnote{Actually, the formula (2.5) refers to singular or de Rham cohomology; but we can establish an isomorphism between de Rham and Čech cohomology in degree 3 by a simple modification of the proof of Proposition 5.5. For the isomorphism between the de Rham–Čech cohomologies in any degree, see [12, 17].} We know also that $\check{H}^2(\mathbb{C}P^m, \mathbb{Z}) = \mathbb{Z}$ (these are the Chern classes found in subsection 5.10), so the image of $2_*$, which equals the kernel of $j_*$, is the subgroup $2\mathbb{Z}$ of “even” Chern classes, and $2_*$ is just multiplication by 2, which is injective. This forces $\partial : \check{H}^1(\mathbb{C}P^m, \mathbb{Z}_2) \to \check{H}^2(\mathbb{C}P^m, \mathbb{Z})$ to be the zero homomorphism, and thus $j_*: \check{H}^1(\mathbb{C}P^m, \mathbb{Z}) \to \check{H}^1(\mathbb{C}P^m, \mathbb{Z}_2)$ is onto; but $\check{H}^1(\mathbb{C}P^m, \mathbb{Z}) = 0$, again by (2.5), so we conclude that $\check{H}^1(\mathbb{C}P^m, \mathbb{Z}_2) = 0$. □

### 7.3 Spin$^c$ structures

What can be done about manifolds like $\mathbb{C}P^2$, which are complex manifolds but do not admit a spin structure? Since our immediate aim is to describe an irreducible module for the complex Clifford bundle over such manifolds, we could relax our requirements slightly by replacing the required structure group of the tangent bundle by Spin$^c(n)$ rather than Spin$(n)$. (Recall that Spin$^c(n)$ is the product of the group Spin$(n)$ and the unitary scalars $U(1)$ within the complex Clifford algebra $\mathbb{C}\ell(V)$. ) We say that a (compact, oriented, Riemannian) manifold $M$ admits a spin$^c$ structure if there is a principal Spin$^c(n)$-bundle $P^c \xrightarrow{\pi} M$ and a bundle map $\tau : P^c \to Q^c$, satisfying $\tau(p \cdot g) = \tau(p) \circ \phi^c(g)$ for $p \in P$ and $g \in \text{Spin}^c(n)$; here $Q^c$ is a principal $SO(n) \times U(1)$-bundle of the form $Q \times R \xrightarrow{\pi} M$, where $Q$ is the $SO(n)$-bundle of oriented orthonormal frames of the tangent bundle, as before, and $R \xrightarrow{\iota} M$ is a principal $U(1)$-bundle which we may choose as we please.

Now a principal $U(1)$-bundle is just the frame bundle of a complex line bundle $L \xrightarrow{\iota} M$, so (up to equivalence) these are classified by $\check{H}^2(M, \mathbb{Z})$. The modulo-2 reduction $j_*[L]$ is an element of $\check{H}^2(M, \mathbb{Z}_2)$, where the second Stiefel-Whitney class also lives. Suppose that $w_2(M) = j_*[L] *$ in $\check{H}^2(M, \mathbb{Z}_2)$, i.e., that $w_2(M) + j_*[L] = 0$; then one can find local sections $\check{h}_{ij} : U_i \cap U_j \to \text{Spin}^c(n)$ such that $\phi^c(\check{h}_{ij}) = \lambda_{ij}\check{g}_{ij}$ are the transition functions of $TM \oplus L \xrightarrow{\iota} M$, which patch together properly to give the required principal bundle $P^c \xrightarrow{\iota} M$.

**Exercise 7.5.** Assume that $w_2(M) + j_*[L] = 0$ and construct $P^c \xrightarrow{\iota} M$ as indicated.

If a spin$^c$ structure exists, $M$ is called a spin$^c$ manifold. The foregoing argument says that this is the case iff $w_2(M) \in j_*(\check{H}^2(M, \mathbb{Z}_2))$. Moreover, inequivalent spin$^c$ structures are parametrized by the classes of complex line bundles, i.e., by $\check{H}^2(M, \mathbb{Z})$.

\footnote{The fibrewise product of principal bundles corresponds, by association, to the Whitney sum of vector bundles; thus, if $Q$ and $R$ are the frame bundles of vector bundles $E$ and $F$ over $M$, with respective structure groups $G$ and $H$, then $Q \times R \xrightarrow{\iota} M$ may be defined as the frame bundle of $E \oplus F \xrightarrow{\iota} M$, which is a principal $G \times H$-bundle.}
Lemma 7.5. The complex projective space $\mathbb{CP}^m$ is a spin* manifold for any positive integer $m$.

Proof. In the proof of Proposition 7.4, we verified the surjectivity of the homomorphism $j_*: \tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}) \to \tilde{H}^2(\mathbb{CP}^m, \mathbb{Z}_2)$, without reference to the parity of $m$. □

In fact, any compact complex manifold carries a spin* structure. This follows from the relation $w_2(M) = j_*(c_1(T_{\text{hol}}M))$, where $T_{\text{hol}}M \to M$ is the holomorphic tangent bundle, i.e., the complex vector bundle whose fibres $T_{x,\text{hol}}M$ are spanned by the tangent vectors $(\partial/\partial z^1)_x, \ldots, (\partial/\partial z^m)_x$. To get a more concrete construction, we first regard $U(m)$ as a subgroup of $SO(2m)$ and consider the homomorphisms $\tau: U(m) \to SO(2m) \times U(1)$ given by $\tau(g) := (g, \det g)$, and $\sigma: U(m) \to \text{Spin}^c(2m)$ defined as follows. For each $g \in U(m)$ there is an orthonormal basis $\{e_1, \ldots, e_m\}$ of $\mathbb{C}^m$ which diagonalizes $g$, i.e., $g(e_j) = e^{i\alpha_j}e_j$; write $f_j := ie_j$ and $a_j := e^{i\alpha_j/2}(\cos \frac{\alpha_j}{2} + (\sin \frac{\alpha_j}{2})e_jf_j) \in \mathbb{C}\ell^+(2m)$; then $\sigma(g) := a_1a_2 \cdots a_m \in \text{Spin}^c(2m)$.

Exercise 7.6. Check that all the $a_j$ commute, that $\sigma$ is a well-defined homomorphism, and that $\phi^c \circ \sigma = \tau$.

Exercise 7.7. The linear map on $\mathbb{R}^{2m}$ determined by $Je_j := f_j, Jf_j := -e_j$ is a complex structure; and $U(m) = \{g \in SO(2m) : gJ = Jg\}$. Write $k := e_1f_1 + \cdots + e_mf_m \in \mathbb{C}\ell^+(2m)$ and check that $\text{exp}(\pi k) = (-1)^m$ and $\phi(\text{exp}(\frac{\pi}{2}k)) = J$. Define the metaunitary group $MU(m)$ as $\{a \in \text{Spin}(2m) : ak = ka\}$ and verify that $\phi(MU(m)) = U(m)$. Show that the centre of the group $MU(m)$ equals $\{\text{exp}(\theta k) : -\pi < \theta < \pi\} \simeq U(1)$ if $m$ is odd, whereas the centre is $\{\pm \text{exp}(\theta k) : 0 < \theta < \pi\} \simeq U(1) \times \mathbb{Z}_2$ if $m$ is even.

Now we can exhibit a spin* structure on a complex manifold $M$. Take any Hermitian metric on $M$ and let $Q' \to M$ be the unitary frame bundle of the holomorphic tangent bundle $T_{\text{hol}}M \to M$. Associate to it, firstly, a principal Spin$^c(2m)$-bundle $P \to M$ via the homomorphism $\sigma$, as in (1.3); and secondly, a principal $U(1)$-bundle $R \to M$ via the homomorphism $\text{det}: U(m) \to U(1)$. The latter is just the frame bundle of the complex line bundle $K^* \to M$ where $K \to M$ is the so-called canonical line bundle on $M$, defined by $\Gamma(K) := \mathcal{A}^{m,0}(M)$. Since $\phi^c \circ \sigma = \tau$, the principal bundle $P \to M$ yields the required spin$^c$ structure.

Of course, a spin manifold is automatically a spin* manifold. It suffices to use the trivial $U(1)$ bundle $R = M \times U(1)$ to build a spin* structure from a given spin structure.

Exercise 7.8. Complete the following construction of the spin structure for $\mathbb{CP}^m$, with $m$ odd, due to Dabrowski and Trautman [21]. Show that $\mathbb{CP}^m$ is diffeomorphic to the homogeneous space $U(m + 1)/(U(1) \times U(m))$ and that the unitary frame bundle is given by $Q' := U(m + 1)/U(1) \times U(m)$. For a fixed orthonormal basis $\{e_1, \ldots, e_m\}$ of $\mathbb{C}^m$, the elements $a_1, \ldots, a_m$ and the scalars $e^{i\alpha_j}1$ generate a subgroup $T$ of Spin$^c(2m)$ which is isomorphic to an $(m + 1)$-dimensional torus $T^{m+1}$; moreover, since $\phi^c(a_1a_2 \cdots a_m)$ is a block diagonal matrix over $\mathbb{R}^{2m}$, the centralizer of $T$ in Spin$^c(2m)$ is $T$ itself, so it is a maximal torus. If $b \in \text{Spin}^c(2m)$, then by pulling back $\phi^c(b)$ to $U(m)$ via $\tau$ and diagonalizing the resulting unitary matrix, one sees that $b$ lies in some conjugate subgroup $aTa^{-1}$. This exemplifies the well-known theorem of Weyl [13] that the maximal tori in a compact connected Lie group are conjugate and cover the whole group.
1)/U(1). Let $MU(m)$ be the metaunitary group of Exercise 7.7 and let $P' := MU(m + 1)/U(1)$, where the subgroup $U(1)$ of $MU(m + 1)$ is given as $\{ \exp(\theta k) : 0 \leq \theta < \pi \}$, the identity component of the centre. Check that the inclusion $MU(m) \subset MU(m + 1)$ drops to a free right action of $MU(m)$ on $P'$, and that the double covering $\phi: MU(m + 1) \to U(m + 1)$ drops to a double covering $\tau': P' \to Q'$, which intertwines the respective actions of $MU(m)$ and $U(m)$. Finally, show how the inclusions $MU(m) \hookrightarrow \text{Spin}(2m)$ and $U(m) \hookrightarrow SO(2m)$ associate to these new principal bundles $P$ and $Q$ together with a double covering $\tau: P \to Q$ which yields the desired spin structure.

Catalogues of spin manifolds and spin$^c$ manifolds are given in several places, e.g., [29, 39]. As well as the classes of manifolds just discussed, it is worth mentioning that any sphere $S^n$ is spin; any simply connected Lie group is spin; any orientable 2-dimensional manifold is spin (e.g., any Riemann surface); any 3-dimensional manifold is spin; and any orientable 4-dimensional manifold is spin$^c$. As counterexamples, we note that $\mathbb{CP}^2$ is an orientable 4-dimensional manifold which is not spin; and the homogeneous space $SU(3)/SO(3)$ is an orientable 5-dimensional manifold which is not spin$^c$.

### 7.4 The spinor module

**Definition 7.7.** Let $M$ be a spin manifold of even dimension $n = 2m$, and let $P \to M$ be the principal $\text{Spin}(n)$-bundle defining its spin structure.$^{10}$ Let $S = \Lambda^*_c W$ be an irreducible Clifford module for $\mathbb{C} \ell(n)$, i.e., a Fock space of complex dimension $2^m$, and let $c: \text{Spin}(n) \to \text{End}_{\mathbb{C}} S$ denote the spin representation. Let $S(M) \to M$ be the complex vector bundle associated to the spin structure $P$ via the spin representation $c$. It is called the spinor bundle over $M$. Its module of sections $\Gamma(S(M))$ is an irreducible Clifford module, and will be called the spinor module for the algebra $\Gamma(\mathbb{C} \ell(M))$.

**Proposition 7.6.** If $M$ is an even-dimensional spin manifold, any Clifford module $\Gamma(F)$ is of the form $\Gamma(\Lambda^r(S(M)))$, where $\Gamma(W)$ is a trivial Clifford module, i.e., the Clifford action is $\kappa \mapsto w$ for $\kappa \in \Gamma(\mathbb{C} \ell(M))$, $w \in \Gamma(W)$.

**Proof.** Let $\Gamma(F)$ be given. If $A = C^\infty(M)$, any $A$-linear map from $\Gamma(S(M))$ to $\Gamma(F)$ is of the form $\tau$, where $\tau: S(M) \to F$ is a bundle map, i.e., $\tau$ is an element of $\Gamma(\text{Hom}(S(M), F))$. Thus any such map which intertwines the two Clifford actions belongs to $\Gamma(W)$, where $W := \text{Hom}_{\mathbb{C} \ell(M)}(S(M), F)$ is the vector bundle over $M$ with fibres $W_x = \text{Hom}_{\mathbb{C} \ell(n)}(S(M)_x, F_x)$.

From Exercise 6.17 it follows that $w_x \otimes \sigma_x \mapsto w_x(\sigma_x)$ gives a vector space isomorphism $W_x \otimes S(M)_x \simeq F_x$. (If $w_x(\sigma_x) = 0$ for any nonzero $\sigma_x$, then $0 = c(a)[w_x(\sigma_x)] = w_x(c(a)\sigma_x)$ for all $a$, so $w_x = 0$ by irreducibility of $S(M)_x$.) The intertwining property $w_x(c(a)\sigma_x) = c(a)[w_x(\sigma_x)]$ shows that $W_x \otimes S(M)_x$ becomes a $\mathbb{C} \ell(n)$-module via the recipe $c(a)[w_x \otimes \sigma_x] := w_x \otimes c(a)\sigma_x$, and the aforementioned isomorphism intertwines this action with the given action on $F_x$. Globally, we obtain an invertible bundle map from $W \otimes S(M)$ to $F$ and hence an $A$-linear isomorphism from $\Gamma(W \otimes S(M))$ to $\Gamma(F)$ which intertwines the action

$$c(\kappa)(w \otimes \sigma) := w \otimes c(\kappa)\sigma$$

$^{10}$If $M$ carries more than one spin structure, we choose and fix a particular one.
with the given Clifford action on $\Gamma(F)$.

**Exercise 7.9.** Prove that $\Gamma(\text{End}_{\mathbb{C}}(M) \ F) \simeq \Gamma(\text{End}_{\mathbb{C}} W)$; in other words, match $A$-linear operators commuting with the Clifford action on $\Gamma(F)$ to $A$-linear operators on $\Gamma(W)$.

**Definition 7.8.** The passage from the irreducible spinor module $\Gamma(S(M))$ to a Clifford module of the form $\Gamma(W \otimes S(M))$ whose Clifford action is given by (7.6) is called a **twisting** by the vector bundle $W \to M$ (which in principle may be any complex vector bundle). We refer to $\Gamma(W \otimes S(M))$ as a **twisted Clifford module**.

Notice, in particular, that any Clifford module $\Gamma(F)$ where $F \to M$ has minimal rank $2^m$ is obtained by twisting the spinor module with a complex line bundle. Clearly, then, the twisting may affect the global topology of $F$, but for algebraic properties of the Clifford action it is enough to study the spinor module.

### 7.5 The spin connection

The next task is to show that the spinor module admits a distinguished connection, to be called the “spin connection”, which satisfies a “Leibniz rule” with respect to the Clifford action. This is not a trivial matter, as the spinor module is not “tensorial”, that is, its existence has topological obstructions not faced by modules of tensors, vector fields, or differential forms. Therefore the algebraic techniques used in Section 5 to construct new connections from old ones are not enough to produce a suitable connection on the spinor bundle.

What is needed is to use the spin representation. This we have defined, in Section 6, as a group representation of Spin$(V,q)$ as unitary operators on $S$ which does not drop to a representation of the special orthogonal group $SO(V,q)$. It may, however, be regarded as a projective (or “double-valued”) representation of $SO(V,q)$. The corresponding infinitesimal representations of Lie algebras are not troubled by this topological problem, and there is a linear isomorphism $\tau : C^2(V,q) \to \mathfrak{so}(V,q)$ between the Lie algebras.

**Definition 7.9.** Let $\dot{\mu} : \mathfrak{so}(V,q) \to \text{End}_{\mathbb{C}}(S)$ be the linear map given by

$$\dot{\mu}(A) := c(\tau^{-1}(A)), \quad (7.7)$$

Note that $\dot{\mu}(A)$ is skewhermitian since the Clifford action is selfadjoint, so $c(b)^\dagger = c(\bar{b}) = -c(b)$ for $b \in C^2(V,q)$ by Lemma 6.5. We call $\dot{\mu}$ the **derived spin representation**.

**Lemma 7.7.** The derived spin representation satisfies the identity

$$[\dot{\mu}(A), c(v)] = c(Av) \quad (7.8)$$

for all $A \in \mathfrak{so}(V,q), v \in V$.

---

11This is because $V$ is finite-dimensional. In the infinite-dimensional case, the covering map $\phi$ has a 1-dimensional kernel (a circle), which gives rise to a “spin anomaly” in the infinitesimal representation of $SO(V,q)$. For details, consult [31].

86
Proof. It is equivalent to show that \([\tau^{-1}(A), v] = Av\) as elements of the Clifford algebra \(\mathcal{C}(V, q)\). However, this follows immediately from the formula of Lemma 6.7 for \(\tau^{-1}(A)\), and from (6.15):

\[
[\tau^{-1}(A), v] = \frac{1}{4} \sum_{j,k=1}^{n} q(Ae_j, e_k) [e_j e_k, v] = \frac{1}{2} \sum_{j,k=1}^{n} q(Ae_j, e_k) (q(e_j, v)e_k - q(e_k, v)e_j) = \frac{1}{2} \sum_{j=1}^{n} q(e_j, v) Ae_j + \frac{1}{2} \sum_{k=1}^{n} q(e_k, v) Ae_k = Av,
\]

on using the antisymmetry of \(A\).

It would be helpful to have a version of (7.8) which is valid when vectors \(v \in V\) are replaced by arbitrary elements \(b \in \mathcal{C}(V, q)\). To accomplish this, one needs to extend the operator \(A\) on \(V\) to a linear operator on \(\mathcal{C}(V, q)\). In fact, \(A\) extends as a derivation of the Clifford algebra: define \(\hat{A}(1) := 0\), \(\hat{A}(v) := Av\) for \(v \in V\), and

\[
\hat{A}(v_1 \ldots v_k) := \sum_{j=1}^{k} v_1 \ldots v_{j-1}(Av_j)v_{j+1} \ldots v_k \quad (7.9)
\]

for \(v_1, \ldots, v_k \in V\).

Exercise 7.10. Check that the definition (7.9) is consistent by first verifying that \(\hat{A}(uv) + \hat{A}(vu) = 0\) for \(u, v \in V\). Then show that (7.8) extends to the identity \([\hat{\mu}(A), c(b)] = c(\hat{A}b)\) for all \(A \in \mathfrak{so}(V, q), b \in \mathcal{C}(V, q)\).

Now we can translate these algebraic identities to relations among sections of bundles over a spin manifold \(M\), by applying them at each fibre. Thus, if \(A \in \Gamma(\text{End} E)\) where \(E \to M\) is a Euclidean vector bundle, and if \((As \mid t) = -(s \mid At)\) for \(s, t \in \Gamma(E)\), then \(A(x) \in \mathfrak{so}(E_x)\) for \(x \in M\), so we may write \(A \in \Gamma(\mathfrak{so}(E))\). In particular, if \(E = TM\) is the tangent bundle, the notation \(A \in \Gamma(\mathfrak{so}(TM))\) means that \(A\) is an operator on \(\Gamma(TM) = \mathfrak{X}(M)\) satisfying \(g(AX, Y) = -g(X, AY)\) for \(X, Y \in \mathfrak{X}(M)\). We write \(\hat{\mu}(A)\) to denote the section \(x \mapsto \hat{\mu}(A(x))\) of the spinor bundle \(S(M) \to M\). This defines a map \(\hat{\mu}; \Gamma(\mathfrak{so}(TM)) \to \Gamma(S(M))\). Furthermore, by tensoring these \(\mathcal{A}\)-modules of sections with \(\mathcal{A}^k(M)\), we obtain maps \(\hat{\mu}; \mathcal{A}^k(M, \mathfrak{so}(TM)) \to \mathcal{A}^k(M, S(M))\).

If \(\nabla\) is a connection on the tangent bundle compatible with the metric, such as the Levi-Civita connection for instance, then, locally at least, \(\nabla = d + \alpha\) where \(\alpha \in \mathcal{A}^1(U, \mathfrak{so}(TM))\) in view of the metric compatibility condition (5.7). Thus \(\hat{\mu}(\alpha)\) makes sense as an element of \(\mathcal{A}^1(U, S(M))\), where \(U\) is the domain of \(\alpha\).

Theorem 7.8. Let \(M\) be an even-dimensional spin manifold and let \(\nabla\) be the Levi-Civita connection on the complex Clifford bundle \(\mathcal{C}(M) \to M\). Then there is a connection \(\nabla^S\) on
the spinor bundle $S(M) \to M$ satisfying the following Leibniz rule:

$$\nabla^S(c(\kappa)\sigma) = c(\nabla^S)\sigma = c(\nabla\kappa)\sigma + c(\kappa)(\nabla^S\sigma),$$

(7.10)

for all $\kappa \in \Gamma(\mathbb{C}\ell(M))$, $\sigma \in \Gamma(S(M))$.

Proof. Let $\mathcal{U} = \{U_j\}$ be a covering of $M$ by chart domains and let $\{f_j\}$ be a smooth partition of unity subordinate to $\mathcal{U}$, with $f_j(x) > 0$ iff $x \in U_j$. Then $f_j \nabla$ is a connection on $U_j$ compatible with the metric $f_j g$. Suppose for the moment that there exist connections $\nabla^S_j$ on each restriction of the spinor bundle $S(M) \to U_j$, satisfying $\nabla^S_j(c(\kappa)\sigma) = c(f_j \nabla\kappa)\sigma + c(\kappa)(\nabla^S_j\sigma)$, whenever $\sigma$ vanishes outside $U_j$; then the recipe $\nabla^S = \sum_j \nabla^S_j(f_j \tau)$, for any $\tau \in \Gamma(S(M))$, defines a connection satisfying (7.10).

Therefore, we may replace $M$ by any chart domain $U_j$; or we may equivalently suppose that the vector bundles $\mathbb{C}\ell(M) \to M$ and $S(M) \to M$ are trivial, and that the Levi-Civita connection may be written as $\nabla = d + \alpha$ with $\alpha \in \mathcal{A}^1(M, \mathfrak{so}(TM))$. The formula $\nabla = d + \alpha$, defined initially on vector fields, is also applicable to Clifford products of vector fields, provided $\alpha \kappa$ is interpreted as the derivation action of $\alpha$ on $\kappa \in \Gamma(\mathbb{C}\ell(M))$, in view of the Leibniz rule (7.4).

Now introduce a connection $\nabla^S$ on $S(M)$ by defining

$$\nabla^S := d + \hat{\mu}(\alpha).$$

(7.11)

From (7.8), extended to Clifford algebra, we get $[\hat{\mu}(\alpha), c(\kappa)] = c(\alpha \kappa)$ on $\Gamma(S(M))$, and then

$$\nabla^S(c(\kappa)\sigma) = d(c(\kappa)\sigma) + \hat{\mu}(\alpha)(c(\kappa)\sigma)$$

$$= c(\alpha \kappa)\sigma + c(\kappa)(d \sigma + [\hat{\mu}(\alpha), c(\kappa)] \sigma + c(\kappa)\hat{\mu}(\alpha)\sigma)$$

$$= c(\nabla^S)\sigma,$$

verifying (7.10).

We have constructed one solution $\nabla^S$ to (7.10). If $\tilde{\nabla}^S$ is another connection on $S(M)$ satisfying the same module-derivation property, then $\tilde{\nabla}^S - \nabla^S$ is given by $\beta \in \mathcal{A}^1(M, \text{End} S(M))$ such that $\beta(c(\sigma)) = c(\kappa)(\beta\sigma)$. That is to say, $\beta$ lies in $\mathcal{A}^1(M, \text{End}_{\mathbb{C}\ell(M)} S(M)) \simeq \mathcal{A}^1(M)$ in view of Exercise 7.9 and the irreducibility of $S(M)$. Therefore, $\nabla^S$ is unique up to addition of a scalar action by a 1-form on $M$.

We may apply the same reasoning to any Clifford module, with the following result.

Proposition 7.9. Let $\Gamma(F) = \Gamma(W \otimes S(M))$ be a Clifford module over a spin manifold $M$, and let $\tilde{\nabla}$ be a connection on $F$ which is a module derivation, i.e., $\tilde{\nabla}(c(\kappa)\tau) = c(\nabla\kappa)\tau + c(\kappa)(\tilde{\nabla}\tau)$. Then there is a unique connection $\nabla^W$ on $W$ such that $\nabla^W \otimes 1 + 1 \otimes \nabla^S = \tilde{\nabla}$.

Proof. Since $\tau \in \Gamma(F)$ is a finite sum of the form $\sum_j w_j \otimes \sigma_j$, we may rewrite the module derivation property of $\tilde{\nabla}$ as $\tilde{\nabla}(w \otimes c(\kappa)\sigma) = w \otimes c(\nabla\kappa)\sigma + c(\kappa)(\tilde{\nabla}(w \otimes \sigma))$. Let $\nabla^W_0$ be an arbitrary connection on $W$. Then we also get

$$[\nabla^W_0 \otimes 1 + 1 \otimes \nabla^S, 1 \otimes c(\kappa)](w \otimes \sigma) = w \otimes c(\nabla\kappa)\sigma,$$

88
so \( \beta := \nabla - (\nabla^W_0 \otimes 1 + 1 \otimes \nabla^S) \) lies in \( A^1(M, \text{End} F) \) and commutes with the Clifford action, that is, \( \beta \in A^1(M, \text{End} \mathfrak{su}(M)) \). By Exercise 7.9 it is of the form \( \beta = \alpha \otimes 1 \) for a unique \( \alpha \in A^1(M, \text{End} W) \). Writing \( \nabla^W := \nabla^W_0 + \alpha \) now gives \( \nabla = \nabla^W \otimes 1 + 1 \otimes \nabla^S \), as desired. \( \square \)

**Lemma 7.10.** The curvature of the spin connection is \( \mu(R) \), where \( R \) denotes the Riemannian curvature, \( R \in A^2(M, \mathfrak{so}(TM)) \).

**Proof.** It suffices to check this on a chart domain \( U \), where we may write \( \nabla^S = d + \mu(\alpha) \). The curvature form of the spin connection is given by (5.10) as \( \omega^S := \mu(d\alpha + \alpha \wedge \alpha) = \mu(R) \). \( \square \)

### 7.6 Local coordinate formulas

**Definition 7.10.** On a chart domain \( U \) of \( M \) with local coordinates \( \{x^1, \ldots, x^n\} \), we shall write the basic vector fields as \( \partial_j \equiv \partial/\partial x^j \). If \( M \) is Riemannian, the **Christoffel symbols** \( \Gamma^k_{ij} \) of the Levi-Civita connection on the chart domain \( U \) are the functions in \( C^\infty(U) \) defined by

\[
\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k,
\]

or equivalently, \( \nabla_{\partial_i} \partial_j = \Gamma^k_{ij} dx^k \otimes \partial_k \). In particular, \( \Gamma^k_{ij} = dx^k(\partial_i \partial_j) \). Notice also that \( \Gamma^k_{ij} \) give the matrix components \( \alpha^k_j \) of \( \alpha \in A^1(U, \text{End} TM) \).

**Exercise 7.11.** Check that the Levi-Civita connection on the cotangent bundle is determined locally by \( \nabla dx^k = -\Gamma^k_{ij} dx^i \otimes dx^j \), or equivalently, \( \nabla_{\partial_i} dx^k = -\Gamma^k_{ij} dx^j \).

**Exercise 7.12.** From the definition (5.27) of the Levi-Civita connection, show that

\[
\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),
\]

using the relations \( [\partial_i, \partial_j] = 0 \).

**Exercise 7.13.** Show that the Christoffel symbols on the sphere \( S^2 \) are given in spherical coordinates by

\[
\Gamma^\phi_{\theta \theta} = \Gamma^\phi_{\phi \theta} = \cot \theta, \quad \Gamma^\theta_{\phi \phi} = -\sin \theta \cos \theta, \quad \Gamma^k_{ij} = 0 \text{ otherwise},
\]

by applying (7.13) to the metric \( g := d\theta^2 + \sin^2 \theta d\phi^2 \).

**Exercise 7.14.** The components of the Riemann curvature tensor may be written [52] as \( R^l_{ijkl} := dx^l(\partial_i \partial_j R_{kl} - \partial_i \partial_k R_{jl}) \). Verify the “Ricci identities”:

\[
R^l_{ijkl} = \partial_k \Gamma^l_{ij} - \partial_l \Gamma^l_{kj} + \Gamma^m_{ij} \Gamma^l_{km} - \Gamma^m_{kj} \Gamma^l_{im},
\]

by using the curvature formula (5.11).

It is somewhat more common to use the components \( R_{ijkl} = g_{im} R^l_{ijkl} = (\partial_i \mid R(\partial_k, \partial_l) \partial_j) \). One sees from (5.11) that, for \( k, l \) fixed, the matrix with \( (i, j) \)-entry \( R_{ijkl} \) is antisymmetric.

**Exercise 7.15.** Verify that the curvature of the spin connection is given locally by

\[
\omega^S(\partial_k, \partial_l) = -\frac{1}{4} R_{ijkl} c(dx^i) c(dx^j),
\]

on account of (7.7).
We now express the spin connection itself in local coordinates. Fix a chart domain $U \subset M$, and write $g = g_{ij} \, dx^i \cdot dx^j$ there. Since the matrix $G = [g_{ij}]$ is positive definite, we can find a matrix $H = [h^i_j]$ of functions in $C^\infty(U)$ such that $H^T H = G$. Indeed, if $G^{1/2}$ is the positive definite square root of $G$, we may take $H = AG^{1/2}$ where $A$ is an orthogonal matrix at each point of $U$ with $A : U \to SO(n)$ smooth. \(^{12}\) Choose and fix such an $H$, and let $H^{-1} = [\tilde{h}_r^s]$ be its inverse matrix. Recall that $g^{-1} = g^{rs} \partial_r \cdot \partial_s$ is the metric on the cotangent bundle $T^*M \to M$; thus we have

$$h^i_\alpha \delta_{\alpha \beta} h^\beta_j = g_{ij}, \quad \tilde{h}^i_\alpha \delta_{\alpha \beta} \tilde{h}^\beta_j = g^{ij}.$$  

Orthonormal bases for $\mathcal{A}^1(U)$ and $\mathcal{X}(U)$ are then given by

$$\theta^\alpha := h^\alpha_j \, dx^j, \quad E_\beta := \tilde{h}^r_\beta \partial_r.$$  

(We use the convention of reserving Latin indices for coordinate bases and Greek indices for orthonormal bases.) By construction, we get

$$g(E_\alpha, E_\beta) = \delta_{\alpha \beta}, \quad g^{-1}(\theta^\alpha, \theta^\beta) = \delta^{\alpha \beta},$$

and also $(\theta^\alpha)^i = E_\alpha$, $(E_\beta)^i = \theta^\beta$ on $U$. \(^{13}\)

Locally, a smooth section of the spinor bundle looks like a smooth map from an open subset $U$ of $M$ (which can actually be taken as the complement of the closure of an arbitrarily small open set in $M$ [19]) to the Hilbert space $S$ of the spinor representation. Let $\{ \gamma^\alpha \equiv \gamma_\alpha : \alpha = 1, \ldots, n \}$ be a fixed set of unitary skewadjoint operators on $S$ with the property

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = -2 \delta^{\alpha \beta}.$$  

[For instance, take $\gamma^\alpha := c(e_\alpha)$ where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $(V, q)$.] We set

$$c(dx^r) := \tilde{h}^r_\gamma \gamma^\gamma.$$  

(7.16)

From (7.7) it is immediate that

$$c(dx^r)c(dx^s) + c(dx^s)c(dx^r) = -2g^{rs} = -2(dx^r \cdot dx^s),$$  

(7.17)

so that (7.16) in fact defines a local Clifford action of $\mathcal{A}^1(U)$ on $\Gamma(U, S)$. Conversely, a given Clifford action on the spinor bundle, which necessarily satisfies (7.17), together with a given fixed set of $\gamma^\alpha$, defines via (7.16) matrices $[\tilde{h}_r^s]$ and $[h^\alpha_i]$ satisfying (7.7). \(^{14}\)

It is convenient to introduce “mixed” or “orthogonal” Christoffel symbols $\hat{\Gamma}^\beta_{\alpha \alpha}, \hat{\Gamma}^\beta_{\alpha r}$ in $C^\infty(U)$ by

$$\nabla_{\alpha} E_\alpha =: \hat{\Gamma}^\beta_{\alpha \alpha} E_\beta; \quad \nabla_{E_\alpha} E_\alpha =: \hat{\Gamma}^\beta_{\alpha r} E_\beta.$$  

---

\(^{12}\) We require $\det A = 1$ so as to preserve the orientation when passing to orthonormal bases in (7.7).

\(^{13}\) These last relations give rise to the common phrasing: “we need not worry about raising and lowering indices, so long as we deal with orthonormal bases”.

\(^{14}\) The possibility of varying the Clifford action by premultiplying $H$ by arbitrary $SO(n)$-valued functions, provided only that these be compatible with changes of local charts, reflects the noncanonical nature of the spinor bundle.
The $\tilde{\Gamma}_{i\alpha}^\beta$ and $\hat{\Gamma}_{\epsilon\alpha}^\beta$ are antisymmetric in the indices $\alpha, \beta$; for instance,

$$\tilde{\Gamma}_{i\alpha}^\beta + \tilde{\Gamma}_{\alpha i}^\beta = g(\nabla_{\partial_j} E_\alpha, E_\beta) + g(E_\alpha, \nabla_{\partial_j} E_\beta) = \partial_i(\delta_{\alpha\beta}) = 0,$$

on account of the compatibility with the metric: $\alpha \in A^1(U, \mathfrak{so}(TM))$.

The components of the spin connection are given by the End $S$-valued functions

$$\omega_i := \frac{1}{4} \tilde{\Gamma}_{i\alpha}^\beta \gamma^\alpha \gamma_\beta, \quad \nu_\epsilon := \frac{1}{4} \hat{\Gamma}_{\epsilon\alpha}^\beta \gamma^\alpha \gamma_\beta = \frac{1}{4} \hat{h}_i^j \tilde{\Gamma}_{i\alpha}^\beta \gamma^\alpha \gamma_\beta. \quad (7.18)$$

**Exercise 7.16.** Show that $h_j^a \tilde{\Gamma}_{ja}^\beta = -\partial_i(h_j^\beta) + \Gamma_{ij}^k h_k^\beta$. Deduce the formula

$$\omega_i = \frac{1}{4} \left( \Gamma_{ij}^k g_{kl} - \partial_i(h_j^\alpha) \delta_{\alpha\beta} h_l^\beta \right) c(dx^j) c(dx^l),$$

that expresses the spin connection in a coordinate basis.

**Exercise 7.17.** Let $X = a^j \partial_j$ be the local expression of a vector field on $M$. Show that the contraction of $X$ with the spin connection is given locally as an operator on $C^\infty(U, S)$ by:

$$\nabla_S^X \sigma = a^j (\partial_j \sigma + \omega_j) \sigma,$$

and, in particular, that $\nabla_S^X E_{\epsilon \sigma} = E_{\epsilon \sigma} + \nu_\epsilon(x) \sigma$.

**Exercise 7.18.** Check that $\nabla^S$ is a hermitian connection on the spinor module.

### 8 Dirac operators and Laplacians

A Dirac operator is an operator of odd parity on a Clifford module over a spin (or spin$^c$) manifold, which is a first-order differential operator whose corresponding Leibniz rule involves Clifford multiplication by differentials of functions. Its square is therefore an even-parity second-order differential operator, which gives a far-reaching generalization of the Laplace-Beltrami operator on a Riemannian manifold: we may call this square a “generalized Laplacian”, using the terminology of [9]. Moreover, any Dirac operator over a compact manifold is elliptic, which implies that its inverse (off its finite-dimensional kernel) is a compact operator, so it has a discrete spectrum of eigenvalues which give precise information about the geometry of the manifold. Even more importantly, as Connes [18] has pointed out, the Dirac operator determines the Riemannian metric, and therefore serves as a gateway for reformulating the entire corpus of Riemannian geometry in analytic terms, allowing its extension to discrete spaces, fractals, spaces of tilings and group orbits, and many other contexts whose geometric structure seemed only a few short years ago to be hopelessly beyond reach.

#### 8.1 Connections and differential forms

**Definition 8.1.** Let $\nabla$ be a connection on the cotangent bundle $T^*M \longrightarrow M$ of a compact manifold $M$; the Leibniz rule $\nabla(\xi \wedge \omega) = (\nabla \xi) \wedge \omega + \xi \wedge (\nabla \omega)$ extends it to a connection on
the exterior product bundle $\Lambda^* T^* M \to M$, so that $\nabla$ maps $\mathcal{A}^* (M)$ to $\mathcal{A}^1 (M) \otimes_A \mathcal{A}^* (M)$.

The exterior product defines an $A$-linear map $\hat{\epsilon} : \mathcal{A}^1 (M) \otimes_A \mathcal{A}^* (M) \to \mathcal{A}^* (M)$ by $\hat{\epsilon} (\omega \otimes \eta) := \omega \wedge \eta$. The composition defines an $A$-linear endomorphism of $\mathcal{A}^* (M)$. In local coordinates $\nabla \omega = dx^j \otimes \nabla_{\partial_j} \omega$, and so $\hat{\epsilon} (\nabla \omega) = \epsilon (dx^j) \nabla_{\partial_j} \omega$ where $\epsilon (dx^j)$ denotes exterior multiplication by $dx^j$.

We introduce also a contraction map $i : \mathcal{X} (M) \otimes_A \mathcal{A}^* (M) \to \mathcal{A}^* (M)$ by $i (X \otimes \eta) := \iota_X \eta$. If $(M, g)$ is a Riemannian manifold, we identify $\mathcal{X} (M)$ with $\mathcal{A}^1 (M)$ via the metric $g$, and write $i (\alpha \otimes \eta) := \iota_{\alpha} \eta$. The composed map $i \circ \nabla$ is another $A$-linear endomorphism of $\mathcal{A}^* (M)$.

**Lemma 8.1.** Let $\nabla$ be a connection on the cotangent bundle $T^* M \to M$; and let $T \in \mathcal{A}^2 (M, TM)$ be the torsion of the dual connection $\nabla^*$ on the tangent bundle. Then for any $\omega \in \mathcal{A}^* (M)$,

$$\hat{\epsilon} (\nabla \omega) = d\omega - \iota_T \omega,$$

where $\iota_T : \mathcal{A}^k (M) \to \mathcal{A}^{k+1} (M)$ denotes contraction with the torsion.

**Proof.** Write $\nabla \omega = \beta^k \otimes \eta_k \in \mathcal{A}^1 (M) \otimes \mathcal{A}^* (M)$; then $\nabla X \omega = \beta^k (X) \eta_k$ by contraction. Therefore, if $X, Y \in \mathcal{X} (M)$ and $\omega \in \mathcal{A}^1 (M)$,

$$\hat{\epsilon} (\nabla \omega) (X, Y) = (\beta^k \wedge \eta_k) (X, Y) = \beta^k (X) \eta_k (Y) - \beta^k (Y) \eta_k (X)$$

$$= (\nabla_X \omega) (Y) - (\nabla_Y \omega) (X)$$

$$= X (\omega (Y)) - Y (\omega (X)) - \omega (\nabla_X Y - \nabla_Y X)$$

$$= d\omega (X, Y) - \omega (\nabla_X Y - \nabla_Y X - [X, Y])$$

$$= d\omega (X, Y) - \omega (T (X, Y)).$$

This establishes (8.1) for 1-forms. Moreover, $\iota_Y \iota_X (\hat{\epsilon} (\nabla \omega)) = \iota_Y \iota_X (d\omega) - \iota_{T (X, Y)} \omega$, for a form $\omega \in \mathcal{A}^k (M)$ of any degree, by an easy extension of the above argument.

**Exercise 8.1.** Generalize (8.2) to higher-degree forms.

**Corollary 8.2.** If $\nabla$ is a connection on the cotangent bundle $T^* M \to M$ whose dual connection is torsion-free, then $\hat{\epsilon} \circ \nabla$ equals the exterior derivation, $d$. [Proof]

In particular, $\hat{\epsilon} \circ \nabla^{LC} = d$ for the Levi-Civita connection $\nabla^{LC}$ (on the cotangent bundle).

There is a kind of dual formula relating the Levi-Civita connection with the codifferential $\delta$. Before revealing that, it is useful to consider the divergence of a vector field on a Riemannian manifold.

---

1 We find it more convenient in this Section to write $\mathcal{A}^r (M, E) = \mathcal{A}^r (M) \otimes_A \Gamma (E)$, reversing the order of the tensor product given in (5.3). The Leibniz rule (5.4) is adapted accordingly, and the contraction operator $\iota_X : \mathcal{A}^1 (M, E) \to \Gamma (E)$ is now given by $\iota_X (\beta \otimes s) := \beta (X) s$. 

92
8.2 Divergence of a vector field

**Definition 8.2.** Let $X$ be a vector field on an oriented manifold $M$ with volume form $\nu$. The **divergence** of $X$ relative to $\nu$ is the function $\text{div}_\nu X \in C^\infty(M)$ determined by

$$(\text{div}_\nu X)\nu := L_X \nu.$$

If $f \in C^\infty(M)$ is never zero, $f \nu$ is another volume form with $(\text{div}_{f\nu} X) f \nu = (X f) \nu + f L_X \nu = (X f + f \text{div}_\nu X) \nu$, so that

$$\text{div}_{f\nu} X = \text{div}_\nu X + \frac{X f}{f}. \quad (8.3)$$

If $(M, g)$ is a Riemannian manifold with Riemannian volume form $\Omega$, we write $\text{div} X \equiv \text{div}_\Omega X$.

Since $L_X \Omega = d(\iota_X \Omega)$ by Cartan’s identity, the identity $\int_M (\text{div} X) \Omega = 0$ (the **divergence theorem**) is an immediate consequence of Stokes’ theorem.

Over a chart domain $U$, we may consider the local volume form $\mu = dx^1 \ wedge \cdots \ wedge dx^n$. If $X = X^j \partial_j \in \mathfrak{X}(U)$, then by computing $L_X (dx^1 \ wedge \cdots \ wedge dx^n)$ we obtain the standard formula $\text{div}_\mu X = \partial_j X^j$. Since $\Omega = \sqrt{\det g} \mu$ on $U$ by (3.3), we derive from (8.3) a local formula for the Riemannian divergence:

$$\text{div} X = \partial_j X^j + \frac{1}{\sqrt{\det g}} \partial_j (X^j \sqrt{\det g}).$$

**Lemma 8.3.** The Riemannian divergence of a vector field is obtained from the Levi-Civita connection on the tangent bundle via the local formula

$$\text{div} X = dx^j (\nabla^\text{LC}_{\partial_j} X). \quad (8.4)$$

**Proof.** The connection $\nabla^\text{LC}$ is determined over a chart domain $U$ by the Christoffel symbols $\Gamma^k_{ij}$ of (7.12), satisfying $\Gamma^k_{ij} = dx^k (\nabla^\text{LC}_{\partial_i} \partial_j)$. The Leibniz rule now yields, for $X = X^j \partial_j \in \mathfrak{X}(U)$,

$$dx^j (\nabla^\text{LC}_{\partial_j} X) = \partial_j X^j + \Gamma^j_{jk} X^k.$$

The verification of (8.4) thus reduces to checking the formula

$$\Gamma^j_{jk} = \partial_k (\ln \sqrt{\det g}).$$

This relation is well known [36], but it is instructive to see how it goes. Recall that the Christoffel symbols are given by (7.13):

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

93
If \([G^{ij}]\) denotes the adjugate matrix to \([g_{ij}]\) (that is, the matrix of cofactors), then by Cramer’s rule \(g^{ij} (\partial_j g_{kl} - \partial_l g_{kj}) = 0\) since the matrix \([g_{ij}]\) is symmetric, and so

\[
\Gamma_{jk}^i = \frac{1}{2} g^{ji} \partial_k g_{jl} = \frac{1}{2} \frac{\partial \det g}{\partial g_{jl}} \partial_k g_{jl} = \frac{1}{2} \frac{\partial \det g}{\partial g_{jl}} \partial_k (\ln \det g),
\]

on using the Cramer expansion \(\det g = g_{jl} G^{jl}\).

\[\square\]

If we choose local orthonormal bases \(\{\theta^a\}\) for \(A^1(U)\) and \(\{E_\alpha\}\) for \(\mathfrak{X}(U)\) with \(E_\alpha = (\theta^a)^\sharp\), we may write, as in (7.7), \(\theta^a = h^a_j \, dx^j\) and \(E_\beta = \hat{h}_\beta \partial \partial\). Then \(\theta^a (\nabla^{LC}_{E_\alpha} X) = h^a_j \, dx^j (\hat{h}_\beta \nabla^{LC}_\beta X)\), so we obtain the alternative formula to (8.4), using local orthonormal bases:

\[
div X = \theta^a (\nabla^{LC}_{E_\alpha} X).
\] (8.5)

Yet another divergence formula, for a particular set of vector fields, is the following.

**Lemma 8.4.** Let \(M\) be an oriented Riemannian manifold. Given \(\zeta \in \mathcal{A}^k(M)\) and \(\eta \in \mathcal{A}^{k-1}(M)\), define \(Z \in \mathfrak{X}(M)\) by \(\omega(Z) := (\zeta \mid \omega \wedge \eta)\) for \(\omega \in \mathcal{A}^1(M)\). The divergence of this vector field is given locally by

\[
div Z = E_\alpha (\zeta \mid \theta^a \wedge \eta),
\]

where \(\{\theta^1, \ldots, \theta^n\}\) and \(\{E_1, \ldots, E_n\}\) are local orthonormal bases of 1-forms and vector fields respectively, with \(E_\alpha = (\theta^a)^\sharp\).

**Proof.** Since \(\omega \mapsto (\zeta \mid \omega \wedge \eta)\) is \(\mathcal{A}\)-linear on \(\mathcal{A}^1(M)\), it indeed defines a vector field \(Z \in \mathfrak{X}(M)\). Since \(\Omega = \theta^1 \wedge \cdots \wedge \theta^n\) locally, we find that \(\iota_Z \Omega = \sum_{\alpha=1}^n (-1)^{\alpha-1} (\zeta \mid \theta^\alpha \wedge \eta) \theta^1 \wedge \cdots \wedge \cdot \wedge \theta^n\), and it follows that \(L_Z \Omega = d(\iota_Z \Omega) = E_\alpha (\zeta \mid \theta^\alpha \wedge \eta) \Omega\).

\[\square\]

### 8.3 The Hodge–Dirac operator revisited

**Lemma 8.5.** If \(\nabla^{LC}\) denotes the Levi-Civita connection on the cotangent bundle of an oriented Riemannian manifold \(M\), and \(\delta : \mathcal{A}^r(M) \to \mathcal{A}^r(M)\) is the codifferential, then \(i \circ \nabla^{LC} = -\delta\).

**Proof.** Choose \(\zeta \in \mathcal{A}^k(M)\) and \(\eta \in \mathcal{A}^{k-1}(M)\) with \(k \geq 1\), and define \(Z \in \mathfrak{X}(M)\) by \(\omega(Z) := (\zeta \mid \omega \wedge \eta)\). Choose local orthonormal bases \(\{\theta^1, \ldots, \theta^n\}\) and \(\{E_1, \ldots, E_n\}\) of 1-forms and vector fields, with \(E^\alpha \equiv E_\alpha = (\theta^a)^\sharp\). Since \((\zeta \mid \beta \wedge \omega) = (\iota_{E^\alpha} \zeta | \omega)\) for any \(\xi, \omega \in \mathcal{A}^*(M)\) and \(\beta \in \mathcal{A}^1(M)\), we obtain

\[
(\zeta \mid d\eta) = (\zeta \mid \xi (\nabla^{LC}_\xi \eta)) = (\zeta \mid \xi (\theta^a) \nabla^{LC}_{E_\alpha} \eta)
\]

\[
= (\iota_{E^\alpha} \zeta \mid \nabla^{LC}_{E_\alpha} \eta) = E_\alpha (\iota_{E^\alpha} \zeta \mid \eta) - (\nabla^{LC}_{E_\alpha} \xi \mid \eta)
\]

\[
= E_\alpha (\zeta \mid \theta^\alpha \wedge \eta) - (\iota_{E^\alpha} \nabla^{LC}_{E_\alpha} \xi \mid \eta)
\]

\[
= \text{div} Z - (i (\nabla^{LC}_\xi \zeta \mid \eta),
\]

94
where we have used Corollary 8.2, the metric compatibility of $\nabla^{LC}$, and the commutation relation $[\nabla_X, \iota_Y] = \iota_{[X,Y]}$ (see Exercise 5.21). On multiplying both sides by $\Omega$ and integrating, the term $\text{div} \, Z$ is killed by Stokes’ theorem, and we obtain $\langle \delta \zeta | \eta \rangle := \langle \zeta | d\eta \rangle = -\langle i(\nabla^{LC} \zeta) | \eta \rangle$; since $\eta$ is arbitrary, this gives $i(\nabla^{LC} \zeta) = -\delta \zeta$.

Corollary 8.6. If $\hat{c} : \Gamma(\mathbb{C}(M)) \otimes \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ denotes the Clifford action on the de Rham algebra of an oriented Riemannian manifold, and if $\nabla^{LC}$ denotes the Levi-Civita connection, then

$$\hat{c} \circ \nabla^{LC} = d + \delta.$$  

Proof. If $\nabla \omega = \beta^k \otimes \eta_k$, then $\hat{c}(\nabla \omega) = c(\beta^k) \eta_k = \epsilon(\beta^k) \eta_k - \iota(\beta^k) \eta_k$, so that $\hat{c} = \hat{\epsilon} - \hat{i}$. Thus (8.6) follows immediately from Corollary 8.2 and Lemma 8.5. □

In subsection 4.5, we referred to $d + \delta$ as the “Hodge–Dirac operator”, which we obtained, in a somewhat ad-hoc fashion, as a square root of the Hodge Laplacian. The formula (8.6) reveals its true nature as the composition of a Clifford action and a connection which is compatible with this action (recall Proposition 7.1). This opens the way for the general definition of Dirac operators.

8.4 Dirac operators

Definition 8.3. Let $\Gamma(F)$ be a selfadjoint Clifford module over a compact oriented Riemannian manifold $M$, that is, let $F \to M$ be a Hermitian vector bundle and let there be given a selfadjoint Clifford action $\hat{c} : \Gamma(\mathbb{C}(M) \otimes F) \to \Gamma(F)$, where we write $\hat{c}(\kappa \otimes \psi) := c(\kappa) \psi$. Let $\nabla$ be a Hermitian connection on $F \to M$ that satisfies the Leibniz rule $\nabla(c(\kappa) \psi) = c(\nabla^{LC} \kappa) \psi + c(\kappa)(\nabla \psi)$, for $\kappa \in \Gamma(\mathbb{C}(E))$, $\psi \in \Gamma(F)$. The Dirac operator associated to the connection $\nabla$ and the Clifford action $\hat{c}$ is

$$\hat{D} := \hat{c} \circ \nabla.$$

Via $\Gamma(F) \xrightarrow{\nabla} \mathcal{A}^1_{\mathbb{C}}(M) \otimes \mathcal{A} \Gamma(F) \xrightarrow{\hat{c}} \Gamma(F)$, this $\hat{D}$ is a $\mathbb{C}$-linear endomorphism of $\Gamma(F)$. In local coordinates, $\hat{D} \psi = c(\partial_j) \nabla \partial_j \psi$.

There are many Dirac operators to be found. First of all, if $M$ is a spin manifold, we may consider the irreducible Clifford module $\Gamma(S)$, i.e., the spinor module, with its spin connection $\nabla^S$. Then $\hat{D}^S := \hat{c} \circ \nabla^S$ is the original Dirac operator.\footnote{$\hat{D}^S$ is usually called the Dirac operator on the spinor module.} Any other Clifford module $\Gamma(S)$ on $M$ is obtained by twisting, that is, $S = W \otimes F$ where $W \to M$ is a Hermitian vector bundle carrying the trivial Clifford action, and the compatible connection is determined, via Proposition 7.9, by an arbitrary Hermitian connection on $W$. The Dirac operator $d + \delta$ on the de Rham algebra arises in this way, since $\Lambda^\bullet T^* M \simeq S \otimes S'$ where $S' \to M$ is the superbundle obtained from the spinor bundle by taking the opposite grading: $(S')^\pm := S^\mp$.  

95
Exercise 8.2. If \( c : \mathcal{C}(V) \to \text{End}_\mathbb{C} \Lambda^*_\mathbb{C}W \) is the Clifford action on Fock space, and if \( v \in V \) is a unit vector, show that \( c'(a) := -c(vav) \) is another selfadjoint Clifford action, whose restriction to \( \text{Spin}(V, q) \) switches the two irreducible subrepresentations of the spin representation. Conclude that there is an irreducible Clifford module \( S' \to M \) on any spin manifold, whose “even” subspace \( \Gamma(S'^+) \) is the “odd” subspace \( \Gamma(S^-) \) of the spinor module, and viceversa.

Exercise 8.3. Define a Clifford action on the conjugate Fock space \( \bar{c} : \mathcal{C}(V) \to \text{End}_\mathbb{C} \Lambda^*_\mathbb{C}\bar{W} \) by setting \( \bar{c}(v)\bar{\alpha} := \epsilon(Pv)\bar{\alpha} - \iota(Pv)\bar{\alpha} \) for \( v \in V_\mathbb{C}, \bar{\alpha} \in \bar{W} \) [compare with (6.13)]. Check that \( \bar{c}(w) = -\iota(w) \) for \( w \in W \) and \( \bar{c}(z) := \epsilon(z) \) for \( z \in \bar{W} \). If \( \gamma \) denotes, as usual, the chirality element of \( \mathcal{C}(V) \), show that \( \bar{c}(\gamma)\bar{\alpha} = (-1)^{k+1}\bar{\alpha} \) whenever \( \bar{\alpha} \in \Lambda^*_\mathbb{C}\bar{W} \) [compare with Proposition 6.3]. If \( S = \Lambda^*_\mathbb{C}W \) is the spinor module for \( \mathcal{C}(V) \), the dual Hilbert space \( S^* \) is identified with \( \Lambda^*_\mathbb{C}\bar{W} \) via

\[
\langle \bar{z}_1 \wedge \cdots \wedge \bar{z}_r, w_1 \wedge \cdots \wedge w_s \rangle := \delta_{rs} \det[2q(z_k, w_l)],
\]

that is, the conjugation \( C : w_1 \wedge \cdots \wedge w_s \mapsto \bar{w}_1 \wedge \cdots \wedge \bar{w}_s \) is the (antilinear) Riesz isomorphism of \( S \) with \( S^* \). Check that \( C \) intertwines the action \( \bar{c} \) on \( S^* \) with the action \( c \) on \( S \), and conclude that \( S^* \) is an irreducible Clifford module for \( \mathcal{C}(V) \) whose \( \mathbb{Z}_2 \)-grading is given by \( (S^*)^+ = C(S^-) \) and \( (S^*)^- = C(S^+) \).

Even if \( M \) is not spin, there may be many Clifford modules with compatible connections; the de Rham algebra with the Levi-Civita connection is again the prime example. In any case, since the obstruction to the existence of a spin structure is global, we can always write \( \Gamma(U, F) = \Gamma(U, W \otimes S) \) over a chart domain (where \( W \) may depend on \( U \)) and from any one compatible connection we can manufacture others by altering \( \nabla^W \) over \( U \) only. Alternatively, we may vary the Clifford action by redefining (7.16) (which amounts to an action of the \( SO(n) \) frame bundle), as explained in subsection 7.6. Moreover, we have the further freedom of making a smooth change in the metric \( g \) and thereby changing the Clifford action \( c \) on \( \Gamma(F) \) (and also the connection, if necessary, to preserve compatibility). Thus any one Dirac operator gives rise to a large family of “smoothly perturbed” Dirac operators on the same Clifford module.

Any Dirac operator is a first-order differential operator. To avoid any possible confusion of terminology, we make a formal definition.

Definition 8.4. Let \( E \to M \) be a vector bundle. Any \( A \in \Gamma(\text{End} E) \) defines a \( \mathbb{C} \)-linear operator on \( \Gamma(E) \) by left multiplication, that is, \( (As)_x := A_x(s_x) \) for \( s \in \Gamma(E), x \in M \). Any connection \( \nabla^E \to M \), when contracted by vector fields \( X \in \mathfrak{X}(M) \), provides other \( \mathbb{C} \)-linear operators \( \nabla^X \) on \( \Gamma(E) \). The differential operators on \( E \to M \) are defined as the elements of the subalgebra \( \mathcal{D}(M, E) \) generated by \( \Gamma(\text{End} E) \) and by \( \{ \nabla^X : X \in \mathfrak{X}(M) \} \).

---

3In more sophisticated terms, the spinor bundles \( S \) and \( S' \) over a spin manifold are associated to the spin structure \( P \to M \) via the representations \( c \) and \( c' \) of \( \text{Spin}(2m) \); these vector bundles are inequivalent since the representations \( c \) and \( c' \) are inequivalent.

4We are indebted to William Ugalde for clarification on this point.

5In view of (5.5), two such connections differ by the action of an element of \( \Gamma(\text{End} E) \), so only a single connection is needed here.
A finite sum of operators of the form $A \nabla_{X_1} \ldots \nabla_{X_r}$, with $r \leq k$ (and $r = k$ for at least one summand) is called a differential operator of order $k$. By a partition-of-unity argument, a differential operator is of order $k$ if it can be written as such a sum in each chart domain separately. Thus a Dirac operator $\mathcal{D} = c(dx^j)\nabla_{\partial_j}$ is a differential operator of first order.

As an operator on the superspace $\Gamma(F)$, the Dirac operator $\mathcal{D}$ is odd, that is, $\mathcal{D}(\Gamma(F^\pm)) \subseteq \Gamma(F^\mp)$, since $\hat{c} : A^2_0(M) \otimes_A \Gamma(F^\pm) \to \Gamma(F^\mp)$. Thus we may write

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}, \quad (8.7)$$

where $\mathcal{D}_\pm : \Gamma(F^\pm) \to \Gamma(F^\mp)$. Its square is an even operator: $\mathcal{D}_\pm^2 : \Gamma(F^\pm) \to \Gamma(F^\pm)$; and

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_-^2 & 0 \\ 0 & \mathcal{D}_+^2 \end{pmatrix}. \quad (8.8)$$

Last but not least, a Dirac operator is essentially selfadjoint. This means that $\mathcal{D} : \Gamma(F) \to \Gamma(F)$ extends uniquely to a selfadjoint operator on the Hilbert space $L^2(F)$ obtained by completing the space of smooth sections $\Gamma(F)$ with respect to the inner product $\langle \langle \phi \mid \psi \rangle \rangle := \int_M (\phi \mid \psi) \Omega$. Regrettably, $\mathcal{D}$ is always an unbounded operator, so this selfadjoint extension is still only densely defined. Rather than worry about identifying the precise domain of the extended $\mathcal{D}$, we will stick to the original domain $\Gamma(F)$; a full proof of essential selfadjointness (see [39], for instance) shows that nothing is thereby lost, as the closure of $\mathcal{D}$ on this domain is the full selfadjoint extension.\footnote{In other words, $\Gamma(F)$ is a “core” for $\mathcal{D}$ [45]. The compactness of $M$ is not indispensable here: essential selfadjointness can be proven under the weaker assumption that the Riemannian manifold $M$ is complete [39, 53, 61].} We shall therefore show merely that $\mathcal{D}$ is formally selfadjoint, that is,

$$\langle \langle \mathcal{D} \phi \mid \psi \rangle \rangle = \langle \langle \phi \mid \mathcal{D} \psi \rangle \rangle \quad \text{for} \quad \phi, \psi \in \Gamma(F). \quad (8.8)$$

**Proposition 8.7.** The Dirac operator $\mathcal{D}$ is formally selfadjoint.

**Proof.** The argument is similar to that of Lemma 8.5. By invoking partitions of unity, we can reduce the problem to verifying (8.8) when $\phi, \psi$ are smooth sections of $F \to M$ which vanish outside some chart domain; thus we write $\mathcal{D} = c(\theta^\alpha)\nabla_{E^\alpha}$, where $\{\theta^1, \ldots, \theta^n\}$ and $\{E_1, \ldots, E_n\}$ are local orthonormal bases of 1-forms and vector fields, with $E^\alpha \equiv E_\alpha = (\theta^\alpha)^\dagger$. Now

$$\langle \mathcal{D} \phi \mid \psi \rangle = (\phi \mid c(\theta^\alpha)\nabla_{E^\alpha} \psi) = - (c(\theta^\alpha) \phi \mid \nabla_{E^\alpha} \psi)$$

$$= - E_\alpha (c(\theta^\alpha) \phi \mid \psi) + (\nabla_{E_\alpha} c(\theta^\alpha) \phi \mid \psi)$$

$$= - E_\alpha (c(\theta^\alpha) \phi \mid \psi) + (c(\nabla^L_{E_\alpha} \theta^\alpha) \phi \mid \psi) + (c(\theta^\alpha) \nabla_{E_\alpha} \phi \mid \psi)$$

$$= - E_\alpha (c(\theta^\alpha) \phi \mid \psi) + (c(\nabla^L_{E_\alpha} \theta^\alpha) \phi \mid \psi) + (\mathcal{D} \phi \mid \psi), \quad (8.9)$$

$$\langle \phi \mid \mathcal{D} \psi \rangle = (\phi \mid c(\theta^\alpha)\nabla_{E^\alpha} \psi) = - (c(\theta^\alpha) \phi \mid \nabla_{E^\alpha} \psi)$$

$$= - E_\alpha (c(\theta^\alpha) \phi \mid \psi) + (\nabla_{E_\alpha} c(\theta^\alpha) \phi \mid \psi)$$

$$= - E_\alpha (c(\theta^\alpha) \phi \mid \psi) + (c(\nabla^L_{E_\alpha} \theta^\alpha) \phi \mid \psi) + (c(\theta^\alpha) \nabla_{E_\alpha} \phi \mid \psi)$$

$$= - E_\alpha (c(\theta^\alpha) \phi \mid \psi) + (c(\nabla^L_{E_\alpha} \theta^\alpha) \phi \mid \psi) + (\mathcal{D} \phi \mid \psi), \quad (8.9)$$
where we have used the skewadjointness of \(c(\theta^a)\), the hermiticity of the connection \(\nabla\), and the compatibility of \(\nabla\) with the Levi-Civita connection on the cotangent bundle. We claim that we can find a vector field \(Z \in \mathfrak{X}(M)\), depending on \(\phi\) and \(\psi\), such that \(\text{div} Z = -E_{\alpha}(c(\theta^\alpha)\phi \mid \psi) + (c(\nabla_{E_{\alpha}}^{LC} \theta^\alpha)\phi \mid \psi)\). Then integration over \(M\) of both sides of (8.9) (multiplied by the Riemannian volume form \(\Omega\)) yields the desired relation (8.8).

The vector field \(Z\) is defined simply by

\[
\omega(Z) := (\phi \mid c(\omega)\psi) = -(c(\omega)\phi \mid \psi),
\]

since the right hand side is \(A\)-linear in \(\omega \in A^1(M)\). Now

\[
-E_{\alpha}(c(\theta^\alpha)\phi \mid \psi) + (c(\nabla_{E_{\alpha}}^{LC} \theta^\alpha)\phi \mid \psi) = E_{\alpha}(\theta^\alpha(Z)) - (\nabla_{E_{\alpha}}^{LC} \theta^\alpha)(Z) = \theta^\alpha(\nabla_{E_{\alpha}}^{LC} Z)
\]

from the definition of a dual connection. The divergence formula (8.5) says that the right hand side equals \(\text{div} Z\), as claimed.

\[\square\]

### 8.5 Laplacians

**Definition 8.5.** Let \(\nabla^E: \Gamma(E) \to A^1(M, E) = \Gamma(T^*M \otimes E)\) be a connection on a vector bundle \(E \to M\) over a Riemannian manifold \((M, g)\), and let \(\nabla^E := \nabla^{LC} \otimes \nabla^E\) be its tensor product with the Levi-Civita connection on the cotangent bundle of \(M\). Then \(\nabla^E\) maps \(\Gamma(T^*M \otimes E)\) to \(A^1(M, T^*M \otimes E) = \Gamma(T^*M \otimes T^*M \otimes E)\), so it can be composed with \(\nabla^E\). Contraction with the metric \(g^{-1}\) on \(T^*M\) gives an \(A\)-linear map \(\text{Tr}_s: \Gamma(T^*M \otimes T^*M \otimes E) \to \Gamma(E)\). The composition of these three maps yields the following operator on \(\Gamma(E)\):

\[
\Delta^E := -\text{Tr}_s \circ \nabla^E \circ \nabla^E,
\]

called the **Laplacian** associated to the connection \(\nabla^E\). The minus sign is a convention\(^7\) which assures that \(\Delta^E\) is a positive operator whenever \(\nabla^E\) is a Hermitian connection.

To express \(\Delta^E\) in a more tractable form, we shall compute the result of contracting the operator \(\nabla^E \circ \nabla^E\) with two vector fields \(X, Y \in \mathfrak{X}(M)\). If \(s \in \Gamma(E)\), we can write \(\nabla^E s = \beta^k \otimes s_k\), and so

\[
\iota_Y \iota_X (\nabla^E \nabla^E s) = \iota_Y \nabla^E_X (\beta^k \otimes s_k) = \iota_Y (\beta^k \otimes \nabla^E_X s_k + \nabla_X \beta^k \otimes s_k)
\]

\[
= \beta^k (Y) \nabla^E_X s_k + (\nabla_X \beta^k)(Y) s_k
\]

\[
= \nabla^E_X (\beta^k(Y)s_k) - \beta^k(\nabla_X Y)s_k = \nabla^E_X (\nabla^E_Y s) - \iota(\nabla_X Y) \nabla^E s
\]

where we have written the Levi-Civita connection simply as \(\nabla\). In other words,

\[
\iota_Y \iota_X (\nabla^E \circ \nabla^E) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X.
\]

\(^7\)Actually, the usual convention is to use the opposite sign; however, this results in operators, such as \(\sum_j \partial_j^2\) on \(\mathbb{R}^n\), which are negative definite.
Since $g^{-1} = g^{ij} \partial_i \cdot \partial^j$ on a chart domain, we get immediately from (7.12) the local expression for the Laplacian:

$$\Delta^E = -g^{ij}(\nabla^E_{\partial_i} \nabla^E_{\partial_j} - \Gamma^k_{ij} \nabla^E_{\partial_k}).$$

(8.10)

With a local orthonormal basis of vector fields $\{e_1, \ldots, e_n\}$, this takes a slightly simpler form:\footnote{We rename the vector fields $E_\alpha$ to $e_\alpha$ temporarily to avoid a notational clash with the exponent $E$ denoting a vector bundle.}

$$\Delta^E = -\sum_{\alpha=1}^n (\nabla^E_{e_\alpha} \nabla^E_{e_\alpha} - \nabla^E_{\nabla^E_{e_\alpha} e_\alpha}).$$

(8.11)

It is immediate from (8.10) or (8.11) that the Laplacian $\Delta^E$ is a second-order differential operator.

Exercise 8.4. Use the formulas of Exercise 7.11 to give an alternative derivation of (8.10).

Exercise 8.5. Show that the Laplacian $\Delta_0$ associated to the standard connection $d$ on the trivial line bundle $\mathbb{S}^2 \times \mathbb{C} \longrightarrow \mathbb{S}^2$ is

$$\Delta_0 = -\left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \phi^2}\right),$$

i.e., the Laplace–Beltrami operator on $A^0(\mathbb{S}^2)$.

Proposition 8.8. Let $\nabla^E$ be a Hermitian connection on a vector bundle $E$ over a Riemannian manifold $(M, g)$. Then its Laplacian is a formally selfadjoint and positive operator on $\Gamma(E)$.

Proof. The strategy of the proof should by now be familiar: we choose sections $s, t \in \Gamma(E)$ and with them construct a vector field $Z \in \mathfrak{X}(M)$ such that $(s \mid \Delta^E t) = \text{div } Z + \text{Tr}(\nabla^E s \mid \nabla^E t)$. (The last term is locally expressed as $\sum_{\alpha=1}^n (\nabla^E_{e_\alpha} s \mid \nabla^E_{e_\alpha} t)$, where $\{e_1, \ldots, e_n\}$ are orthonormal vector fields.) Multiplying these functions by $\Omega$ and integrating over $M$, we get the relation

$$\langle s \mid \Delta^E t \rangle = \langle \nabla^E s \mid \nabla^E t \rangle,$$

(8.12)

where these brackets denote integrated inner products on $\Gamma(E)$ and $A^1(M, E)$ respectively.\footnote{On account of (8.12), the Laplacian is often denoted $\nabla^* \nabla$, as in [39], for instance.}

Positivity follows from setting $s = t$, and formal selfadjointness follows by repeating the argument with the rôles of $s$ and $t$ interchanged. The vector field $Z$ is defined by

$$g(Z, X) := -(s \mid \nabla^E_X t) \quad \text{for} \quad X \in \mathfrak{X}(M).$$

Its divergence may be computed with the formula (8.5):

$$\text{div } Z = \theta^\alpha (\nabla^E_{e_\alpha} Z) = g(\nabla^E_{e_\alpha} Z, e_\alpha) = -g(Z, \nabla^E_{e_\alpha} e_\alpha) + e_\alpha g(Z, e_\alpha) = (s \mid \nabla^E_{\nabla^E_{e_\alpha} e_\alpha} t) - e_\alpha (s \mid \nabla^E_{e_\alpha} t) = (s \mid \Delta^E t) - (\nabla^E_{e_\alpha} s \mid \nabla^E_{e_\alpha} t),$$

(with summation over $\alpha$), so $\text{div } Z = (s \mid \Delta^E t) - \text{Tr}(\nabla^E s \mid \nabla^E t)$ as claimed. \qed
We already know a second-order differential operator on the bundle $\Lambda^\bullet T^\ast M \to M$, namely the “Hodge Laplacian”, which we now write $\Delta_{\text{Hodge}}$. It is natural to ask whether this operator equals the Laplacian $\Delta_{\text{LC}}$ associated to the Levi-Civita connection on the exterior bundle. It turns out that they are not the same: indeed, they differ by a differential operator of order zero, that is essentially the Ricci tensor of Riemannian geometry [8, 36, 39].

**Definition 8.6.** The **Ricci tensor** on a Riemannian manifold is the symmetric tensor $\text{Ric}$ of bidegree $(2, 0)$ obtained from the Riemann curvature tensor $R$ by defining $\text{Ric}(X, Y) = \text{tr} (X \mapsto R(Y, X) X)$. In terms of the components of the Riemann curvature tensor, we get $\text{Ric}(\partial_i, \partial_j) = \sum_k R^k_{ikj}$. By contracting with the metric, we obtain the **curvature scalar** $K := \text{tr} g \circ \text{Ric}$ of the Riemannian manifold $(M, g)$; notice that $K \in C^\infty(M)$. Locally, $K = g^{ij} R^k_{ikj}$.

**Exercise 8.6.** Show that the Riemannian curvature tensor on the sphere $S^2$ satisfies $R_{\theta \phi \theta \phi} = 1$ and $R^\theta_{\theta \phi \phi} = \sin^2 \theta$, and deduce that $\text{Ric} = g$: the Ricci tensor coincides with the metric on $S^2$. Conclude that $K = 2$ on $S^2$: the sphere is a surface of constant (Gaussian) curvature.

We can identify $\text{Ric}$ with a tensor of bidegree $(0, 2)$ on $M$, also called $\text{Ric}$, via the metric $g$; or better, with the element of $\Gamma(\text{End} T^\ast M)$ defined by $\text{Ric}(\omega) := \text{Ric}(\omega^\sharp, X)$. The relation between the Hodge Laplacian and the connection Laplacian, as operators on $\mathcal{A}^\bullet(M)$, is then given by the so-called *Weitzenböck formula*:

$$\Delta_{\text{Hodge}} = \Delta_{\text{LC}} + \text{Ric}. \quad (8.13)$$

We shall not prove this here, though we have the means to do so; consult [9] or [39] for the details.

Two things are notable about the formula (8.13). First of all, $\Delta_{\text{Hodge}} = (d + \delta)^2$ is the square of the Dirac operator $d + \delta$ on the Clifford module $\mathcal{A}^\bullet(M)$, so the formula says that, for the de Rham complex at least, the square of the Dirac operator is almost, but not quite, a Laplacian; more precisely, it differs from the Laplacian by a differential operator of lower order that, by the way, depends only on the curvature of the connection which defines both the Dirac operator and the Laplacian. This is one of a family of formulae due variously to Bochner, Weitzenböck and Lichnerowicz, which say that $\mathcal{D}^2 - \Delta$ is a curvature-dependent multiplication operator.

The second noteworthy feature is that $\Delta_{\text{Hodge}}$ and $\Delta_{\text{LC}}$ are positive (formally) selfadjoint operators. What happens if the Ricci operator is positive too? For one thing, both $\Delta_{\text{LC}}$ and $\text{Ric}$ must then vanish on the kernel of the Hodge Laplacian, namely, on the harmonic forms. Thus, for example, on $S^2$ the Ricci operator kills 0-forms and 2-forms but acts as the identity on 1-forms since the Ricci tensor coincides with the metric; the upshot of (8.13) is then that any harmonic 1-form on $S^2$ must be zero (as we already noted in Section 4), and therefore $H^1_{\text{dr}}(S^2) = 0$. This result is an example of a *vanishing theorem*, whereby certain cohomology groups reduce to zero — a topological result — on account of positivity properties of certain analytic operators.

---

10The curvature scalar is also called the *Gaussian curvature* when $\dim M = 2$. 100
8.6 The Lichnerowicz formula

**Proposition 8.9.** Let $M$ be a compact spin manifold, let $\mathcal{D}^S = \hat{c} \circ \nabla^S$ denote the Dirac operator on the irreducible spinor module $\Gamma(S)$, and let $\Delta^S$ be the Laplacian associated to the spin connection $\nabla^S$. Then

$$(\mathcal{D}^S)^2 = \Delta^S + \frac{K}{4},$$

where $K$ is the curvature scalar of $M$.

**Proof.** It suffices to prove this on any chart domain; that is, we must show that, in a local coordinate basis,

$$(c(dx^i) \nabla^S_{\partial_i})^2 = -g^{ij}(\nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij} \nabla^S_{\partial_k}) + \frac{1}{4} K.$$

The left hand side is

$$c(dx^i) \nabla^S_{\partial_i} c(dx^j) \nabla^S_{\partial_j} = c(dx^i) c(dx^j) (\nabla^S_{\partial_i} \nabla^S_{\partial_j} + c(dx^i) c(\nabla^S_{\partial_i} dx^k) \nabla^S_{\partial_k}$$

$$= c(dx^i) c(dx^j) (\nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij} \nabla^S_{\partial_k})$$

$$= \frac{1}{2} [c(dx^i), c(dx^j)](\nabla^S_{\partial_i} \nabla^S_{\partial_j} - \Gamma^k_{ij} \nabla^S_{\partial_k}) + \frac{1}{2} c(dx^i) c(dx^j)[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}],$$

where we have used the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ due to the zero torsion $\nabla_{\partial_i} \partial_j = \nabla_{\partial_j} \partial_i$ of the Levi-Civita connection. Now $[c(dx^i), c(dx^j)] = -2g^{ij}$ — recall (7.17) — and $[\nabla^S_{\partial_i}, \nabla^S_{\partial_j}] = \omega^S(\partial_k, \partial_l) = -\frac{1}{4} R_{ijkl} c(dx^i) c(dx^j)$ by (7.15), so by taking (8.10) into account we arrive at

$$(\mathcal{D}^S)^2 = \Delta^S - \frac{1}{4} R_{ijkl} c(dx^k) c(dx^l) c(dx^i) c(dx^j).$$

It remains only to check that the second term on the right reduces to $\frac{1}{4} K$. We may rewrite it as $\frac{1}{8} R_{ijkl} c(dx^k) c(dx^l) c(dx^i) c(dx^j)$ since $R_{ijkl}$ is antisymmetric in the indices $i, j$ by (5.29). Since the antisymmetrization of $R_{ijkl}$ in the indices $i, k, l$ vanishes by (5.28), it follows from Exercise 8.7 below that this term reduces to

$$\frac{1}{8} R_{ijkl} (-g^{kl} c(dx^k) c(dx^l) + g^{il} c(dx^k) c(dx^i) - g^{ik} c(dx^l) c(dx^j)),$$

and since $R_{ijkl} g^{kl} = 0$ by antisymmetry of $R_{ijkl}$ in $k, l$, again by (5.29), this in turn reduces to

$$\frac{1}{4} R_{ijkl} g^{kl} c(dx^i) c(dx^j) = \frac{1}{4} R^k_{ijkl} c(dx^i) c(dx^j) = \frac{1}{4} g^{ij} R^k_{ijkl} = \frac{1}{4} K,$$

due to the symmetry $R^k_{ijkl} := R^k_{ijk}$ of the Ricci tensor. \hfill \Box

**Exercise 8.7.** If $u, v, w$ are three vectors in a Euclidean vector space $(V, q)$, denote their antisymmetrized product by $a := \frac{1}{6}(uvw + wvu + wuv - uvw - vwu - wuv) \in \text{Cl}(V, q)$. Show that $uvw = a - q(v, w) u + q(u, w) v - q(u, v) w$.

The identity (8.14) is due to Lichnerowicz [40]. The proof generalizes in a straightforward manner [9, 27, 39] to the case of a twisted Clifford module. The only differences consist in replacing the Laplacian $\Delta^S$ by $\Delta^F$, where $F = W \otimes S$, and the curvature term $\frac{1}{2} c(dx^i) c(dx^j) \omega^S(\partial_i, \partial_j)$ in (8.15) by $\frac{1}{2} c(dx^i) c(dx^j) \omega^F(\partial_i, \partial_j)$. Now by Proposition 7.9, the
curvature $\omega^F$ can be decomposed as $\omega^F = \omega^W + \omega^S$, where $\omega^W$ is the curvature of some compatible connection on $W$. This yields an extra term of the form $\frac{1}{2} c(dx^i)c(dx^j)\omega^W(\partial_i, \partial_j)$, which, in view of (6.10), we may write as $Q(\omega^W)$, where $Q: \mathcal{A}^\bullet(M, W) \rightarrow \mathcal{C}(\ell(M))$ is the extension of the quantization map of Definition 6.12 to the (trivial) Clifford module $\mathcal{A}^\bullet(M, W)$. The result is a generalization of (8.14) known as the Bochner–Weitzenböck formula:

$$(\mathcal{D}^F)^2 = \Delta^F + \frac{1}{4} K + Q(\omega^W).$$

Exercise 8.8. When $F = \Lambda^\bullet T^* M \simeq S \otimes S$, verify (8.13) by showing that $\text{Ric} - \frac{1}{4} K = Q(\omega_0)$ for a suitable $\omega_0 \in \mathcal{A}^2(M, S)$.

9 The Dirac operator on the Riemann sphere

This section is devoted to a detailed exploration of a single but fundamental example: the Dirac operator on the irreducible spinor module over the sphere $S^2$. While the sphere is undoubtedly the simplest possible even-dimensional compact spin manifold, its Dirac operator exemplifies the full complexity of the general case while remaining directly accessible by elementary computations. We give here an account of its action on spinors, show its equivariance under the Lie group $SU(2)$ of symmetries of the spinor module, compute its spectrum and exhibit a full set of eigenspinors.

Surprisingly, such an account is not to be found in the current literature on spinors, so this exposition breaks some new ground. The ingredients have been available for a long time, and there is no reason why this story could not have been told thirty years ago. Indeed, before the geometrical theory of Dirac operators was developed at all, the eigenspinors for the Dirac operator on the sphere were considered (in 1938) by Schrödinger [47], who put his finger on the basic module property (7.10) of the spin connection (albeit not in so many words). A generation later, Newman and Penrose [43] introduced a family of functions on the sphere that they called “spinor harmonics”, which generalize the ordinary spherical harmonics and constitute the eigenspinors, as we shall see.

In order to keep the development of the Dirac operator on $S^2$ fairly self-contained, we begin by reviewing some elementary notions. Everything can be done with elementary calculus, provided one takes great care to get the signs and the constants right from the beginning.

9.1 Coordinates on the Riemann sphere

The most direct way to reveal the Dirac operator on the sphere is to regard it as the Riemann sphere $\mathbb{CP}^1$, and use complex coordinates. Recall that $\mathbb{CP}^1$ may be described by homogeneous coordinates $[z^0: z^1]$ with $z^0, z^1$ not both zero, and is covered by two chart domains $U_0 := \{[z^0: z^1] : z^0 \neq 0\}$ and $U_1 := \{[z^0: z^1] : z^1 \neq 0\}$. We shall use the local complex coordinates $z := z^1/z^0$ on $U_0$, $\zeta := z^0/z^1$ on $U_1$.

The coordinate change is just $z = \zeta^{-1}$, $\zeta = z^{-1}$ on $U_0 \cap U_1$. 

102
We identify $\mathbb{CP}^1$ with the sphere $S^2$ of unit vectors $(u^1, u^2, u^3) \in \mathbb{R}^3$ by the stereographic projections:

$$z = \frac{u^1 + iu^2}{1 - u^3} \quad \text{on} \quad S^2 \setminus \{N\}, \quad \zeta = \frac{u^1 - iu^2}{1 + u^3} \quad \text{on} \quad S^2 \setminus \{S\}$$

where $N = (0, 0, 1)$ and $S = (0, 0, -1)$ are the north and south poles. This identifies $U_0$ with $S^2 \setminus \{N\}$ and $U_1$ with $S^2 \setminus \{S\}$. It is important to notice that both stereographic projections are orientation reversing.

We may also use the standard spherical coordinates $(\theta, \phi)$, satisfying $u^1 = \sin \theta \cos \phi$, $u^2 = \sin \theta \sin \phi$, $u^3 = \cos \theta$. The stereographic projections are then expressed as

$$z = e^{i\phi} \cot \frac{\theta}{2} = e^{i\phi} \frac{\sin \theta}{1 - \cos \theta} = e^{i\phi} \frac{1 + \cos \theta}{\sin \theta},$$

$$\zeta = e^{-i\phi} \tan \frac{\theta}{2} = e^{-i\phi} \frac{\sin \theta}{1 + \cos \theta} = e^{-i\phi} \frac{1 - \cos \theta}{\sin \theta}.$$  \hfill (9.1)

The positive functions (2.6) on $U_0$ and $U_1$ are given by

$$Q_0(z) := 1 + z\bar{z} = \frac{2}{1 - \cos \theta}, \quad Q_1(\zeta) := 1 + \zeta\bar{\zeta} = \frac{2}{1 + \cos \theta}. \hfill (9.2)$$

Observe that $Q_0(z)/Q_1(z^{-1}) = z\bar{z}$.

The local 1-forms $dz$, $d\bar{z}$ on $U_0$ and $d\zeta$, $d\bar{\zeta}$ on $U_1$ are given —recall (2.14)— by

$$dz = \frac{e^{i\phi}}{1 - \cos \theta} (-d\theta + i \sin \theta \, d\phi), \quad d\zeta = \frac{e^{-i\phi}}{1 + \cos \theta} (d\theta - i \sin \theta \, d\phi),$$

and their complex conjugates; notice that $d\zeta = -e^{-2i\phi} Q_1/Q_0 \, dz$ on $U_0 \cap U_1$. The basic local vector fields are given by

$$\frac{\partial}{\partial z} = e^{-i\phi} \frac{1 - \cos \theta}{2} \left( -\frac{\partial}{\partial \theta} - \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right), \quad \frac{\partial}{\partial \zeta} = e^{i\phi} \frac{1 + \cos \theta}{2} \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right), \hfill (9.3)$$

and their complex conjugates; now $\partial/\partial \zeta = -e^{+2i\phi} Q_0/Q_1 \, \partial/\partial z$ on $U_0 \cap U_1$.

The Riemannian metric $g$ on $S^2$ is the usual one:

$$g = d\theta^2 + \sin^2 \theta \, d\phi^2 = 4dz \cdot d\bar{z}(1 + z\bar{z})^2 = 4d\zeta \cdot d\bar{\zeta}(1 + \zeta\bar{\zeta})^2 \hfill (9.4)$$

(although (2.13) differs from this by a factor of two). The Riemannian volume form is $\Omega = \sin \theta \, d\theta \wedge d\phi = -2i \, Q_0^{-2} \, dz \wedge d\bar{z} = -2i \, Q_1^{-2} \, d\zeta \wedge d\bar{\zeta}$, where the minus sign [compare with (2.12)] indicates the reversal of orientation in passing from $(\theta, \phi)$ coordinates to $(z, \bar{z})$ or $(\zeta, \bar{\zeta})$ coordinates.
9.2 Sections and gauge transformations

A section of a complex line bundle \( L \to \mathbb{S}^2 \) is given by a pair of local sections over \( U_0, U_1 \) respectively. Once we have chosen basic sections \( s_0 \in \Gamma(U_0, L) \) and \( s_1 \in \Gamma(U_1, L) \) which are nonvanishing, any global section \( s \in \Gamma(L) \) is determined by a pair of smooth functions \( f_0, f_1 \) on \( \mathbb{C} \) such that

\[
s(x) = f_0(z) s_0(x) \quad \text{for } x \in U_0, \quad s(x) = f_1(\zeta) s_1(x) \quad \text{for } x \in U_1,
\]

where \( z, \zeta \) are the respective coordinates of the point \( x \in \mathbb{S}^2 \). The basic local sections are related by the transition function of the line bundle \( s_0 = g_{01}s_1 \) (since the sphere is covered by only two charts, one transition function suffices), which implies a corresponding relation between \( f_0(z) \) and \( f_1(\zeta) \), called a gauge transformation. In fine: a global section is determined by a related pair of functions, and the line bundle is identified by the particular gauge transformation relating them.

We start with a brief mention of the holomorphic line bundles. The tautological line bundle \( L \to \mathbb{CP}^1 \) is defined, as in (5.12), by the fibres \( L_x := \{ (\lambda z^0, \lambda z^1) \in \mathbb{C}^2 : \lambda \in \mathbb{C} \} \) for \( x = [z^0 : z^1] \); its basic sections are defined as \( \tilde{s}_0(x) := (1, z) \) for \( x \in U_0 \), \( \tilde{s}_1(x) := (\zeta, 1) \) for \( x \in U_1 \). This yields \( \tilde{s}_1(x) = z^{-1} \tilde{s}_0(x) \) for \( x \in U_0 \cap U_1 \); the transition functions are therefore \( \tilde{g}_{01}(z) = z^{-1}, \tilde{g}_{01}(\zeta) = \zeta^{-1} \), which of course are holomorphic on \( U_0 \cap U_1 \).

The hermitian metric on \( L \) is obtained from the natural inclusion of the fibres in \( \mathbb{C}^2 \); this means that \( \langle \tilde{s}_0 | \tilde{s}_0 \rangle = \|(1, z)\|^2 = 1 + z\bar{z} = Q_0(z) \) and \( \langle \tilde{s}_1 | \tilde{s}_1 \rangle = ||(\zeta, 1)||^2 = 1 + \zeta\bar{\zeta} = Q_1(\zeta) \). Its dual, the hyperplane bundle \( H \to \mathbb{CP}^1 \) has local sections \( \tilde{\sigma}_0, \tilde{\sigma}_1 \) with \( \langle \tilde{\sigma}_0 | \tilde{\sigma}_0 \rangle = Q_0^{-1} \) on \( U_0 \), \( \langle \tilde{\sigma}_1 | \tilde{\sigma}_1 \rangle = Q_1^{-1} \) on \( U_1 \) — recall (5.13) — and so they extend smoothly to global sections on \( \mathbb{S}^2 \) just by setting \( \tilde{\sigma}_0(N) := 0, \tilde{\sigma}_1(S) := 0 \). The (holomorphic) transition functions are \( \tilde{g}_{01}(\zeta) = \zeta \) and \( \tilde{g}_{01}(\zeta) = z \). A global holomorphic section \( \tilde{\sigma} \in \mathcal{O}(\mathbb{CP}^1, H) \) is given by a pair of holomorphic functions \( f_0, f_1 \) with \( \tilde{\sigma}(x) = f_0(z) \tilde{\sigma}_0(x) \) and \( \tilde{\sigma}(x) = f_1(\zeta) \tilde{\sigma}_1(x) \) for all \( x \). That means that \( f_0, f_1 \) are entire functions whose possible singularities at infinity can only be poles, and \( f_0(z) = z f_1(z^{-1}) \) for \( z \in \mathbb{C}^\times \); this relation identifies a Taylor series and a Laurent series, and can only hold if \( f_0, f_1 \) are of the form \( f_0(z) = a + bz, f_1(\zeta) = b + a\zeta \) for some \( a, b \in \mathbb{C} \); we have once again established that \( \mathcal{O}(\mathbb{CP}^1, H) \simeq \mathbb{C}^2 \).

From now on, we shall normalize all basic sections and thus use only \( U(1) \)-valued transition functions. Thus we replace the basic sections of \( L \) by

\[
s_0 := Q_0(z)^{-1/2} \tilde{s}_0, \quad s_1 := Q_1(\zeta)^{-1/2} \tilde{s}_1,
\]

and likewise \( \sigma_0 := Q_0(z)^{1/2} \tilde{\sigma}_0, \sigma_1 := Q_0(\zeta)^{1/2} \tilde{\sigma}_1 \). This gives

\[
s_1(x) = z^{-1} \sqrt{\frac{Q_0(z)}{Q_1(z^{-1})}} s_0(x) = \frac{z \bar{z}}{z} s_0(x) = (\bar{z}/z)^{1/2} s_0(x),
\]

so the transition function is \( g_{01}(z) = (\bar{z}/z)^{1/2} = e^{-i\phi} \). A section \( s : \mathbb{CP}^1 \to L \) is then given by a pair of functions \( (f_0, f_1) \) such that \( f_0(z)s_0(x) \equiv f_1(\zeta)s_1(x) \) on \( U_0 \cap U_1 \), i.e., such that

\[
f_0(z) \equiv (\bar{z}/z)^{1/2} f_1(z^{-1}), \quad f_1(\zeta) \equiv (\bar{\zeta}/\zeta)^{1/2} f_0(\zeta^{-1}), \quad (9.5)
\]

104
where the second equation is of course redundant. The formula (9.5) exhibits the $U(1)$ gauge transformation of the tautological line bundle.

It is clear from (9.5) that $f_0$ and $f_1$ cannot both be holomorphic, and in general neither is holomorphic. Therefore, it would be more correct to write $f_0(z, \bar{z})$ and $f_1(\zeta, \bar{\zeta})$ to signal the dependence of these smooth functions on both real coordinates. We shall do so whenever the need arises.

For the hyperplane bundle $H \to \mathbb{CP}^1$, the very same argument shows that a global section is given by a pair of functions $(h_0, h_1)$ such that $h_0(z)\sigma_0(x) \equiv h_1(\zeta)\sigma_1(x)$ on $U_0 \cap U_1$, and which are therefore related by the gauge transformation

$$h_0(z) \equiv (z/\bar{z})^{1/2}h_1(z^{-1}), \quad h_1(\zeta) \equiv (\zeta/\bar{\zeta})^{1/2}h_0(\zeta^{-1}). \quad (9.6)$$

**Exercise 9.1.** Show that for any Hermitian line bundle $E \to \mathbb{CP}^1$, the sections in $\Gamma(E)$ are described by pairs of functions $(f_0, f_1)$ satisfying the relation $f_0(z) \equiv (z/\bar{z})^{k/2}f_1(z^{-1})$, where $k \in \mathbb{Z}$ and $k[H]$ is the Chern class of $E$.

We may decompose the complexified tangent bundle as $T_\mathbb{C}\mathbb{S}^2 = T^{1,0}\mathbb{S}^2 \oplus T^{0,1}\mathbb{S}^2$, where the sections of the holomorphic tangent bundle $T^{1,0}\mathbb{S}^2$ are locally of the form $f_0(z, \bar{z}) \partial/\partial z$ or $f_1(\zeta, \bar{\zeta}) \partial/\partial \zeta$. [The sections of the "antiholomorphic tangent bundle" $T^{0,1}\mathbb{S}^2$ are of the form $h_0(z, \bar{z}) \partial/\partial \bar{z}$ or $h_1(\zeta, \bar{\zeta}) \partial/\partial \bar{\zeta}$.] Then $T^{1,0}\mathbb{S}^2 \to \mathbb{S}^2$ is a Hermitian line bundle, under the metric determined by

$$\langle \partial/\partial z | \partial/\partial z \rangle := g(\partial/\partial \bar{z}, \partial/\partial z) = \frac{4}{(1 + z\bar{z})^2}.$$

As normalized local sections we take

$$E_z := \frac{1}{2}Q_0(z) \frac{\partial}{\partial z} \quad \text{over } U_0, \quad -E_\zeta := -\frac{1}{2}Q_1(\zeta) \frac{\partial}{\partial \zeta} \quad \text{over } U_1.$$

Since $z = \zeta^{-1}$ gives $\partial/\partial z = -\zeta^2 \partial/\partial \zeta$, we find that $E_z = -\zeta^2 Q_0(z)/Q_1(\zeta) E_\zeta = (\bar{z}/z)(-E_\zeta)$. Thus a section of the holomorphic tangent bundle is given by a pair of functions $f_0, \ f_1$ satisfying $f_0(z)E_z \equiv f_1(\zeta)(-E_\zeta)$ on $U_0 \cup U_1$, or equivalently

$$f_0(z) \equiv (z/\bar{z}) f_1(z^{-1}).$$

This establishes that $[T^{1,0}\mathbb{S}^2] = 2[H]$ in $\tilde{H}(\mathbb{S}^2, \mathbb{Z})$, i.e., the line bundles $T^{1,0}\mathbb{S}^2$ and $H \otimes H$ are equivalent.

**Exercise 9.2.** Conclude from $\bar{E}_z = (z/\bar{z})(-\bar{E}_\zeta)$ that the complex line bundle $T^{0,1}\mathbb{S}^2$ is equivalent to $L \otimes L$.

**Exercise 9.3.** Write $T_\mathbb{C}\mathbb{S}^2 = \Lambda^{1,0}T^*\mathbb{S}^2 \oplus \Lambda^{0,1}T^*\mathbb{S}^2$, where $\mathcal{A}^{1,0}(\mathbb{S}^2) = \Gamma(\Lambda^{1,0}T^*\mathbb{S}^2)$ has elements $f_0(z, \bar{z}) \, dz = f_1(\zeta, \bar{\zeta}) \, d\zeta$ and $\mathcal{A}^{0,1}(\mathbb{S}^2) = \Gamma(\Lambda^{0,1}T^*\mathbb{S}^2)$ consists of all $h_0(z, \bar{z}) \, d\bar{z} = h_1(\zeta, \bar{\zeta}) \, d\bar{\zeta}$. Use $g^{-1}$ to define Hermitian metrics on both these line bundles, write down suitable normalized local sections, and verify that $\Lambda^{1,0}T^*\mathbb{S}^2 \sim L \otimes L$ and $\Lambda^{0,1}T^*\mathbb{S}^2 \sim H \otimes H$. 

105
**Definition 9.1.** Let $S \rightarrow S^2$ denote the irreducible spinor bundle. Since $T^*\mathcal{S}^2 \sim S \otimes S^*$, we expect that $S \sim L \oplus H$, and that $S^+ \sim L$, $S^- \sim H$. Due to the noncanonical nature of the spinor bundle, we bypass the construction of basic local sections and define a spinor $\psi$ over $S^2$ directly as a pair of functions on each chart, denoted $\psi^+(z, \bar{z})$ and $\psi^-(\zeta, \bar{\zeta})$ respectively, satisfying the gauge transformation rules:

\[
\psi^+_N(z, \bar{z}) \equiv (z/\bar{z})^{1/2}\psi^+_S(z^{-1}, \bar{z}^{-1}), \quad \psi^-_N(z, \bar{z}) \equiv (z/\bar{z})^{1/2}\psi^-_S(z^{-1}, \bar{z}^{-1}),
\]

\[
\psi^+_S(\zeta, \bar{\zeta}) \equiv (\zeta/\bar{\zeta})^{1/2}\psi^+_S(\zeta^{-1}, \bar{\zeta}^{-1}), \quad \psi^-_S(\zeta, \bar{\zeta}) \equiv (\zeta/\bar{\zeta})^{1/2}\psi^-_S(\zeta^{-1}, \bar{\zeta}^{-1}). \tag{9.7}
\]

It is immediate from (9.5) and (9.6) that $(\psi^+_N, \psi^+_S)$ determines a section of $L$ and $(\psi^-_N, \psi^-_S)$ determines a section of $H$.

### 9.3 The spin connection over the sphere

**Lemma 9.1.** The Levi-Civita connection on the sphere is determined by the local formulae

\[
\nabla_{\partial_i} \partial_j = -\frac{2}{1 + x_1^2 + x_2^2}(x_i \partial_j + x_j \partial_i - \delta_{ij} x_k \partial_k), \tag{9.8}
\]

with $x_1 \equiv x^1$, $x_2 \equiv x^2$, where $x^1 + ix^2 = z$ on $U_0$ and $x^1 + ix^2 = \zeta$ on $U_1$. 

**Proof.** The metric (9.4) is given by $g = 4(1 + x_1^2 + x_2^2)^{-2}(dx_1^2 + dx_2^2)$ on both charts, so $g_{ij} = 4(1 + x_1^2 + x_2^2)^{-2}\delta_{ij}$ and $g^{ij} = \frac{1}{4}(1 + x_1^2 + x_2^2)^2\delta^{ij}$. Also, $\partial_k g_{ij} = -16x_k(1 + x_1^2 + x_2^2)^{-3}\delta_{ij}$.

From (7.13) we get

\[
\Gamma^k_{ij} = -\frac{2}{1 + x_1^2 + x_2^2}(x_i \delta^k_j + x_j \delta^k_i - x^k \delta_{ij}),
\]

from which (9.8) follows. \qed

Local orthonormal bases of vector fields (on both charts) are given by $E_1 := \frac{1}{2}(1 + x_1^2 + x_2^2)\partial/\partial x^1$, $E_2 := \frac{1}{2}(1 + x_1^2 + x_2^2)\partial/\partial x^2$. With these bases, we get

\[
\nabla_{\partial_\alpha} E_\alpha = x_\alpha \partial_\alpha + \frac{1}{2}(1 + x_1^2 + x_2^2)\nabla_{\partial_\alpha} \partial_\alpha = \frac{2}{1 + x_1^2 + x_2^2}(\delta_{i\alpha} x^\beta E_\beta - x_\alpha E_i),
\]

or equivalently,

\[
\tilde{\Gamma}^\beta_{i\alpha} = \frac{2}{1 + x_1^2 + x_2^2}(\delta_{i\alpha} x^\beta - \delta^\beta_i x_\alpha).
\]

The spin connection components are given by (7.18) as $\omega_i = \frac{1}{4}\tilde{\Gamma}^\beta_{i\alpha} \gamma^\alpha \gamma_\beta$, which yields

\[
\omega_1 = \frac{1}{2(1 + x_1^2 + x_2^2)}(x^\beta \gamma^\alpha \gamma_\beta - x_\alpha \gamma^\alpha \gamma_1) = \frac{x_2}{1 + x_1^2 + x_2^2} \gamma^1 \gamma_2,
\]

\[
\omega_2 = \frac{1}{2(1 + x_1^2 + x_2^2)}(x^\beta \gamma^\alpha \gamma_\beta - x_\alpha \gamma^\alpha \gamma_2) = -\frac{x_1}{1 + x_1^2 + x_2^2} \gamma^1 \gamma_2, \tag{9.9}
\]

\footnote{We label the local spinor coefficients $N$ and $S$ (north and south) in order to reduce the clutter of numerical indices.}
To return to complex notation, we notice that \((\omega_1 + i\omega_2)(z) = -\frac{i\zeta}{1 + \zeta}\gamma^1\gamma_2\) over \(U_0\) while \((\omega_1 + i\omega_2)(\zeta) = -\frac{i\zeta}{1 + \zeta}\gamma^1\gamma_2\) over \(U_1\). These are related by a gauge transformation: \((\omega_1 + i\omega_2)(z) = (z/\bar{z}) (\omega_1 + i\omega_2)(\zeta)\).

9.4 The Dirac operator over the sphere

**Definition 9.2.** We fix a local Clifford action on the spinor module by choosing a particular matrix function \(\tilde{H} := [\tilde{h}_{ij}(z)]\) such that \(\tilde{H}^I \tilde{H} = G^{-1} \equiv \{g^{ij}(z)\} = \frac{1}{2}(1 + z\bar{z})^2 I\) over \(U_0\). Let us (arbitrarily) choose the positive square root \(\tilde{H} := G^{-1/2} = \frac{1}{2}(1 + z\bar{z}) I\). The **Dirac operator** on the chart \(U_0\) is then defined as

\[
\mathcal{D}_N := c(dx^i) \nabla_{\partial_j} = \tilde{h}_{ij}(z) \gamma^j (\partial_j + \omega_j(z)) \\
= \frac{1}{2}(1 + z\bar{z}) (\gamma^j \partial_j + \gamma^j \omega_j(z)) \\
= \frac{1}{2}(1 + z\bar{z}) (\gamma^1 \partial/\partial x^1 + \gamma^2 \partial/\partial x^2) - \frac{1}{2}(x_1\gamma^1 + x_2\gamma^2).
\]

Here we have used the identities \(\gamma^1\gamma^1\gamma^2 = -\gamma^2\) and \(\gamma^2\gamma^1\gamma^2 = +\gamma^1\).

It turns out that the Dirac operator over the chart \(U_1\), which we shall write as \(\mathcal{D}_S\), is now completely determined: the Dirac operator on the spinor module is determined by its restriction to a single chart, no matter how small! Indeed, this is a characteristic feature of the spinor module [19]. However, in order to preserve the illusion of a symmetrical treatment of both charts, we shall anticipate the corresponding formulas for \(\mathcal{D}_S\). We therefore choose the matrix \([\tilde{h}_{ij}(\zeta)]\) to be the **negative** square root of \(G^{-1} \equiv [g^{ij}(\zeta)] = \frac{1}{2}(1 + \zeta\bar{\zeta})^2 I\), that is, \(\tilde{h}_{ij}(\zeta) := -\frac{1}{2}(1 + \zeta\bar{\zeta}) \delta_{ij}\). This leads to

\[
\mathcal{D}_S = -\frac{1}{2}(1 + \zeta\bar{\zeta}) (\gamma^1 \partial/\partial x^1 + \gamma^2 \partial/\partial x^2) + \frac{1}{2}(x_1\gamma^1 + x_2\gamma^2). \tag{9.10}
\]

Explicitly, \(\partial/\partial x^1 = \partial/\partial z + \partial/\partial \bar{z}\) and \(\partial/\partial x^2 = i(\partial/\partial z - \partial/\partial \bar{z})\) on \(U_0\), and similar formulas hold on \(U_1\) with \(z\) replaced by \(\zeta\). This allows us to express the Dirac operators properly in complex coordinates:

\[
\mathcal{D}_N = \frac{1}{2}(1 + z\bar{z}) \left( (\gamma^1 + i\gamma^2) \frac{\partial}{\partial z} + (\gamma^1 - i\gamma^2) \frac{\partial}{\partial \bar{z}} \right) - \frac{1}{4}(z(\gamma^1 - i\gamma^2) + \bar{z}(\gamma^1 + i\gamma^2)),
\]

\[
\mathcal{D}_S = -\frac{1}{2}(1 + \zeta\bar{\zeta}) \left( (\gamma^1 + i\gamma^2) \frac{\partial}{\partial \zeta} + (\gamma^1 - i\gamma^2) \frac{\partial}{\partial \bar{\zeta}} \right) + \frac{1}{4}(\zeta(\gamma^1 - i\gamma^2) + \bar{\zeta}(\gamma^1 + i\gamma^2)). \tag{9.11}
\]

These formulae are still a bit cumbersome; to obtain a simpler picture, we need to select a particular representation for the operators \(\gamma^1\) and \(\gamma^2\).

\footnote{On more general compact spin manifolds, the restriction of the Dirac operator to a single chart determines its restriction to neighbouring charts through its interaction with the gauge transformations; this in turn determine their neighbours, and so on.}
**Definition 9.3.** The Fock space of $\mathbb{R}^2$ is the two-dimensional complex superspace $\Lambda^0 \mathbb{C} \oplus \Lambda^1 \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. We may represent $\gamma^1$ and $\gamma^2$ as anticommuting odd operators of square $-1$; a suitable choice is

$$\gamma^1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$  

The grading operator is $i\gamma^1 \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since $\gamma^1 + i\gamma^2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $\gamma^1 - i\gamma^2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$,

the expressions (9.11) simplify to

$$\mathcal{D}/N = \begin{pmatrix} 0 & \bar{z} \\ -z & 0 \end{pmatrix}, \quad \mathcal{D}/S = \begin{pmatrix} 0 & -\bar{\zeta} \\ \bar{\zeta} & 0 \end{pmatrix},$$  

(9.13)

where $\bar{z}$ is the first-order differential operator

$$\bar{z} := (1 + z\bar{z}) \frac{\partial}{\partial z} - \frac{1}{2} \bar{z}.$$  

(9.14)

The operator $\bar{z}$ was introduced by Newman and Penrose [43] and further studied by Goldberg et al. [30], with particular attention to its eigenfunctions. Since

$$\bar{z}\psi = (1 + z\bar{z}) \frac{\partial}{\partial z} - \frac{1}{2} \bar{z}(1 + z\bar{z})\psi = (1 + z\bar{z})^{3/2} \frac{\partial}{\partial z} ((1 + z\bar{z})^{-1/2}\psi),$$  

(9.15)

we get the identity $\bar{z} = Q_0^{3/2}(\partial/\partial z)Q_0^{-1/2}$ in the algebra of differential operators on $U_0$.

The formal selfadjointness of $\mathcal{D}$ is not perhaps apparent from (9.13), since the term $-\frac{1}{2} \bar{z}$ in (9.14) seems to have the adjoint $-\frac{1}{2} z$ rather than $+\frac{1}{2} z$. However, the inner product of spinors $\langle \phi | \psi \rangle$ involves integration over the sphere, i.e., integration over $\mathbb{C}$ with respect to to the area form $2i(1 + z\bar{z})^{-2} dz \wedge d\bar{z}$. Thus, if $\phi, \psi$ are two spinors, then (9.15) gives

$$(1 + z\bar{z})^{-2} \bar{z}\psi_N = (1 + z\bar{z})^{-1/2} \bar{z}\psi_N \frac{\partial}{\partial z} ((1 + z\bar{z})^{-1/2}\psi_N),$$

and it follows that $\langle \phi^+ | \bar{z}\psi^- \rangle = -\langle \bar{z}\phi^+ | \psi^- \rangle$ on integrating by parts.

**Proposition 9.2.** The restriction of the Dirac operator to $U_1$ is determined by its restriction to $U_0$.

**Proof.** We must show that the form (9.13) of the operator $\mathcal{D}/S$ is completely determined by that of $\mathcal{D}/N$. This is possible because these operators act respectively on functions $\psi_N^\pm$ and $\psi_S^\pm$ which are linked by the gauge transformations (9.7), and because the Dirac operator is odd, so the gauge transformations for both spinor parities must be invoked.

---

3 The letter $\bar{\partial}$, from the Icelandic alphabet, is pronounced “edth”.

---

108
Given a spinor $\psi$ with components $\psi^+$ and $\psi^-$, let $\phi = \mathcal{D}\psi$. Then $(\partial_z \phi_N^*)(z, \bar{z}) = \phi_N^*(z, \bar{z}) = (\bar{z}/z)^{1/2}\phi_S^*(z^{-1}, \bar{z}^{-1})$. On the other hand, with $\zeta = z^{-1}$ we get

$$\frac{\partial \psi_N^*}{\partial z}(z, \bar{z}) = \frac{\partial}{\partial z}((z/\bar{z})^{1/2}\psi_S^*(z^{-1}, \bar{z}^{-1})) = \frac{1}{2}(\bar{z}z)^{-1/2}\psi_S^*(-1, \bar{z}^{-1}) - (z/\bar{z})^{1/2}z^{-2}\frac{\partial \psi_S^*}{\partial \zeta}(z^{-1}, \bar{z}^{-1}) = (\zeta \bar{\zeta})^{1/2}\left(-\zeta \frac{\partial \psi_S^*}{\partial \zeta}(\zeta, \bar{\zeta}) + \frac{i}{2} \psi_S^*(\zeta, \bar{\zeta})\right),$$

and so the operator $\partial_z$ transforms as follows:

$$(\partial_z \psi_N^*)(z, \bar{z}) = (1 + z\bar{z}) \frac{\partial \psi_N^*}{\partial z}(z, \bar{z}) - \frac{i}{2} \bar{z} \psi_N^*(z, \bar{z}) = (\zeta \bar{\zeta})^{-1/2}(1 + \zeta \bar{\zeta})\left(-\zeta \frac{\partial \psi_S^*}{\partial \zeta}(\zeta, \bar{\zeta}) + \frac{i}{2} \psi_S^*(\zeta, \bar{\zeta})\right) - \frac{(\zeta / \bar{\zeta})^{1/2}}{2\zeta} \psi_S^*(\zeta, \bar{\zeta}) = (\zeta / \bar{\zeta})^{1/2}\left(-1 + \frac{1}{\zeta} \psi_S^*(\zeta, \bar{\zeta}) + \frac{1}{2}(\zeta^{-1} + \bar{\zeta}) \psi_S^*(\zeta, \bar{\zeta}) - \frac{i}{2} \psi_S^*(\zeta, \bar{\zeta})\right) = -(\zeta / \bar{\zeta})^{1/2} \frac{\partial \psi_S^*}{\partial \zeta}(\zeta, \bar{\zeta}) = -(\bar{z}/z)^{1/2} \frac{\partial \psi_S^*}{\partial \zeta}(z^{-1}, \bar{z}^{-1}).$$

We conclude that $\phi_S^+ = -\bar{\zeta} \psi_S^-$. If we apply complex conjugation to (9.16) and replace $\psi_N^*$ by $\psi_N^-$ and $\psi_S^-$ by $\psi_S^+$, we find that

$$\phi_N^*(z, \bar{z}) = -(\partial_z \psi_N^+)(z, \bar{z}) = (z/\bar{z})^{1/2} \frac{\partial \psi_S^+}{\partial \zeta}(z^{-1}, \bar{z}^{-1}),$$

and it follows that $\phi_S^- = \bar{\zeta} \psi_S^+$. We have recovered the expression of (9.13) for $\mathcal{D}_S$ from the corresponding for $\mathcal{D}_N$ and from (9.7), without using the recipe (9.10) for $\mathcal{D}_S$. This means that the “choice” $\hat{h}_S^\gamma(\zeta) := -\frac{1}{2}(1 + \zeta \bar{\zeta})\delta^\gamma_\gamma$ which led to (9.10) is actually forced by the action of $\mathcal{D}$ on spinors. 

**Exercise 9.4.** Use (9.1) and (9.3) to obtain expressions for $\partial_z$ and $\partial_\zeta$ in spherical coordinates $(\theta, \phi)$. Check that $e^{i\phi} \partial_z = -e^{-i\phi} \partial_\zeta - \csc \theta$ on $U_0 \cap U_1$.

Proposition 9.2 shows that the only freedom in the choice of the Dirac operator, once the metric and the spin connection are given, lies in selecting the matrix function $\hat{H} = [\hat{h}_S^\gamma(z)]$ which gives the local Clifford action on $U_0$. The condition $\hat{H}^{i}\hat{H} = G^{-1}$ fixes the matrix $\hat{H}$ up to premultiplication by an arbitrary $SO(2)$ matrix function on $U_0$. We may alternatively think of this as the freedom to select any local orthonormal bases of tangent vectors compatible with the orientation of $S^2$, since this amounts to picking a section of the $SO(2)$-frame bundle. To express this in our complex-coordinate notation, we must bear in mind that the stereographic projections $(\theta, \phi) \mapsto (z, \bar{z})$ and $(\theta, \phi) \mapsto (\zeta, \bar{\zeta})$ are orientation reversing; such a local orthonormal basis is therefore given by

$$\begin{pmatrix} E_1 \\ E_2 \end{pmatrix} := \begin{pmatrix} -\cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \partial / \partial \theta \\ \csc \theta \partial / \partial \phi \end{pmatrix},$$

where $\alpha(\theta, \phi)$ is a smooth real-valued function which we call the spin gauge [25].
Exercise 9.5. Check that \( E_z = \frac{1}{2}(E_1 - iE_2) \) if and only if \( \alpha \equiv -\phi \), and \( -E_\zeta = \frac{1}{2}(E_1 - iE_2) \) if and only if \( \alpha \equiv \phi \).

Exercise 9.6. Compute the mixed Christoffel symbols \( \tilde{\Gamma}^a_{i\alpha} \) with the spin gauge (9.17) (use Exercise 7.16) and verify that \( \tilde{\Gamma}^2_{\theta 1} = \partial \alpha / \partial \theta, \tilde{\Gamma}^2_{\phi 1} = \partial \alpha / \partial \phi - \cos \theta \).

Exercise 9.7. Use the results of the Exercises 9.5 and 9.6 to obtain the following local expressions in spherical coordinates for the spin connection \( \nabla^S \) on the chart domains \( U_0 \) and \( U_1 \):

\[
\nabla^S_\theta = \frac{\partial}{\partial \theta}, \quad \nabla^S_\phi = \frac{\partial}{\partial \phi} \mp \frac{1}{2}(1 \pm \cos \theta)^1 \gamma^1 \gamma^2 \quad (9.18)
\]

where the upper signs are for \( U_0 \) and the lower signs are for \( U_1 \).

Exercise 9.8. Compute the Dirac operator in spherical coordinates with the spin gauge (9.17), using the results of Exercise 7.16: check that

\[
\begin{align*}
D_- &= e^{i\alpha} \left( -\frac{\partial}{\partial \theta} - i \frac{\partial}{\sin \theta \partial \phi} + \frac{1}{2 \sin \theta} \left( \frac{\partial \alpha}{\partial \phi} - \cos \theta \right) - i \frac{\partial \alpha}{2 \partial \theta} \right), \\
D_+ &= e^{-i\alpha} \left( \frac{\partial}{\partial \theta} - i \frac{\partial}{\sin \theta \partial \phi} - \frac{1}{2 \sin \theta} \left( \frac{\partial \alpha}{\partial \phi} - \cos \theta \right) - i \frac{\partial \alpha}{2 \partial \theta} \right),
\end{align*}
\]

with the conventions of (8.7) and (9.12).

9.5 The spinor Laplacian

Lemma 9.3. The spinor Laplacian \( \Delta^S \) on the sphere \( S^2 \) is given locally by

\[
\Delta^S = -\left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \phi^2} \right) + \left( \frac{1 \pm \cos \theta}{2 \sin \theta} \right)^2 \pm \frac{1 \pm \cos \theta}{\sin^2 \theta} \gamma^1 \gamma^2 \frac{\partial}{\partial \phi}, \quad (9.19)
\]

where the upper signs are for \( U_0 \) and the lower signs are for \( U_1 \).

Proof. It suffices to check that the general local formula (8.10) for the Laplacian specializes, in view of (7.14) and (9.18), to

\[
\Delta^S = -\frac{\partial^2}{\partial \theta^2} - \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} \mp \frac{1}{2}(1 \pm \cos \theta)^1 \gamma^2 \right)^2 - \frac{1}{\sin^2 \theta} \left( \sin \theta \cos \theta \frac{\partial}{\partial \theta} \right),
\]

from which (9.19) is immediate. \( \square \)

It is convenient to rewrite the spinor Laplacian in complex coordinates. From (9.3) we derive

\[
\frac{\partial}{\partial \phi} = i \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) = -i \left( \zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right) \quad \text{on } U_0 \cap U_1.
\]

\[\text{The formulae (9.18) have been obtained by Dray [25] as the expressions of the Newman–Penrose operators } -\bar{\delta} \text{ and } \delta \text{ on quantities of “spin-weights” } -\frac{1}{2} \text{ and } +\frac{1}{2} \text{ respectively.}\]
Using (9.2) and (9.3) it is readily checked that

\[(1 + z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} = (1 + \zeta\bar{\zeta})^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \phi^2}.
\]

Since \((1 + \cos \theta/\sin \theta)^2 = z\bar{z}\) and \((1 - \cos \theta/\sin \theta)^2 = \zeta\bar{\zeta}\) by (9.1), we arrive at

\[
\Delta^S = -(1 + z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4} z\bar{z} + \frac{1}{2}(1 + z\bar{z}) i\gamma^1 \gamma^2 \left( \frac{\partial}{\partial z} - \frac{z}{\bar{z}} \frac{\partial}{\partial \bar{z}} \right) \text{ over } U_0,
\]

\[
= -(1 + \zeta\bar{\zeta})^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} + \frac{1}{4} \zeta\bar{\zeta} + \frac{1}{2}(1 + \zeta\bar{\zeta}) i\gamma^1 \gamma^2 \left( \frac{\partial}{\partial \zeta} - \frac{\zeta}{\bar{\zeta}} \frac{\partial}{\partial \bar{\zeta}} \right) \text{ over } U_1. \tag{9.20}
\]

**Lemma 9.4.** The Dirac operator and the spinor Laplacian on \(S^2\) are related by

\[\slashed{D}^2 = \Delta^S + \frac{1}{2} \gamma^1 \gamma^2.\]

**Proof.** This follows, of course, from the Lichnerowicz formula (8.14), since the sphere has constant scalar curvature \(K \equiv 2\); but it is instructive to make a direct verification. From (9.14) we get

\[
-\bar{z} \frac{\partial}{\partial \bar{z}} (1 + z\bar{z}) = \left( \frac{1}{4} z\bar{z} + \frac{1}{2}(1 + z\bar{z}) i\gamma^1 \gamma^2 \left( \frac{\partial}{\partial z} - \frac{z}{\bar{z}} \frac{\partial}{\partial \bar{z}} \right) \right) \text{ over } U_0,
\]

\[
= -\bar{z} \frac{\partial}{\partial \bar{z}} (1 + z\bar{z}) = \left( \frac{1}{4} \zeta\bar{\zeta} + \frac{1}{2}(1 + \zeta\bar{\zeta}) i\gamma^1 \gamma^2 \left( \frac{\partial}{\partial \zeta} - \frac{\zeta}{\bar{\zeta}} \frac{\partial}{\partial \bar{\zeta}} \right) \right) \text{ over } U_1. \tag{9.21}
\]

whereas

\[
-\bar{z} \frac{\partial}{\partial \bar{z}} (1 + z\bar{z}) = \left( \frac{1}{4} \zeta\bar{\zeta} + \frac{1}{2}(1 + \zeta\bar{\zeta}) i\gamma^1 \gamma^2 \left( \frac{\partial}{\partial \zeta} - \frac{\zeta}{\bar{\zeta}} \frac{\partial}{\partial \bar{\zeta}} \right) \right) \text{ over } U_1. \tag{9.22}
\]

by a similar calculation. The different signs of the last terms on the right in (9.21) and (9.22) signify the presence of the grading operator \(i\gamma^1 \gamma^2\) in the representation (9.12). Thus

\[
\slashed{D}^2 = \left( \begin{array}{cc} -\bar{z} \frac{\partial}{\partial \bar{z}} & 0 \\ 0 & -z \frac{\partial}{\partial z} \end{array} \right) = \left( \begin{array}{cc} -\bar{z} \frac{\partial}{\partial \bar{z}} & 0 \\ 0 & -z \frac{\partial}{\partial z} \end{array} \right) = -\bar{z} \frac{\partial}{\partial \bar{z}} (1 + z\bar{z}) = \left( \frac{1}{4} \zeta\bar{\zeta} + \frac{1}{2}(1 + \zeta\bar{\zeta}) i\gamma^1 \gamma^2 \left( \frac{\partial}{\partial \zeta} - \frac{\zeta}{\bar{\zeta}} \frac{\partial}{\partial \bar{\zeta}} \right) \right) \text{ over } U_1. \tag{9.23}
\]

There is an identical formula for \(\slashed{D}_S\), on replacing \(z\) by \(\zeta\). Now a glance at (9.20) gives the Lichnerowicz formula \(\slashed{D}^2 = \Delta^S + \frac{1}{2}\). \(\square\)
9.6 The SU(2) action on the spinor bundle

**Definition 9.4.** The Lie group SU(2) of unitary matrices of determinant 1,
\[ g = \left( \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right) \quad \text{with} \quad \alpha \bar{\alpha} + \beta \bar{\beta} = 1, \]
acts transitively on the sphere \( S^2 \) by rotations. The identification \( g \leftrightarrow (\alpha, \beta) \in \mathbb{C}^2 \) shows that SU(2) is topologically the sphere \( S^3 \), so it is compact. If \( [z^0 : z^1] \) are homogeneous coordinates of a point in \( \mathbb{C}P^1 \), the action is given by \( g \cdot [z^0 : z^1] = [\alpha z^0 + \beta z^1 : -\bar{\beta} z^0 + \bar{\alpha} z^1] \).

On the charts \( U_0 \) and \( U_1 \), the action of (9.25) is given by Möbius transformations:
\[ g \cdot z = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}}, \quad g' \cdot \zeta = \frac{\bar{\alpha} \zeta - \bar{\beta}}{\beta \zeta + \alpha}, \quad (9.24) \]
which clearly satisfies \( (g \cdot z)^{-1} = g' \cdot \zeta \). Notice that \( g \mapsto g' \) is an (inner) automorphism of SU(2), since
\[ g' := \left( \begin{array}{cc} \bar{\alpha} & -\bar{\beta} \\ \beta & \alpha \end{array} \right) = \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right) \left( \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right) \left( \begin{array}{cc} 0 & -i \\ -i & 0 \end{array} \right). \]

**Exercise 9.9.** Any isometry (rotation or reflection) of the sphere \( S^2 \) takes circles to circles and takes antipodal pairs of points to antipodal pairs of points. Conversely, any smooth bijective transformation of \( S^2 \) with these two geometrical properties is either a rotation or a reflection. Prove this converse, using the fact that a circle-preserving transformation of \( S^2 \) corresponds, under stereographic projection, to a circle-preserving transformation of \( \mathbb{C}_\infty \), which is either a Möbius transformation \( z \mapsto (az + \beta)/(\gamma z + \delta) \) or a conjugate Möbius transformation \( z \mapsto (\alpha z + \beta)/(\gamma \bar{z} + \bar{\delta}) \); and that a Möbius transformation is determined by the images of three points in \( \mathbb{C}_\infty \).

**Exercise 9.10.** The antipode of \( z = e^{i\psi} \cot \frac{1}{2} \theta \) is \( e^{i(\pi + \phi)} \cot \frac{1}{2}(\pi - \theta) = -1/\bar{z} \). Use this fact and the preceding exercise to prove that any rotation of \( S^2 \) is given by a Möbius transformation of the form \( z \mapsto (az + \beta)/(\bar{\beta} z + \bar{\alpha}) \).

**Definition 9.5.** The elements of SU(2) are conveniently described in terms of the Pauli matrices:
\[ \sigma_1 := \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 := \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \]

If \( \vec{n} = (n^1, n^2, n^3) \) denote Cartesian coordinates in \( \mathbb{R}^3 \) of a point \( \vec{n} \in \mathbb{S}^2 \), we write \( \vec{n} \cdot \vec{\sigma} := n^1 \sigma^1 + n^2 \sigma^2 + n^3 \sigma^3 \); then any \( g \in SU(2) \) may be written in the form
\[ g = \exp \left( \frac{1}{2} i \psi \vec{n} \cdot \vec{\sigma} \right) = \cos \frac{1}{2} \psi + i \sin \frac{1}{2} \psi \vec{n} \cdot \vec{\sigma} \quad (9.25) \]
with \( \vec{n} \in \mathbb{S}^2 \) and \( -\pi < \psi \leq \pi \). This is called the angle-axis parametrization of SU(2). We may identify the sphere \( \mathbb{S}^2 \) with the submanifold \( \{ g : \text{Tr} g = 0 \} \) of SU(2), where \( \vec{n} \in \mathbb{S}^2 \) corresponds to \( i \vec{n} \cdot \vec{\sigma} \in SU(2) \); the rotation action \( \rho \) of SU(2) on the sphere is given by conjugation: \( i \vec{n} \cdot \vec{\sigma} \mapsto g(i \vec{n} \cdot \vec{\sigma}) g^{-1} =: \rho(g) i \vec{n} \cdot \vec{\sigma} \). The isotropy subgroup of the point \( \vec{n} \) consists of elements of the form (9.25) with \( \psi \) arbitrary, which form a subgroup isomorphic to U(1). Thus \( \mathbb{S}^2 \approx SU(2)/U(1) \) as a homogeneous space.

112
Exercise 9.11. Show that \( \rho(g)\bar{n} = \bar{n} \) if and only if \( g \) is of the form \( (9.25) \) for some \( \psi \in \mathbb{R} \).

Exercise 9.12. Show that there is a group isomorphism between \( \text{Spin}(3) \) and \( SU(2) \), such that if \( \{e_1, e_2, e_3\} \) is an orthonormal basis of \( \mathbb{R}^3 \), then the elements \( e_2e_3, e_3e_1, e_1e_2 \in \text{Spin}(3) \) correspond respectively to \( i\sigma_1, i\sigma_2, i\sigma_3 \in SU(2) \).

The quotient mapping \( \eta: SU(2) \to S^2 \) may be described more economically by regarding \( S^2 \) as \( CP^1 \) and proceeding as follows.

**Definition 9.6.** The Hopf fibration \( \eta: SU(2) \to CP^1 \) is the map given by

\[
\eta\left(\begin{array}{c}
\alpha \\
-\beta \\
\bar{\alpha}
\end{array}\right) := \frac{\beta}{\bar{\alpha}}. \quad (9.26)
\]

It is immediate that \( SU(2) \overset{\eta}{\to} CP^1 \) is a principal \( U(1) \)-bundle, where the free right action of \( U(1) \) on the fibres \( \eta^{-1}(\beta/\bar{\alpha}) \) is given simply by multiplication on the right by the diagonal elements of \( SU(2) \).

Exercise 9.13. The Hopf fibration decomposes the sphere \( S^3 \approx SU(2) \) into a disjoint union of circles (the fibres), any two of which are linked. Regard \( S^3 \) as \( \mathbb{R}^3 \cup \{\infty\} \) via the stereographic projection \( (\alpha, \beta) \mapsto (w, t) \) where

\[
w := \frac{\alpha}{1 - 3\beta} \in \mathbb{C}, \quad t := \frac{3\bar{\beta}}{1 - 3\beta} \in \mathbb{R}.
\]

Check that \( \chi(w, t) := (2t + i(ww + t^2 - 1))/2\bar{w} \in \mathbb{C}_\infty \) is the expression in \( (w, t) \)-coordinates of the Hopf fibration. Deduce that the equation \( \chi(w, t) = z \) represents the circle obtained by cutting the sphere \( w\bar{w} + t^2 - i(\bar{z}w - z\bar{w}) = 1 \) with the equatorial plane \( \bar{z}w + z\bar{w} - 2t = 0 \); in particular, \( \chi(w, t) = \infty \) is the \( t \)-axis (including \( \infty \)) and \( \chi(w, t) = 0 \) is the unit circle in the \( w\bar{w} \)-plane. Show that any other circle \( \chi(w, t) = z \) cuts the \( w\bar{w} \)-plane obliquely at two points, one inside and one outside the unit circle; conclude that the circles \( \chi(w, t) = 0 \) and \( \chi(w, t) = z \) are linked.

The group \( SU(2) \) acts on itself by left translations \( \lambda(g): h \mapsto gh \); since these commute with right translations by \( U(1) \), they define a left action on the quotient manifold \( CP^1 \), which is just the aforementioned rotation action. Indeed, if \( h = (1 + z\bar{z})^{-1}\left(\begin{array}{c}
1 \\
z \\
\bar{z}
\end{array}\right) \), then \( \eta(h) = z \), and from

\[
\frac{1}{1 + z\bar{z}}\left(\begin{array}{c}
\alpha \\
-\beta \\
\bar{\alpha}
\end{array}\right) \left(\begin{array}{c}
1 \\
z \\
\bar{z}
\end{array}\right) = \frac{1}{1 + z\bar{z}}\left(\begin{array}{cc}
\alpha - \beta\bar{z} & \alpha z + \beta \\
-\bar{\beta} - \bar{\alpha} & -\beta z + \bar{\alpha}
\end{array}\right)
\]

it follows that \( \eta(gh) = g \cdot \eta(h) \equiv \rho(g)\eta(h) \) for \( g, h \in SU(2) \). Schematically, we get a commutative diagram

\[
\begin{array}{ccc}
SU(2) & \overset{\lambda(g)}{\longrightarrow} & SU(2) \\
\downarrow & & \downarrow \\
S^2 & \overset{\rho(g)}{\longrightarrow} & S^2
\end{array}
\]

113
which says that each pair of maps \((\lambda(g), \rho(g))\) is a morphism of the principal \(U(1)\)-bundle \(SU(2) \rightarrow S^1\).

We may say that the group \(SU(2)\) acts “equivariantly” on the principal bundle. The corresponding type of group action on vector bundles is defined as follows.

**Definition 9.7.** A **homogeneous vector bundle** with symmetry group \(G\) (a Lie group) is a vector bundle \(E \rightarrow M\) together with a pair of (left) actions \(\tau : G \times E \rightarrow E\) and \(\rho : G \times M \rightarrow M\) such that each \((\tau(g), \rho(g))\) is a vector bundle morphism on \(E \rightarrow M\). We shall call such a pair a bundle action of \(G\).

A **Hermitian homogeneous vector bundle** is a Hermitian vector bundle with a bundle action for which each \(\tau(g) \in \text{End}(E_x)\) is unitary. If \(E = E^+ \oplus E^-\) is a superbundle, we say the bundle action of \(G\) is even if \(\tau(g) \in \text{End}^+(E_x)\) for each \(x \in M\); in other words, if both subbundles \(E^\pm \rightarrow M\) are \(G\)-homogeneous under the bundle action \((\tau, \rho)\).

We seek to define an even action of \(SU(2)\) on the spinor bundle \(S \rightarrow S^2\). This can be pictured as a commutative diagram

\[
\begin{array}{ccc}
S^\pm & \xrightarrow{\tau(g)} & S^\pm \\
\pi^\pm \downarrow & & \uparrow \pi^\pm \\
S^2 & \xrightarrow{\rho(g)} & S^2
\end{array}
\]

which, in terms of the spinor components \(\psi^\pm \in \Gamma(S^\pm)\), means that

\[
\begin{align*}
\tau(g)\psi^+_N(z, \bar{z}) &= A_N^+(g, z) \psi^+_N(g^{-1} \cdot z, (g^{-1} \cdot z)^-), \\
\tau(g)\psi^+_S(\zeta, \bar{\zeta}) &= A_S^+(g', \zeta) \psi^+_S(g'^{-1} \cdot \zeta, (g'^{-1} \cdot \zeta)^-),
\end{align*}
\]

(9.27)

for \(g \in SU(2)\). By unitarity, the multipliers \(A^\pm\) must be \(U(1)\)-valued functions; and they must satisfy the following consistency conditions in order that (9.27) define a group action:

\[
\begin{align*}
A_N^+(gh, z) &= A_N^+(g, z)\ A_N^+(h, g^{-1} \cdot z), \\
A_S^+(g'h', \zeta) &= A_S^+(g', \zeta)\ A_S^+(h', g'^{-1} \cdot \zeta).
\end{align*}
\]

(9.28)

(9.29)

**Exercise 9.14.** Show that \(A(g, z) := (\beta \bar{z} + \alpha)^k / (\bar{\beta} z + \alpha)^k\) is a formal solution to the equation \(A(gh, z) = A(g, z)A(h, g^{-1} \cdot z)\), for \(g, h \in SU(2), z \in \mathbb{C}_\infty\); and that this solution is well-defined provided \(2k\) is an integer.

Bearing in mind that

\[
g^{-1} \cdot z = \frac{\alpha \bar{z} - \beta}{\bar{\beta} z + \alpha}, \quad g'^{-1} \cdot \zeta = \frac{\alpha \zeta + \bar{\beta}}{-\bar{\beta} \zeta + \alpha},
\]

the gauge transformation rule (9.7) for the spinors \(\tau(g)\psi\) yields

\[
A_S^+(g', \zeta)\psi^+_S\left(\frac{\alpha \zeta + \beta}{-\bar{\beta} \zeta + \alpha}, \frac{\alpha \bar{\zeta} + \beta}{-\bar{\beta} \zeta + \alpha}\right) = (\bar{\zeta} / \zeta)^{1/2} A_N^+(g, \zeta^{-1})\psi^+_N\left(\frac{\bar{\alpha} - \beta \zeta}{\beta + \alpha \zeta}, \frac{\alpha - \bar{\beta} \zeta}{\beta + \alpha \zeta}\right).
\]

(9.30)

---

\(^5\)This is often called a **G-vector bundle**, for short.
shows that the coefficients in (9.30) must satisfy the relation
(9.7); this leads to the choice of
\( A \)
or equivalently
The gauge transformation:
\[ \Gamma(\alpha \zeta + \beta, -\beta \zeta + \alpha) = \left( \frac{\bar{\alpha} \zeta + \beta}{-\bar{\beta} \zeta + \alpha} \right) = \left( \frac{\alpha \zeta + \beta}{-\beta \zeta + \alpha} \right)^{1/2} \psi_N^+ \left( \frac{\alpha - \bar{\beta} \zeta}{\beta + \alpha \zeta} \right) \]
shows that the coefficients in (9.30) must satisfy the relation
\[
\frac{A^+_S(g', \zeta)}{A^+_N(g, \zeta^{-1})} = \left( \frac{\bar{\zeta}(\alpha \zeta + \beta)(-\bar{\beta} \zeta + \alpha)}{\zeta(-\beta \zeta + \alpha)(\bar{\alpha} \zeta + \beta)} \right)^{1/2} = \left( \frac{(\alpha + \bar{\beta} \zeta^{-1})(-\bar{\beta} \zeta + \alpha)}{(-\beta \zeta + \alpha)(\bar{\alpha} + \beta \zeta^{-1})} \right)^{1/2},
\]
or equivalently
\[
\frac{A^+_S(g', \zeta)}{A^+_N(g, z)} = \left( \frac{-\bar{\beta} \zeta + \alpha}{-\beta \zeta + \alpha} \right)^{1/2} = \left( \frac{\beta \bar{z} + \bar{\alpha}}{\beta z + \alpha} \right)^{1/2}.
\]
(9.31)
The following solution of (9.29) is therefore consistent with the spinor gauge transformations:
\[
A^+_N(g, z) := \left( \frac{\beta \bar{z} + \bar{\alpha}}{\beta z + \alpha} \right)^{1/2}, \quad A^+_S(g', \zeta) := \left( \frac{-\bar{\beta} \zeta + \alpha}{-\beta \zeta + \alpha} \right)^{1/2}.
\]
Substituting these in (9.27) yields a bundle action of \( SU(2) \) on \( S^+ \to S^2 \).
The same procedure leads to a bundle action of \( SU(2) \) on \( S^- \to S^2 \). One need only replace the term \( \frac{\bar{\zeta}}{\zeta} \) in (9.30) by \( \frac{\zeta}{\bar{\zeta}} \) when invoking the gauge transformation rule (9.7); this leads to the choice of \( A_N(g, z) \) and \( A_S(g', \zeta) \) as the complex conjugates of (9.32):
\[
A_N(g, z) := \left( \frac{\beta \bar{z} + \bar{\alpha}}{\beta z + \alpha} \right)^{-1/2}, \quad A_S(g', \zeta) := \left( \frac{-\bar{\beta} \zeta + \alpha}{-\beta \zeta + \alpha} \right)^{-1/2}.
\]
(9.33)
We summarize the foregoing in a definition.

**Definition 9.8.** The Lie group \( SU(2) \) acts on the spinor bundle via \( (\tau, \rho) \), where \( \rho \) is the rotation action (9.24) on the Riemann sphere, and \( \tau \) is given by (9.27), where the multipliers \( A_N^\pm \) and \( A_S^\pm \) are defined by (9.32) and (9.33).

### 9.7 Equivariance of the Dirac operator

**Lemma 9.5.** Let \( T \in \text{End}^+(\Gamma(U_0, S)) \) be a transformation of the form
\[
(T \psi_N^+)(z, \bar{z}) := a(z, \bar{z})^{1/2} \psi_N^+(b(z), \bar{b}(z)),
\]
\[
(T \psi_N^-)(z, \bar{z}) := a(z, \bar{z})^{-1/2} \psi_N^-(b(z), \bar{b}(z)).
\]
(9.34)
where \( a \) is a smooth function on \( \mathbb{C}^\times \) and \( b \) is a rational function on \( \mathbb{C}_\infty \). Then \( T \partial \bar{z} = \partial \bar{z} T \) as operators from \( \Gamma(U_0, S^-) \) to \( \Gamma(U_0, S^+) \) if and only if \( a, b \) satisfy the pair of differential equations
\[
(1 + \bar{z} \bar{z}) \frac{db}{dz} = a(z, \bar{z})(1 + b(z)\bar{b}(z)),
\]
\[
(1 + \bar{z} \bar{z}) \frac{\partial a}{\partial z} = a(z, \bar{z})\bar{b}(z) - \bar{a}(z, \bar{z}).
\]
(9.35)

115
Proof. It is enough to notice that
\[ T(\bar{\partial}_z \psi_N^-)(z, \bar{z}) = a^{1/2}(1 + b\bar{b}) \frac{\partial \psi_N^-}{\partial z}(b, \bar{b}) - \frac{1}{2}a^{1/2}b\psi_N^-(b, \bar{b}), \]
whereas
\[ \partial_z(T\psi_N^-)(z, \bar{z}) = (1 + z\bar{z}) \left\{ -\frac{1}{2}a^{-3/2} \frac{\partial a}{\partial z} \psi_N^- (b, \bar{b}) + a^{-1/2} \frac{db}{dz} \psi_N^- (b, \bar{b}) \right\} - \frac{1}{2}z a^{-1/2} \psi_N^- (b, \bar{b}), \]
using both halves of (9.34); and then to equate coefficients of \( \psi_N^- \) and \( \partial \psi_N^- / \partial z \).

**Proposition 9.6.** The Dirac operator on the sphere is equivariant under the action of \( SU(2) \) on the spinor bundle, i.e., \( \tau(g)\mathcal{D} = \mathcal{D}\tau(g) \) on \( \Gamma(S) \), for all \( g \in SU(2) \).

**Proof.** To show that \( \tau(g)\mathcal{D} = \mathcal{D}\tau(g) \) for all \( g \), it is enough to check this for \( g \) belonging to a collection of one-parameter subgroups which generate \( SU(2) \). Since \( \alpha\bar{\alpha} + \beta\bar{\beta} = 1 \), we can write \( \alpha = \exp(\frac{i}{2}\phi + \frac{i}{2}\psi)\cos \frac{1}{2} \theta, \beta = \exp(\frac{i}{2}\phi - \frac{i}{2}\psi)\sin \frac{1}{2} \theta; \) we thereby see that any \( g \in SU(2) \) is of the form \( k(\phi)h(\theta)k(\psi) \), where

\[
k(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \quad h(\theta) = \begin{pmatrix} \cos \frac{1}{2} \theta & \sin \frac{1}{2} \theta \\ -\sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta \end{pmatrix}.
\]

(9.36)

Now \( \tau(k(t)) \) is of the form (9.34) with \( a(z, \bar{z}) = e^{-it}, b(z) = e^{-it}z \), so the equations (9.35) reduce to the identities
\[
(1 + z\bar{z})e^{-it} = e^{-it}(1 + z\bar{z}), \quad 0 = e^{-2it}e^{it}z - z e^{-it}.
\]

On the other hand, \( \tau(h(t)) \) is of the form (9.34) with
\[
a(z, \bar{z}) = \frac{\bar{z}\sin \frac{1}{2}t + \cos \frac{1}{2}t}{\bar{z}\sin \frac{1}{2}t + \cos \frac{1}{2}t}, \quad b(z) = \frac{z\cos \frac{1}{2}t - \sin \frac{1}{2}t}{z\sin \frac{1}{2}t + \cos \frac{1}{2}t}.
\]

(9.37)

for which
\[
\frac{\partial a}{\partial z} = -\sin \frac{1}{2}t \frac{\bar{z}\sin \frac{1}{2}t + \cos \frac{1}{2}t}{(z\sin \frac{1}{2}t + \cos \frac{1}{2}t)^2}, \quad \frac{db}{dz} = \frac{1}{(z\sin \frac{1}{2}t + \cos \frac{1}{2}t)^2}.
\]

(9.38)

From this it is easy to check that (9.37) satisfies the equations (9.35) for all \( t \in \mathbb{R} \).

Thus \( \tau(g)\bar{\partial}_z \psi_N^- = \bar{\partial}_z\tau(g)\psi_N^- \) for all \( g \in SU(2) \). By applying complex conjugation, we obtain \( \bar{\partial}_z \tau(g^{-1}) = \tau(g^{-1})\bar{\partial}_z \) on functions \( \psi_N^+ \), and so \( \tau(g)\mathcal{D}_N = \mathcal{D}_N\tau(g) \) for all \( g \). Replacing \( z \) by \( \zeta \) and \( g \) by \( g' \) and adjusting a few signs, the same calculations show that \( \tau(g)\mathcal{D}_S = \mathcal{D}_S\tau(g) \) for all \( g \), as expected.

**Exercise 9.15.** Verify directly that (9.37) satisfies the equations (9.35).

---

The parameters \( (\phi, \theta, \psi) \) in this product are the so-called Euler angles for the group \( SU(2) \).
9.8 Angular momentum operators

Definition 9.9. The homomorphisms \( t \mapsto g(\vec{n}; t) := \exp(\frac{1}{2} it \vec{n} \cdot \vec{\sigma}) = \cos \frac{1}{2} t + i \sin \frac{1}{2} t \vec{n} \cdot \vec{\sigma} \), for each \( \vec{n} \in S^2 \), yield all one-parameter subgroups of \( SU(2) \). The infinitesimal generators of these subgroups are \(-\frac{1}{2} \vec{n} \cdot \vec{\sigma} \in \mathfrak{su}(2)\), where \( \mathfrak{su}(2) \) is the Lie algebra of antihermitian \( 2 \times 2 \) matrices. The corresponding generator \( J_{\vec{r}} \) of the spinor action of this subgroup is

\[
- i (J_{\vec{r}} \psi^\pm_N)(z, \bar{z}) := \lim_{t \to 0} \frac{d}{dt} \tau(g(\vec{n}; t)) \psi^\pm_N(z, \bar{z}) = \lim_{t \to 0} \frac{d}{dt} \left[ a_i(z, \bar{z}) (\pm 1)^2 \psi^\pm_N(b_t(z), \bar{b}_t(z)) \right],
\]

where the coefficient \(-i\) is inserted for convenience, so that \( J_{\vec{r}} \) is formally selfadjoint (rather than skewadjoint).

For the three cardinal directions, where \( \vec{n} \cdot \vec{\sigma} = n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3 \), we write the generators simply as \( J_1, J_2, J_3 \) respectively; these are commonly called the angular momentum generators.\(^7\) As before, we write \( J_\pm := J_1 \pm i J_2 \).

With the notations \( \hat{a}_0(z, \bar{z}) := \frac{d}{dt} \big|_{t=0} a_t(z, \bar{z}) \) and \( \hat{b}_0(z) := \frac{d}{dt} \big|_{t=0} b_t(z) \), the definition (9.39) simplifies to

\[
J_{\vec{r}} = -i \left( \hat{b}_0(z) \frac{\partial}{\partial z} + \hat{b}_0(z) \frac{\partial}{\partial \bar{z}} \right) + \frac{1}{2} \hat{a}_0(z, \bar{z}) \gamma^1 \gamma^2
\]

on recalling that \( i \gamma^1 \gamma^2 = \pm 1 \) on \( \Gamma(S^2) \).

Thus, for the one-parameter subgroup \( \{ k(-t) : t \in \mathbb{R} \} \), where \( a_t(z, \bar{z}) = e^{it} \) and \( b_t(z) = e^{it} z \), we get \( \hat{a}_0(z, \bar{z}) = \frac{1}{2}(z - \bar{z}) \) and \( \hat{b}_0(z) = \frac{1}{2}(z^2 + 1) \), leading to

\[
J_3 = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} + \frac{1}{2} i \gamma^1 \gamma^2 = -i \frac{\partial}{\partial \phi} + \frac{1}{2} i \gamma^1 \gamma^2.
\]

For the one-parameter subgroup \( \{ h(-t) : t \in \mathbb{R} \} \), \( a_t(z, \bar{z}) \) and \( b_t(z) \) are given by (9.37) with \( t \) replaced by \(-t\). In this case \( \hat{a}_0(z, \bar{z}) = \frac{1}{2}(z - \bar{z}) \) and \( \hat{b}_0(z) = \frac{1}{2}(z^2 + 1) \), leading to

\[
J_2 = \frac{i}{2}(z^2 + 1) \frac{\partial}{\partial z} + \frac{1}{2}(z^2 + 1) \frac{\partial}{\partial \bar{z}} - \frac{1}{4}(z - \bar{z}) \gamma^1 \gamma^2.
\]

The one-parameter subgroup \( \{ \cos \frac{1}{2} t + i \sin \frac{1}{2} t \sigma_1 : t \in \mathbb{R} \} \) is generated by \( \frac{i}{2} \sigma_1 \), and \( a_t(z, \bar{z}) \), \( b_t(z) \) are now given by

\[
a_t(z, \bar{z}) = \frac{\cos \frac{1}{2} t + i \bar{z} \sin \frac{1}{2} t}{\cos \frac{1}{2} t - i z \sin \frac{1}{2} t}, \quad b_t(z) = \frac{z \cos \frac{1}{2} t - i \sin \frac{1}{2} t}{\cos \frac{1}{2} t - i z \sin \frac{1}{2} t},
\]

for which \( \hat{a}_0(z, \bar{z}) = \frac{1}{2}(\bar{z} + z) \) and \( \hat{b}_0(z) = \frac{1}{4}(z^2 - 1) \). Therefore

\[
J_1 = -\frac{1}{2}(z^2 - 1) \frac{\partial}{\partial z} + \frac{1}{2}(z^2 - 1) \frac{\partial}{\partial \bar{z}} - \frac{i}{4}(z + \bar{z}) \gamma^1 \gamma^2.
\]

\(^7\)The suitability of interpreting these generators as angular momentum operators for a magnetic monopole is discussed at length in [10].

117
The operators \( J_\pm \) are therefore given by:

\[
J_+ = J_1 + iJ_2 = -z^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} - \frac{1}{2} z i \gamma^1 \gamma^2,
\]

\[
J_- = J_1 - iJ_2 = \frac{\partial}{\partial z} + \bar{z}^2 \frac{\partial}{\partial \bar{z}} - \frac{1}{2} \bar{z} i \gamma^1 \gamma^2.
\]

(9.42)

It follows from this and from (9.41) that \([J_+, J_-] = 2J_3\). One also obtains that

\[
J_1^2 + J_2^2 = J_- J_+ + \frac{1}{2}[J_+, J_-] = J_- J_+ + J_3.
\]

Exercise 9.16. Show that the operators \( J_\pm \) are given in spherical coordinates by

\[
J_\pm = e^{\pm i\phi} \left( \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} - \frac{1 + \cos \theta}{2 \sin \theta} i \gamma^1 \gamma^2 \right).
\]

These operators arise in the theory of the magnetic monopole [10, 25] when the monopole parameter \( \mu = eg/\hbar c \) takes the value \( \mu = \frac{1}{2} \).

Exercise 9.17. Show that over \( U_1 \) the angular momentum generators satisfy the analogue of (9.40), with \( z \) replaced by \( \zeta \). Verify that

\[
J_3 = -\zeta \frac{\partial}{\partial \zeta} + \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} - \frac{1}{2} i \gamma^1 \gamma^2,
\]

\[
J_+ = \frac{\partial}{\partial \zeta} + \zeta^2 \frac{\partial}{\partial \bar{\zeta}} - \frac{1}{2} \bar{\zeta} i \gamma^1 \gamma^2,
\]

\[
J_- = -\zeta^2 \frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}} - \frac{1}{2} \zeta i \gamma^1 \gamma^2,
\]

(9.43)

in the coordinates \((\zeta, \bar{\zeta})\) over \( U_1 \).

There is one more operator worthy of mention, namely that corresponding to the “Casimir element” \( X_1^2 + X_2^2 + X_3^2 \), where \( \{X_1, X_2, X_3\} \) is an orthonormal basis for the Lie algebra \( su(2) \).

[The Casimir element belongs to the enveloping algebra \( \mathcal{U}(su(2)) \).] The image of this element under a representation of the Lie algebra is called a “Casimir operator”.

Definition 9.10. The **Casimir operator** for the spinor bundle action of \( SU(2) \) is the operator defined by

\[
C := J_1^2 + J_2^2 + J_3^2 = J_- J_+ + J_3(J_3 + 1) \quad \text{on} \quad \Gamma(S). \]

Proposition 9.7. The Casimir operator satisfies the relations

\[
C = \Delta^8 + \frac{1}{4} = \mathcal{D}^2 - \frac{1}{4}.
\]

(9.44)

---

8The constants are \( c \), the speed of light; \( h \), Planck’s constant; \( e \), the electric charge of a particle whose total angular momentum is \( J \); and \( g \), the magnetic charge of the monopole. The condition that \( 2\mu \) be an integer is the quantization condition of Dirac [24], and arises from the necessary description of monopoles by complex line bundles over the sphere.

9Strictly speaking the Casimir operator should be \(-C\), on account of the factor \((-i)\) in (9.39); but we change the sign to obtain a positive operator. On \( \Gamma(S) \), it is formally selfadjoint.
Proof. Using (9.41) and (9.42), we compute that

\[ C = J_+ J_+ + J_3 (J_3 + 1) \]

\[ = -(1 + \bar{z} z)^2 \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4} (1 + \bar{z} z) + \frac{1}{2} (1 + \bar{z} z) i \gamma^1 \gamma^2 \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \]

over \( U_0 \), with an analogous formula over \( U_1 \). A glance at (9.20) and (9.23) is enough to verify (9.44).

This result shows that even in the simplest example of a spinor bundle over a compact manifold, the three operators commonly referred to as “the Laplacian” are distinct, and must be carefully distinguished. Though they only differ by constants, this has the important consequence that their spectra are not the same. A similar shifting of the Laplacian occurs in harmonic analysis on compact Lie groups [59], where the Casimir satisfies \( C = \Delta + \frac{1}{12} \dim G \).

For \( SU(2) \), a 3-dimensional group, this leads us to expect \( C = \Delta + \frac{1}{4} \), which is nicely confirmed by Proposition 9.7.

### 9.9 Spinor harmonics

We come, finally, to the matter of diagonalizing the Dirac operator by finding an explicit basis of eigenspinors for \( \slashed{D} \). In view of the \( SU(2) \) symmetry, we may suspect, by analogy with the diagonalization of the Hodge–Dirac operator in Section 4, that the members of this basis should be closely related to the spherical harmonics \( Y_{lm}(\theta, \phi) \) on \( S^2 \). We may also anticipate that at some point we shall need to use some heavy artillery from the representation theory of \( SU(2) \). However, we begin in a fairly pedestrian manner, with some polynomial calculations over the chart \( U_0 \).

**Lemma 9.8.** The identity

\[ \frac{\partial}{\partial z} \left( (1 + z \bar{z})^{-l} z^r (-\bar{z})^s \right) = (1 + z \bar{z})^{-l} ((l + \frac{1}{2} - r) z^r (-\bar{z})^{s+1} + rz^{-1} (-\bar{z})^s) \]

holds for all \( r, s \in \mathbb{N} \) and \( l \in \mathbb{R} \).

**Exercise 9.18.** Prove Lemma 9.8, and show also that the conjugate identity holds:

\[ -\bar{\partial}_z \left( (1 + z \bar{z})^{-l} z^r (-\bar{z})^s \right) = (1 + z \bar{z})^{-l} ((l + \frac{1}{2} - s) z^r (-\bar{z})^{s+1} + sz^{-1} (-\bar{z})^{s-1}) \]

These calculations show one method of finding eigenspinors: take for \( \psi_N \) a linear combination of several terms of the form \( (1 + z \bar{z})^{-l} z^r (-\bar{z})^s \), with a common value for the difference of exponents \( r - s \), and choose the coefficients cleverly enough that the result of applying \( \partial_z \) closely resembles a multiple of the original function. However, since we wish these functions to be components of spinors, we must first consider the effect of the gauge transformation rules.

**Lemma 9.9.** Let \( \phi: \mathbb{C} \to \mathbb{C} \) be a smooth function of the form

\[ \phi(z, \bar{z}) := (1 + z \bar{z})^{-l} \sum_{r,s \in \mathbb{N}} a(r, s) z^r (-\bar{z})^s. \]
Then \( \phi \) represents a section in \( \Gamma(U_0,S^+) \) if and only if \( l + \frac{1}{2} \) is a positive integer, and \( a(r,s) = 0 \) for \( r > l \mp \frac{1}{2} \) or \( s > l \pm \frac{1}{2} \). Moreover, the coefficients must satisfy the symmetry relations \( a(r,s) = (-1)^{-\frac{1}{2} l} a(l \mp \frac{1}{2} - r, l \pm \frac{1}{2} - s) \).

**Proof.** Suppose that \( \phi \) represents a section in \( \Gamma(U_0,S^+) \). Then by (9.7) we obtain

\[
\phi(z, \bar{z}) = (\bar{z}/z)^{1/2} \phi(z^{-1}, \bar{z}^{-1}) = z^{-1/2 \bar{z}^{1/2}(z \bar{z})^{1/2}} (1 + z \bar{z})^{-l} \sum_{r,s \in \mathbb{N}} a(r,s) z^{-r} (-z)^{-s}
\]

\[
= (1 + z \bar{z})^{-l} (-1)^{l+\frac{1}{2}} \sum_{r,s \in \mathbb{N}} a(r,s) z^{l-\frac{1}{2} - r} (-z)^{l+\frac{1}{2} - s},
\]

where the exponents in the sum on the right hand side must also be nonnegative integers. Thus \( l - \frac{1}{2} \in \mathbb{N} \), and the nonnegativity of the exponents on the right guarantees that \( r \in \{0, 1, \ldots, l - \frac{1}{2} \} \) while \( s \in \{0, 1, \ldots, l + \frac{1}{2} \} \). The argument for sections in \( \Gamma(U_0,S^-) \) is similar.

The structure of the symmetry relations among the coefficients, and the allowed ranges of the exponents, suggests the introduction of the following spinors.

**Definition 9.11.** For each \( l \in \mathbb{N} + \frac{1}{2} = \{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \} \), and for each \( m \in \{-l, -l + 1, \ldots, l - 1, l\} \),

let \( Y_{lm}^+ \in \Gamma(S) \) be the spinors whose components over \( U_0 \) are \( 2^{-l/2} Y_{lm}^\pm(z, \bar{z}) \), where

\[
Y_{lm}^+(z, \bar{z}) := C_{lm} (1 + z \bar{z})^{-l} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r} \binom{l+\frac{1}{2}}{s} z^r (-z)^s,
\]

\[
Y_{lm}^-(z, \bar{z}) := C_{lm} (1 + z \bar{z})^{-l} \sum_{r-s=m+\frac{1}{2}} \binom{l+\frac{1}{2}}{r} \binom{l-\frac{1}{2}}{s} z^r (-z)^s,
\]

where the constants \( C_{lm} \) are defined as

\[
C_{lm} := (-1)^{l-m} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)! (l-m)!}{(l+\frac{1}{2})! (l-\frac{1}{2})!}}.
\]

Also, let \( Y_{lm}'' \in \Gamma(S) \) be the spinor whose components are \( 2^{-1/2} Y_{lm}^+(z, \bar{z}) \) and \(-2^{-1/2} Y_{lm}^-(z, \bar{z}) \).

The simplest examples of (9.45) are

\[
Y_{\frac{1}{2} \frac{1}{2}}' := \frac{1}{\sqrt{4\pi}} \left( 1/\sqrt{1 + z \bar{z}} \right), \quad Y_{\frac{1}{2} \frac{1}{2}}'' := \frac{1}{\sqrt{4\pi}} \left( z/\sqrt{1 + z \bar{z}} \right),
\]

\[
Y_{\frac{1}{2} -\frac{1}{2}}' := \frac{1}{\sqrt{4\pi}} \left( \bar{z}/\sqrt{1 + z \bar{z}} \right), \quad Y_{\frac{1}{2} -\frac{1}{2}}'' := \frac{1}{\sqrt{4\pi}} \left( -1/\sqrt{1 + z \bar{z}} \right).
\]

---

\(^{10}\) The range of allowed values of \( m \) is obtained by listing the possibilities for \((r-s)\) in (9.45) and adjusting by \( \frac{1}{2} \). Notice that each \( m \) is a half-integer.

\(^{11}\) The precise form of the constants \( C_{lm} \) is obtained by looking ahead to the normalization \( \langle Y_{lm}^\prime | Y_{lm}'' \rangle = 1 \); for the present, we need only that the constants for \( Y_{lm}^\prime \) and \( Y_{lm}'' \) be the same.
The functions $Y_{lm}^+$ and $Y_{lm}^-$ were introduced by Newman and Penrose [43], using the notations $-\frac{1}{2}Y_{lm}$ and $\frac{1}{2}Y_{lm}$ respectively. In fact, these appear as a subfamily of functions $sY_{lm}$ with $s \in \{-l, -l+1, \ldots, l-1, l\}$ which, for $l, m, s$ integers, they called “spin-$s$ spherical harmonics”; for $s = 0$ they reduce to the everyday spherical harmonics $Y_{lm}$ on $S^2$. Newman and Penrose also noted that their formulas make sense when $l, m, s$ are all half-integers, and christened such functions “spinor harmonics”; they were investigated further by Goldberg et al [30]. Later, Dray [25] showed that these same functions occur as the spinor components in the theory of the magnetic monopole [10].

**Lemma 9.10.** $\bar{\partial}_z Y_{lm}^- = (l + \frac{1}{2})Y_{lm}^+$ and $-\bar{\partial}_z Y_{lm}^+ = (l + \frac{1}{2})Y_{lm}^-.

**Proof.** Using Lemma 9.8, we find that $\bar{\partial}_z Y_{lm}^-(z, \bar{z})$ equals

$$C_{lm}(1 + z\bar{z})^{-l} \sum_{r-s=m+\frac{1}{2}} \left(\begin{array}{c} l + \frac{1}{2} \\ r \\
\end{array}\right) \left(\begin{array}{c} l - \frac{1}{2} \\ s \\
\end{array}\right) \{(l + \frac{1}{2} - r)z^r(-\bar{z})^{s+1} + rz^{r-1}(-\bar{z})^s\}

= C_{lm}(1 + z\bar{z})^{-l} \sum_{j-k=m-\frac{1}{2}} \left\{ (l + \frac{1}{2} - j) \left(\begin{array}{c} l + \frac{1}{2} \\ j \\
\end{array}\right) \left(\begin{array}{c} l - \frac{1}{2} \\ k - 1 \\
\end{array}\right)

+ (j + 1) \left(\begin{array}{c} l + \frac{1}{2} \\ j + 1 \\
\end{array}\right) \left(\begin{array}{c} l - \frac{1}{2} \\ k \\
\end{array}\right) \right\} z^j(-\bar{z})^k. \hspace{1cm} (9.47)

The term in braces can be simplified, using the binomial identities

$$k \left(\begin{array}{c} r \\ k \\
\end{array}\right) = r \left(\begin{array}{c} r - 1 \\ k - 1 \\
\end{array}\right), \hspace{0.5cm} (r - k) \left(\begin{array}{c} r \\ k \\
\end{array}\right) = r \left(\begin{array}{c} r - 1 \\ k \\
\end{array}\right),$$

so the right hand side of (9.47) equals $(l + \frac{1}{2})Y_{lm}^+$.

**Corollary 9.11.** The spinors $Y_{lm}'$ and $Y_{lm}''$ are eigenspinors for the Dirac operator, with nonzero integer eigenvalues $\pm(l + \frac{1}{2})$:

$$\not{\partial}Y_{lm}' = (l + \frac{1}{2})Y_{lm}', \hspace{1cm} \not{\partial}Y_{lm}'' = -(l + \frac{1}{2})Y_{lm}'$$

and each eigenvalue $\pm(l + \frac{1}{2})$ has multiplicity $(2l + 1)$.

**Proof.** Just observe that

$$\begin{pmatrix} 0 & \bar{\partial}_z \\ -\bar{\partial}_z & 0 \end{pmatrix} \begin{pmatrix} Y_{lm}^+ \\ Y_{lm}^- \end{pmatrix} = \begin{pmatrix} (l + \frac{1}{2})Y_{lm}^+ \\ (l + \frac{1}{2})Y_{lm}^- \end{pmatrix}, \hspace{1cm} \begin{pmatrix} 0 & \bar{\partial}_z \\ -\bar{\partial}_z & 0 \end{pmatrix} \begin{pmatrix} Y_{lm}^+ \\ -Y_{lm}^- \end{pmatrix} = \begin{pmatrix} -(l + \frac{1}{2})Y_{lm}^+ \\ (l + \frac{1}{2})Y_{lm}^- \end{pmatrix}.$$

The multiplicity is just the number of possibilities for the index $m$, i.e., the $(2l + 1)$ elements of $\{-l, -l+1, \ldots, l-1, l\}$. 

**Exercise 9.19.** For a fixed $l \in \mathbb{N} + \frac{1}{2}$, express $J_3Y_{lm}'$, $J_4Y_{lm}'$ and $J_-Y_{lm}'$ as linear combinations of the spinors $Y_{ln}$ with $n \in \{-l, -l+1, \ldots, l-1, l\}$.
Proposition 9.12. The spinors \( \psi \) shows that the functions \( D \) of the Euler angles as \( \theta \) associated to each spinor \( \psi \).

Proof. We associate to each spinor \( \psi = k(\alpha)h(\beta)k(\gamma) \in SU(2) \) by

\[
D^j_{mn}(\alpha, \beta, \gamma) := \sqrt{(j + n)! (j - n)! (j + m)! (j - m)!} e^{i(n\alpha + m\beta)} (\sin \frac{1}{2} \beta)^2 \times \sum_r (-1)^{j + m - r} \binom{j + m}{r} \binom{j - m}{r - m - n} (\cot \frac{1}{2} \beta)^{2r - m - n}.
\]  

(9.48)

The Hilbert space \( L^2(SU(2)) \) is described by defining the Haar measure on \( SU(2) \) in terms of the Euler angles as \( dg = (16\pi^2)^{-1} \sin \beta \, d\alpha \, d\beta \, d\gamma \), and the Parseval–Plancherel formula [37] shows that the functions \( D^j_{mn} \) form an orthonormal basis for this Hilbert space:

\[
\int_{SU(2)} |h(g)|^2 \, dg = \sum_{2j=0}^\infty (2j + 1) \sum_{m,n=-j}^j |(D^j_{mn} | h)|^2.
\]

On comparing the definitions (9.45), (9.46) of the functions \( Y^\pm_{lm} \) with (9.48), we see from \( \psi = e^{i\phi} \cot \frac{1}{2} \theta \) that

\[
Y^+_{lm}(z, \bar{z}) = \sqrt{\frac{2l + 1}{4\pi}} D^1_{\frac{1}{2},m}(\phi, \theta, \phi), \quad Y^-_{lm}(z, \bar{z}) = \sqrt{\frac{2l + 1}{4\pi}} D^1_{\frac{1}{2},m}(\phi, \theta, -\phi).
\]

Exercise 9.20. Show that, with \( \zeta = e^{-i\phi} \tan \frac{1}{2} \theta \), the formulae

\[
Y^+_{lm}(\zeta, \bar{\zeta}) = \sqrt{\frac{2l + 1}{4\pi}} D^1_{\frac{1}{2},m}(\phi, \theta, -\phi), \quad Y^-_{lm}(\zeta, \bar{\zeta}) = \sqrt{\frac{2l + 1}{4\pi}} D^1_{\frac{1}{2},m}(\phi, \theta, -\phi)
\]

express \( Y^\pm_{lm} \) in terms of the \( SU(2) \) representative functions over \( U_1 \).

We now (at last!) fix the normalization of the inner product of spinors by

\[
\|\psi\|^2 = \langle\psi | \psi\rangle := \frac{1}{4\pi} \int_{S^2} (\bar{\psi}^+ \psi^+ + \bar{\psi}^- \psi^-) \sin \theta \, d\theta \wedge d\phi.
\]  

(9.49)

Proposition 9.12. The spinors \( \{ Y^l_{l,m}, Y^l_{\ell,m} : l \in \mathbb{N} + \frac{1}{2}, m \in \{-l, \ldots, l\} \} \) form an orthonormal basis for the Hilbert space \( L^2(S) \).

Proof. We associate to each spinor \( \psi \in \Gamma(S) \) a pair of functions \( h^\pm \) on \( SU(2) \) by

\[
h^\pm(\phi, \theta, \psi) := \sqrt{4\pi} e^{\pm i(\phi - \psi)/2} \psi^\pm_N(z, \bar{z}) = \sqrt{4\pi} e^{\pm i(\phi + \psi)/2} \psi^\pm_S(\zeta, \bar{\zeta}).
\]

122
By integration over the $\psi$ variable, we see that $h^+$ is orthogonal to $D_{jm}^i$ unless $m = -\frac{1}{2}$, and $h^-$ is orthogonal to $D_{jm}^i$ unless $m = +\frac{1}{2}$. The Parseval–Plancherel formula then shows that

$$\|\psi\|^2 = \frac{1}{4\pi} \int_{SU(2)} \left( |h^+(g)|^2 + |h^-(g)|^2 \right) dg$$

$$= \sum_{l,m} \frac{(2l + 1)}{4\pi} \left( |\langle D_{-\frac{1}{2},m}^i | h^+ \rangle|^2 + |\langle D_{\frac{1}{2},m}^i | h^- \rangle|^2 \right)$$

$$= \sum_{l,m} \frac{1}{4\pi} \int_{S^2} \overline{Y}_{lm}^+ \psi^+ \Omega \bigg| + \frac{1}{4\pi} \int_{S^2} \overline{Y}_{lm}^- \psi^- \Omega \bigg|^2$$

$$= \sum_{l,m} \left| \langle Y'_{lm} | \psi \rangle \bigg|^2 + \left| \langle Y''_{lm} | \psi \rangle \bigg|^2. \quad (9.50)$$

This is a Parseval identity for the orthonormal family $\{Y'_{lm}, Y''_{lm}\}$, and so establishes completeness of this family in $L^2(S)$.

**Corollary 9.13.** The spectra of the Dirac operator, its square, the Casimir operator and the spinor Laplacian are given by

- $\text{sp}(\mathcal{D}) = \{ \pm(l + \frac{1}{2}) : l \in \mathbb{N} + \frac{1}{2} \} = \mathbb{Z} \setminus \{0\}$,
- $\text{sp}(\mathcal{D}^2) = \{ (l + \frac{1}{2})^2 : l \in \mathbb{N} + \frac{1}{2} \}$,
- $\text{sp}(C) = \{ l(l + 1) : l \in \mathbb{N} + \frac{1}{2} \}$,
- $\text{sp}(\Delta^S) = \{ l^2 + l - \frac{1}{4} : l \in \mathbb{N} + \frac{1}{2} \}$.

The respective multiplicities are: $2l + 1$ for the eigenvalue $\pm(l + \frac{1}{2})$ of $\mathcal{D}$, and $2(2l + 1)$ for each listed eigenvalue of $\mathcal{D}^2$, $C$ and $\Delta^S$.

**Proof.** The eigenvalues of $\mathcal{D}$ are those given by Corollary 9.11; the completeness relation (9.50) shows that there are no others. The eigenvalues of $C$ and $\Delta^S$ follow from (9.43). Notice that the Casimir eigenvalues have the form $l(l + 1)$ (compare the spectrum of the Hodge Laplacian), but with $l$ half-integral in the present case.

We recall from (4.16) the definition of the index of (any) Dirac operator:

$$\text{ind} \mathcal{D} := \dim(\ker \mathcal{D}^+) - \dim(\ker \mathcal{D}^-),$$

We end with an important result.

**Corollary 9.14.** For the spinor module over $S^2$, the index of the Dirac operator is zero.

The Atiyah–Singer index theorem [9, 28, 39, 42] asserts the existence of a characteristic class whose integral coincides with the index of the Dirac operator. In fact, the characteristic form is given by $\hat{A}(R) := \det^{-1/2}(j(R))$, where $R$ is the Riemannian curvature and $j(x) := (\sinh \frac{1}{2}x)/\frac{1}{2}x$. Since the power series $j(x)$ is even, however, only forms of degree $4k$ appear in

123
the expansion of $\hat{A}(R)$ by degrees; in particular, the second-degree component is zero. Thus for $S^2$ or indeed any compact two-dimensional manifold $M$, the integral $(2\pi i)^{-1} \int_M \hat{A}(R)$ (called the “$\hat{A}$-genus of $M$”) vanishes, and on that basis the vanishing of the index of the Dirac operator is to be expected.

To get a Dirac operator with a nontrivial kernel, one only has to twist the irreducible spinor module with a complex line bundle, such as $H$ or $L$. In that context, the eigenspinors (9.45) yield explicit solutions of the Seiberg–Witten equations [60].

10 Construction of representations of $SU(2)$

The theory of the Dirac operator on the Riemann sphere, developed in the previous chapter, has a direct application to the construction of the irreducible unitary representations of the group $SU(2)$. There are three main routes to that goal. In view of the Peter–Weyl theorem, an irreducible representation of a compact Lie group is finite-dimensional, unitarizable (that is, the representation space may be provided with an inner product that is invariant under the group action) and lies within the left regular representation of the group. The first route to the irreducible unitary representations is an algebraic construction called the “theorem of the highest weight” [13, 35, 37] which yields an abstract description of all such representations up to unitary equivalence, but does not exhibit them concretely. Secondly, the famous Borel–Weil theorem [11, 50] constructs the representation spaces as spaces of holomorphic sections of certain line bundles over coadjoint orbits of the compact Lie group in question; this construction depends on the fact that these orbits are complex manifolds, and therefore carry a spin$^c$ structure. It turns out that the maximal-dimensional coadjoint orbits in fact carry a spin$^c$ structure, compatible with the group action, and this opens a third path to the construction of representations, whereby the representation spaces are the kernel spaces of certain Dirac operators. (Indeed, this last recipe is related to the Borel–Weil–Bott construction by a “twisting” with a certain line bundle, so the second and third routes are thereby equivalent.) Here we shall not attempt to give the construction for all compact groups, but rather shall build up the particular example of $SU(2)$, that contains all the features of the general theory.

10.1 Characters of the maximal torus

Any compact connected Lie group $G$ contains a maximal torus $T$, and any two such tori are conjugate, by a basic theorem of Weyl [13]. For $G = SU(2)$, we may take $T$ to be the subgroup of diagonal unitary matrices $k(\phi)$ of (9.36), so that $T \simeq U(1)$. Its characters are

$$\chi_m(k(\phi)) = \chi_m \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = e^{im\phi/2}$$  \hspace{1cm} (10.1)

where $m$ is necessarily an integer in order that $\chi_m$ be a well-defined homomorphism from $T$ to $U(1)$. The Hopf map $\eta$ of (9.26) allows us to regard $S^2$ as the homogeneous space $SU(2)/T$, whereby $SU(2) \rightarrow S^2$ becomes a principal $U(1)$-bundle.
Let \( x \in \mathbb{S}^2 \); if it is not the north or south pole, let \( \zeta = z^{-1} \) be its local coordinates. Two particular local sections of this principal bundle are \( \gamma_0 \in \Gamma(U_0, SU(2)) \) and \( \gamma_1 \in \Gamma(U_1, SU(2)) \), given by

\[
\gamma_0(x) := \frac{1}{\sqrt{1 + z\bar{z}}} \begin{pmatrix} \bar{z} & 1 \\ -1 & z \end{pmatrix}, \quad \gamma_1(x) := \frac{1}{\sqrt{1 + \zeta\bar{\zeta}}} \begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix}.
\]

It is immediate that \( \eta(\gamma_0(x)) = z^{-1} = \zeta = \eta(\gamma_1(x)) \) for \( x \in U_0 \cap U_1 \), which verifies that these are sections and implies that \( \gamma_0(x) = \gamma_1(x)h(\zeta) \) with \( h(\zeta) \in T \); a quick calculation yields

\[
h(\zeta) = \begin{pmatrix} \sqrt{\zeta/\bar{\zeta}} & 0 \\ 0 & \sqrt{\zeta/\bar{\zeta}} \end{pmatrix},
\]

so that \( \chi_m(h(\zeta)) = (\zeta/\bar{\zeta})^{m/2} \).

Every complex line bundle over \( \mathbb{S}^2 \) may be constructed as an associated bundle to this principal bundle via a suitable character of the maximal torus \( T \). This can be seen directly by exhibiting these associated bundles, since we have already classified all line bundles over \( \mathbb{S}^2 \) in Proposition 5.13. However, it should be said that one can take a more principled point of view. On each such line bundle, we shall eventually construct an equivariant bundle action of \( SU(2) \); the equivalence classes of such actions generate a commutative semiring (under the Whitney sum and tensor product of equivariant vector bundles), and by formal subtraction (the Grothendieck construction) they generate an abelian group, denoted \( K_{SU(2)}(\mathbb{S}^2) \) [48].

The characters of the maximal torus \( T \) form a ring \( R(T) \), isomorphic to \( \mathbb{Z} \), and the matching of equivariant line bundles and characters yields a canonical isomorphism \( K_{SU(2)}(\mathbb{S}^2) \cong R(T) \).

The line bundle \( L^m \to \mathbb{S}^2 \) associated to \( SU(2) \to \mathbb{S}^2 \) via the character \( \chi_m \) of \( T \) is given by (1.2): the space \( L^m \) has elements \( [g, v] \) with \( g \in SU(2), v \in \mathbb{C} \), and \( [gh, v] = [g, \chi_m(h)v] \) whenever \( h \in T \). The local sections \( s_0 \in \Gamma(U_0, L^m) \) and \( s_1 \in \Gamma(U_1, L^m) \) defined by \( s_j(x) := [\gamma_j(x), 1] \) suffice to determine the class of \( L^m \). Indeed,

\[
s_0(x) = [\gamma_1(x)h(\zeta), 1] = [\gamma_1(x), \chi_m(h(\zeta))] = \chi_m(h(\zeta))s_1(x) = (\zeta/\bar{\zeta})^{m/2}s_1(x),
\]

so the transition function of \( L^m \) is \( g_{01}(\zeta) = (\zeta/\bar{\zeta})^{m/2} = e^{-im\phi/2} \). As before, we represent a general section \( s: \mathbb{S}^2 \to L^m \) by a pair of functions \( (f_0, f_1) \) satisfying \( f_0(z)s_0(x) = f_1(\zeta)s_1(x) \) on \( U_0 \cap U_1 \). We therefore get the gauge transformation rule

\[
f_1(\zeta) \equiv (\bar{\zeta}/\zeta)^{m/2}f_0(\zeta^{-1}).
\]

From Exercise 9.1 (compare also (9.5) and (9.6), which are particular cases), we obtain that the Chern class of \( L^m \) is \( m[L] = -m[H] \), so that \( L^m \) is indeed equivalent to the \( m \)-th tensor power of the tautological line bundle, as the notation had anticipated.\(^1\)

\(^1\)The isomorphism \( K_{SU(2)}(\mathbb{S}^2) \cong R(T) \) is thus given by \( m[L] \mapsto \chi_m \).
10.2 Twisting connections

In subsection 9.3 we have seen that the spin connection $\nabla^S$ is defined over $U_1$ (say) by $\nabla^S_\partial = \partial_t + \omega_1$, where $\zeta = x^1 + ix^2$ and $\omega_1, \omega_2$ are given by (9.9). Since $\partial/\partial \zeta = \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial/\partial \bar{\zeta} = \frac{1}{2}(\partial_1 + i\partial_2)$, we may rewrite (9.9) as $^2$

$$\nabla^S_{\partial/\partial \zeta} = \frac{\partial}{\partial \zeta} + \frac{i\bar{\zeta}}{2(1 + \zeta \bar{\zeta})} \gamma^1 \gamma_2, \quad \nabla^S_{\partial/\partial \bar{\zeta}} = \frac{\partial}{\partial \bar{\zeta}} - \frac{i\bar{\zeta}}{2(1 + \zeta \bar{\zeta})} \gamma^1 \gamma_2.$$  

If we use the presentation (9.12) of $\gamma^1$ and $\gamma^2$, in which the grading operator $i\gamma^1 \gamma^2$ is a diagonal matrix, these become

$$\nabla^{S^\pm}_{\partial/\partial \zeta} = \frac{\partial}{\partial \zeta} \pm \frac{\bar{\zeta}}{2(1 + \zeta \bar{\zeta})}, \quad \nabla^{S^\pm}_{\partial/\partial \bar{\zeta}} = \frac{\partial}{\partial \bar{\zeta}} \mp \frac{\bar{\zeta}}{2(1 + \zeta \bar{\zeta})}, \quad (10.2)$$

Now $S^+ = L$, the tautological bundle over $S^2$. If $f_1, \ldots, f_m$, with $m \in \mathbb{N}$, are local sections in $\Gamma(U_1, L)$, then their product is a local section in $\Gamma(U_1, L^m)$. Denote by $\nabla^{(m)}$ the tensor product of $m$ copies of $\nabla^{S^+}$; this is a connection on the vector bundle $L^m \rightarrow S^2$, for which

$$\nabla^{(m)}_X (f_1 \cdots f_m) = \sum_{j=1}^m f_1 \cdots f_{j-1} \nabla^{S^+}_X (f_j) f_{j+1} \cdots f_m,$$

from which it follows that

$$\nabla^{(m)}_{\partial/\partial \zeta} = \frac{\partial}{\partial \zeta} + \frac{m\bar{\zeta}}{2(1 + \zeta \bar{\zeta})}, \quad \nabla^{(m)}_{\partial/\partial \bar{\zeta}} = \frac{\partial}{\partial \bar{\zeta}} - \frac{m\bar{\zeta}}{2(1 + \zeta \bar{\zeta})}, \quad (10.3)$$

on $\Gamma(U_1, L^m)$.

Exercise 10.1. If $m$ is a negative integer, let $\nabla^{(m)}$ be the tensor product of $(-m)$ copies of $\nabla^{-S^+}$, i.e., the connection on $L^{-m} = H^m$ dual to the connection $\nabla^{(-m)}$ on $L^{-m}$; show that (10.3) holds also in this case.

Exercise 10.2. The curvature $\omega_m$ of the connection $\nabla^{(m)}$ is determined by the relation $\omega_m(\partial/\partial \zeta, \partial/\partial \bar{\zeta}) = [\nabla^{(m)}_{\partial/\partial \zeta}, \nabla^{(m)}_{\partial/\partial \bar{\zeta}}]$. Check that $\omega_m = -m(1 + \bar{\zeta})^{-2} d\zeta \wedge d\bar{\zeta}$.

Definition 10.1. Let $S \rightarrow S^2$ denote the irreducible spinor bundle; let $L^m \rightarrow S^2$ be the complex line bundle with first Chern class $m[L]$; we call $S \otimes L^m \rightarrow S^2$ a twisted spinor bundle. Clearly $S \otimes L^m \sim L^{m+1} \oplus L^{m-1}$. The twisted spin connection $\tilde{\nabla} := \nabla^S \otimes 1 + 1 \otimes \nabla^{(m)}$ is determined, in view of (10.2) and (10.3), by

$$\tilde{\nabla}^{\pm}_{\partial/\partial \zeta} = \frac{\partial}{\partial \zeta} + \frac{(m \pm 1)\bar{\zeta}}{2(1 + \zeta \bar{\zeta})}, \quad \tilde{\nabla}^{\pm}_{\partial/\partial \bar{\zeta}} = \frac{\partial}{\partial \bar{\zeta}} - \frac{(m \pm 1)\bar{\zeta}}{2(1 + \zeta \bar{\zeta})}. \quad (10.4)$$

$^2$In this Section, we shall write explicit formulas over the chart domain $U_1$ with local coordinates $(\zeta, \bar{\zeta})$; we leave the corresponding formulas for $U_0$ to the reader.
10.3 Twisted Dirac operators

Definition 10.2. The Dirac operator $\mathcal{D}_m$ on the twisted spinor bundle $S \otimes L^m \rightarrow S^2$ is given, in accordance with Definition 8.3, by $\mathcal{D}_m := c(dx^1)\nabla_{\partial_1} + c(dx^2)\nabla_{\partial_2} = c(d\zeta)\nabla_{\partial/\partial\zeta} + c(d\bar{\zeta})\nabla_{\partial/\partial\bar{\zeta}}$. From (9.10), we get

\[
c(d\zeta) = -\frac{1}{2}(1 + \zeta \bar{\zeta})(\gamma^1 + i\gamma^2) = \begin{pmatrix} 0 & -(1 + \zeta \bar{\zeta}) \\ 0 & 0 \end{pmatrix},
\]
\[
c(d\bar{\zeta}) = -\frac{1}{2}(1 + \zeta \bar{\zeta})(\gamma^1 - i\gamma^2) = \begin{pmatrix} 0 & 0 \\ 1 + \zeta \bar{\zeta} & 0 \end{pmatrix},
\]
so that $\mathcal{D}_m = c(d\zeta)\nabla_{\partial/\partial\zeta} + c(d\bar{\zeta})\nabla_{\partial/\partial\bar{\zeta}}$; explicitly,

\[
\mathcal{D}_m = \begin{pmatrix} 0 & -\frac{1}{2}(1 + \zeta \bar{\zeta}) \partial/\partial\zeta - \frac{1}{2}(m + 1)\zeta \\ \partial_\zeta - \frac{1}{2}m\zeta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{D}_m^- \\ \mathcal{D}_m^+ & 0 \end{pmatrix}.
\]

From (9.15) we get $\delta_\zeta \psi + \frac{1}{2}m\bar{\zeta}\psi = Q_1(\zeta)\partial/\partial\zeta + \frac{1}{2}(m - 1)(\partial Q_1/\partial\zeta)\psi$, where we have written $Q_1(\zeta) = 1 + \zeta \bar{\zeta}$. Also, $\delta_\zeta \psi - \frac{1}{2}m\bar{\zeta}\psi = Q_1(\zeta)\partial/\partial\bar{\zeta} - \frac{1}{2}(m + 1)(\partial Q_1/\partial\bar{\zeta})\psi$. Thus,

\[
\mathcal{D}_m^- = -Q_1(\zeta)^{-m-1/2} \frac{\partial}{\partial\zeta} Q_1(\zeta)^{1/2},
\]
\[
\mathcal{D}_m^+ = Q_1(\zeta)^{m+1/2} \frac{\partial}{\partial\bar{\zeta}} Q_1(\zeta)^{-1/2}.
\]

The kernel of $\mathcal{D}_m^-$ is a subspace of $\Gamma(L^{m-1})$ and the kernel of $\mathcal{D}_m^+$ is a subspace of $\Gamma(L^{m+1})$. Suppose that $\psi \in \Gamma(L^{m+1})$, and let $\psi_S(\zeta, \bar{\zeta})$ and $\psi_N(z, \bar{\zeta})$ be its component functions over $U_1$ and $U_0$ respectively. These are related by the gauge transformation rule $\psi_N(\zeta^{-1}, \bar{\zeta}^{-1}) = (\zeta/\bar{\zeta})^{1/2} \psi_S(\zeta, \bar{\zeta})$. From (10.6), $\psi \in \ker \mathcal{D}_m^+$ iff

\[
a(\zeta) := (1 + \zeta \bar{\zeta})^{-(m+1)/2} \psi_S(\zeta, \bar{\zeta})
\]

is an entire holomorphic function on $\mathbb{C}$. Moreover,

\[
\psi_N(\zeta^{-1}, \bar{\zeta}^{-1}) = ((1 + \zeta \bar{\zeta})\zeta/\bar{\zeta})^{1/2}a(\zeta)
\]

must be regular at $\zeta = \infty$. Since $Q_1(\zeta)^{1/2} = O(|\zeta|^{m+1})$, for nonnegative $m$ this is only possible if $a(\zeta) \equiv 0$. If $m$ is negative, the entire function $a(\zeta)$ is $O(|\zeta|^{-m-1})$ as $|\zeta| \to \infty$, so that $a(\zeta)$ is a polynomial of degree at most $|m| - 1$. To sum up:

\[
\dim \ker \mathcal{D}_m^+ = \begin{cases} 0, & \text{if } m \geq 0, \\ |m|, & \text{if } m < 0. \end{cases}
\]
The kernel of \( \mathcal{D}_m^- \) is found similarly. Suppose that \( \phi \in \Gamma(L^{m-1}) \), with component functions \( \phi_S(\zeta, \bar{\zeta}) \) and \( \phi_N(z, \bar{z}) \) related by \( \phi_N(\zeta^{-1}, \bar{\zeta}^{-1}) \equiv (\zeta/\bar{\zeta})^{(m-1)/2} \phi_S(\zeta, \bar{\zeta}) \). Then \( \phi \in \ker \mathcal{D}_m^- \) iff

\[
b(\bar{\zeta}) := (1 + \zeta \bar{\zeta})^{(m-1)/2} \phi_S(\zeta, \bar{\zeta})
\]

is antiholomorphic. The regularity at \( \infty \) of \( \phi_N(\zeta^{-1}, \bar{\zeta}^{-1}) = ((1 + \zeta \bar{\zeta}) \zeta/\bar{\zeta})^{-(m-1)/2} b(\bar{\zeta}) \), together with \( Q_1(\zeta)^{-(m-1)/2} = O(|\zeta|)^{1-m} \) shows that \( b(\zeta) \equiv 0 \) for \( m \) negative or zero, while for \( m \) positive \( b(\zeta) \) is a polynomial in \( \zeta \) of degree at most \( m - 1 \). In fine:

\[
\dim \ker \mathcal{D}_m^- = \begin{cases} 
  m, & \text{if } m > 0, \\
  0, & \text{if } m \leq 0.
\end{cases}
\]

We have in particular shown the following result.

**Proposition 10.1.** The index of the Dirac operator \( \mathcal{D}_m \) on the twisted spinor bundle \( S \otimes L^m \to \mathbb{S}^2 \) is given by

\[
\text{ind} \mathcal{D}_m = \text{dim} \ker \mathcal{D}_m^+ - \text{dim} \ker \mathcal{D}_m^- = -m,
\]

that is, by the integer that labels the first Chern class of the twisting line bundle \( L^m \). \( \square \)

### 10.4 The group action on the twisted spinor bundles

**Proposition 10.2.** A bundle action of \( SU(2) \) on \( S \otimes L^m \to \mathbb{S}^2 \) is given by the formula (9.27), where \( \psi^\pm \in \Gamma(L^{m \pm 1}) \), and where the multipliers \( A_N^\pm, A_S^\pm \) are determined by

\[
A_N^\pm(g, z) := \left( \frac{\beta \bar{z} + \alpha}{\bar{\beta} z + \alpha} \right)^{(m \pm 1)/2}, \quad A_S^\pm(g', \zeta) := \left( \frac{-\bar{\beta} \zeta + \alpha}{-\beta \zeta + \alpha} \right)^{(m \pm 1)/2}.
\]

**Proof.** The analysis of the \( SU(2) \)-action on the untwisted spinor bundle, given in Section 9.6, may be repeated verbatim, except that the gauge transformation factor \((\zeta/\bar{\zeta})^{1/2}\) in (9.30) must be replaced by \((\zeta/\bar{\zeta})^{(m \pm 1)/2}\). This leads to the necessary and sufficient condition (9.31), with the exponent \( 1/2 \) replaced by \((m \pm 1)/2\), and the cocycles (10.2) satisfy that condition. \( \square \)

**Proposition 10.3.** The Dirac operator \( \mathcal{D}_m \) is equivariant under the action of \( SU(2) \) on the twisted spinor bundle \( S \otimes L^m \to \mathbb{S}^2 \) determined by (10.2).

**Proof.** Let \( T \in \text{End}^+(\Gamma(S \otimes L^m)) \) be an operator that commutes with \( \mathcal{D}_m \) and is of the form \((T \psi^\pm_S)(\zeta, \bar{\zeta}) = c_\pm(\zeta, \bar{\zeta}) \psi^\pm_S(b(\zeta), \bar{\zeta}) \). Then, as in Lemma 9.5, the identity \( T \mathcal{D}_m^- = \mathcal{D}_m^- T \) —in \( \text{Hom}(\Gamma(L^{m-1}), \Gamma(L^{m+1})) \)— yields the relations

\[
Q_1(\zeta)c_-((\zeta, \bar{\zeta}) b'((\zeta)) = Q_1(b(\zeta))c_+(\zeta, \bar{\zeta}),
\]

\[
Q_1(\zeta)\frac{\partial c_-}{\partial \zeta} = \frac{1}{2}(m - 1)(\overline{b(\zeta)} c_+(\zeta, \bar{\zeta}) - \bar{\zeta} c_-((\zeta, \bar{\zeta})).
\]

(10.9)
Similarly, from $T\mathcal{D}_m^+ = \mathcal{D}_m^+ T$ in $\text{Hom}(\Gamma(L^{m+1}), \Gamma(L^{m-1}))$ we get

$$Q_1(\zeta) c_+(\zeta, \bar{\zeta}) \partial c_+ = Q_1(b(\zeta)) c_-(\zeta, \bar{\zeta}),$$

$$Q_1(\zeta) \frac{\partial c_+}{\partial \zeta} = -\frac{1}{2} (m + 1)(b(\zeta) c_-(\zeta, \bar{\zeta}) - c_+(\zeta, \bar{\zeta})). \tag{10.10}$$

We check that these conditions are satisfied when $T = T(g)$ is the operator corresponding to the diagonal element $g = k(\phi)$ of $SU(2)$, determined by (10.2). Here $b(\zeta) = g^{-1} \cdot \zeta = e^{i\phi} \zeta$. Clearly $Q_1(b(\zeta)) \equiv Q_1(\zeta)$, so that (10.9) and (10.10) reduce to the conditions $c_+(\zeta, \bar{\zeta}) = e^{i\phi} c_-(\zeta, \bar{\zeta})$, $\partial c_- / \partial \zeta = 0$, and $\partial c_+ / \partial \zeta = 0$, which together imply that $c_-$ and $c_+$ are independent of $\zeta$ and $\bar{\zeta}$. Since the prescription $c_+(\zeta, \bar{\zeta}) := e^{i(m+1)\phi/2}$, dictated by (10.2), obeys $c_+ = e^{i\phi} c_-$, we see that $T(g)$ commutes with $\mathcal{D}_m$.

In the same way, it is easy to show that the operator $T(h(\theta))$, associated to $h(\theta)$ of (9.36) by $b(\zeta) = h(\theta)^{-1} \cdot \zeta$ and by (10.2), commutes with $\mathcal{D}_m$. Since elements of the form $k(\phi)$ and $h(\theta)$ generate $SU(2)$, and since $g \mapsto T(g)$ is a homomorphism on account of the cocycle relations (9.29), we conclude that $T(g) \mathcal{D}_m = \mathcal{D}_m T(g)$ for all $g \in SU(2)$. \qed

**Exercise 10.3.** Complete the proof of the previous Proposition by verifying that the coefficient functions of $T(h(\theta))$ satisfy (10.9) and (10.10).

**Corollary 10.4.** The bundle action of $SU(2)$ on $S \otimes L^m \rightarrow \mathbb{S}^2$ determined by (9.27) and (10.2) restricts to a finite-dimensional unitary representation $\rho_m$ of $SU(2)$ on the kernel of the Dirac operator $\mathcal{D}_m$.

To exhibit the representation $\rho_m$, it is useful to notice that with $g^{-1} \cdot \zeta = (\alpha \zeta + \bar{\beta})/(-\beta \zeta + \bar{\alpha})$, one has $Q_1(g^{-1} \cdot \zeta) = Q_1(\zeta) - \beta \zeta + \bar{\alpha} c_-$ on account of $\alpha \bar{\alpha} + \beta \bar{\beta} = 1$. If $m > 0$, then for $\psi \in \ker \mathcal{D}_m \subset \Gamma(L^{m-1})$, the bundle action (9.27) specializes to

$$\rho_m \left( \begin{array}{c} \alpha \\ -\beta \\ \bar{\alpha} \end{array} \right) \psi_S(\zeta, \bar{\zeta}) = \left( \begin{array}{c} -\bar{\beta} \bar{\zeta} + \alpha \\ -\beta \zeta + \bar{\alpha} \end{array} \right)^{(m-1)/2} \psi_S \left( \begin{array}{c} \alpha \zeta + \bar{\beta} \\ -\beta \zeta + \bar{\alpha} \end{array} \right). \tag{10.11}$$

Using (10.8), we can write $\psi_S(\zeta, \bar{\zeta}) = Q_1(\zeta)^{-(m-1)/2} b(\zeta)$ with $b$ a polynomial in $\zeta$ of degree less than $m$, and the previous formula simplifies to

$$\rho_m \left( \begin{array}{c} \alpha \\ -\beta \\ \bar{\alpha} \end{array} \right) \left[ Q_1(\zeta)^{-(m-1)/2} b(\zeta) \right] = Q_1(\zeta)^{-(m-1)/2} (-\bar{\beta} \bar{\zeta} + \alpha)^{m-1} b \left( \frac{-\bar{\alpha} \bar{\zeta} + \beta}{-\beta \zeta + \bar{\alpha}} \right).$$

The right hand side is clearly also $Q_1^{-(m-1)/2}$ times a polynomial in $\zeta$ of degree less than $m$. If we define $\psi_k(\zeta, \bar{\zeta}) := Q_1(\zeta)^{-(m-1)/2} \bar{\zeta}^k$, for $k = 0, 1, \ldots, m-1$, these form an orthogonal basis for $\ker \mathcal{D}_m$. For these basis vectors,

$$\rho_m \left( \begin{array}{c} \alpha \\ -\beta \\ \bar{\alpha} \end{array} \right) \psi_k(\zeta, \bar{\zeta}) = Q_1(\zeta)^{-(m-1)/2} (\bar{\alpha} \bar{\zeta} + \beta)^k (-\bar{\beta} \bar{\zeta} + \alpha)^{m-k-1}, \tag{10.12}$$

from which the matrix elements of the representation may be computed at once.
When \( m < 0 \), we obtain similar formulas for \( \rho_m \). Indeed, for \( \psi \in \ker \mathcal{D}^-_m \subset \Gamma(L^{m+1}) \), we obtain (10.11) with the exponent \((m - 1)/2\) replaced by \((m + 1)/2\). Since, by (10.7), \( \psi_S(\zeta, \bar{\zeta}) = Q_1(\zeta)^{(m+1)/2} a(\zeta) \) with \( a \) a polynomial in \( \zeta \) of degree less than \( m \), we get

\[
\rho_m \left( \frac{\alpha}{-\bar{\alpha}}, \frac{\beta}{\bar{\alpha}} \right) [Q_1(\zeta)^{(m+1)/2} a(\zeta)] = Q_1(\zeta)^{(m+1)/2}(-\beta \zeta + \bar{\alpha})^{m-1} a \left( \frac{\alpha \zeta + \bar{\beta}}{-\beta \zeta + \bar{\alpha}} \right),
\]

(10.13)

and on the orthogonal basis \( \psi_k(\zeta, \bar{\zeta}) := Q_1(\zeta)^{-(m-1)/2} \zeta^k \) \((k = 0, 1, \ldots, m - 1)\) for \( \ker \mathcal{D}^-_m \), we find that

\[
\rho_m \left( \frac{\alpha}{-\bar{\alpha}}, \frac{\beta}{\bar{\alpha}} \right) \psi_k(\zeta, \bar{\zeta}) = Q_1(\zeta)^{-(m-1)/2}(\alpha \zeta + \bar{\beta})(-\beta \zeta + \bar{\alpha})^{m-k-1}.
\]

(10.14)

From the explicit formulae (10.12) and (10.14), it is evident that \( \rho_{-m} \) is the conjugate representation to \( \rho_m \).

**Proposition 10.5.** The representation \( \rho_m \) is irreducible.

**Proof.** If \( m = 0 \), there is nothing to prove. The case \( m < 0 \) mirrors the case \( m > 0 \), on interchanging \( \mathcal{D}^+_m \) with \( \mathcal{D}^-_m \); thus we may take \( m > 0 \). Then \( \ker \mathcal{D}_m = \ker \mathcal{D}^-_m \) is an \( m \)-dimensional Hilbert space with the orthogonal basis \( \{ \psi_k : k = 0, 1, \ldots, m - 1 \} \). It is immediate from (10.12) that this basis consists of joint eigenvectors for \( \{ \rho_m(g) : g \in T \} \); indeed, with \( g = \bar{k}(\phi) \) we get

\[
\rho_m(\bar{k}(\phi)) \psi_k = e^{i(m-1)\phi/2} e^{-ik\phi} \psi_k,
\]

(10.15)

so the eigenvalues are generally distinct, and therefore any operator \( S \) on \( \ker \mathcal{D}^-_m \) that commutes with each \( \rho_m(g) \) must have a diagonal matrix in this basis. On the other hand, the one-parameter subgroup of operators \( t \mapsto \rho_m(h(t)) \) mingles the basis elements \( \psi_k \): for \( h(t) \), one has \( \alpha = \cos \frac{1}{2} t \), \( \beta = \sin \frac{1}{2} t \), and from (10.12) we get \( (d/dt)|_{t=0} \rho_m(h(t)) \psi_k = \frac{1}{2} k \psi_{k-1} - \frac{1}{2} (m - k - 1) \psi_{k+1} \) if \( 1 \leq k \leq m - 2 \). A similar calculation with the subgroup \( h'(t) \) for which \( \alpha = \cos \frac{1}{2} t \) and \( \beta = i \sin \frac{1}{2} t \) gives \( (d/dt)|_{t=0} \rho_m(h'(t)) \psi_k = \frac{1}{2} i k \psi_{k-1} + \frac{1}{2} i (m - k - 1) \psi_{k+1}. \) Thus if \( S \) commutes with every \( \rho_m(g) \), it commutes with the “ladder operator” \( \psi_k \mapsto k \psi_{k-1} \), and hence \( S \) is scalar. Irreducibility of \( \rho_m \) now follows from Schur’s lemma. \( \square \)

In order to identify the unitary irreducible representation \( \rho_m \) of \( SU(2) \), it suffices to compute its character. Since the character is a class function, it is determined by its restriction to a maximal torus, so for \( m > 0 \) we obtain it immediately from (10.15):

\[
\chi_m(\bar{k}(\phi)) = \sum_{k=0}^{m-1} e^{i(m-1)\phi/2} e^{-ik\phi} = \frac{e^{i(m+1)\phi/2}}{e^{i\phi/2}} \frac{1 - e^{-im\phi}}{1 - e^{-i\phi}} = \frac{e^{i(m+1)\phi/2} - e^{-im\phi/2}}{e^{i\phi/2} - e^{-i\phi/2}}.
\]

When \( m < 0 \), we similarly get from (10.14)

\[
\rho_m(\bar{k}(\phi)) \psi_k = e^{i(m+1)\phi/2} e^{ik\phi} \psi_k,
\]

130
and so its character is:

\[ \chi_m(k(\phi)) = \sum_{k=0}^{\lfloor m/2 \rfloor} e^{i(m+1)\phi/2} e^{ik\phi} = \frac{1 - e^{im\phi}/2}{1 - e^{i\phi}/2} \left( e^{im\phi}/2 - e^{-im\phi}/2 \right) . \]

Therefore the representations \( \rho_m \) and \( \rho_{-m} \) are equivalent.

This equivalence is to be expected on general grounds. The Weyl group of \( SU(2) \), namely the normalizer subgroup of the maximal torus \( T \) modulo \( T \) itself, is just the two-element group \( W \simeq \mathbb{Z}_2 \), since the only nontrivial isomorphism \( \sigma \) of the diagonal subgroup \( T \) is the interchange of diagonal elements, that can be implemented by conjugating with \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

By (10.1), \( \chi_m \circ \sigma = \chi_{-m} \); so two characters of \( T \) lead to equivalent representations of \( SU(2) \) iff they lie in the same orbit under the action of the Weyl group. This exemplifies the general relation [13, 49]:

\[ R(G) \simeq R(T)^W. \]

This construction of the unitary irreducible representations of \( SU(2) \) directly from the equivariant Dirac operator exemplifies the “universal quantization map” of Vergne [55]:

\[ Q: K_{SU(2)}(S^2) \to R(SU(2)), \]

which is already an instance of the index theorem of Atiyah, Segal and Singer [5, 6].

10.5 The Borel–Weil theorem

The construction of the irreducible unitary representations of \( SU(2) \) on the kernel spaces of Dirac operators, developed in the preceding sections, produces two families of equivalent representations, according as one twists the standard bundle with a tensor power of the tautological bundle \( (L^m, \text{with } m > 0) \), or with a tensor power of the hyperplane bundle \( (H^n \sim L^{-n}, \text{with } n > 0) \). Now only the latter line bundles admit nonzero holomorphic sections, while only the former admit nonzero antiholomorphic sections. The celebrated construction of Borel and Weil does not deal with spinor bundles, but rather uses the fact that the coadjoint orbits of compact Lie groups are complex manifolds and constructs the desired representations on the spaces of holomorphic sections of holomorphic line bundles over those orbits. By a simple application of Liouville’s theorem — see Section 5.6 for the details in the case \( M = \mathbb{C}P^m \) — the compactness of the orbit implies that the space of holomorphic sections is finite-dimensional. Indeed, in many cases, that of the trivial line bundle for instance, the space of holomorphic sections reduces to zero. A variant of the Borel–Weil construction produces representations on space of antiholomorphic sections of the respective dual line bundles.

There is a simple relationship between the spinor-based and the Borel–Weil constructions: one passes from one to the other by twisting or untwisting with a fixed line bundle that can be obtained directly from the spin structure of the maximal coadjoint orbit \( G/T \). Indeed, this relationship is parallel to the replacement of the spin structure by a spin\(^c\) structure.
on a spin manifold: for that, one needs to identify a particular principal $U(1)$-bundle that combines with the spin structure to yield a spin$^c$ structure. (See the discussion in Section 7.3, and also Appendix D of [39].)

If $M$ is a complex manifold of complex dimension $m$, let $K := \Lambda^{m,0}T^*M$; then $K \longrightarrow M$ is the so-called canonical line bundle, and $\Gamma(K) = \mathcal{A}^{m,0}(M)$. The first Chern class of the dual bundle $K^*$, turns out [41] to be the element of $\check{H}^2(M, \mathbb{Z})$ whose modulo-2 reduction is the Stiefel–Whitney class $w_j$; thus $w_2(M) + j_*(c_1(K)) = 0$ in $\check{H}^2(M, \mathbb{Z}_2)$, in the notation of Section 7.3. If $M$ is spin, so that $w_2(M) = 0$, then $j_*(c_1(K)) = 0$ also, so that $c_1(K)$ is even; in other words, there is a complex line bundle $K^{1/2} \longrightarrow M$ such that $K^{1/2} \otimes K^{1/2} \sim K$. In principle, the “square root of the canonical bundle” $K^{1/2}$ may depend on the spin structure chosen for $M$.

For the case $M = \mathbb{S}^2$, we know that $K \sim L \otimes L = L^2$ by Exercise 9.3. Thus we can take $K^{1/2} := L$. Its dual line bundle is $K^{-1/2} := H$. Therefore we find that

$$S \otimes K^{-1/2} = (L \oplus H) \otimes H \sim E \oplus H^2 \sim \Lambda^0 \cdot T^* \mathbb{S}^2$$

where $E = \mathbb{S}^2 \times \mathbb{C}$ denotes the trivial line bundle, and $\Lambda^0 \cdot T^* \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is the complex vector bundle whose smooth sections form the module $\mathcal{A}^{0,0}(\mathbb{S}^2) = \mathcal{A}^{0,0}(\mathbb{S}^2) \oplus \mathcal{A}^{0,1}(\mathbb{S}^2)$. Over $U_1$, such a section may be written as $f(\zeta, \bar{\zeta}) + h(\zeta, \bar{\zeta})d\bar{\zeta}$.

Thus $\mathcal{A}^{0,\ast}(\mathbb{S}^2) = \Gamma(S \otimes L^{-1})$ is the domain of the twisted Dirac operator $\bar{\nabla}_{-1}$. Of course, (10.5) for $m = -1$ simplifies to

$$\bar{\nabla}_{-1} = \begin{pmatrix} 0 & -Q_1(\zeta) \partial/\partial \bar{\zeta} + \bar{\zeta} \\ -Q_1(\zeta) \partial/\partial \zeta & 0 \end{pmatrix}.$$ 

The operator $\bar{\nabla}^+_{-1} : \mathcal{A}^{0,0}(\mathbb{S}^2) \rightarrow \mathcal{A}^{0,1}(\mathbb{S}^2)$ has as kernel the holomorphic functions in $\mathcal{A}^{0,0}(\mathbb{S}^2) = C^\infty(\mathbb{S}^2)$, which are just the constant functions, by Liouville’s theorem. The factor $Q_1(\zeta) = 1 + \zeta \bar{\zeta}$ is a normalization factor, that takes account of the metric on $\mathcal{A}^{0,1}(\mathbb{S}^2)$.

To see that, consider the twisted spinor bundle $S \otimes L^m$ for any negative $m$, say $m = -(2j + 1)$ with $j$ a nonnegative half-integer. Clearly $S \otimes L^m = \Lambda^0 \cdot T^* S^2 \otimes H^{2j} = H^{2j} \oplus H^{2j+2}$, so that $S^+ \otimes L^m = H^{2j}$ admits holomorphic sections. If $s_1$ denotes a section of $H$ normalized by $(s_1|s_1) = 1$, then $s_1^2$ is a normalized section of $H^{2j}$. We may select $s_1$ as $s_1 := Q_1(\zeta)^{1/2} \sigma_1$, where $\sigma_1$ is the holomorphic section of $H \longrightarrow \mathbb{C}P(1)$ of Section 5.6: the normalization follows from (5.13), whereby we have $(\sigma_1|\sigma_1) = Q_1(\zeta)^{-1}$.

It is convenient to choose the metric $4g^{-1} = Q_1(\zeta)^2(\partial/\partial \zeta) \cdot (\partial/\partial \bar{\zeta})$ on $T^c_\zeta M$, so that the normalized section of $\mathcal{A}^{0,1}(\mathbb{S}^2)$ is just $s_1^2 = Q_1(\zeta)^{-1} d\bar{\zeta}$. It follows that $\sigma_1^2 = Q_1(\zeta)^{-2} d\bar{\zeta}$.

Any section of $\mathcal{A}^{0,1}(\mathbb{S}^2)$ is of the form $h(\zeta, \bar{\zeta})d\bar{\zeta} = Q_1(\zeta)h(\zeta, \bar{\zeta})s_1^2$, and a section of $S \otimes L^{-1} = L^0 \oplus L^{-2}$ can be written as $\phi = f(\zeta, \bar{\zeta}) + h(\zeta, \bar{\zeta})d\bar{\zeta}$. Then the Dirac operator $\bar{\nabla}_{-1}$ can be identified as follows. Firstly,

$$\bar{\nabla}_{-1} f = \left(Q_1(\zeta) \frac{\partial f}{\partial \zeta} \right) s_1^2 = \frac{\partial f}{\partial \bar{\zeta}} d\bar{\zeta} = \bar{\partial} f$$

---

3By now it should be clear that it is preferable to label representations of $SU(2)$ by nonnegative half-integers $j$ rather than the integers $m$.

4Without this normalization, the operator $\bar{\partial}$ in the following formulae should be replaced by $\sqrt{2} \bar{\partial}$; this convention is adopted in [28], for instance.
where \( \bar{\partial} \) is the Dolbeault operator of Section 2.1.

The twisted Dirac operator \( \mathcal{D}_m \), with \( m = -(2j + 1) \), is

\[
\mathcal{D}_m = \begin{pmatrix}
0 & -Q_1(\zeta)\partial/\partial \zeta + (j + 1)\bar{\zeta} \\
Q_1(\zeta)\partial/\partial \bar{\zeta} + j\bar{\zeta} & 0
\end{pmatrix},
\]

by rewriting (10.5). An element of \( \Gamma(H^{2j}) \) can be written (over \( U_1 \)) as \( f(\zeta, \bar{\zeta}) \sigma_1^{2j} \). Then

\[
\mathcal{D}_m^+(f(\zeta, \bar{\zeta}) \sigma_1^{2j}) = \mathcal{D}_m^+(Q_1(\zeta)^{-j} f(\zeta, \bar{\zeta}) s_1^{2j})
= \left( Q_1(\zeta)\frac{\partial}{\partial \zeta} + j\bar{\zeta} \right) [Q_1(\zeta)^{-j} f(\zeta, \bar{\zeta})] s_1^{2j+2} = Q_1(\zeta)^{-j+1} \frac{\partial f}{\partial \zeta} s_1^{2j+2}
= Q_1(\zeta)^2 \frac{\partial f}{\partial \zeta} \sigma_1^{2j+2} = \frac{\partial f}{\partial \zeta} d\zeta \otimes \sigma_1^{2j} = \bar{\partial} f \otimes \sigma_1^{2j}.
\]

The Dolbeault operator extends to \( \mathcal{A}^{0, \bullet}(S^2, H^{2j}) := \mathcal{A}^{0, \bullet}(S^2) \otimes \Gamma(H^{2j}) \) by setting \( \bar{\partial}(\omega \otimes s) := (\bar{\partial}\omega) \otimes s \) for \( s \in \Gamma(H^{2j}) \). With this understanding,\(^5\) we have the relation \( \mathcal{D}_{-2j-1} = \bar{\partial} \) as operators from \( \mathcal{A}^{0,0}(S^2, H^{2j}) \) to \( \mathcal{A}^{0,1}(S^2, H^{2j}) \).

In order to identify \( \mathcal{D}_{-m} \), we must compute the adjoint of the Dolbeault operator. We consider first the case \( m = -1 \).

**Definition 10.3.** The adjoint of \( \bar{\partial} \colon \mathcal{A}^{0,0}(S^2) \to \mathcal{A}^{0,1}(S^2) \) is the operator \( \bar{\partial}^* \colon \mathcal{A}^{0,1}(S^2) \to \mathcal{A}^{0,0}(S^2) \) given by

\[
\langle\langle \bar{\partial}^*(h \, d\bar{\zeta}) \mid f \rangle\rangle := \langle h \, d\bar{\zeta} \mid \bar{\partial} f \rangle
\]

where \( \langle\langle \cdot \mid \cdot \rangle\rangle \) denotes the integrated inner product in either space of sections, defined as in (9.49) by integrating the appropriate pairing of sections over \( S^2 \) with respect to the volume form \((4\pi)^{-1} \sin \theta \, d\theta \wedge d\phi = (2\pi i)^{-1} Q_1(\zeta)^{-2} d\zeta \wedge d\bar{\zeta} \). We find that

\[
\langle\langle \bar{\partial}^*(h \, d\bar{\zeta}) \mid f \rangle\rangle = \langle h \, d\bar{\zeta} \mid \bar{\partial} f \rangle = \frac{1}{2\pi i} \int_C 4g^{-1} \left( h \, d\bar{\zeta}, \frac{\partial f}{\partial \zeta} \right) Q_1(\zeta)^{-2} d\zeta \wedge d\bar{\zeta}
= \frac{1}{2\pi i} \int_C \left( -h \frac{\partial f}{\partial \zeta} d\zeta \wedge d\bar{\zeta} - \frac{\partial h}{\partial \zeta} f d\zeta \wedge d\bar{\zeta} \right)
= -\langle\langle Q_1(\zeta)^2 \partial h/\partial \zeta \mid f \rangle\rangle,
\]

on integrating by parts, so that \( \bar{\partial}^*(h \, d\bar{\zeta}) = -(1 + \zeta\bar{\zeta})^2 \partial h/\partial \zeta \).

From this it follows easily that

\[
\mathcal{D}_{-1}(h \, d\bar{\zeta}) = \mathcal{D}_{-1}(Q_1 h \, s_1^{2j}) = (-Q_1 \partial/\partial \zeta + \bar{\zeta})(Q_1 h) = -Q_1^2 \partial h/\partial \zeta = \bar{\partial}^*(h \, d\bar{\zeta}).
\]

Thus \( \mathcal{D}_{-1} = \bar{\partial}^* \) on \( \mathcal{A}^{0,1}(S^2) \).

**Exercise 10.4.** Show \( \mathcal{D}_{-2j-1} = \bar{\partial}^* \) as operators from \( \mathcal{A}^{0,1}(S^2, H^{2j}) \) to \( \mathcal{A}^{0,0}(S^2, H^{2j}) \).

\(^5\)We could have written \( \bar{\partial}(2j) \) to denote the extended operator, but we suppress the index to reduce notational clutter.
We summarize these calculations as follows.

**Proposition 10.6.** The Dirac operator on the twisted spinor bundle $S \otimes L^{-2j-1}$ equals the sum of the Dolbeault operator and its adjoint on the twisted bundle $N^{0,•}S^2 \otimes H^{2j}$.

**Proof.** The formulae (10.16) and (10.17), plus the previous exercise, establish that $\mathcal{D}_{-2j-1} = \bar{\partial} + \partial^*$, thus

$$\mathcal{D}_{-2j-1} = \bar{\partial} + \partial^*$$

on the module $\mathcal{A}^{0,•}(S^2, H^{2j})$.

The kernel of the Dirac operator $\mathcal{D}_{-2j-1}$ thus coincides with $\ker \bar{\partial} \oplus \ker \partial^*$. The second summand is zero since $\ker \mathcal{D}_{-2j-1} = 0$. Thus $\ker \mathcal{D}_{-2j-1} = \ker \bar{\partial}$ consists of sections $\psi_S(\zeta, \bar{\zeta}) \sigma_1^{2j} = f(\zeta, \bar{\zeta}) \sigma_1^{2j} \in \mathcal{A}^{0,0}(S^2, H^{2j})$ for which $\partial f / \partial \zeta = 0$; these are precisely the holomorphic sections $f(\zeta) \sigma_1^{2j} \in \mathcal{O}(H^{2j})$. Since

$$\psi_S(\zeta, \bar{\zeta}) \sigma_1^{2j} = f(\zeta) \sigma_1^{2j} = f(\zeta) Q_1(\zeta)^{-j} \sigma_1^{2j}$$

it follows from (10.7) that $f(\zeta)$ is a polynomial of degree at most $2j$. That is to say, we have shown that $\dim \mathcal{O}(H^{2j}) = 2j + 1$. By suppressing the factors $Q_1(\zeta)^{-j}$ in (10.13) and (10.14), we arrive at the following representation $\pi_j$ of $SU(2)$ on $\mathcal{O}(H^{2j})$, that is by construction equivalent to $\rho_{-2j-1}$:

$$\pi_j \left( \begin{array}{ccc} \alpha & \beta \\ -\beta & \bar{\alpha} \end{array} \right) [f(\zeta)] = (-\beta \zeta + \bar{\alpha})^{2j} f \left( \frac{\alpha \zeta + \bar{\beta}}{-\beta \zeta + \bar{\alpha}} \right),$$

and on the orthogonal basis $\{ \xi_k := \zeta^k \sigma_1^{2j} : k = 0, 1, \ldots, 2j \}$ for $\mathcal{O}(H^{2j})$, we find that

$$\pi_j \left( \begin{array}{ccc} \alpha & \beta \\ -\beta & \bar{\alpha} \end{array} \right) \xi_k = (\alpha \zeta + \bar{\beta})^k (-\beta \zeta + \bar{\alpha})^{2j-k} \sigma_1^{2j}.$$

This completes the passage from the Dirac-operator-kernel construction of the irreducible representations of $SU(2)$ to their realization on finite-dimensional spaces of holomorphic sections of line bundles over $S^2$. The content of the foregoing construction is the **Borel–Weil theorem** for the compact group $SU(2)$, that we may now state as follows.

**Theorem 10.7.** Every irreducible unitary representation of $SU(2)$ can be realized on a space of holomorphic sections of a holomorphic Hermitian line bundle over $S^2$. This line bundle is determined, up to equivalence, by a character $\chi$ of a maximal torus of $SU(2)$ modulo the action of the Weyl group $\mathbb{Z}_2$. Moreover, if $E \rightarrow S^2$ is a holomorphic Hermitian line bundle whose Chern class is $m[H]$ with $m > 0$, then $\mathcal{O}(E)$ carries the representation corresponding to the character $\chi_m$ of the maximal torus.

This result is of course well known; what we have done is to show how it arises from the equivariant index of the Dirac operator on the flag manifold $S^2$. We remark that our construction is equivalent to the standard induced representation recipe for each $\pi_j$, though this is usually achieved by studying the complexification of the compact Lie group in question [56].
A  Calculus on manifolds

In this Appendix, we briefly review the concepts and notations of calculus on manifolds, with emphasis on the algebraic formulae which arise in differential geometry. Proofs are left to the reader. General references are the books of Abraham, Marsden and Ratiu [1], Crampin and Pirani [20], Singer and Thorpe [51], and Spivak [52].

A.1 Differential manifolds

Definition A.1. A differential manifold of finite dimension n is a paracompact Hausdorff topological space M together with a family (or “atlas”) of local charts \{(U_j, \phi_j) : j \in J\} such that \(\mathcal{U} := \{U_j : j \in J\}\) is a locally finite open covering of M, \(\phi_j : U_j \rightarrow \mathbb{R}^n\) is a homeomorphism onto an open subset of \(\mathbb{R}^n\), and the transition functions

\[
\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \rightarrow \phi_i(U_i \cap U_j)
\]

are smooth. A second atlas \{(V_k, \psi_k) : k \in K\} is declared equivalent to the first if every \(\phi_i \circ \psi_k^{-1}\) is smooth (the “differentiable structure” of M is actually an equivalence class of atlases). If M is compact, a finite atlas may be chosen.

If \(n = 2m\) is even, we can regard the chart maps \(\phi_j\) as having images in \(\mathbb{C}^m\). We say that M is a complex manifold if the transition functions are holomorphic maps between open subsets of \(\mathbb{C}^m\).

Definition A.2. If M, N are two differential manifolds, a continuous map \(f : M \rightarrow N\) is smooth if for any pair of local charts \((U, \phi)\) for M and \((V, \psi)\) for N, the composite map \(\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \rightarrow \psi(V)\) is smooth. When \(N = \mathbb{R}\), the set of all smooth functions on M is a commutative algebra over \(\mathbb{R}\), which we denote by \(C^\infty(M, \mathbb{R})\); when \(N = \mathbb{C}\), the smooth complex-valued functions on M forms a commutative \(\mathbb{C}\)-algebra, \(C^\infty(M, \mathbb{C})\). We often write simply \(C^\infty(M)\), if it is clear from the context whether real-valued or complex-valued functions are to be used.

A diffeomorphism between M and N is a bijective smooth function \(f : M \rightarrow N\) whose inverse \(f^{-1} : N \rightarrow M\) is also smooth. If such an \(f\) exists, we say that M and N are diffeomorphic.

If \((U_j, \phi_j)\) is a local chart for M, we define \(x^1, \ldots, x^n \in C^\infty(U_j, \mathbb{R})\) by \(x^k := \text{pr}_k \circ \phi_j\), where \(\text{pr}_k : \mathbb{R}^n \rightarrow \mathbb{R}\) is the k-th coordinate projector. We say \((x^1, \ldots, x^n)\) is a system of local coordinates for M on the chart domain \(U_j\).

The following two lemmas show that smooth functions are abundant.

Lemma A.1. If M is a differential manifold, and if \(\overline{V} \subset W\), there exists \(f \in C^\infty(M, \mathbb{R})\) such that \(\text{supp } f \subset W\), \(f \equiv 1\) on V, and \(0 \leq f \leq 1\) on \(W \setminus \overline{V}\).

Lemma A.2. If M is a differential manifold, with atlas \\{\((U_j, \phi_j) : j \in J\)\}, there exists a smooth partition of unity subordinate to the locally finite covering \(\mathcal{U}\), that is, a family \(\{f_j : j \in J\} \subset C^\infty(M, \mathbb{R})\) with \(0 \leq f_j \leq 1\) and \(\text{supp } f_j \subset U_j\) for each \(j\), such that \(\sum_{j \in J} f_j(x) = 1\) for all \(x \in M\) (the sum is finite for each \(x\)).
If $M$, $N$ are two manifolds with respective atlases $\{(U_j, \phi_j)\}$, $\{(V_k, \psi_k)\}$, the product manifold is the cartesian product $M \times N$ with atlas $\{(U_j \times V_k, \phi_j \times \psi_k)\}$; its dimension is $\dim(M \times N) = \dim M + \dim N$.

### A.2 Tangent spaces

**Definition A.3.** Let $M$ be a differential manifold, $x \in M$. Let $C^\infty(M, x)$ be the set of all smooth functions $f: V \to \mathbb{R}$ whose domain is an open neighbourhood of $x$ in $M$; this is a commutative algebra that includes $C^\infty(M)$. Indeed, $C^\infty(M) = \bigcap_{x \in M} C^\infty(M, x)$.

A tangent vector at $x$ is an $\mathbb{R}$-linear map $v: C^\infty(M, x) \to \mathbb{R}$ which satisfies the “local Leibniz rule”:

$$v(fg) = v(f)g(x) + f(x)v(g), \quad \text{for all } f, g \in C^\infty(M, x).$$

These form a real vector space $T_xM$.

If $(U_j, \phi_j)$ is a local chart for $M$, with local coordinates $(x^1, \ldots, x^n)$, then the directional derivatives \( \frac{\partial}{\partial x^i}|_x : f \mapsto D_j(f \circ \phi^{-1})(\phi(x)) \) form a basis for $T_xM$; in particular, $\dim T_xM = n$.

If $\gamma: I \to M$ is a smooth curve, whose domain is an interval $I \subseteq \mathbb{R}$, with $\gamma(t_0) = x$, its velocity vector at $x$ is $\gamma'(t_0) \in T_xM$ defined by $\gamma'(t_0)(f) := (f \circ \gamma)'(t_0)$.

**Definition A.4.** If $f: M \to N$ is smooth, and $x \in M$, the tangent mapping $T_xf : T_xM \to T_{f(x)}N$ is the $\mathbb{R}$-linear map

$$T_xf(v) : h \mapsto v(h \circ f),$$

for $v \in T_xM$, $h \in C^\infty(N, f(x))$. If $(y^1, \ldots, y^n)$ is a system of local coordinates near $f(x) \in N$, the matrix of $T_xf$ is has entries $\frac{\partial f^k}{\partial x^j}|_x := \frac{\partial}{\partial y^j}|_x (y^k \circ f)$.

**Definition A.5.** A smooth mapping $f: M \to N$ is an immersion if for all $x \in M$, the tangent map $T_xf : T_xM \to T_{f(x)}N$ is injective. If each $T_xf$ is surjective, $f$ is a submersion.

If $f$ is both an immersion and a submersion, then $\dim M = \dim N$, and the Jacobian matrix of $T_xf$ (in local coordinates) is invertible, for each $x$; by the inverse function theorem, $f$ is a diffeomorphism between a neighbourhood of $x$ and a neighbourhood of $f(x)$, for each $x$; we say $f$ is a local diffeomorphism. However, $f$ need not be injective or surjective on all of $M$, so it need not be a global diffeomorphism.

### A.3 Vector fields

**Definition A.6.** Let $M$ be a differential manifold. A vector field on $M$ is a $\mathbb{R}$-linear operator $X: C^\infty(M) \to C^\infty(M)$, which is a derivation of this algebra, that is, it satisfies the Leibniz rule:

$$X(fg) = X(f)g + f(Xg), \quad \text{for all } f, g \in C^\infty(M).$$

These derivations form a vector space denoted by $\mathfrak{X}(M)$. This is in fact a module for the algebra $C^\infty(M)$, if we define $fX \in \mathfrak{X}(M)$ by $fX(h) := f(Xh)$.

\[1\text{There is a notational difficulty here, since expressions like } fXh, \text{ while not ambiguous, are confusing. For this reason, the derivation action of the vector field } X \text{ on the function } h \text{ is sometimes written } X \cdot h \text{ rather than } Xh; \text{ then the module structure can be defined by } (fX) \cdot h := f(X \cdot h), \text{ and so forth.}\]
If $Y \in \mathfrak{X}(M)$ and $x \in M$, the recipe $Y_x f := (Yf)(x)$ defines a tangent vector $Y_x \in T_x M$.

If $(U, \phi)$ is a local chart of $M$, any vector field $X \in \mathfrak{X}(M)$ determines a vector field $X|_U \in \mathfrak{X}(U)$ by restriction. If $(x^1, \ldots, x^n)$ is the local coordinate system for this chart, we obtain $n$ linearly independent local vector fields $\frac{\partial}{\partial x^j} \in \mathfrak{X}(U)$ by writing

$$\frac{\partial}{\partial x^j}(f) := D_j(f \circ \phi^{-1}) \circ \phi.$$ 

These form a basis for the $C^\infty(U)$-module $\mathfrak{X}(U)$, that is, every $X \in \mathfrak{X}(U)$ is of the form $X = \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}$, with $a^1, \ldots, a^n \in C^\infty(U)$.

**Definition A.7.** The **Lie bracket** of two vector fields $X, Y \in \mathfrak{X}(M)$ is defined as

$$[X, Y] : f \mapsto X(Yf) - Y(Xf).$$

It is easy to check that this is a derivation of $C^\infty(M)$; it is clearly skewsymmetric, and it satisfies the **Jacobi identity**:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (A.1)$$

Thus $\mathfrak{X}(M)$ is an (infinite-dimensional) Lie algebra.

**Definition A.8.** If $\tau : M \to N$ is a diffeomorphism, and $X \in \mathfrak{X}(M)$, we define the **pushout** $\tau_* X \in \mathfrak{X}(N)$ by:

$$\tau_* X(h) := X(h \circ \tau) \circ \tau^{-1} \quad \text{for all} \quad h \in C^\infty(N).$$

For each $x \in M$, we find that $(\tau_* X)_{\tau(x)} = T_x \tau(\dot{x})$. Note that this last formula makes sense if $\tau : M \to N$ is smooth and surjective, but not necessary invertible.

**Lemma A.3.** The pushout $\tau_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is a Lie algebra homomorphism, i.e., $\tau_*$ is linear and $\tau_* [X, Y] = [\tau_* X, \tau_* Y]$ for $X, Y \in \mathfrak{X}(M)$. If $\sigma : N \to R$ is a diffeomorphism, then $(\sigma \circ \tau)_* = \sigma_* \circ \tau_*$. \hfill \Box

**Definition A.9.** An **integral curve** of a vector field $X \in \mathfrak{X}(M)$ is a smooth curve $\gamma : I \to M$ such that $\gamma'(t) = X_{\gamma(t)}$ for all $t \in I$.

One can always find a unique integral curve $\gamma_x$ for $X$ satisfying $\gamma_x(0) = x$ in some maximal interval $I_x \ni 0$, by the existence and uniqueness theorem for first-order ordinary differential equations. We say the vector field $X$ is **complete** if $I_x = \mathbb{R}$ for all $x \in M$; if $M$ is compact, every vector field is complete. Write $\phi_t(x) := \gamma_x(t)$; then $\phi_t : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$, and $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s$.

The one-parameter group of diffeomorphisms $\{\phi_t\}$ is called the **flow** generated by the vector field $X$. The vector field may be recovered from the flow by noticing that

$$X f = \lim_{t \to 0} \frac{f \circ \phi_t - f}{t} = \frac{d}{dt} \bigg|_{t=0} (f \circ \phi_t).$$

137
### A.4 Lie groups

**Definition A.10.** A Lie group is a differential manifold $G$ which is also a group, for which the multiplication $(g, h) \mapsto gh : G \times G \to G$ and the inversion $g \mapsto g^{-1} : G \to G$ are smooth maps.

The left translations $\lambda_g : h \mapsto gh$ and the right translations $\rho_g : h \mapsto hg$ are diffeomorphisms from $G$ onto $G$. We usually write $e$ to denote the identity element of $G$.

A finite-dimensional (real or complex) vector space $V$ is an additive Lie group. The group $\text{GL}_R(V)$ of invertible linear operators on $V$ is a Lie group (since it is a dense open subset of the vector space $\text{End}_R(V)$); if $V$ is a complex vector space, $\text{GL}_C(V)$ is a Lie group. We write $\text{GL}(n, \mathbb{R}) := \text{GL}_R(\mathbb{R}^n)$ and $\text{GL}(m, \mathbb{C}) := \text{GL}_C(\mathbb{C}^m)$.

**Definition A.11.** A vector field $X \in \mathfrak{X}(G)$ on a Lie group $G$ is left-invariant if $(\lambda_g)_* X = X$ for all $g \in G$. In that case, $X$ is determined by its value at the identity, $X_e \in T_e G$.

A Lie algebra is a real vector space with a skewsymmetric bilinear operation $[\cdot, \cdot]$ satisfying the Jacobi identity (A.1). Since the Lie bracket $[X, Y]$ of two left-invariant vector fields $X, Y$ is also left-invariant, $T_e G$ becomes a (finite-dimensional) Lie algebra by defining $[X_e, Y_e] := [X, Y]_e$. We usually write $\mathfrak{g} := T_e G$ to denote this Lie algebra.

**Definition A.12.** If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, and if $X \in \mathfrak{g}$, let $\gamma_X$ be the integral curve of the corresponding left-invariant vector field such that $\gamma_X(0) = e$. Then $\gamma_X(s + t) = \gamma_X(s) \gamma_X(t)$ for all $s, t \in \mathbb{R}$, so that $t \mapsto \gamma_X(t)$ is a one-parameter subgroup of $G$; also $\gamma_X(1) = \gamma_X(t)$ for $t \in \mathbb{R}$.

We write $\exp X := \gamma_X(1)$. This defines the exponential map $\exp : \mathfrak{g} \to G$, which satisfies $\exp t X = \gamma_X(t)$, and thus $t \mapsto \exp t X$ is a homomorphism.

Unless $G$ is abelian, $\exp$ is not a homomorphism of the additive group $\mathfrak{g}$ into $G$. However, there is the important Campbell–Baker–Hausdorff formula:

$$\exp tX \exp tY = \exp(t(X + Y) + \frac{1}{2} t^2 [X, Y] + O(t^3)),$$

and its corollary:

$$\exp tX \exp tY \exp(-tX) = \exp(tY + t^2 [X, Y] + O(t^3)). \quad (A.2)$$

If $G$ is a closed subgroup of $\text{GL}_R(V)$, the Lie algebra can be identified with the subspace of operators $\{ X \in \text{End}_R(V) : \exp tX \in G \text{ for all } t \in \mathbb{R} \}$, and the Lie bracket becomes $[X, Y] = XY - YX \in \text{End}_R(V)$. If $W$ is a complex vector space, the Lie algebra of a subgroup of $\text{GL}_C(W)$ is likewise identified to a subspace of $\text{End}_C(W)$.

**Definition A.13.** A smooth left action of a Lie group $G$ on a differential manifold $M$ is a smooth map $\Phi : G \times M \to M : (g, x) \mapsto \Phi(g, x) \equiv g \cdot x$, such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$, for $g, h \in G$, $x \in M$.

Thus $\Phi_g : x \mapsto g \cdot x$ is a diffeomorphism of $M$ for each $g \in G$.

If $x \in M$, the isotropy subgroup for $x$ is $G_x := \{ h \in G : h \cdot x = x \}$, and the orbit of $x$ is $G \cdot x := \{ g \cdot x \in M : g \in G \} \subseteq M$. The natural bijection $g \cdot x \mapsto gG_x$ between $G \cdot x$ and the left-coset space $G/G_x$ is a diffeomorphism when $G/G_x$ is given a natural differential structure for which the quotient map $\eta : G \to G/G_x$ is a submersion.
We say that the action of $G$ on $M$ is free if no group element except $e$ leaves any point fixed; in that case, all isotropy groups are trivial and all orbits are diffeomorphic to $G$.

We say that that the action of $G$ on $M$ is transitive if there is only one orbit. If so, and if $H$ is the isotropy subgroup of some point of $M$, then $M$ is diffeomorphic to $G/H$ (and the action of $G$ on $M$ corresponds to permutation of the left-cosets $gH \mapsto gg'H$). A manifold with a transitive $G$-action is called a homogeneous space for the group $G$.

**Definition A.14.** A smooth right action of a Lie group $G$ on a differential manifold $M$ is similarly defined, as a smooth map $M \times G \to M: (x,g) \mapsto x \cdot g$ such that $x \cdot e = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$, for $g, h \in G, x \in M$.

If $H$ is a closed subgroup of $G$, then $H$ acts (on the right) on $G$ by right translations $g \mapsto gh$; the orbits of this action are the cosets $gH$.

**Definition A.15.** The adjoint action of a Lie group $G$ on its Lie algebra $\mathfrak{g}$ is the map $(g,X) \mapsto \operatorname{Ad}(g)X$ given by

\[ \operatorname{Ad}(g)X := \frac{d}{dt} \bigg|_{t=0} g(\exp tX)g^{-1}. \]

It can be deduced from (A.2) that $\operatorname{Ad}(g)X \in \mathfrak{g}$. The map $g \mapsto \operatorname{Ad}(g)$ is a homomorphism $\operatorname{Ad}: G \to GL(\mathfrak{g})$.

**Definition A.16.** A representation of a Lie group $G$ on a vector space $V$ is a homomorphism $\rho: G \to GL(V)$ for which $(g,v) \mapsto \rho(g)v$ is a smooth map from $G \times V$ to $V$.

The derived representation of its Lie algebra $\mathfrak{g}$ is the linear map $\dot{\rho}: \mathfrak{g} \to \text{End}(V)$ given by

\[ \dot{\rho}(X)v := \frac{d}{dt} \bigg|_{t=0} \rho(\exp tX)v. \]

In particular, the derived representation $\text{ad}$ of the adjoint representation $\operatorname{Ad}$ is given, on account of (A.2), as $\text{ad}(X)Y = [X,Y]$.

### A.5 Fibre bundles

**Definition A.17.** A fibre bundle is a triple $(E,M,\pi)$, more usually written as $E \xrightarrow{\pi} M$, where $E$ and $M$ are differential manifolds (called, respectively, the total space and the base space), and $\pi: E \to B$ is a surjective submersion,\(^2\) such that each fibre $E_x := \pi^{-1} \{x\}$ is diffeomorphic to a fixed manifold $F$, called the “typical fibre”, and which is locally trivial in the following sense. If $U = \{ U_j : j \in J \}$ is a covering of the base space by chart domains, there are diffeomorphisms $\psi_j: \pi^{-1}(U_j) \to U_j \times F$ (called “local trivializations”) such that $\pi(\psi_j^{-1}(x,v)) = x$ for all $x \in U_j, v \in F$.

\(^2\)We shall often omit explicit mention of the submersion $\pi$ and simply write $E \longrightarrow M$ to denote the fibre bundle.
Definition A.18. If $E \overset{\pi}{\longrightarrow} M$ and $E' \overset{\pi'}{\longrightarrow} M'$ are two fibre bundles, a **bundle morphism** is a pair of smooth maps $(\tau, \sigma)$, with $\tau: E \to E'$ and $\sigma: M \to M'$, such that $\pi' \circ \tau = \sigma \circ \pi$. (We also say that $\tau$ is a lifting of the map $\sigma$ between the base spaces.)

A **bundle equivalence** is a bundle morphism $(\tau, \sigma)$ such that both $\tau$ and $\sigma$ are diffeomorphisms.

A fibre bundle $E \to M$ is **trivial** if it is equivalent to the product bundle $M \times F \overset{pr_1}{\to} M$ via a bundle morphism $(\tau, \text{id}_M)$. Thus (A.3) says that any fibre bundle is locally a trivial bundle.

Definition A.19. A **vector bundle** is a fibre bundle $E \overset{\pi}{\longrightarrow} M$ whose typical fibre is a (real or complex) vector space $V$, and whose fibres $E_x$ are vector spaces of the same dimension, such that the maps $V \to E_x : v \mapsto \psi^{-1}_x(x,v)$ are linear isomorphisms.

The dimension $\dim V$ is called the **rank** of the vector bundle.

The **tangent bundle** $TM \to M$ of a manifold $M$ has total space $TM := \{(x,v) : v \in T_xM \}$, with $\pi(x,v) := x$. The local trivialization $\psi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^n$ is given by $\psi_j(x,v) := (x;\, v^1, \ldots, v^n)$, with $v = \sum_k v^k \frac{\partial}{\partial x^k}|_x$. The atlas $\{(\pi^{-1}(U_j), (\phi_j \circ \text{id}) \circ \psi_j) : j \in J\}$ makes $TM$ a $2n$-dimensional manifold. The fibre at $x \in M$ is the tangent space $T_xM$.

The **cotangent bundle** $T^*M \to M$ is formed similarly; its fibres are the dual spaces $T^*_xM := (T_xM)^\ast$. We define $\{dx^1|_x, \ldots, dx^n|_x\}$ as the dual basis in $T^*_xM$ to the basis $\{\frac{\partial}{\partial x^1}|_x, \ldots, \frac{\partial}{\partial x^n}|_x\}$ of $T_xM$, and if $\xi = \sum_k \xi_k \, dx^k|_x$, then $\psi_j(x,\xi) := (x;\, \xi_1, \ldots, \xi_n)$.

Definition A.20. A smooth **section** of a fibre bundle $E \overset{\pi}{\longrightarrow} M$ is a smooth map $s: M \to E$ such that $\pi \circ s = \text{id}_M$, i.e., $s(x) \in E_x$ for each $x \in M$. We denote the totality of smooth sections by $\Gamma(M, E)$, or simply by $\Gamma(E)$ if the base space $M$ is understood; it is a $C^\infty(M)$-**module**, where the action of $C^\infty(M)$ is just scalar multiplication on each fibre:

$$(fs)(x) := f(x) \, s(x),$$

for $s \in \Gamma(E)$, $f \in C^\infty(M)$.

If $U \subset M$ is open, a smooth map $s: U \to \pi^{-1}(U)$ satisfying $\pi(s(x)) = x$ for $x \in U$ is called a **local section** of $E \overset{\pi}{\longrightarrow} M$; all such maps form a vector space $\Gamma(U, E)$, which is a module over $C^\infty(U)$.

From the definition, it is easy to see that a smooth section of the tangent bundle $TM \to M$ can be written in local coordinates on a chart domain $U_j$ as $X = \sum_{k=1}^n a^k \frac{\partial}{\partial x^k}$, with each $a^k \in C^\infty(U)$. Thus $X$ is nothing other than a vector field on $M$; we have $\Gamma(TM) = \mathfrak{X}(M)$ as $C^\infty(M)$-modules.

A section of a trivial fibre bundle $M \times F \overset{pr_1}{\longrightarrow} M$ is of the form $s(x) = (x, f(x))$ where $f: M \to F$ is a smooth map. Thus sections of more general bundles can be thought of as “functions” which take values in different sets at each point of their domains.
A.6 Tensors and differential forms

Definition A.21. A differential 1-form on a differential manifold $M$ is a map $\alpha : \mathfrak{X}(M) \to C^\infty(M)$ that is $C^\infty(M)$-linear, that is:

$$\alpha(X + Y) = \alpha(X) + \alpha(Y), \quad \alpha(fX) = f \alpha(X),$$

for $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. These form a real vector space $\mathcal{A}^1(M)$, which becomes a $C^\infty(M)$-module on defining $f \alpha : X \mapsto f \alpha(X)$.

If $x \in M$, then $\alpha(Y)(x) = f \alpha(Y)(x)$ if $f$ is any smooth function with $f(x) = 1$, whose support is an (arbitrary small) neighbourhood of $x$. Thus $\alpha(Y)(x)$ depends only on $Y_x$, and so $Y_x \mapsto \alpha(Y)(x)$ is an element $\alpha_x$ of the dual space $T^*_x M$ of $T_x M$; by definition, $\alpha(Y)(x) = \alpha_x(Y_x)$.

Hence $\alpha$ can be identified with the section $x \mapsto \alpha_x$ of the cotangent bundle $T^*M \rightarrow M$; and $\Gamma(T^*M) = \mathcal{A}^1(M)$ as $C^\infty(M)$-modules. In local coordinates over $U$, we can write $\alpha = \sum_{k=1}^n f_k dx^k$ with each $f_k \in C^\infty(U)$.

Definition A.22. A tensor of bidegree $(p, q)$ on a manifold $M$ is a multilinear map

$$T : \mathfrak{X}(M)^p \times \mathcal{A}^1(M)^q \to C^\infty(M)$$

such that $T(X_1, \ldots, X_p, \alpha^1, \ldots, \alpha^q)$ is $C^\infty(M)$-linear in each $X_j$ and each $\alpha^k$. The tensor is called covariant if $q = 0$, or contravariant if $p = 0$. Any such tensor $T$ defines a smooth section of a vector bundle over $M$ whose fibre at $x \in M$ is $(T^*_x M)^{\otimes p} \otimes (T_x M)^{\otimes q}$.

Definition A.23. A Riemannian metric on $M$ is a tensor $g$ of bidegree $(2, 0)$ that is symmetric, i.e., $g(X, Y) = g(Y, X)$ for all $X, Y \in \mathfrak{X}(M)$, and positive definite: $g(X, X) > 0$ for nonzero $X \in \mathfrak{X}(M)$. Locally, we may take $g = g_{ij} dx^i \cdot dx^j$, where $[g_{ij}]$ is a positive-definite symmetric matrix of elements of $C^\infty(U)$ and $dx^i \cdot dx^j := \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$; a Riemannian metric may be defined globally on $M$ by taking $g = \sum_j f_j g^{(j)}$, where $g^{(j)}$ is a metric on the chart domain $U_j$, and the functions $f_j$ form a smooth partition of unity on $M$.

The pair $(M, g)$, consisting of a differential manifold with a Riemannian metric $g$, is called a Riemannian manifold.

A Hermitian metric on $M$ is a tensor $h$ of bidegree $(2, 0)$ with values in $C^\infty(M, \mathbb{C})$, such that $h(X, Y) = \overline{h(Y, X)}$ for $X, Y \in \mathfrak{X}(M)$ and $h$ is positive definite. One often writes $(X | Y)$ instead of $h(X, Y)$. The same partition-of-unity argument shows that any manifold can be given a Hermitian metric.

Definition A.24. A differential $k$-form on $M$ is a covariant tensor $\omega : \mathfrak{X}(M)^k \to C^\infty(M)$ that is alternating, which means that $\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = (-1)^\sigma \omega(X_1, \ldots, X_k)$ for $\sigma \in S_k$.

The totality of $k$-forms on $M$ is denoted $\mathcal{A}^k(M)$, and is a $C^\infty(M)$-module.

Let $\Lambda^k T^* M \rightarrow M$ be the vector bundle whose fibre at $x$ is $\Lambda^k T^*_x M$, the $k$-th exterior power of $T^*_x M$; then $\mathcal{A}^k(M) = \Gamma(\Lambda^k T^* M)$.

The direct sum $\mathcal{A}^\bullet(M) := \bigoplus_{k=0}^n \mathcal{A}^k(M) = \Gamma(\Lambda^\bullet T^* M)$ is a $\mathbb{Z}$-graded $C^\infty(M)$-module. The zero-degree term is $\mathcal{A}^0(M) := C^\infty(M)$. Under the exterior product, $\mathcal{A}^\bullet(M)$ is an algebra;
in fact, it is $\mathbb{Z}_2$-graded by the parity of the degree $k$, and is a supercommutative superalgebra, since this property holds in each fibre of $\Lambda^\bullet T^* M \to M$. Thus $\omega \wedge \eta = (-1)^{k\|\omega\|\eta} \wedge \omega$ where $\|\omega\| = k$ for $\omega \in \mathcal{A}^k(M)$.

Thus if $\omega \in \mathcal{A}^k(M)$, $\eta \in \mathcal{A}^l(M)$, and $X_1, \ldots, X_{k+l} \in \mathfrak{X}(M)$, then
\[
(\omega \wedge \eta)(X_1, \ldots, X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{\|\sigma\|} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}),
\]
and in particular, $(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$ for $\alpha, \beta \in \mathcal{A}^1(M)$. Moreover, if $\alpha^1, \ldots, \alpha^k \in \mathcal{A}^1(M)$ then $\alpha^1 \wedge \cdots \wedge \alpha^k \in \mathcal{A}^k(M)$, with
\[
(\alpha^1 \wedge \cdots \wedge \alpha^k)(X_1, \ldots, X_k) = \det[\alpha^i(X_j)].
\]

In local coordinates, an element of $\mathcal{A}^k(U)$ is of the form $\omega = \sum_{|J|=k} f_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}$, where $J = \{j_1, \ldots, j_k\} \subseteq \{1, \ldots, n\}$.

### A.7 Calculus of differential forms

**Definition A.25.** If $\tau : M \to N$ is a diffeomorphism, and $\omega \in \mathcal{A}^k(N)$, we define the pullback $\tau^* \omega \in \mathcal{A}^k(M)$ by:
\[
\tau^* \omega(X_1, \ldots, X_k) := \omega(\tau_* X_1, \ldots, \tau_* X_k) \circ \tau.
\]

In particular, $\tau^* \beta(X) := \beta(\tau_* X) \circ \tau$ for $\beta \in \mathcal{A}^1(N)$. Thus $(\tau^* \beta)_x(X) := \beta_{\tau(x)}(T_{\tau(x)} \tau(X))$, so that the linear map $\beta_{\tau(x)} \to (\tau^* \beta)_x : T^*_{\tau(x)} N \to T^*_x M$ is the transpose of the tangent map $T_{\tau(x)} : T_x M \to T_{\tau(x)} N$.

If $\tau : M \to N$ is any smooth map, not necessarily a diffeomorphism, the pullback $\tau^* \omega$ of $\omega \in \mathcal{A}^k(N)$ is likewise defined by transposition:
\[
(\tau^* \omega)_x((X_1)_x, \ldots, (X_k)_x) := \omega_{\tau(x)}(T_{\tau(x)}((X_1)_x), \ldots, T_{\tau(x)}((X_k)_x)).
\]

For $k = 0$, we get simply: $\tau^* f := f \circ \tau$.

**Lemma A.4.** The pullback $\tau^* : \mathcal{A}^\bullet(N) \to \mathcal{A}^\bullet(M)$ is a degree-preserving homomorphism of exterior algebras, that is, $\tau^*$ is linear and $\tau^* (\omega \wedge \eta) = \tau^* \omega \wedge \tau^* \eta$ for $\omega, \eta \in \mathcal{A}^\bullet(N)$. If $\sigma : N \to R$ is a smooth map, then $(\sigma \circ \tau)^* = \tau^* \circ \sigma^*$.

**Definition A.26.** The **contraction** of a $k$-form $\omega \in \mathcal{A}^k(M)$ with a vector field $X \in \mathfrak{X}(M)$ is the $(k-1)$-form $\iota(X)\omega \equiv \iota_X \omega$ defined as
\[
\iota_X \omega(X_1, \ldots, X_{k-1}) := \omega(X, X_1, \ldots, X_{k-1}).
\]

For $f \in \mathcal{A}^0(M)$, we set $\iota_X f := 0$. If $\alpha \in \mathcal{A}^1(M)$, then $\iota_X \alpha = \alpha(X) \in C^\infty(M)$.
Lemma A.5. Contraction with $X \in \mathfrak{X}(M)$ is an odd derivation of the graded algebra $\mathcal{A}^\bullet(M)$, that is,

$$\iota_X(\omega \wedge \eta) = \iota_X \omega \wedge \eta + (-1)^{\text{deg} \omega} \omega \wedge \iota_X \eta.$$  

In particular,

$$\iota_X(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{j=1}^k (-1)^{j-1} \iota_X^j(\omega) (\alpha_1 \wedge \cdots \wedge \hat{\alpha_j} \wedge \cdots \wedge \alpha_k)$$

for $\alpha_1, \ldots, \alpha_k \in \mathcal{A}^1(M)$.

Proposition A.6. There is a unique operator $d: \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$, called exterior derivation, such that:

1. $d$ is an odd derivation of degree $+1$, that is, $d(\mathcal{A}^k(M)) \subset \mathcal{A}^{k+1}(M)$, and

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\text{deg} \omega} \omega \wedge d\eta;$$

2. $df(X) = Xf$ for $f \in \mathcal{A}^0(M) = C^\infty(M)$;

3. $d^2 \equiv d \circ d = 0$;

4. $d$ is natural with respect to restrictions, that is, if $U \subset M$ is open and $\omega \in \mathcal{A}^\bullet(U)$, then

$$d(\omega|_U) = (d\omega)|_U.$$  

In local coordinates, $d(\sum_j f_j \, dx^j \wedge \cdots \wedge dx^k) = \sum_j df_j \wedge dx^j \wedge \cdots \wedge dx^k$, where, for $f \in C^\infty(U)$, $df = \sum_j \left(\frac{\partial f}{\partial x^j}\right) dx^j$.

Lemma A.7. If $\omega \in \mathcal{A}^k(M)$, its exterior derivative $d\omega \in \mathcal{A}^{k+1}(M)$ is given by

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} X_j(\omega(X_1, \cdots, \hat{X_j}, \cdots, X_{k+1}))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X_j}, \cdots, X_{k+1}]).$$

In particular, if $\alpha \in \mathcal{A}^1(M)$, then $d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$.

Lemma A.8. The exterior derivative commutes with pullbacks: if $\tau: M \to N$ is a smooth map, and $\omega \in \mathcal{A}^k(N)$, then $d(\tau^*\omega) = \tau^*(d\omega)$ in $\mathcal{A}^{k+1}(M)$.

Definition A.27. If $X \in \mathfrak{X}(M)$ is a complete vector field, whose flow is $\{\phi_t\}$, the Lie derivative of a differential form $\omega \in \mathcal{A}^\bullet(M)$ is the form $\mathcal{L}_X \omega$ defined by

$$\mathcal{L}_X \omega := \frac{d}{dt} \bigg|_{t=0} \phi_t^* \omega = \lim_{t \to 0} \frac{\phi_t^* \omega - \omega}{t}. \quad (A.4)$$

Notice that $\mathcal{L}_X f = Xf$ for $f \in \mathcal{A}^0(M)$.  

143
If $R$ is a contravariant tensor, we interpret $\phi_t^* R$ as the pushout $\phi^{-t} R$ by the inverse diffeomorphism $\phi^{-t} = \phi_t^{-1}$; with this convention, (A.4) makes sense when $\omega$ is any tensor, and $\mathcal{L}_X \omega$ is a tensor of the same bidegree. In particular, if $Y$ is a vector field,

$$\mathcal{L}_X Y := \left. \frac{d}{dt} \right|_{t=0} \phi_t^{-t} Y = \left. \frac{d}{dt} \right|_{t=0} Y(\bullet \circ \phi^{-t}) \circ \phi = [X,Y].$$

**Lemma A.9.** The Lie derivative $\mathcal{L}_X : \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ is an $\mathbb{R}$-linear map satisfying:

1. $\mathcal{L}_X$ is an even derivation of degree 0, that is, $\mathcal{L}_X(\mathcal{A}^k(M)) \subset \mathcal{A}^k(M)$, and

$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta; \quad (A.5)$$

2. in particular, $\mathcal{L}_X(f \omega) = (Xf) \omega + f \mathcal{L}_X \omega$;

3. $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$; for all $\omega, \eta \in \mathcal{A}^\bullet(M)$, $f \in \mathcal{C}^{\infty}(M)$.

**Lemma A.10.** The Lie derivative $\mathcal{L}_X : \mathcal{A}^\bullet(M) \to \mathcal{A}^\bullet(M)$ is given by Cartan’s formula:

$$\mathcal{L}_X \omega = \iota_X(d\omega) + d(\iota_X \omega),$$

i.e., $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$, for $X \in \mathfrak{X}(M)$.

**Corollary A.11.** For any $\omega \in \mathcal{A}^k(M)$, $\mathcal{L}_X \omega$ is given by the formula:

$$\mathcal{L}_X \omega(X_1, \ldots, X_k) = X(\omega(X_1, \ldots, X_k)) - \sum_{j=1}^{k} \omega(X_1, \ldots, [X,X_j], \ldots, X_k)).$$

In particular, $\mathcal{L}_X \alpha(Y) = X(\alpha(Y)) - \alpha([X,Y])$ for $\alpha \in \mathcal{A}^1(M)$, $X, Y \in \mathfrak{X}(M)$.

Note that this last formula can be rearranged as: $\mathcal{L}_X(\alpha(Y)) = \mathcal{L}_X \alpha(Y) + \alpha(\mathcal{L}_X Y)$. In other words, the Leibniz rule (A.5) for $\mathcal{L}_X$ is valid not only for exterior products of forms, but also for pairings of forms and vector fields. In passing, we note also that the Jacobi identity (A.1) for vector fields can be written as $\mathcal{L}_X([Y,Z]) = [\mathcal{L}_X Y, Z] + [Y, \mathcal{L}_X Z]$.

**Lemma A.12.** The map $X \mapsto \mathcal{L}_X$ from $\mathfrak{X}(M)$ to $\text{End}(\mathcal{A}^\bullet(M))$ is a Lie algebra homomorphism; that is,

$$\mathcal{L}_{[X,Y]} \omega = \mathcal{L}_X(\mathcal{L}_Y \omega) - \mathcal{L}_Y(\mathcal{L}_X \omega) \quad (A.6)$$

for all $X, Y \in \mathfrak{X}(M)$, $\omega \in \mathcal{A}^\bullet(M)$.

Note that if we substitute a vector field $Z$ for $\omega$ in (A.6), we again get the Jacobi identity.
A.8 The de Rham complex

**Definition A.28.** A cochain complex \((C^\bullet, d)\) is a sequence of abelian groups and homomorphisms

\[
C^0 \xrightarrow{d_0} C^1 \rightarrow \cdots \rightarrow C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} C^{n+2} \rightarrow \cdots
\]
such that \(d_{n+1} \circ d_n = 0\) for all \(n\). (The complex may terminate: it may happen that for some \(N\), \(C^n = 0\) for \(n > N\).) We usually suppress the index of \(d\) and write simply that \(d^2 = 0\) at all stages. The elements of \(C^n\) are called “\(n\)-cochains”: \(c \in C^n\) is an \(n\)-cochain if \(dc = 0\); it is an \(n\)-coboundary if \(c = db\) for some \(b \in C^{n-1}\). The condition \(d^2 = 0\) says that the totality of \(n\)-coboundaries \(B^n(C^\bullet)\) is a subgroup of the group \(Z^n(C^\bullet)\) of \(n\)-cocycles. The quotient group

\[
H^n(C^\bullet) := Z^n(C^\bullet)/B^n(C^\bullet)
\]
is called the \(n\)-th cohomology group of the complex.

A *morphism* between two complexes \((C^\bullet, d)\) and \((K^\bullet, d')\) is a set of homomorphisms \(f_n: C^n \rightarrow K^n\) which intertwines the \(d\)-maps, i.e., \(f_{n+1} \circ d_n = d'_n \circ f_n\) for all \(n\). Thus \(f_n(Z^n(C^\bullet)) \subseteq Z^n(K^\bullet)\) and \(f_n(B^n(C^\bullet)) \subseteq B^n(K^\bullet)\), so that \(f\) induces a homomorphism \(H^n(f): H^n(C^\bullet) \rightarrow H^n(K^\bullet)\).

The components \(C^n\) of a complex may have more structure than that of an abelian group: they could be vector spaces, modules over a commutative ring, etc.³ The cohomology groups \(H^n(C^\bullet, d)\) inherit a similar structure.

**Definition A.29.** The de Rham complex of an \(n\)-dimensional differential manifold \(M\) is the terminating complex

\[
\mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \rightarrow \cdots \rightarrow \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \rightarrow \cdots \xrightarrow{d} \mathcal{A}^n(M)
\]
of \(\mathcal{A}^\infty(M)\)-modules; here \(d\) is the exterior derivation. We say a \(k\)-form \(\omega\) is *closed* if \(d\omega = 0\), and that \(\omega\) is *exact* if \(\omega = d\eta\) for some \((k-1)\)-form \(\eta\); thus \(Z^k_{\text{dR}}(M) := Z^k(\mathcal{A}^\bullet(M), d)\) comprises the closed \(k\)-forms and \(B^k_{\text{dR}}(M) := B^k(\mathcal{A}^\bullet(M), d)\) comprises the exact \(k\)-forms. The \(k\)-th de Rham cohomology group

\[
H^k_{\text{dR}}(M) := H^k(\mathcal{A}^\bullet(M), d)
\]
is a real vector space. We shall denote by \([\omega] \in H^k_{\text{dR}}(M)\) the class of \(\omega \in Z^k_{\text{dR}}(M)\).

By the de Rham theorems —see, for instance, [23]— if \(M\) is compact then \(H^k_{\text{dR}}(M) \cong H^k(M, \mathbb{R})\), where the latter is the \(k\)-th singular cohomology group, which is a finite-dimensional real vector space depending only on the topology of \(M\).

The most important single fact about de Rham cohomology is the following proposition, often called the Poincaré lemma.

**Proposition A.13.** Suppose that \(U\) is a contractible manifold, i.e., for some \(x_0 \in U\), there is a smooth map \(f: [0, 1] \times U \rightarrow U\) such that \(f(0, x) = x_0\) and \(f(1, x) = x\), for \(x \in U\). Then \(H^0_{\text{dR}}(U) = \mathbb{R}\) and \(H^k_{\text{dR}}(U) = 0\) for \(k > 0\).

³In fact, a complex can be formed from objects \(C^n\) and morphisms \(d_n\) in any abelian category.
Proof. A contractible manifold has an atlas with a single chart \((U, \phi)\), so we can suppose that \(U \subseteq \mathbb{R}^n\), that \(U\) is star-shaped about \(x_0\), and that \(f(t, x) = (1 - t)x_0 + tx\). If \(\omega \in \mathcal{A}^k(U)\), let \(\eta := \omega_{\nuU}(f^*\omega) \in \mathcal{A}^{k-1}([0, 1] \times U)\). Now define \(h_k: \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)\) by \(h_k(\omega) := \int_0^1 \eta dt\). Then one can check that the maps \(h_k\) form a “cochain homotopy”, i.e., that \(h_{k+1} \circ d + d \circ h_k = \text{id}\) for each \(k > 0\), and \(h_0 \circ d = \text{id} - \delta_0\). The triviality of the cohomology groups follows at once, since \(\omega = 0\) implies \(\omega = d(h_k\omega)\) for \(k > 0\), and \(df = 0\) implies \(f = f(x_0)\) for \(f \in \mathcal{A}^0(M)\). \(\square\)

A.9 Volume forms and integrals

Definition A.30. A volume form on an \(n\)-dimensional manifold \(M\) is a real \(n\)-form \(\nu \in \mathcal{A}^n(M)\) which is nonvanishing, i.e., \(\nu_x \neq 0\) in \(\Lambda^nT^*_x M\) for all \(x \in M\). A volume form need not exist; we say that \(M\) is orientable if one exists.

We say that two volume forms \(\mu, \nu\) on \(M\) are equivalent if \(\mu = f \nu\) for some \(f \in C^\infty(M)\) with \(f(x) > 0\) for all \(x \in M\). (This is clearly an equivalence relation.) An equivalence class for this relation is called an orientation on \(M\); a pair \((M, \nu)\) consisting of a manifold and a volume form \(\nu\) in a given equivalence class is called an oriented manifold.

In a local coordinate system on a chart domain \(U\), we have \(\nu = h \, dx^1 \wedge \cdots \wedge dx^n\) with \(h \in C^\infty(U, \mathbb{R})\) nonvanishing. Under a change of local coordinates in the overlap of two chart domains, \(h\) is multiplied by a Jacobian factor, \(\det[\partial y^i/\partial x^j]\).

Lemma A.14. A differential manifold \(M\) is orientable if and only if \(M\) has an atlas \(\{(U_j, \phi_j)\}\) all of whose transition functions \(\phi_i \circ \phi_j^{-1}\) have positive Jacobians. \(\square\)

On an oriented manifold \((M, \nu)\), we therefore may and shall always choose an atlas such that in every local coordinate system we have \(\nu = h \, dx^1 \wedge \cdots \wedge dx^n\) with \(h > 0\). (We say that the corresponding charts are “positively oriented”.)

Proposition A.15. Let \((M, \nu)\) be an oriented manifold. Then there is a unique linear form \(\int_M: \mathcal{A}^n(M) \to \mathbb{R}\), called the integral over \(M\), such that if \(\eta \in \mathcal{A}^n(M)\) vanishes outside the domain of a positively oriented chart \((U, \phi)\) with local coordinate system \((x^1, \ldots, x^n)\), and if \(\eta = f \, dx^1 \wedge \cdots \wedge dx^n\), then

\[
\int_M \eta = \int_{\phi(U)} f(x^1, \ldots, x^n) \, dx^1 \cdots dx^n,
\]

where the right hand side is a Lebesgue integral on \(\mathbb{R}^n\).

Proof. Uniqueness of \(\int_M\) follows from the change-of-variables formula for multiple Lebesgue integrals; existence follows by writing \(\eta = \sum_j f_j \eta_j\) where each \(\eta_j\) is supported in a chart domain and \(\{f_j\}\) is a partition of unity. \(\square\)

Lemma A.16. If \((M, \nu)\) and \((N, \rho)\) are two oriented \(n\)-dimensional manifolds and if \(\tau: M \to N\) is an orientation-preserving diffeomorphism (i.e., \(\tau^* \rho\) is equivalent to \(\nu\)), then \(\int_M \tau^* \eta = \int_N \eta\) for all \(\eta \in \mathcal{A}^n(N)\). \(\square\)
An $n$-form on $M$ is closed, since $\mathcal{A}^{n+1}(M) = 0$. If $\eta = d\zeta$ is an exact $n$-form, with compact support in the domain of an oriented chart $(U, \phi)$, then

$$
\int_M \eta = \int_U d\zeta = \int_{\phi(U)} \psi^*(d\zeta) = \int_{\phi(U)} d(\psi^*\zeta),
$$

where $\psi = \phi^{-1} : \phi(U) \to U$. Since $\psi^*\zeta = \sum_j g_j \, dx^1 \wedge \cdots \wedge dx^n$ for some $g_j \in C^\infty(U)$ having compact supports in $U$, we conclude that $d(\psi^*\zeta) = (\sum_j \partial g_j / \partial x^j) \, dx^1 \wedge \cdots \wedge dx^n$; therefore, $\int_U d\zeta = 0$ by the fundamental theorem of calculus. By a partition-of-unity argument, we obtain $\int_M d\zeta = 0$ for any $\zeta \in \mathcal{A}^{n-1}(M)$, so the integral vanishes on exact $n$-forms. (This is the “boundaryless” case of Stokes’ theorem.) In consequence, $[\eta] \mapsto \int_M \eta$ is a well-defined linear form on $H^n_{dR}(M)$. 

147
References


150


